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A NONSTANDARD UNIFORM FUNCTIONAL LIMIT LAW FOR THE INCREMENTS OF THE MULTIVARIATE EMPIRICAL DISTRIBUTION FUNCTION

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A nonstandard uniform functional limit law for the increments of the multivariate empirical distribution function

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Abstract

Let $(Z_i)_{i\geq 1}$ be an independent, identically distributed sequence of random variables on $\mathbb{R}^d$. Under mild conditions on the density of Z_1, we provide a nonstandard uniform functional limit law for the following processes on $[0,1)^d$:

$$\Delta_n(z,h_n,\cdot) := s \mapsto \frac{\sum_{i=1}^{n} 1_{[0,s_1]\times\ldots\times[0,s_d]}\left(\frac{Z_i - z}{h_n^{1/d}}\right)}{c\log n},\ s \in [0,1)^d,$$

along a sequence $(h_n)_{n\geq 1}$ fulfilling $h_n \downarrow 0$, $nh_n \uparrow$, $nh_n/\log c \to c > 0$. Here z ranges through a compact set of $\mathbb{R}^d$. This result is an extension of a theorem of Deheuvels and Mason [5] to the multivariate, non uniform case.

Key words: Empirical processes, Erdös-Rényi law of large numbers, Kernel density estimation.
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1 Introduction and statement of the result

In this paper, we consider an independent, identically distributed sequence of random vectors $(Z_i)_{i\geq 1}$ having a density f on an open set $O \subset \mathbb{R}^d$. We make the following assumption on f:

(Hf) $\quad f$ is continuous and strictly positive on O.

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Throughout this article, $s, s' \in \mathbb{R}^d$, we shall write $s \prec s'$ when $s_i \leq s'_i$ for each $i = 1, \ldots, n$. Intervals and semi intervals are implicitly understood as product of intervals or semi intervals, namely

$$\begin{aligned}[s, s'] :=& \{u \in \mathbb{R}^d,\ s \prec u \prec s'\} \\ =& [s_1, s'_1] \times \ldots \times [s_d, s'_d],\ s = (s_1, \ldots, s_d),\ s' = (s'_1, \ldots, s'_d). \end{aligned} \tag{1.1}$$

We shall also write $a \prec s$ (resp. $s \prec a$) for $s \in \mathbb{R}^d$ and $a \in \mathbb{R}$ when $a \leq s_i$ (resp. $s_i \leq a$) for each $i = 1, \ldots, d$. For fixed $0 < h < 1$ and $z \in O$, we define the following process on $[0, 1)^d$:

$$\mathbf{\Delta}_n(z, h, s) := \frac{1}{n} \sum_{i=1}^{n} 1_{[0,s]} \left(\frac{Z_i - z}{h^{1/d}} \right),\ s \in [0, 1)^d.$$

These processes, usually called functional increments of the empirical distribution function, have been intensively investigated in the literature (see, e.g., Shorack and Wellner [12], Van der Vaart and Wellner [13], Deheuvels and Mason [5,3], Einmahl and Mason [7], Mason [10]). A particular domain of investigation of these increments is when their almost sure behavior is studied along a sequence of bandwidths $(h_n)_{n\geq 1}$ satisfying the following conditions:

$(HVE1)$ $\quad 0 < h_n < 1,\ h_n \downarrow 0,\ nh_n \uparrow \infty,$
$(HVE2)$ $\quad nh_n / \log n \to c.$

Here, $c > 0$ denotes a finite constant. Such conditions on the sequence $(h_n)_{n\geq 1}$ are called Erdös-Rényi conditions, since these two authors have given a pioneering result in this domain (see [8]). Deheuvels and Mason [5] showed that, whenever the $(Z_i)_{i\geq 1}$ are uniformly distributed on $[0, 1]$, and under $(HVE1) - (HVE2)$, the increments $n\mathbf{\Delta}_n(z, h, .)/(c \log n)$ have a nonstandard almost sure behaviour. Before citing their result, we need to introduce the following notations. Set $B([0, 1)^d)$ as the cone of all bounded increasing functions g on $[0, 1)^d$ (implicitly with respect to the order $\prec$), satisfying $g(0) = 0$. We shall endow this cone with the topology spawned by the usual sup-norm $|| g || := \sup_{s\in[0,1)^d} | g(s) |$. Define the usually called Chernoff function h as

$$h(x) := \begin{cases} x \log x - x + 1, & \text{for x>0;} \\ 1, & \text{for x=0;} \\ \infty, & \text{for x<0.} \end{cases} \tag{1.2}$$

That function is known to play an important role in the large deviation of Poisson processes on $[0, 1]$ (see, e.g., [9]). Define the following (rate) function on $B([0, 1)^d)$. Whenever $g \in B([0, 1)^d)$ is absolutely continuous with respect to the Lebesgue measure on $[0,1)^d$, we set

$$I(g) := \int_{[0,1)^d} h(g'(s))ds, \tag{1.3}$$

g' denoting (a version of) the derivative of g with respect to the Lebesgue measure. Whenever g fails to be absolutely continuous, we set $I(g) = \infty$. Also define, for any $a > 0$,

$$\Gamma_a := \left\{ g \in B([0,1)^d),\ I(g) \le 1/a \right\}. \tag{1.4}$$

In a pioneering work, Deheuvels and Mason ([5]) established the following non standard uniform functional limit law for the $\Delta_n(z, h_n, \cdot)$, when the (Z_i) are uniform on $[0,1]$.

Theorem 1 (Deheuvels, Mason, 1992) *Assume that $d = 1$ and that the $(Z_i)_{i\ge1}$ are uniformly distributed on $[0,1]$. Let $0 \le a < b < 1$ be two real numbers, and let $(h_n)_{n\ge1}$ be a sequence of positive constants satisfying $(HVE1) - (HVE2)$ for some constant $c > 0$. Then we have almost surely*

$$\lim_{n\to\infty} \sup_{z\in[0,1-h_n]} \inf_{g\in\Gamma_c} \left|\left| \frac{n}{c\log n} \mathbf{\Delta}_n(z, h_n, \cdot) - g \right|\right| = 0,$$
$$\forall g \in \Gamma_c,\ \lim_{n\to\infty} \inf_{z\in[0,1-h_n]} \left|\left| \frac{n}{c\log n} \mathbf{\Delta}_n(z, h_n, \cdot) - g \right|\right| = 0.$$

As a corollary, the authors showed that, when the sequence of bandwidth $(h_n)_{n\ge1}$ satisfies $(HVE1)-(HVE2)$, the Parzen-Rosenblatt kernel density estimator is **not** uniformly strongly consistent. They proved this non-consistency result by making use of some optimisation techniques on Orlicz balls (see Deheuvels and Mason [4]). The aim of the present paper is to provide a generalisation of the former result to the case where the $(Z_i)_{i\ge1}$ take values in $\mathbb{R}^d$. This generalisation can be stated as follows.

Theorem 2 *Assume that the $(Z_i)_{i\ge1}$ have a density f satisfying (Hf). Let $H \subset O$ be a compact set with nonempty interior. Let $(h_n)_{n\ge1}$ be a sequence of positive constants fulfilling $(HVE1)$ and $(HVE2)$. Then we have almost surely*

$$(i)\ \forall z \in H, \forall g \in \Gamma_{cf(z)}, \lim_{n\to\infty} \inf \left\{ \left|\left| \Delta_n(z', h_n, \cdot) - g \right|\right|,\ z' \in H \right\} = 0, \tag{1.5}$$

$$(ii) \lim_{n\to\infty} \sup_{z\in H} \inf \left\{ \left|\left| \Delta_n(z, h_n, \cdot) - g \right|\right|,\ g \in \Gamma_{cf(z)} \right\} = 0. \tag{1.6}$$

Denote by $f_n(K, z, h_n)$ the usual kernel density estimator with bandwidth h_n and kernel K. A consequence of Theorem 2 is that, under $(HVE1)-(HVE2)$,

$f_n(K,z,h_n)$ is not uniformly consistent (in a strong sense) over (say) an hypercube of $\mathbb{R}^d$.
Corollary: Let K be a kernel with compact support and bounded variation. Assume (Hf) and $(HVE1)-(HVE2)$. Let $H \subset O$ be a compact with nonempty interior. Then the following event holds with probability one:

$$\exists \epsilon > 0,\ \exists n_0,\ \forall n \geq n_0,\ \sup_{z \in H} \mid f_n(K,z,h_n) - f(z) \mid > \epsilon.$$

Proof: The proof follows exactly the lines of Deheuvels and Mason (see [5], Theorem 4.2) and is based on some optimisation results on Orlicz Balls that have been provided in Deheuvels and Mason [4]. □
From now on, we shall make use of the following notation

$$\Delta_n(z,h_n,s) := \frac{\sum_{i=1}^{n} 1_{[0,s]}\left(\frac{Z_i - z}{h_n^{1/d}}\right)}{cf(z)\log n},\ s \in [0,1)^d.$$

Remark 1

Deheuvels and Mason [6] have already given a nonstandard functional limit law for a single increment $\Delta_n(z_0,h_n,\cdot)$ when $(HVE2)$ is replaced by $nh_n/\log\log n \to c > 0$. Their result is presented in a more general setting, considering the $\Delta_n(z_0,h_n,\cdot)$ as random measures indexed by a class of sets.
The remainder of this paper is organised as follows. In §2 we provide some tools in large deviation theory, which are consequences of results of Arcones [1] and Lynch and Sethuraman [9]. In §3, a uniform large deviation principle for "poissonized" versions of the $\Delta_n(z,h_n,\cdot)$ is established. In §4 and §5, we make use of the just-mentioned uniform large deviation principle to prove Theorem 2.

2 Uniform large deviation principles

The main tool we shall make use of in §4 and §5 is a uniform large deviation principle for a triangular array of compound Poisson processes. We must first remind some usual notions in large deviation theory. Let (E,d) be a metric space. A real function $J : E \to [0,\infty]$ is said to be a rate function (implicitly for (E,d)) when the sets $\{x \in E :\ J(x) \leq a\}$, $a \geq 0$, are compact sets of (E,d). We shall first show that I is a rate function on $\Big(B([0,1)^d), ||\cdot||\Big)$ by approximating it by suitably chosen simple rate functions.

Given $g \in B([0,1)^d)$ and a Borel set A, we shall write

$$g(A) := \int_{[0,1)^d} 1_A dg, \tag{2.1}$$

which is valid as soon as either g or 1_A has bounded variation. For any integer $p \geq 1$ and for each $1 \prec \mathbf{i} \prec 2^p$ set

$$A^p_{\mathbf{i}} := 2^{-p} [\mathbf{i} - 1, \mathbf{i}), \tag{2.2}$$

with the notation $\mathbf{i} - 1 := (i_1 - 1, \dots, i_d - 1)$. Recall that h is given in (1.2), and that λ is the Lebesgue measure on $[0,1)^d$. The following functions will play the role of approximations of I (given in (1.3)), as $p \to \infty$:

$$\begin{aligned} I_p(g) &:= \sum_{1 \prec \mathbf{i} \prec 2^p} 2^{-pd} h\left(2^{pd} g(A^p_{\mathbf{i}})\right) \qquad (2.3) \\ &= \sum_{1 \prec \mathbf{i} \prec 2^p} \lambda\left(A^p_{\mathbf{i}}\right) h\left(\frac{g(A^p_{\mathbf{i}})}{\lambda(A^p_{\mathbf{i}})}\right), \; g \in B([0,1)^d). \end{aligned}$$

We point out the following properties of the function I.

Proposition 2.1 *For each $g \in B([0,1)^d)$, we have*

$$\lim_{p \to \infty} I_p(g) = I(g). \tag{2.4}$$

Moreover, I is a rate function on $\left(B([0,1)^d), ||\cdot||\right)$.

Proof: Choose $g \in B([0,1)^d)$ arbitrarily and assume that $I(g) > 0$ (nontrivial case). In a first time, we suppose that g has bounded variation, so that it can be interpreted as a finite measure. Denote by $\mathcal{T}_p$ the σ-algebra of $[0,1)^d$ spawned by the sets $A^p_{\mathbf{i}}$, $1 \prec \mathbf{i} \prec 2^p$. Clearly, for all $p \geq 1$, the measure g is absolutely continuous with respect to the (trace of the) Lebesgue measure λ on $\mathcal{T}_p$. Furthermore, the corresponding Radon-Nicodym derivative is given by the following equality.

$$L_p := \frac{dg}{d\lambda} |_{\mathcal{T}_p} = \sum_{1 \prec \mathbf{i} \prec 2^p} 1_{A^p_{\mathbf{i}}} \frac{g(A^p_{\mathbf{i}})}{\lambda(A^p_{\mathbf{i}})}. \tag{2.5}$$

Clearly the σ-algebra spawned by the (increasing) sequence $(\mathcal{T}_p)_{p \geq 1}$ is equal to the Borel σ-algebra of $[0,1)^d$. Assume first that g is absolutely continuous with respect to λ. According to Dacunha-Castelle and Duflo [2], p. 63, the sequence L_p converges $\lambda + g$ almost everywhere to a positive function L satisfying $L =$

g' ($\lambda+g$ almost everywhere). Now select $0 < l < I(g)$ arbitrarily. By definition of I, there exists $\epsilon > 0$ satisfying

$$\int_{\epsilon<L<1/\epsilon} h(L)d\lambda > l.$$

Since $L_p \to L$ ($\lambda + g$ almost everywhere as $p \to \infty$) and since h is continuous, we have

$$\liminf_{p\to\infty} h(L_p)1_{\{\epsilon<L_p<1/\epsilon\}} \geq h(L)1_{\{\epsilon<L<1/\epsilon\}} \quad \lambda + g \textit{ almost everywhere}$$

Hence by an application of Fatou's lemma,

$$\liminf_{p\to\infty} \int_{\epsilon<L_p<1/\epsilon} h(L_p)d\lambda \geq \int_{\epsilon<L<1/\epsilon} h(L)d\lambda > l.$$

Since $\sup_{p\geq 1} I_p(g) \leq I(g)$ by a straightforward use of Jensen's inequality, and since $l < I(g)$ was chosen arbitrarily, we readily infer that $I_p(g) \to I(g)$ as $p \to \infty$. Now assume that $I(g) = \infty$ and that g is not absolutely continuous with respect to λ. According to Dacunha-Castelle and Duflo [2], p. 63, the sequence L_p converges $\lambda + g$ almost everywhere to a positive function L satisfying $(\lambda + g)(\{L = \infty\}) =: \tau > 0$. Define

$$\ell(x) := x^{-1}h(x) = \log(x) - 1 + x^{-1},\ x > 0.$$

Clearly, $\ell(x) \to \infty$ as $\mid x \mid\to \infty$. Now select $l > 0$ arbitrarily, and choose $A > 0$ satisfying

$$\inf_{x>A} \ell(x) > \frac{2l}{\tau}.$$

Since $L_p \to L$ ($\lambda + g$ almost everywhere as $p \to \infty$) we have $g(L_p > A) > \tau/2$ for all large p, whence

$$\begin{aligned} I_p(g) \geq & \int_{L_p\in(A,\infty)} \ell(L_p)L_p \, d\lambda \\ = & \int_{L_p\in(A,\infty)} \ell(L_p)dg \\ \geq & \frac{2l}{\tau} g(L_p > A) \\ > & l. \end{aligned} \tag{2.6}$$

We have shown that (2.4) is true for each g with bounded variation. Whenever g has infinite variation, then it can be shown that $I_p(g) \to \infty$ by a discrete version of the argument that have just been invoked to obtain (2.6). We omit details for sake of briefness.

Since all the functions I_p are $||\cdot||$-continuous and since $I_p(g) \uparrow I(g)$ for all

$g \in B([0,1)^d)$, we conclude that I is lower-semicontinuous for $||\cdot||$. Hence, I is a rate function if and only if the set Γ_a is totally bounded for each $a > 0$ (recall (1.4)). Since $x^{-1}h(x) \to \infty$ as $|x| \to \infty$, we have, for some constant $M > 0$,

$$|x| \leq |x| 1_{|x|\leq M} + h(x), \tag{2.7}$$

from where we readily infer that

$$\int_{[0,1)^d} |g'| \, d\lambda \leq M + 1/a \text{ for each } a > 0 \text{ and } g \in \Gamma_a. \tag{2.8}$$

Applying the Arzela-Ascoli criterion, we conclude that, for each $a > 0$, the closed set Γ_a is totally bounded, which entails that I is a rate function on $\big(B([0,1)^d), ||\cdot||\big)$. This concludes the proof of Proposition 2.1.□

2.2 Uniform large deviations in $\big(B([0,1)^d), ||\cdot||\big)$

We shall now give a definition of a large uniform large deviation principle in the metric space $\big(B([0,1)^d), ||\cdot||\big)$. In the sequel, $(\epsilon_{n,i})_{n\geq 1, i\leq m_n}$ will always denote a triangular array of positive numbers satisfying $\max_{i\leq m_n} \epsilon_{n,i} \to 0$ as $n \to \infty$. Let $(X_{n,i})_{n\geq 1,\, i\leq m_n}$ be a triangular array of random elements on probability space $(\omega, \mathcal{T}', \mathbb{P})$, taking values in $B([0,1)^d)$. In order to handle carefully the notions of inner and outer probabilities, we shall that each $X_{n,i}$ is a suitable projection mapping from $(\Omega, \mathcal{T}')$ to E, where

$$\Omega := \prod_{n=1}^{\infty} \prod_{i=1}^{p} B([0,1)^d), \quad \mathcal{T}' := \bigotimes_{n=1}^{\infty} \bigotimes_{i=1}^{p} \mathcal{T},$$

and $\mathcal{T}$ is the Borel σ-algebra of $\big(B([0,1)^d), ||\cdot||\big)$. From now on, outer and inner probabilities $\mathbb{P}^*$ and $\mathbb{P}_*$ are understood with $(\Omega, \mathcal{T}')$ as the underlying probability space. We say that $(X_{n,i})_{n\geq 1,\, i\leq m_n}$ satisfies the Uniform Large Deviation Principle (ULDP) for $(\epsilon_{n,i})_{n\geq 1,\, i\leq m_n}$ and for a rate function J whenever the two following conditions hold.

- For any $||\cdot||$-open set $O \subset B([0,1)^d)$ we have

$$\liminf_{n\to\infty} \min_{i\leq m_n} \epsilon_{n,i} \log\left(\mathbb{P}_*\left(X_{n,i}(\cdot) \in O\right)\right) \geq -J(O). \tag{2.9}$$

- For any $||\cdot||$-closed set $F \subset B([0,1)^d)$ we have

$$\limsup_{n\to\infty} \max_{i\leq m_n} \epsilon_{n,i} \log\left(\mathbb{P}^*\left(X_{n,i}(\cdot) \in F\right)\right) \leq -J(F). \tag{2.10}$$

Remark 2

The same definition holds for triangular arrays of random variables taking values in $\mathbb{R}^p,\ p \geq 1$. The norm $||\cdot||$ can then be replaced by any norm.

Arcones [1] provided a powerful tool to establish Large Deviation Principles for sequences of bounded stochastic processes. Some verifications lead to the conclusion that the just-mentioned tool can be used in our context. Recall that the sets $A^p_{\mathbf{i}}$ have been define by (2.2). Consider the following finite grid, for $p \geq 1$:

$$s_{\mathbf{i},p} := 2^{-p}(\mathbf{i}-1),\ 1 \prec \mathbf{i} \prec 2^p. \tag{2.11}$$

Given, $p \geq 1$ and $g \in B([0,1)^d)$, we write

$$g^{(p)} = \sum_{1 \prec \mathbf{i} \prec 2^p} 1_{A^p_{\mathbf{i}}} g(s_{\mathbf{i},p}).$$

Proposition 2.2 *Let $(X_{n,i})_{n\geq 1,\, i\leq m_n}$ be a triangular array of random elements taking values in $(B([0,1)^d))$ almost surely, and let $(\epsilon_{n,i})_{n\geq 1,\, i\leq m_n}$ be a triangular array of positive real numbers. Assume that the following conditions are satisfied.*

(1) The triangular array of stochastic process $(X^{(p)}_{n,i})_{n\geq 1,\, i\leq m_n}$ satisfies the ULDP for $(\epsilon_{n,i})_{n\geq 1,\, i\leq m_n}$ and for the rate function I_p on $\Big(B([0,1)^d), ||\cdot||\Big)$.

(2) For each $\tau > 0$ and $M > 0$ there exists $p \geq 1$ satisfying

$$\limsup_{n\to\infty} \max_{i\leq m_n} \epsilon_{n,i} \log\left(\mathbb{P}^*\left(\max_{1\prec \mathbf{i} \prec 2^p} \sup_{s\in A^p_{\mathbf{i}}} \mid X_{n,i}(t) - X_{n,i}(s^p_{\mathbf{i}}) \mid \geq \tau\right)\right) \leq -M.$$

Then $(X_{n,i})_{n\geq 1,\, i\leq m_n}$ satisfies the ULDP for $(\epsilon_{n,i})_{n\geq 1,\, i\leq m_n}$ and for the following rate function.

$$J(g) := \sup_{p\geq 1} I_p\left(g^p\right),\ g \in B([0,1)^d).$$

Proof: The proof follows exactly the same lines as in the proof of Theorem 3.1 of Arcones [1]. Using theses arguments in our context remains possible since the cone $B([0,1)^d)$ is a closed subset of $L^\infty([0,1)^d)$ for the usual sup norm $||\cdot||$. We avoid writing the proof for sake of briefness. □
Another tool we shall make an intensive use of is a ULDP for random vectors with mutually independent coordinates.

Proposition 2.3 *Let $(X_{n,i})_{n\geq 1,\, 1\leq i\leq m_n}$ and $(Y_{n,i})_{n\geq 1,\, 1\leq i\leq m_n}$ be two triangular arrays of random vectors taking values in $\mathbb{R}^d$ and $\mathbb{R}^{d'}$ respectively, and satisfying $X_{n,i} \perp\!\!\!\perp Y_{n,i}$ for each $n \geq 1,\ 1 \leq i \leq m_n$. Assume that both $(X_{n,i})_{n\geq 1,\, 1\leq i\leq m_n}$ and $(Y_{n,i})_{n\geq 1,\, 1\leq i\leq m_n}$ satisfy the ULDP for a triangular array $(\epsilon_{n,i})_{n\geq 1,\, i\leq m_n}$ and for two rate functions J_1 and J_2 respectively. Then the triangular array $(X_{n,i}, Y_{n,i})_{n\geq 1, i\leq m_n}$ satisfies the ULDP for $(\epsilon_{i,n})_{n\geq 1,\, i\leq m_n}$ and for the following* *rate function.*

$$J(z_1, z_2) := J_1(z_1) + J_2(z_2),\ z_1 \in \mathbb{R}^d,\ z_2 \in \mathbb{R}^{d'}.$$

Proof: The proof follows the same lines as Lemma 2.6 and Corollary 2.9 in Lynch and Sethuraman [9]. In the just-mentioned article, the authors make use of the notions of Weak Large Deviation Principle and of LD-tightness for sequences of random variables in a Polish space. These notions can be easily extended to the frame of triangular arrays of random variables. □
The following proposition is nothing else than the contraction principle in the framework of ULDP (see, e.g., [1], Theorem 2.1 for the most general version of that principle).

Proposition 2.4 *Let $(X_{n,i})_{n\geq 1,\, i\leq m_n}$ be a triangular arrays of $\mathbb{R}^p$ valued random vectors satisfying the ULDP for a triangular array $(\epsilon_{n,i})_{n\geq 1,\, i\leq m_n}$ and for a rate function J. Let $\mathcal{R}$ be a continuous mapping from $\mathbb{R}^d$ to $\left(B([0,1)^d), ||\cdot||\right)$. Then $(\mathcal{R}(X_{n,i}))_{n\geq 1,\, i\leq m_n}$ satisfies the ULDP for $(\epsilon_{n,i})_{n\geq 1,\, i\leq m_n}$ and for the following rate function.*

$$J_{\mathcal{R}}(g) := \inf\{J(x),\ \mathcal{R}(x) = g\},\ g \in B([0,1)^d),$$

with the convention $\inf \emptyset = \infty$.

Proof: Straightforward. □
The following proposition shall be useful in our the proof of our Lemma 4.

Proposition 2.5 *Let $(X_{n,i})_{n\geq 1, i\leq m_n}$ be a triangular array of real random variables and let $(\epsilon_{n,i})_{n\geq 1, i\leq m_n}$ be a triangular array of positive real numbers. Assume that there exists a strictly convex positive function J on $\mathbb{R}$ and a real number μ such that $J(\mu) = 0$ and*

$$\forall a > \mu,\ \lim_{n\to\infty} \max_{i\leq m_n} \left|\epsilon_{n,i} \log\left(\mathbb{P}\left(X_{n,i} \geq a\right)\right) - J(a)\right| = 0, \qquad (2.12)$$

$$\forall a < \mu,\ \lim_{n\to\infty} \max_{i\leq m_n} \left|\epsilon_{n,i} \log\left(\mathbb{P}\left(X_{n,i} \leq a\right)\right) - J(a)\right| = 0. \qquad (2.13)$$

Then $\left(X_{n,i}\right)_{n\geq 1,\, i\leq m_n}$ satisfies the ULDP for $\left(\epsilon_{n,i}\right)_{n\geq 1,\, i\leq m_n}$ and for J.

Proof: The proof is routine calculus.□

3 A ULDP for poissonised versions of the $\Delta_n(z, h_n, \cdot)$

Define the following process, for each integer $n \geq 1$.

$$\Delta\Pi_n(z, h_n, s) := \frac{\sum_{i=1}^{\eta_n} 1_{[0,s]}\left(\frac{Z_i - z}{h_n^{1/d}}\right)}{cf(z)\log n},\ s \in [0,1)^d. \tag{3.1}$$

Here η_n is a Poisson random variable independent of $(Z_i)_{i\geq 1}$, with expectation n. These "poissonized" versions of the processes $\Delta_n(z, h_n, \cdot)$ can be identified to random (Poisson) measures by the following relation

$$\Delta\Pi_n(z, h_n, A) := \int_{[0,1)^d} 1_A(s) d\Delta\Pi_n(z, h_n, s),\ A \text{ Borel}. \tag{3.2}$$

The key of our proof of Theorem 2 is the following ULDP.

Proposition 3.1 *Let $(z_{i,n})_{n\geq 1,\, 1\leq i\leq m_n}$ be a triangular array of elements of H. Under the assumptions of Theorem 2, the triangular array of processes $(\Delta\Pi_n(z_{i,n}, h_n, \cdot))_{n\geq 1,\, 1\leq i\leq m_n}$ satisfies the ULDP in $(B([0,1)^d), ||\cdot||)$ for the rate function I and for the following triangular array*

$$\epsilon_{n,i} := \frac{1}{cf(z_{i,n})\log n},\ n \geq 1,\ 1 \leq i \leq m_n. \tag{3.3}$$

Remark 3

Proposition 3.1 is true whatever the constant $c > 0$ appearing in assumption (HVE1). This remark will show up to be useful in Lemma 6 in §5.
Proof: To prove proposition 3.1, we shall make use of Proposition 2.2. We hence have to check conditions 1, 2 and 3 of the just-mentioned proposition. This will be achieved through several lemmas.

3.1 A preliminary lemma

Recall notation (2.1). To check condition 2 of Proposition 2.2, we need first to establish the following lemma.

Lemma 4 *Assume that the hypothesis of Theorem 2 are satisfied. Then, for each $p \geq 1$ and for each $1 \prec \mathbf{i}_0 \prec 2^p$, the triangular array of random variables $(\Delta\Pi_n(z_{i,n}, h_n, A^p_{\mathbf{i}_0}))_{n\geq 1,\, 1\leq i\leq m_n}$ satisfies the ULDP in $[0,\infty)$ for the triangular array $(\epsilon_{n,i})_{n\geq 1,\, i\leq m_n}$ and for the following rate function:*

$$\widetilde{I}_p(x) := 2^{-pd} h\left(\frac{x}{2^{-pd}}\right) = \lambda\left(A^p_{\mathbf{i}_0}\right) h\left(\frac{x}{\lambda\left(A^p_{\mathbf{i}_0}\right)}\right),\ x \geq 0. \tag{3.4}$$

Proof: Fix once for all $p \geq 1$ and $1 \prec \mathbf{i}_0 \prec 2^d$. We shall make use of Proposition 2.5, with $J := \widetilde{I}_p$ and $\mu := 2^{-pd}$. We give details only for the proof of (2.12), as proving (2.13) is very similar. Fix $a > 2^{-pd}$. For each integers $n \geq 1$ and $1 \leq i \leq m_n$, we set (recall (3.2))

$$\begin{aligned} V_{i,n,\mathbf{i}_0} :=& cf(z_{i,n})(\log n)\Delta\Pi_n(z_{i,n}, h_n, A^p_{\mathbf{i}_0}), \\ p_{i,n,\mathbf{i}_0} :=& \mathbb{P}\left(Z_1 \in z_{i,n} + h_n^{1/d} A^p_{\mathbf{i}_0}\right). \end{aligned}$$

Clearly $V_{i,n,\mathbf{i}_0}$ is a Poisson random variable with expectation $np_{i,n,\mathbf{i}_0}$. Since the density f satisfies (Hf) and since $\lambda(A^p_{\mathbf{i}_0}) = 2^{-pd}$, we have

$$\lim_{n\to\infty} \max_{1\leq i\leq m_n} \left| \frac{p_{i,n,\mathbf{i}_0}}{f(z_{i,n})2^{-pd}h_n} - 1 \right| = 0. \tag{3.5}$$

Hence according to (HVE2) we have, ultimately as $n \to \infty$,

$$\min_{1\leq i\leq m_n} \frac{acf(z_{i,n}) \log n}{np_{i,n,\mathbf{i}_0}} > 1. \tag{3.6}$$

We then make use of Chernoff's inequality for Poisson random variables to get, for all large n (satisfying (3.6)) and for all $1 \leq i \leq m_n$,

$$\begin{aligned} \mathbb{P}\left(\Delta\Pi_n(z_{i,n}, h_n, A^p_{\mathbf{i}_0}) \geq a\right) &= \mathbb{P}\left(V_{i,n,\mathbf{i}_0} \geq acf(z_{i,n}) \log n\right) \\ &\leq \exp\left(-np_{i,n,\mathbf{i}_0} h\left(\frac{acf(z_{i,n}) \log n}{np_{i,n,\mathbf{i}_0}}\right)\right). \end{aligned} \tag{3.7}$$

But (3.7) in combination with (3.5) entails

$$\limsup_{n\to\infty} \max_{1\leq i\leq m_n} \frac{p_{i,n,\mathbf{i}_0}}{f(z_{i,n})h_n} h\left(\frac{acf(z_{i,n}) \log n}{np_{i,n,\mathbf{i}_0}}\right) \leq 2^{-pd} h\left(\frac{a}{2^{-pd}}\right), \tag{3.8}$$

which, together with (3.7) leads to

$$\limsup_{n\to\infty} \max_{1\leq i\leq m_n} \epsilon_{n,i} \log\left(\mathbb{P}\left(\Delta\Pi_n(z_{i,n}, h_n, A^p_{\mathbf{i}_0}) \geq a\right)\right) \leq -\widetilde{I}_p(a). \tag{3.9}$$

Now select $y > a$ arbitrarily. If we could show that

$$\liminf_{n\to\infty} \min_{1\leq i\leq m_n} \epsilon_{n,i} \log\left(\mathbb{P}\left(\Delta\Pi_n(z_{i,n}, h_n, A^p_{\mathbf{i}_0}) \geq a\right)\right) \geq -\widetilde{I}_p(y),$$

then, as $y > a$ was chosen arbitrarily, and since $\widetilde{I}_p$ is increasing on $[a, \infty)$, we should be able to conclude the proof of (2.12)with $J = \widetilde{I}_p$. Now set $\phi(t) := \exp\left(\exp(t) - 1\right),\ t \in \mathbb{R}$ and notice that $h(z) = \max_{u\in\mathbb{R}} zu - \log\left(\phi(u)\right)$ for each $z > 0$. Set $u_0 := \log(2^{pd}y)$, so as

$$h(2^{pd}y) = 2^{pd}yu_0 - \log\left(\phi(u_0)\right). \tag{3.10}$$

Denote by F the distribution function of a Poisson random variable with expectation 1, and define F_0 by

$$dF_0(x) := \phi(u_0)^{-1} \exp(u_0 x) dF(x). \tag{3.11}$$

Let "*" be the convolution operator for infinitely divisible laws and notice that, for each $L > 0$, we have

$$\begin{aligned} dF_0^{*L}(\cdot) =& \phi(u_0)^{-L} \exp(u_0 \cdot) dF^{*L}(\cdot), & (3.12)\\ \mathbb{E}_{F_0^{*L}}(X) =& 2^{pd} L y, & (3.13)\\ \mathrm{Var}_{F_0^{*L}}(X) =& L \mathrm{Var}_{F_0}(X) & (3.14) \end{aligned}$$

Here we have written $\mathbb{E}_F(X)$ as the expectation of a random variable with distribution F. Now fix $\delta > 0$ satisfying $[y-\delta, y+\delta] \subset [a, \infty[$ arbitrarily. Obviously, $F^{*np_{i,n,\mathbf{i}_0}}$ is the distribution function of $cf(z_{i,n})(\log n)\, \Delta\Pi_n(z_{i,n}, h_n, A^p_{\mathbf{i}_0})$, whence

$$\begin{aligned} &\mathbb{P}\left(\Delta\Pi_n(z_{i,n}, h_n, A^p_{\mathbf{i}_0}) \geq a\right)\\ &\geq \mathbb{P}\left(\Delta\Pi_n(z_{i,n}, h_n, A^p_{\mathbf{i}_0}) \in [y-\delta, y+\delta]\right)\\ &= \int_{\frac{x}{cf(z_{i,n})\log n} \in [y-\delta, y+\delta]} dF^{*np_{i,n,\mathbf{i}_0}}(x)\\ &\geq \exp\left(-u_0(y+\delta) cf(z_{i,n}) \log n\right) \times \int_{\frac{x}{cf(z_{i,n})\log n} \in [y-\delta, y+\delta]} \exp(u_0 x) dF^{*np_{i,n,\mathbf{i}_0}}(x)\\ &\geq \exp\left(-cf(z_{i,n})(\log n) u_0(y+\delta) + np_{i,n,\mathbf{i}_0} \log\left(\phi(u_0)\right)\right)\\ &\quad \times \int_{\frac{x}{cf(z_{i,n})\log n} \in [y-\delta, y+\delta]} dF_0^{*np_{i,n,\mathbf{i}_0}}(x) \qquad (3.15)\\ &:= a_{i,n,\mathbf{i}_0,\delta} \times b_{i,n,\mathbf{i}_0,\delta}. \end{aligned}$$

Here (3.15) is a consequence of (3.12), with $L := np_{i,n,\mathbf{i}_0}$. Now let $n \geq 1$ be an integer large enough to fulfill (recall (3.5))

$$\max_{1 \leq i \leq m_n} \left| \frac{np_{i,n,\mathbf{i}_0}}{2^{-pd} cf(z_{i,n}) \log n} - 1 \right| \leq u_0 \log\left(\phi(u_0)\right)^{-1} \delta, \tag{3.16}$$

which enables us to write the following chain of inequalities.

$$\begin{aligned} &cf(z_{i,n})(\log n) u_0(y+\delta) - np_{i,n,\mathbf{i}_0} \log\left(\phi(u_0)\right)\\ &\leq 2^{-pd}(y+\delta) cf(z_{i,n}) \log n \ \left(u_0 2^{pd} - \log\left(\phi(u_0)\right) + u_0\delta\right)\\ &\leq 2^{-pd} cf(z_{i,n}) \log n \left(h\left(2^{pd} y\right) + u_0(2^{pd}+1)\delta\right)\\ &= cf(z_{i,n}) \log n \left(\widetilde{I}_p(y) + 2^{-pd}\left(2^{pd}+1\right) u_0\delta\right)\\ &\leq cf(z_{i,n}) \log n \left(\widetilde{I}_p(y) + 2u_0\delta\right). \qquad (3.17) \end{aligned}$$

Therefore we have, for all large n and for all $1 \leq i \leq m_n$,

$$a_{i,n,\mathbf{i}_0,\delta} \geq \exp\left(-cf(z_{i,n}) \log n \left(\widetilde{I}_p(y) + 2u_0\delta\right)\right), \tag{3.18}$$

where $u_0 = \log(2^{pd}y)$ depends on $y > a$ only. It remains to show that

$$\lim_{n\to\infty} \min_{1\leq i\leq m_n} b_{i,n,\mathbf{i}_0,\delta} = 1. \tag{3.19}$$

Consider n large enough to fulfill (recall (3.5))

$$\frac{y-\delta}{y+2^{-pd}\delta} < \min_{1\leq i\leq m_n} \frac{np_{i,n,\mathbf{i}_0}}{2^{-pd}cf(z_{i,n})\log n} \leq \max_{1\leq i\leq m_n} \frac{np_{i,n,\mathbf{i}_0}}{2^{-pd}cf(z_{i,n})\log n} < \frac{y+\delta}{y-2^{-pd}\delta},$$

so as, for all $1 \leq i \leq m_n$,

$$\frac{np_{i,n,\mathbf{i}_0}}{2^{-pd}cf(z_{i,n})\log n} \times [y - 2^{-pd}\delta, y + 2^{-pd}\delta] \subset]y-\delta, y+\delta[, \tag{3.20}$$

and hence

$$b_{i,n\mathbf{i}_0,\delta} \geq \int\limits_{\frac{x}{np_{i,n,\mathbf{i}_0}} \in [2^{pd}y-\delta, 2^{pd}y+\delta]} dF_0^{*np_{i,n,\mathbf{i}_0}}(x).$$

Recalling (3.13) and (3.14) we get, by the Bienaymé-Tchebychev inequality,

$$1 - b_{i,n,\mathbf{i}_0,\delta} \leq \frac{\mathrm{Var}_{F_0}(X)}{\delta np_{i,n,\mathbf{i}_0}}. \tag{3.21}$$

By assumption (Hf) we infer that the $h_n^{-1}p_{i,n,\mathbf{i}_0}$ are bounded away from zero, from where (3.15) follows. Then (3.15), (3.18) and (3.19) entail

$$\liminf_{n\to\infty} \min_{1\leq i\leq m_n} \epsilon_{n,i} \log\left(\mathbb{P}\left(\Delta\Pi_n(z_{i,n}, h_n, A^p_{\mathbf{i}_0}) \geq a\right)\right) \geq -\widetilde{I}_p(y) - 2u_0\delta. \tag{3.22}$$

Assertion (2.12) is then proved by combining (3.9) with (3.22), as $\delta > 0$ is arbitrary. □

3.2 Verification of condition 2 of Proposition 2.2

For $n \geq 1$ and $1 \leq i \leq m_n$, define the following $\mathbb{R}^{2^{pd}}$ valued random vector:

$$\begin{aligned} X_{n,i} &:= (X_{\mathbf{i}_0,n,i})_{1\prec\mathbf{i}_0\prec 2^p} \\ &:= \left(\Delta\Pi_n(z_{i,n}, h_n, A^p_{\mathbf{i}_0})\right)_{1\prec\mathbf{i}_0\prec 2^p}. \end{aligned}$$

Notice that the random variables $X_{i_0,n,i}$, $1 \prec \mathbf{i}_0 \prec 2^p$ are mutually independent for fixed $n \geq 1$ and $1 \leq i \leq m_n$ by usual properties of Poisson random measures. Hence, by Lemma 4 together with Proposition 2.3 we deduce that the triangular array $(X_{n,i})_{n\geq 1,\, i\leq m_n}$ satisfies the ULDP with $(\epsilon_{n,i})_{n\geq 1,\, i\leq m_n}$ and with the following rate function.

$$I'_p(x) := \sum_{1\prec \mathbf{i}\prec 2^p} 2^{-pd} h\left(\frac{x_{\mathbf{i}}}{2^{-pd}}\right),\ x\in[0,\infty)^{2^{pd}}. \tag{3.23}$$

Here we have written $x := (x_{\mathbf{i}})_{1\prec \mathbf{i}\prec 2^p}$. We now define the following mappings from $[0,\infty)^{2^{pd}}$ to $\Big(B([0,1)^d)\Big)$

$$\begin{aligned}\mathcal{R}_p(x) : [0,1)^d &\mapsto [0,\infty)\\ s &\to \sum_{A^p_{\mathbf{i}}\subset[0,s]} x_{\mathbf{i}}.\end{aligned}$$

Denote by $[x]$ the integer part of a real number x ($[x]\leq x<[x]+1$), and write $[s] := ([s_1],\ldots,[s_d])$ for any $s=(s_1,\ldots,s_d)\in\mathbb{R}^d$. We point out that with probability one (recall the notations of Proposition 2.2)

$$\begin{aligned}\mathcal{R}_p(X_{n,i})(s) =&\Delta\Pi_n\left(z_{i,n}, h_n, 2^{-p}[2^p s]\right)\\ =&\Delta\Pi_n\left(z_{i,n}, h_n, s\right)^{(p)},\ s\in[0,1)^d.\end{aligned}$$

For fixed $p\geq 1$, we make use of the contraction principle (Proposition 2.4) to conclude that $(\mathcal{R}_p(X_{n,i}))_{n\geq 1,\, i\leq m_n}$ satisfies the ULDP for $(\epsilon_{n,i})_{n\geq 1,\, i\leq m_n}$ and for the following rate function.

$$\overline{I}_p(g) := \inf\left\{I'_p(x),\ x\in[0,\infty)^{2^{pd}},\ \mathcal{R}_p(x)=g\right\},\ g\in B([0,1)^d), \tag{3.24}$$

with the convention $\inf\emptyset=\infty$. Obviously, the set appearing in (3.24) is non void if and only if g is the cumulative distribution function of a purely atomic measure with atoms belonging to the grid $\{s_{\mathbf{i},p},\ 1\prec\mathbf{i}\prec 2^p\}$. In that case we have

$$\overline{I}_p(g) = \sum_{1\prec \mathbf{i}\prec 2^p} 2^{-pd} h\left(\frac{g(A^p_{\mathbf{i}})}{2^{-pd}}\right) = I_p(g).$$

Here, we have identified g to a positive finite measure on $[0,1)^d$ (recall (2.1)). Assumption 2 of Proposition 2.2 is then satisfied.

Fix $\tau > 0$ and $M > 0$. We have to prove that, provided that p is large enough,

$$\begin{aligned}
&\limsup_{n\to\infty} \max_{1\le i\le m_n} \epsilon_{n,i} \\
&\log\left(\mathbb{P}\left(\max_{1\prec \mathbf{i}\prec 2^p} \sup_{s\in A_{\mathbf{i}}^p}\left|\Delta\Pi_n\left(z_{i,n},h_n,s\right)-\Delta\Pi_n\left(z_{i,n},h_n,2^{-p}(\mathbf{i}-1)\right)\right|\ge\tau\right)\right)\\
\le& -M. \qquad (3.25)
\end{aligned}$$

For fixed $p\ge 1,\ n\ge 1,\ 1\le i\le m_n$, a rough upper bound gives

$$\begin{aligned}
&\mathbb{P}\left(\max_{1\prec \mathbf{i}\prec 2^p} \sup_{s\in A_{\mathbf{i}}^p}\left|\Delta\Pi_n\left(z_{i,n},h_n,s\right)-\Delta\Pi_n\left(z_{i,n},h_n,2^{-p}(\mathbf{i}-1)\right)\right|\ge\tau\right)\\
\le& 2^{pd}\max_{1\prec \mathbf{i}\prec 2^p}\mathbb{P}\left(\sup_{\substack{2^{-p}(\mathbf{i}-1)\prec s\\ \prec 2^{-p}\mathbf{i}}}\left|\Delta\Pi_n\left(z_{i,n},h_n,s\right)-\Delta\Pi_n\left(z_{i,n},h_n,2^{-p}(\mathbf{i}-1)\right)\right|\ge\tau\right)\\
\le& \mathbb{P}\left(\Delta\Pi_n\left(z_{i,n},h_n,2^{-p}\mathbf{i}\right)-\Delta\Pi_n\left(z_{i,n},h_n,2^{-p}(\mathbf{i}-1)\right)\ge\tau\right)\\
=:& \mathbb{P}_{i,n,\mathbf{i},p}. \qquad (3.26)
\end{aligned}$$

We shall now write

$$\begin{aligned}
W_{i,n,\mathbf{i},p} :=& cf(z_{i,n})\log n\left(\Delta\Pi_n\left(z_{i,n},h_n,2^{-p}\mathbf{i}\right)-\Delta\Pi_n\left(z_{i,n},h_n,2^{-p}(\mathbf{i}-1)\right)\right),\\
\mu_{i,n,\mathbf{i},p} :=& \mathbb{P}\left(\frac{Z_1-z_{i,n}}{h_n^{1/d}}\in[0,2^{-p}\mathbf{i})-[0,2^{-p}(\mathbf{i}-))\right),\ \text{and}\\
\nu_{\mathbf{i},p} :=& \lambda\left([0,2^{-p}\mathbf{i})-[0,2^{-p}(\mathbf{i}-))\right)\le d2^{-p}. \qquad (3.27)
\end{aligned}$$

Clearly, $W_{i,n,\mathbf{i},p}$ is a Poisson random variable with expectation $n\mu_{i,n\mathbf{i},p}$. Moreover, by assumption (Hf) we have

$$\lim_{n\to\infty}\min_{\substack{1\le i\le m_n,\\ 1\prec\mathbf{i}\prec 2^p}}\frac{cf(z_{i,n})(\log n)\nu_{\mathbf{i},p}}{n\mu_{i,n,\mathbf{i},p}}=1. \qquad (3.28)$$

Recall that $x^{-1}h(x)\to\infty$ as $x\to\infty$. We can then choose $A_{M,\tau}>1$ large enough to satisfy

$$\inf_{x\ge A_{M,\tau}}\frac{h(x)}{x}>\frac{8M}{\tau}. \qquad (3.29)$$

By (3.27) we can choose p large enough to fulfill

$$\min_{1\prec\mathbf{i}\prec 2^p}\frac{\tau}{2\nu_{\mathbf{i},p}}>A_{\tau,M}. \qquad (3.30)$$

Assertion (3.28) together with (3.30) leads to the following inequality, for all large n, for all $1 \leq i \leq m_n$ and for all $1 \prec \mathbf{i} \prec 2^p$.

$$\frac{cf(z_{i,n})\tau \log n}{n\mu_{i,n,\mathbf{i},p}} \geq \frac{\tau}{2\nu_{\mathbf{i},p}} > A_{\tau,M} > 1. \tag{3.31}$$

Applying Chernoff's inequality to the Poisson random variables $W_{i,n,\mathbf{i},p}$ we get, for all large n and for all $1 \leq i \leq m_n$,

$$\begin{aligned}\mathbb{P}_{i,n,\mathbf{i},p} &= \mathbb{P}\left(W_{i,n,\mathbf{i},p} \geq \tau cf(z_{i,n}) \log n\right) \\ &\leq \exp\left(-n\mu_{i,n,\mathbf{i},p} h\left(\frac{cf(z_{i,n})\tau \log n}{n\mu_{i,n,\mathbf{i},p}}\right)\right).\end{aligned}$$

Therefore, recalling (3.28) and (3.31), the following inequality holds for all large n, for all $1 \leq i \leq m_n$ and for all $1 \prec \mathbf{i} \prec 2^p$.

$$\begin{aligned}\mathbb{P}_{i,n,\mathbf{i},p} &\leq \exp\left(-\frac{1}{2} cf(z_{i,n})\nu_{\mathbf{i},p}(\log n) h\left(\frac{\tau}{2\nu_{\mathbf{i},p}}\right)\right) \\ &\leq \exp\left(-cf(z_{i,n}) 2M \log n\right). \end{aligned} \tag{3.32}$$

Here, (3.32) is a consequence of (3.30). By combining (3.32) with and (3.26) we get, for all large n and for each $1 \leq i \leq m_n$,

$$\begin{aligned}&\mathbb{P}\left(\max_{1\prec \mathbf{i} \prec 2^p} \sup_{s \in A_{\mathbf{i}}^p} \left|\Delta\Pi_n\left(z_{i,n}, h_n, s\right) - \Delta\Pi_n\left(z_{i,n}, h_n, 2^{-p}(\mathbf{i}-1)\right)\right| \geq \tau\right) \\ &\leq \exp\left(-2Mcf(z_{i,n}) \log n + \log(2^{pd})\right),\end{aligned}$$

which proves (3.25) and shows that condition 3 of Proposition 2.2 is satisfied, as f is bounded away from zero on H. We can now make use of the just-mentioned proposition in combination with Proposition 2.1 to conclude the proof of Proposition 3.1. □

4 Proof of part (i) of Theorem 2

Denote by $\mathrm{Int}(H)$ the interior of H, and fix $z \in \mathrm{Int}(H)$, $g \in \Gamma_{cf(z)}$, and $\epsilon > 0$. We set

$$g^{\epsilon} := \left\{g' \in B([0,1)^d),\ || g' - g || < \epsilon\right\}. \tag{4.1}$$

By lower semi continuity of I in $\left(B([0,1)^d), ||\cdot||\right)$ (recall Proposition 2.1), there exists $\alpha_1 > 0$ satisfying

$$I\left(g^{\epsilon}\right) = \frac{1 - 3\alpha_1}{cf(z)}. \tag{4.2}$$

Now choose an hypercube with nonempty interior $H' := [a_1, b_1] \times \ldots \times [a_p, b_p]$ fulfilling $H' \subset H$, $\mathbb{P}(Z_1 \in H') \leq 1/2$ and

$$\inf_{z' \in H'} \frac{f(z')}{f(z)} > \frac{1 - 2\alpha_1}{1 - \alpha_1}. \tag{4.3}$$

Such a choice is possible since H has a nonempty interior by assumption. We now divide H' into disjoint hypercubes $z_{i,n} + h_n^{1/d}[0,1)^d$, $1 \leq i \leq m_n$, where m_n is the maximal number of disjoint hypercubes we can construct without violating

$$\bigcup_{i=1}^{m_n} \left\{ z_{i,n} + h_n^{1/d}[0,1)^d \right\} \subset H'. \tag{4.4}$$

Notice that, as $n \to \infty$,

$$m_n = h_n^{-1+o(1)} = n^{(1+o(1))}. \tag{4.5}$$

Now recall (3.1). By making use of a well-known "poissonization" technique (see, e.g., Mason [11], Fact 6), we get the following upper bound for all large n.

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{z' \in H} \{ \Delta_n(z', h_n, \cdot) \notin g^\epsilon \} \right) &\leq \mathbb{P}\left(\bigcap_{i=1}^{m_n} \{ \Delta_n(z_{i,n}, h_n, \cdot) \notin g^\epsilon \} \right) \\
&\leq 2\mathbb{P}\left(\bigcap_{i=1}^{m_n} \{ \Delta\Pi_n(z_{i,n}, h_n, \cdot) \notin g^\epsilon \} \right) \qquad (4.6) \\
&= 2 \prod_{i=1}^{m_n} \left(1 - \mathbb{P}\left(\Delta\Pi_n(z_{i,n} h_n, \cdot) \in g^\epsilon \right)\right) \qquad (4.7) \\
&\leq 2 \exp\left(-m_n \min_{1 \leq i \leq m_n} \mathbb{P}\left(\Delta\Pi_n(z_{i,n}, h_n, \cdot) \in g^\epsilon \right) \right) \\
&\qquad (4.8)
\end{aligned}$$

The transition between (4.6) and (4.7) is a classical property of Poisson random measures, while inequality (4.8) is a consequence of $1 - u \leq \exp(-u)$, $u \geq 0$. We now make use of Proposition 3.1 (with the open ball g^ϵ) to get, for all large n (recall (4.2)),

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{z' \in H} \{ \Delta_n(z', h_n, \cdot) \notin g^\epsilon \} \right) &\leq 2 \exp\left(-m_n \min_{1 \leq i \leq m_n} n^{-\frac{f(z_{i,n})}{f(z)}(1 - 2\alpha_1)} \right) \\
&\leq \exp\left(-n^{\alpha_1} \right),
\end{aligned}$$

which is a consequence of (4.3) and (4.5). Hence we conclude by the Borel-Cantelli lemma that, almost surely,

$$\lim_{n \to \infty} \inf \{ || \, \Delta_n(z', h_n, \cdot) - g \, ||, \; z' \in H \} \leq \epsilon.$$

As $\epsilon > 0$ was chosen arbitrarily, the proof of part (i) of Theorem 2 is concluded for each $z \in \text{Int}(H)$. Now the case where $z \in H$ does not belong to $\text{Int}(H)$ is treated by making use of the following argument: for each $z_1 \in H,\ g_1 \in \Gamma_{cf(z_1)}$ and $\epsilon > 0$, there exists $z_2 \in \mathrm{Int}(H)$ and $g_2 \in \Gamma_{cf(z_2)}$ satisfying $|| g_1 - g_2 ||< \epsilon$. Such an argument is valid by (Hf) and by Lemma 5 (see below).□

5 Proof of part (ii) of Theorem 2

We shall make use of somewhat usual blocking arguments along the following subsequence $n_k := [\exp(k/\log k)]\,,\ k \geq 3$ and its associated blocks $N_k := \{n_{k-1}+1, \ldots, n_k\}$. Given $A \subset B([0,1)^d)$ and $\epsilon > 0$ we shall write

$$A^\epsilon := \left\{ g \in B([0,1)^d),\ \inf_{g' \in A} || g - g' ||< \epsilon \right\}. \tag{5.1}$$

The following lemma shall come in handy.

Lemma 5 *For any $\epsilon > 0$ and $L > 0$ there exists $\eta > 0$ satisfying, for each, $L' \in [(1+\eta)^{-1}L, L]$, $\Gamma_{L'} \subset \Gamma^\epsilon_L$.*

Proof: The proof is routine analysis.□
Now fix $\epsilon > 0$. Since I is lower-semi continuous on $\Big(B([0,1)^d), || \cdot ||\Big)$ (recall Proposition 2.1) we deduce that, given $z \in H$, there exists $\alpha_z > 0$ satisfying

$$I\left(B([0,1)^d) - \Gamma^\epsilon_{cf(z)}\right) = \frac{1+3\alpha_z}{cf(z)}. \tag{5.2}$$

By (Hf) and Lemma 5 we can construct an hypercube H_z with nonempty interior satisfying the following conditions.

$$z \in H_z,\quad H_z \subset O, \tag{5.3}$$
$$\inf_{z_1, z_2 \in H_z} \frac{f(z_1)}{f(z_2)} \geq \frac{1+\alpha_z}{1+2\alpha_z}, \tag{5.4}$$
$$\bigcup_{z' \in H_z} \Gamma_{cf(z')} \subset \Gamma^\epsilon_{cf(z)}, \tag{5.5}$$
$$\mathbb{P}\left(Z_1 \in \bigcup_{z \in H_z} \left\{ z + [0, {h_{n_k}}^{1/d})^d \right\} \right) \leq 1/2. \tag{5.6}$$

The compact set H is included in the union of the interiors of $H_z,\ z \in H$, from where we can extract a finite union, noted as

$$H \subset \bigcup_{l=1}^{L} \mathrm{Int}H_{z_l} \subset \bigcup_{l=1}^{L} H_{z_l} \subset O. \tag{5.7}$$

Our problem is now reduced to showing that, for fixed $l = 1, \dots, L$,

$$\limsup_{n\to\infty} \sup_{z\in H_{z_l}} \inf_{g\in\Gamma_{cf(z_l)}} \| \Delta_n(z, h_n, \cdot) - g \| \leq 10\epsilon \ \textit{almost surely}. \tag{5.8}$$

We now fix $1 \leq l \leq L$, and we write $H_{z_l} =: [a_1, b_1] \times \dots \times [a_d, b_d]$. We now introduce a parameter $\delta > 0$ that will be chosen in function of ϵ in the sequel. For each $k \geq 1$, we cover H_{z_l} by hypercubes

$$H_{z_l} \subset \bigcup_{1\leq i\leq m_{n_k}} C_{i,n_k} \subset O,, \tag{5.9}$$

with

$$\begin{aligned} C_{i,n_k} :=& z_{i,n_k} + [0, (\delta h_{n_k})^{1/d})^d, \ k \geq 1, \ 1 \leq i \leq m_{n_k} \text{ and} \\ m_{n_k} :=& \prod_{p=1}^{d} \left(\left[\frac{b_p - a_p}{(\delta h_{n_k})^{1/d}} \right] + 1 \right). \end{aligned} \tag{5.10}$$

Now define, for each $k \geq 1$, $n \in N_k$, $z \in H$,

$$\mathcal{H}_n(z, s) := \frac{1}{c \log n_k} \sum_{i=1}^{n} 1_{[0,s)} \left(\frac{Z_i - z}{h_{n_k}{}^{1/d}} \right), \ s \in [0, 1)^d.$$

We shall first show that, for any choice $\delta > 0$, we have almost surely

$$\limsup_{n\to\infty} \sup_{1\leq i\leq m_{n_k}} \inf_{g\in\Gamma_{cf(z_l)}} \| \mathcal{H}_n(z_{i,n_k}, \cdot) - g \| \leq 2\epsilon. \tag{5.11}$$

Consider the following probabilities for all large k.

$$\mathbb{P}_k := \mathbb{P} \left(\bigcup_{1\leq i\leq m_{n_k}} \bigcup_{n\in N_k} \mathcal{H}_n(z_{i,n_k}, \cdot) \notin \Gamma^{2\epsilon}_{cf(z_l)} \right).$$

We have, ultimately as $k \to \infty$,

$$\mathbb{P}_k \leq m_k \max_{1\leq i\leq m_{n_k}} \mathbb{P} \left(\bigcup_{n\in N_k} \mathcal{H}_n(z_{i,n_k}, \cdot) \notin \Gamma^{\epsilon}_{cf(z_l)} \right). \tag{5.12}$$

We now make use of a well-known maximal inequality (see, e.g., Deheuvels and Mason [5], Lemma 3.4) to get, for all large k and for all $1 \leq i \leq m_{n_k}$,

$$\mathbb{P} \left(\bigcup_{n\in N_k} \mathcal{H}_n(z_{i,n_k}, \cdot) \notin \Gamma^{2\epsilon}_{cf(z_l)} \right) \leq 2\mathbb{P} \left(\mathcal{H}_{n_k}(z_{i,n_k}, \cdot) \notin \Gamma^{\epsilon}_{cf(z_l)} \right). \tag{5.13}$$

We point out that the conditions of Lemma 3.4 in [5] are satisfied since, by a straightforward use of Markov's inequality we have, ultimately as $k \to \infty$,

$$\sup_{z\in H} \max_{n\in N_k} \mathbb{P} \left(\| \mathcal{H}_{n_k}(z, \cdot) - \mathcal{H}_n(z, \cdot) \| \geq \epsilon \right) \leq \frac{1}{2}.$$

Making use of (5.13) in (5.12), we obtain, for all large k,

$$\begin{aligned}\mathbb{P}_k &\leq 2m_k \max_{1\leq i\leq m_{n_k}} \mathbb{P}\left(\mathcal{H}_{n_k}(z_{i,n_k},\cdot) \notin \Gamma^{\epsilon}_{cf(z_l)}\right)\\ &= 2m_{n_k} \max_{1\leq i\leq m_{n_k}} \mathbb{P}\left(\Delta_{n_k}(z_{i,n_k}, h_{n_k},\cdot) \notin \Gamma^{\epsilon}_{cf(z_l)}\right)\\ &\leq 4m_{n_k} \max_{1\leq i\leq m_{n_k}} \mathbb{P}\left(\Delta\Pi_{n_k}(z_{i,n_k}, h_{n_k},\cdot) \notin \Gamma^{\epsilon}_{cf(z_l)}\right). \qquad (5.14)\end{aligned}$$

The last inequality is a consequence of usual poissonization techniques (see, e.g., Mason [11], Fact 6). We now make use of Proposition 3.1, which, together with (5.2) leads to the following inequality, ultimately as $k \to \infty$,

$$\mathbb{P}_k \leq 4m_{n_k} \max_{1\leq m_k} \exp\left(-\frac{f(z_{i,n_k})}{f(z_l)}(1+2\alpha_{z_l})\log n_k\right).$$

Moreover (5.4) entails $\mathbb{P}_k \leq 4m_{n_k} \exp\left(-(1+\alpha_{z_l})\log n_k\right)$. Since $m_{n_k} = h_{n_k}^{-1+o(1)} = n_k^{1+o(1)}$ as $k \to \infty$ (recall (5.10)), the sumability of $\mathbb{P}_k$ follows, which proves (5.11) by the Borel-Cantelli lemma. We point out that (5.11) is true whatever the choice of $\delta > 0$ (recall (5.9)). We now focus on showing that, for a small value of $\delta > 0$ we have

$$\limsup_{k\to\infty} \sup_{z\in H_{z_l}} \min_{1\leq i\leq m_{n_k}} \max_{n\in N_k} \| \mathcal{H}_n(z_{i,n_k},\cdot) - \Delta_n(z, h_n,\cdot) \| \leq 7\epsilon \ \ a.s, \qquad (5.15)$$

which will be achieved through two separate lemmas.

Lemma 6 *Assume that the conditions of Theorem 2 are fulfilled. There exists* $\delta_\epsilon > 0$ *such that, for any choice of* $0 < \delta < \delta_\epsilon$ *we have almost surely*

$$\limsup_{k\to\infty} \max_{n\in N_k} \max_{1\leq i\leq m_{n_k}} \sup_{z\in C_{i,n_k}} \left\| \mathcal{H}_n(z_{i,n_k},\cdot) - \frac{f(z)}{f(z_{i,n_k})}\mathcal{H}_n(z,\cdot)\right\| \leq \epsilon.$$

Proof: For all large k we have

$$\begin{aligned}&\mathbb{P}\left(\max_{n\in N_k} \max_{1\leq i\leq m_{n_k}} \sup_{z\in C_{i,n_k}} \left\| \mathcal{H}_n(z_{i,n_k},\cdot) - \frac{f(z)}{f(z_{i,n_k})}\mathcal{H}_n(z,\cdot)\right\| > \epsilon\right)\\ =&\mathbb{P}\left(\bigcup_{1\leq i\leq m_{n_k}} \bigcup_{n\in N_k} \sup_{z\in C_{i,n_k}} \left\| \mathcal{H}_n(z_{i,n_k},\cdot) - \frac{f(z)}{f(z_{i,n_k})}\mathcal{H}_n(z,\cdot)\right\| > \epsilon\right)\\ \leq& m_{n_k} \max_{1\leq i\leq m_{n_k}} \mathbb{P}\left(\bigcup_{n\in N_k} \sup_{z\in C_{i,n_k}} \left\| \mathcal{H}_n(z_{i,n_k},\cdot) - \frac{f(z)}{f(z_{i,n_k})}\mathcal{H}_n(z,\cdot)\right\| > \epsilon\right) \qquad (5.16)\end{aligned}$$

Fix $k \geq 1$, $1 \leq i \leq m_{n_k}$ and $z \in z_{i,n_k} + (\delta h_{n_k})^{1/d}[0,1)^d$. We write $z_{i,n_k} := (z^1_{i,n_k},\ldots,z^d_{i,n_k})$, $z := (z^1,\ldots,z^d)$ and $Z_j := (Z^1_j,\ldots,Z^d_j)$, $j \geq 1$. Notice that for each $p = 1,...,d$ we have $z^p_{i,n_k} \leq z^p \leq z^p_{i,n_k} + (\delta h_{n_k})^{1/d}$. Hence, in virtue of the equality $| 1_A - 1_B | = 1_{A-B} + 1_{B-A}$ we have, for each integer j we have almost surely, for each $(s_1, \ldots, s_d) \in [0,1)^d$,

$$\begin{aligned}
&\Big| 1_{[0,s)}\left(\frac{Z_j - z}{h_{n_k}^{1/d}}\right) - 1_{[0,s)}\left(\frac{Z_j - z_{i,n_k}}{h_{n_k}^{1/d}}\right) \Big| \\
=& 1_{\left\{\left[z, z+h_{n_k}^{1/d}s\right) - \left[z_{i,n_k}, z_{i,n_k}+h_{n_k}^{1/d}s\right)\right\}}(Z_j) + 1_{\left\{\left[z_{i,n_k}, z_{i,n_k}+h_{n_k}^{1/d}s\right) - \left[z, z+h_{n_k}^{1/d}s\right)\right\}}(Z_j) \\
\leq& \sum_{l=1}^{d} 1_{\left[z_{i,n_k}^l + s_l h_{n_k}^{1/d},\, z_{i,n_k}^l + h_{n_k}^{1/d}(s_l + \delta^{1/d})\right]}(Z_j^l) \prod_{1 \leq p \neq l \leq d} 1_{\left[z_{i,n_k}^p,\, z_{i,n_k}^p + h_{n_k}^{1/d}(s_p + \delta^{1/d})\right]}(Z_j^p) \\
&+ \sum_{l=1}^{d} 1_{\left[z_{i,n_k}^l,\, z_{i,n_k}^l + (\delta h_{n_k})^{1/d}\right]}(Z_j^l) \prod_{1 \leq p \neq l \leq d} 1_{\left[z_{i,n_k}^p,\, z_{i,n_k}^p + h_{n_k}^{1/d} s_p\right]}(Z_j^p) && (5.17) \\
=:& X_{j,k,i,\delta}(s). && (5.18)
\end{aligned}$$

Here (5.17) follows from $z_{i,n_k}^l \leq z^l \leq z_{i,n_k}^l + \delta^{1/d} h_{n_k}^{1/d}$, $l = 1, \ldots, d$. As the $X_{j,k,i,\delta}(\cdot)$ are positive processes almost surely, (5.18) entails, for all large k and for all $1 \leq i \leq m_{n_k}$,

$$\begin{aligned}
&\mathbb{P}\left(\bigcup_{n \in N_k} \sup_{z \in C_{i,n_k}} \left\| \mathcal{H}_n(z_{i,n_k}, \cdot) - \frac{f(z)}{f(z_{i,n_k})}\mathcal{H}_n(z, \cdot) \right\| > \epsilon\right) \\
\leq& \mathbb{P}\left(\bigcup_{n \in N_k} \sup_{z \in C_{i,n_k}} \sup_{s \in [0,1)^d} \right. \\
&\left. \sum_{j=1}^{n} \Big| 1_{[0,s)}\left(\frac{Z_j - z}{h_{n_k}^{1/d}}\right) - 1_{[0,s)}\left(\frac{Z_j - z_{i,n_k}}{h_{n_k}^{1/d}}\right) \Big| \geq \epsilon c f(z_{i,n_k}) \log n_k \right) \\
\leq& \mathbb{P}\left(\bigcup_{n=1}^{n_k} \sup_{s \in [0,1)^d} \sum_{j=1}^{n} X_{j,k,i,\delta}(s) \geq \epsilon c f(z_{i,n_k}) \log n_k\right) \\
\leq& \mathbb{P}\left(\left\| \sum_{j=1}^{n_k} X_{j,k,i,\delta}(\cdot) \right\| \geq \epsilon c f(z_{i,n_k}) \log n_k\right). && (5.19)
\end{aligned}$$

But a close look at (5.17) leads to the conclusion that, almost surely, for each $s \in [0,1)^d$,

$$\begin{aligned}
0 \leq& \sum_{j=1}^{n_k} X_{j,k,i,\delta}(s) \\
\leq& 2dcf(z_{i,n_k})(\log n_k) \sup_{s,s' \in [0,2)^d,\ ||s'-s||_d < \delta^{1/d}} \Big| \Delta_{n_k}(z_{i,n_k}, h_{n_k}, s') - \Delta_{n_k}(z_{i,n_k}, h_{n_k}, s) \Big|. && (5.20)
\end{aligned}$$

Here we have written $| s |_d := \max\{| s_j |,\ j = 1, \dots, p\}$. Now (5.20) together with (5.19) entails

$$\mathbb{P}\left(\bigcup_{n\in N_k} \sup_{z\in C_{i,n_k}} \left|\left|\mathcal{H}_n(z_{i,n_k},\cdot) - \frac{f(z)}{f(z_{i,n_k})}\mathcal{H}_n(z,\cdot)\right|\right| > \epsilon\right)$$
$$\leq \mathbb{P}\left(\sup_{\substack{s,s'\in[0,2)^d,\\ ||s'-s||_d<\delta^{1/d}}} \left|\Delta_{n_k}(z_{i,n_k}, h_{n_k}, s') - \Delta_{n_k}(z_{i,n_k}, h_{n_k}, s)\right| > \epsilon(2d)^{-1}\right)$$
$$\leq 2\mathbb{P}\left(\sup_{\substack{s,s'\in[0,1)^d,\\ ||s'-s||_d<\delta^{1/d}/2}} \left|\Delta\Pi_{n_k}(z_{i,n_k}, h_{n_k}, 2s') - \Delta\Pi_{n_k}(z_{i,n_k}, h_{n_k}, 2s)\right| > \epsilon(2d)^{-1}\right). \tag{5.21}$$

Here (5.21) follows from poissonization techniques. Now consider the following sequence $\mathfrak{h}_n := 2^d h_n,\ n \geq 1$. Clearly, $(\mathfrak{h}_n)_{n\geq 1}$ satisfies (HVE1) and (HVE2), replacing c by $\mathfrak{c} := 2^d c$. Moreover, for each $k \geq 1$, $1 \leq i \leq m_{n_k}$ we have almost surely, for all $s \in [0, 1)^d$,

$$\Delta\Pi_{n_k}(z_{i,n_k}, h_{n_k}, 2s) = \Delta\Pi_{n_k}(z_{i,n_k}, \mathfrak{h}_{n_k}, s). \tag{5.22}$$

Applying Proposition 3.1 we deduce that the triangular array of processes

$$U_{k,i}(\cdot) := \Delta\Pi_{n_k}(z_{i,n_k}, \mathfrak{h}_{n_k}, 2\cdot),\ k \geq 1,\ 1 \leq i \leq m_{n_k}$$

satisfies the ULDP in $(B([0, 1)^d), ||\cdot||)$ (see §2) for the rate function I and for the following triangular array:

$$\epsilon_{k,i} := (c2^d f(z_{i,n_k}) \log n_k)^{-1} k \geq 1,\ 1 \leq i \leq m_{n_k}.$$

Now consider the following set

$$\Gamma := \left\{g \in \mathcal{M}([0, 1)^d),\ I(g) \leq \frac{4}{2^d c\beta}\right\}.$$

By proposition 2.1, there exists $\delta_\epsilon > 0$ such that

$$\sup_{g\in 2^d\Gamma} \sup_{s,s'\in[0,2)^d, ||s'-s||_d\leq\delta_\epsilon^d/2} | g(s') - g(s) | < (4d)^{-1}\epsilon. \tag{5.23}$$

Now choose $0 < \delta < \delta_\epsilon$ arbitrarily for the construction of the z_{i,n_k}, $k \geq 1, 1 \leq i \leq m_{n_k}$ (recall (5.9)). By lower-semicontinuity of I, the closed set

$$F := \left\{g \in \mathcal{M}([0, 2)^d),\ \inf_{g'\in\Gamma} || g - g' ||_{[0,2)^d} \geq \frac{2^{-d}\epsilon}{8d}\right\}$$

satisfies $I(F) > 4/(2^d c\beta)$. Hence, (5.21) together with (5.23) leads to the following inequalities for all large k and for each $1 \le i \le m_{n_k}$.

$$\begin{aligned}
&\mathbb{P}\left(\bigcup_{n\in N_k}\sup_{z\in C_{i,n_k}}\left\|\mathcal{H}_n(z_{i,n_k},\cdot)-\frac{f(z)}{f(z_{i,n_k})}\mathcal{H}_n(z,\cdot)\right\|>\epsilon\right)\\
\le&2\mathbb{P}\left(\sup_{s,s'\in[0,1)^d,||s'-s||_d<\delta^{1/d}/2}\left|U_{k,i}(s')-U_{k,i}(s)\right|>\epsilon(2d)^{-1}\right)\\
\le&2\mathbb{P}\left(\Delta\Pi_{n_k}(z_{i,n_k},\mathfrak{h}_{n_k},\cdot)\in F\right)\\
\le&2\exp\left(-\frac{3}{4}I\left(F\right)\mathfrak{c}f(z_{i,n_k})\log n_k\right)\\
\le&2\exp\left(-3\times\frac{c2^d f(z_{i,n_k})}{\beta c2^d}\log n_k\right)\\
\le&2\exp\left(-3\log n_k\right). && (5.24)
\end{aligned}$$

Now (5.24) in combination with (5.16) entails, for all large k,

$$\mathbb{P}\left(\max_{n\in N_k}\max_{1\le i\le m_{n_k}}\sup_{z\in C_{i,n_k}}\left\|\mathcal{H}_n(z_{i,n_k},\cdot)-\frac{f(z)}{f(z_{i,n_k})}\mathcal{H}_n(z,\cdot)\right\|>\epsilon\right)\le 2\frac{m_{n_k}}{{n_k}^3}. \tag{5.25}$$

But for fixed $\delta > 0$ we have $m_{n_k} = h_{n_k}^{-1+o(1)} = {n_k}^{1+o(1)}$ as $k \to \infty$. The proof of Lemma 6 is concluded by applying the Borel-Cantelli lemma to (5.25). □

Lemma 7 *Under the assumptions of Theorem 2, for any choice of $\delta > 0$, we have almost surely*

$$\limsup_{k\to\infty}\max_{1\le i\le m_{n_k}}\sup_{z\in C_{i,n_k}}\max_{n\in N_k}\left\|\Delta_n(z,h_n,\cdot)-\frac{f(z)}{f(z_{i,n_k})}\mathcal{H}_n(z,\cdot)\right\|\le 6\epsilon.$$

Proof: For all large k and for all $1 \le i \le m_{n_k}$, $z \in C_{i,n_k}$, $n \in N_k$ we have almost surely, for each $s \in [0,1)^d$,

$$\Delta_n(z,h_n,s)-\frac{f(z)}{f(z_{i,n_k})}\mathcal{H}_n\left(z,s\right)=T_{n,i,k}\frac{f(z)}{f(z_{i,n_k})}\mathcal{H}_n\left(z,\rho_{n,k}s\right)-\frac{f(z)}{f(z_{i,n_k})}\mathcal{H}_n(z,s), \tag{5.26}$$

with $T_{i,n,k} : f(z_{i,n_k})\log n_k/f(e)\log n$ and $\rho^d_{n,k} := h_{n_k}/h_n$. First notice that

$$\lim_{k\to\infty}\max_{1\le i\le m_{n_k}}\sup_{z\in C_{i,n_k}}\mid T_{n,i,k}-1\mid=0,\quad \lim_{k\to\infty}\max_{n\in N_k}\mid\rho_{n,k}-1\mid=0.$$

Moreover, by Proposition 2.1 we have

$$\lim_{T\to1,\rho\to1}\sup_{g\in\Gamma_{cf(z_l)}}||\,Tg(\rho^{1/d}\cdot)-g(\cdot)\,||=0. \tag{5.27}$$

Finally, by (5.11) and by Lemma 6 we have, for all large k and for all $1 \leq i \leq m_{n_k}$, $z \in C_{i,n_k}$, $n \in N_k$,

$$\inf_{g\in\Gamma_{cf(z_l)}} \left|\left|\frac{f(z)}{f(z_{i,n_k})}\mathcal{H}_n(z,\cdot) - g\right|\right| < 3\epsilon \ \ \textit{almost surely.} \qquad (5.28)$$

Hence, combining (5.26), (5.27), (5.27), (5.28) and the triangle inequality, we obtain almost surely, for all large k and for all $n \in N_k$:

$$\left|\left|\Delta_n(z,h_n,\cdot) - \frac{f(z)}{f(z_{i,n_k})}\mathcal{H}_n(z,\cdot)\right|\right|$$
$$\leq 6\epsilon,$$

which proves Lemma 7. □

End of the proof of part(ii) of Theorem 2: By combining Lemma 7 with Lemma 6 we conclude that (5.15) is true for $\delta > 0$ small enough. Now (5.15) together with (5.11) leads to

$$\limsup_{n\to\infty} \sup_{z\in H_{z_l}} \inf_{g\in\Gamma_{cf(z_l)}} || \Delta_n(z,h_n,\cdot) - g || \leq 9\epsilon \ \ \textit{almost surely.}$$

Whence, recalling (5.5),

$$\limsup_{n\to\infty} \sup_{z\in H_{z_l}} \inf_{g\in\Gamma_{cf(z)}} || \Delta_n(z,h_n,\cdot) - g || \leq 10\epsilon \ \ \textit{almost surely.} \qquad (5.29)$$

Repeating (5.29) for each $l = 1,\ldots,L$ (recall (5.7)) we get

$$\limsup_{n\to\infty} \sup_{z\in H} \inf_{g\in\Gamma_{cf(z)}} || \Delta_n(z,h_n,\cdot) - g || \leq 10\epsilon \ \ \textit{almost surely.}$$

As $\epsilon > 0$ was chosen arbitrarily, the proof of part(ii) of Theorem 2 is concluded.□

References

[1] M.A. Arcones. The large deviation principle of stochastic processes, Part 1 . *Theory Probab. Appl.*, 47(4):567–583, 2003.

[2] D. Dacunha-Castelle and M. Duflo. *Probabilités et statistiques II: problèmes à temps mobile.* Masson, 1993.

[3] P. Deheuvels and D.M. Mason. Nonstandard functional laws of the iterated logarithm for tail empirical and quantile processes. *Ann. Probab.*, 18:1693–1722, 1990.

[4] P. Deheuvels and D.M. Mason. A tail empirical process approach to some nostandard laws of the iterated logarithm. *J. Theoret. Probab.*, 4:53–85, 1991.

[5] P. Deheuvels and D.M. Mason. Functional laws of the iterated logarithm for the increments of empirical and quantile processes. *Ann. Probab.*, 20:1248–1287, 1992.

[6] P. Deheuvels and D.M. Mason. Nonstandard local empirical processes indexed by sets. *J. Statist. Plann. Inference*, 45:91–112, 1995.

[7] J. Einmahl and D.M. Mason. Strong limit theorems for weighted quantile processes. *Ann. Probab.*, 16(4):1626, 1988.

[8] P. Erdös and A. Rényi. On a new law of large numbers. *J. Analyse Math.*, 23:103–111, 1970.

[9] J. Lynch and J. Sethuraman. Large deviations for processes with independent increments. *Ann. Probab.*, 15(2):610–627, 1987.

[10] D.M. Mason. A strong invariance principle for the tail empirical process. *Ann. Inst. H. Poincaré Probab. Statist.*, 24:491–506, 1988.

[11] D.M. Mason. A uniform functional law of the iterated logarithm for the local empirical process. *Ann. Probab.*, 32(2):1391–1418, 2004.

[12] G.R. Shorack and J.A. Wellner. *Empirical Processes and applications to statistics.* Springer, 1986.

[13] A.W. Van der Vaart and J.A. Wellner. *Weak convergence and empirical processes.* Springer, 1996.