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Distribution of Desirability Index in Multicriteria Optimization using Desirability Functions based on the Cumulative Distribution Function of the Standard Normal

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1 Introduction

Multiresponse optimization is a common problem in industries. Each response defining the quality of the product, Y_i $(i = 1, 2, \dots, p)$, is assumed to be related to the same set of varying factors, x_j's $(j = 1, 2, \dots, k)$. The objective is to find the factors settings $\mathbf{x} = (x_1, x_2, \dots, x_k)$ that simultaneously optimize the p responses $\mathbf{Y} = (Y_1, Y_2, \dots, Y_k)$.

The quality of a compromise between the responses can be measured by the *desirability* concept (**?**). Using *desirability functions*, the adequacy of each of the p responses, Y_i, are first quantified by a value between 0 and 1 (the higher, the better). Those p values are then aggregated in a *desirability index* providing, for any combination of factors levels, a value between 0 and 1 measuring the desirability of the resulting product quality.

As industries can not face testing all possible combinations of factors levels and measure the quality of resulting products, a model capturing the relationship between each response and factors is assumed over the domain of interest, denoted χ, through an equation of the form

$$Y_i = f_i\left(\mathbf{x}, \boldsymbol{\beta_i}\right) + \epsilon_i \quad \text{with} \quad \epsilon_i \sim N(0, \sigma^2_{\epsilon_i}). \tag{1}$$

We assume that the link function f_i and the model parameters $\boldsymbol{\beta_i}$ are known as well as the error term variance $\sigma^2_{\epsilon_i}$.

The most well-known class of models is the multiple linear regression. The link between the i^{th} response Y_i and transformed factors $\mathbf{z} = g(\mathbf{x})$ is assumed to follow an equation of the form

$$Y_i = \mathbf{z}'\boldsymbol{\beta_i} + \epsilon_i \quad \textit{with} \quad \epsilon_i \sim N(0, \sigma^2_{\epsilon_i}). \tag{2}$$

For given factors settings $\mathbf{x}$, each response $Y_i|\mathbf{x}$ is supposed to be a random variable with known distribution:

$$Y_i|\mathbf{x} \sim N(E\left[Y_i|\mathbf{x}\right], \sigma^2_{\epsilon_i}). \tag{3}$$

The usual way to deal with those p random variables is to optimize the desirability of the expected quality responses, $E[Y_i|\mathbf{x}]$ (**??**). An other treatment of randomness could be, as proposed by **?**, to optimize the expected desirability of responses considering the desirability index as a random variable. This is possible if

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the distribution of the desirability index is known.

This paper discusses the derivation of the desirability index distribution using the density transformation theorem (?). First, desirability functions and desirability indexes are reviewed. A focus is made on desirability functions based on the standard Normal distribution function. Then, the desirability functions distribution is studied and formulas derived in the case of desirability functions based on the standard Normal distribution function. Finally, the question of desirability index distribution is treated. The practical case of unknown models and parameters estimation is also discussed.

2 Desirability functions and desirability index

The concept of *desirability* was introduced by **?** to provide a solution to multiresponse optimization problems. It allows to balance the optimized properties, Y_i's, one against the other, taking into account their target value, their relative importance and their scale.

Harrington proceeds in two steps. First, each response Y_i is transformed to the same scale using a *desirability function*, denoted by d_i, such that $d_i(Y_i) \in [0,1]$. If $d_i(Y_i) = 0$, the product is not at all acceptable according to the specifications of the i^{th} property and if $d_i(Y_i) = 1$, the product fullfields them perfectly. Secondly, the properties transformed by desirability functions are aggregated in a single value still in the $[0,1]$ interval, the *desirability index*, representing the overall desirability of the product. The weighted geometric mean or the weighted arithmetic mean of the desirability functions, as well as their minimum are the three most often used desirability index, denoted by D:

$$D(\mathbf{Y}) = \prod_{i=1}^{p} (d_i(Y_i))^{w_i}, \quad D(\mathbf{Y}) = \sum_{i=1}^{p} w_i \cdot d_i(Y_i) \quad \text{or} \quad D(\mathbf{Y}) = \min d_i(Y_i) \quad \text{with} \sum_{i=1}^{p} w_i = 1 \tag{4}$$

The most well-known desirability functions are the Harrington's ones (1965) based on the exponential function of a linear transformation of the Y_i's and the Derringer and Suich's ones (1980) based on a power of a linear transformation of the Y_i's. **?** proposed also smoother and differentiable desirability functions using the logit function. These three types of desirability functions are presented in Table 1 for the cases where the response Y (the sub-index i has been removed to simplify notations) must be maximized, minimized or reach a target value. Desirability functions depend on parameters (a, b, s or T) that have to be fixed by a specialist of the product to define which responses values are desirable. As it can be seen on Figure **??**, the three desirability functions may provide similar desirability curves.

	Maximum	Minimum	Target Value
Harrington (1965)	$exp(-exp(-a-bY))$	$1-exp(-exp(-a-bY))$	$exp(-\lvert\frac{Y-T}{b}\rvert^n)$
Derringer and Suich (1980)	$\begin{cases} 0 & \text{if } Y<a \\ (\frac{Y-a}{b-a})^s & \text{if } a\leq Y\leq b \\ 1 & \text{if } Y>b \end{cases}$	$\begin{cases} 1 & \text{if } Y<b \\ (\frac{a-Y}{a-b})^s & \text{if } b\leq Y\leq a \\ 0 & \text{if } Y>a \end{cases}$	$\begin{cases} 0 & \text{if } Y<a_1 \\ (\frac{Y-a_1}{T-a_1})^{s_1} & \text{if } a_1\leq Y\leq T \\ (\frac{a_2-Y}{a_2-T})^{s_2} & \text{if } T\leq Y\leq a_2 \\ 0 & \text{if } Y>a_2 \end{cases}$
Gibb *et al* (2001)	$(1+exp(-\frac{Y-a}{b}))^{-1}$	$(1-exp(-\frac{Y-a}{b}))^{-1}$	$exp(-\frac{1}{2}(\frac{Y-T}{b})^2)$

Table 1: Examples of desirability functions. Y is a response; the target value T and the parameters a, b and s have to be adjusted according to the specifications.

We propose a new classe of desirability functions to transform a quality property Y in the $(0,1)$ interval. It is based on the cumulative distribution function, abbreviated *cdf*, of the standard Normal:

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp\left(-\frac{t^2}{2}\right) dt$$

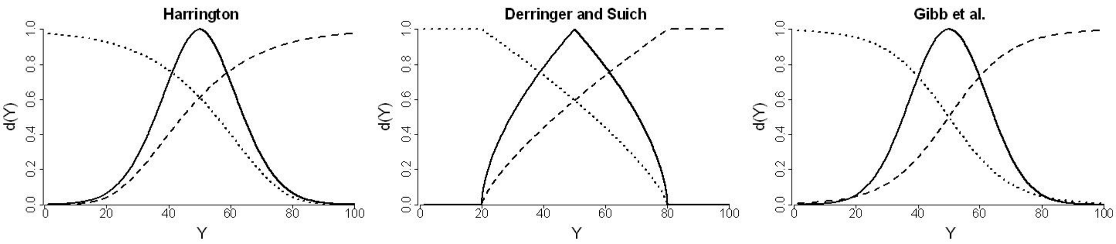

Figure 1: Different desirability functions. The continuous lines represent cases of a targeted property and the dotted and dashed lines represent cases of a minimized and a maximized property respectively.

This function is continuous and differentiable, which will be usefull to apply the density transformation theorem further. According to the desirable values of the property Y, we will use the following transformations as depicted on Figure ??:

$$d(Y) = \Phi\left(\frac{Y-a}{b}\right) \quad \text{if} Y \text{ has to be maximized,} \tag{5}$$

$$d(Y) = 1-\Phi\left(\frac{Y-a}{b}\right) \quad \text{if} Y \text{ has to be minimized,} \tag{6}$$

$$d(Y) = \sqrt{\Phi\left(\frac{Y-a_1}{b_1}\right)\cdot\left[1-\Phi\left(\frac{Y-a_2}{b_2}\right)\right]} \quad \text{if} Y \text{ has to reach some optimal value.} \tag{7}$$

The parameters a and b have to be chosen according to the knowledge of both the statistician and the specialist of the product. They can be fixed using one of the two simple following rules, :

- If the specialist notifies that under the value LL (Lower Limit) and over the value UL (Upper Limit) changes for the property Y have no additional interest, the parameters a and b can be chosen as a function of those limits according to the properties of the standard Normal distribution:

  - If Y has to be maximized, take $a = \frac{LL+UL}{2}$ and $b = \frac{UL-\frac{LL+UL}{2}}{2}$ such that $d(LL) \approx 0.023$ and $d(UL) \approx 0.977$

  - If Y has to be minimized, take $a = \frac{LL+UL}{2}$ and $b = \frac{UL-\frac{LL+UL}{2}}{2}$ such that $d(LL) \approx 0.977$ and $d(UL) \approx 0.023$

  - If Y has to reach some optimal value T, take $a_1 = \frac{LL+T}{2}$, $b_1 = \frac{T-\frac{LL+T}{2}}{2}$, $a_2 = \frac{T+UL}{2}$ and $b_2 = \frac{UL-\frac{T+UL}{2}}{2}$ such that $d(LL) = d(UL) \approx 0.151$ and $d(T) = 0.977$

- If the specialist does not notify any limit for the property Y and just desires to maximize or minimize it, or to reach a target value T, the two parameters a and b can be chosen according to the values of the response Y observed on a small sample of the experimental domain.

  - If Y has to be maximized or minimized, take a and b respectively as the sample average and the sample standard deviation.

  - If Y has to reach some optimal value T, divide the sample into a first subset with smaller values than T and a second subset with higher value than T. Take a_1 and b_1, as the arithmetic mean and the standard deviation of the first subset and a_2 and b_2, as the arithmetic mean and the standard deviation of the second subset respectively.

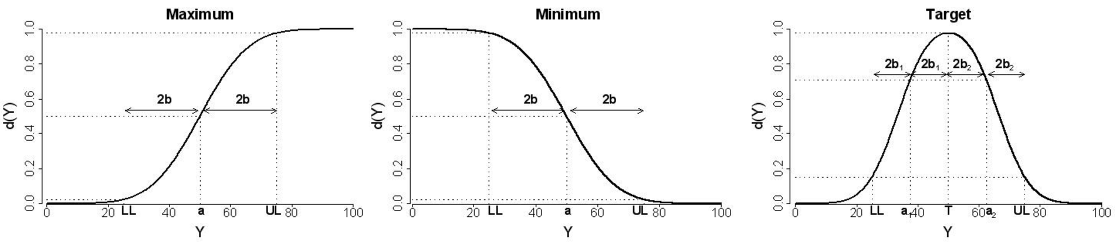

Figure 2: Desirability functions based on the standard Normal *cdf*. The three graphs correspond to the case of maximized property (left), minimized property (center) and targeted property (right). The dashed lines correspond to the limits (LL and USL). Decreasing or increasing parameter a shifts the desirability curves respectively to the left or to the right and decreasing or increasing parameter b makes the curves respectively more or less stiff.

If statistical models of the form (**??**) are assumed, there are two possibilities to associate a desirability value to any factors settings x: applying desirability functions on expected responses or taking the expectation of the desirability index. The first case is the classical use of desirability, $D^C(x)$, as Harrington and Derringer and Suich proposed in their papers and the second case is the new concept introduced by **?**, $D^N(x)$.

If for instance the weighted geometric mean is used as desirability index (**??**) the optimization problem can be formalised as below:

$$\max_{\mathbf{x}\in\chi} D^C(x) = \max_{\mathbf{x}\in\chi} D\left(E\left[\mathbf{Y}|\mathbf{x}\right]\right) = \max_{\mathbf{x}\in\chi} \prod_{i=1}^{p} \left[d_i\left(E[Y_i|\mathbf{x}]\right)\right]^{w_i} = \max_{\mathbf{x}\in\chi} \prod_{i=1}^{p} \left[d_i\left(f_i\left(\mathbf{x}, \boldsymbol{\beta_i}\right)\right)\right]^{w_i} \quad \text{or} \tag{8}$$

$$\max_{\mathbf{x}\in\chi} D^N(x) = \max_{\mathbf{x}\in\chi} E\left[D(\mathbf{Y}|\mathbf{x})\right] = \max_{\mathbf{x}\in\chi} E\left[\prod_{i=1}^{p} \left[d_i\left(Y_i|\mathbf{x}\right)\right]^{w_i}\right] = \max_{\mathbf{x}\in\chi} E\left[\prod_{i=1}^{p} \left[d_i\left(f_i\left(\mathbf{x}, \boldsymbol{\beta_i}\right) + \epsilon_i\right)\right]^{w_i}\right] \tag{9}$$

As the expectation of a random variable function is the function of the random variable expectation if and only if the transformation is linear, most of the time, $D\left(E\left[\mathbf{Y}|\mathbf{x}\right]\right) \neq E\left[D(\mathbf{Y}|\mathbf{x})\right]$ and the corresponding optima, $\mathbf{x}_{opt}^C$ and $\mathbf{x}_{opt}^N$, are different.

The idea of maximizing the expected DI instead of the DI of expected responses is the same as in the utility theory field (**?**). The expected utility function, u, is maximized instead of the utility function of expected results, R, to take risk factors into account: $\max E\left[u(R)\right]$ instead of $\max u\left(E[R]\right)$. Similarly, as it takes into account the propagation of the uncertainty of the response ϵ_i on the desirability index, Steuer proposes to associate to each factors setting $\mathbf{x}$ the average desirability of the resulting product quality, $E\left[D\left(\mathbf{Y}|\mathbf{x}\right)\right]$.

Steuer approximates for each design point $\mathbf{x} \in \chi$ the distribution of $D(\mathbf{Y}|\mathbf{x})$ by Monte-Carlo simulations on the basis of the model error distribution often assumed to be Normal. This is an heavy procedure, especially if there is more than two optimized properties and a huge experimental domain χ to explore.

To avoid intensive use of Monte-Carlo simulations, **?**, suggest to derive analytically the distribution of $D(\mathbf{Y}|\mathbf{x})$ on the basis of the model error distribution using the density transformation theorem. If the probability density function (abbreviated *pdf*) of $D(\mathbf{Y}|\mathbf{x})$ is known, its expectation $E\left[D(\mathbf{Y}|\mathbf{x})\right]$ can be computed by analytical or numerical integration. They derived the analytical expression of the Harrington DFs distribution and deduced, for special cases, the DI distribution. In this paper we derive analytically the distribution of DFs based on the Normal *cdf*.

3 Distribution of desirability functions

In this section, the *pdf* of the DFs $d_i(Y_i|\mathbf{x})$ and of the weighted DFs $[d_i(Y_i|\mathbf{x})]^{w_i}$ are derived analytically in the case of maximized, minimized and targeted properties. For simplicity we remove the i indice standing for the i^{th} property Y_i.

3.1 *Pdf* of the desirability function $d(Y|\mathbf{x})$

Most of the following results are based on the univariate density transformation theorem.

Theorem 1 (Density transformation). *Let Z have a pdf $f_Z(z)$. If $h(z)$ is either increasing or decreasing for all z such that $f_Z(z) > 0$, then $U = h(Z)$ has a pdf given by*

$$f_U(u) = f_Z\big(h^{-1}(u)\big) \cdot \left|\frac{d\left(h^{-1}(u)\right)}{du}\right| \tag{10}$$

Given the assumption that the error term ϵ for each model (**??**) follows a $N(0, \sigma_\epsilon^2)$ distribution, we know that $Y|\mathbf{x} \sim N(E[Y|\mathbf{x}], \sigma_\epsilon^2)$ and $\frac{Y|\mathbf{x}-a}{b} \sim N\left(\frac{E[Y|\mathbf{x}]-a}{b}, \frac{\sigma_\epsilon^2}{b^2}\right)$. As the DFs based on the Normal *cdf* for the maximization and the minimization cases (see equations (**??**-**??**)) are respectively increasing and decreasing functions of $\frac{Y|\mathbf{x}-a}{b}$, we can derive their *pdf*'s using the density transformation theorem.

In the following $E[Y|\mathbf{x}]$ is abbreviated μ_Y, $\phi_{M;S^2}$ and $\Phi_{M;S^2}$ denote respectively the *pdf* and the *cdf* of the Normal with expectation M and variance S^2 and ϕ and Φ without any indice denote respectively the *pdf* and the *cdf* of the standard Normal.

Proposition 1 (DF - maximization). *The pdf of $U \equiv d(Y|\mathbf{x}) = \Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)$ is given by*

$$f_U(u) = \begin{cases} \dfrac{\phi_{\frac{\mu_Y-a}{b};\frac{\sigma_\epsilon^2}{b^2}}\left(\Phi^{-1}(u)\right)}{\phi(\Phi^{-1}(u))} & \text{if } 0 < \text{u} < 1 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Proposition 2 (DF - minimization). *The pdf of $U \equiv d(Y|\mathbf{x}) = 1 - \Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)$ is given by*

$$f_U(u) = \begin{cases} \dfrac{\phi_{\frac{\mu_Y-a}{b};\frac{\sigma_\epsilon^2}{b^2}}\left(\Phi^{-1}(1-u)\right)}{\phi(\Phi^{-1}(1-u))} & \text{if } 0 < \text{u} < 1 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

For both maximization and minimization cases, the DFs *pdf*'s are the ratio of two Normal *pdf*'s and are then easy to compute in practice with any statistical software. The shapes of those two densities are similar and vary in the same way with parameters μ_Y, a, b and σ_ϵ^2 as presented in Figure **??**.

The *pdf* of the DF in the case of a targeted property can not be derived directly using the density transformation theorem (Theorem **??**) as $U \equiv d(Y|\mathbf{x}) = \sqrt{\Phi\left(\frac{Y|\mathbf{x}-a_1}{b_1}\right) \cdot \left[1 - \Phi\left(\frac{Y|\mathbf{x}-a_2}{b_2}\right)\right]}$ does not satisfy the monotonicity assumption. Nevertheless, $d(Y|\mathbf{x})$ is strictly increasing in $Y|\mathbf{x}$ below the value target T and strictly decreasing above the value target T. Thanks to this particular shape, as represented in Figure **??**, realizations of U smaller than a given u correspond either to realizations of $Y|\mathbf{x}$ smaller than a certain $y_1 \in (-\infty, T]$ such that $d(y_1) = u$, or realizations of $Y|\mathbf{x}$ higher than a certain $y_2 \in [T, \infty)$ such that $d(y_2) = u$. This allows to compute first the *cdf* of $d(Y|\mathbf{x})$ and then derive this analytical expression to obtain its *pdf*.

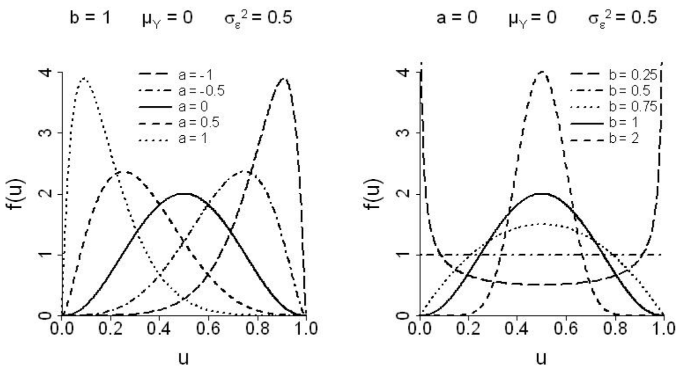

Figure 3: Different shapes for the *pdf* of DFs based on the standard Normal *cdf* in the case of maximized property. The first graphs shows how the shape varies with a (inverse behaviour with μ_Y) and the second graph shows how the shape varies with b (inverse behaviour with σ_ϵ^2).

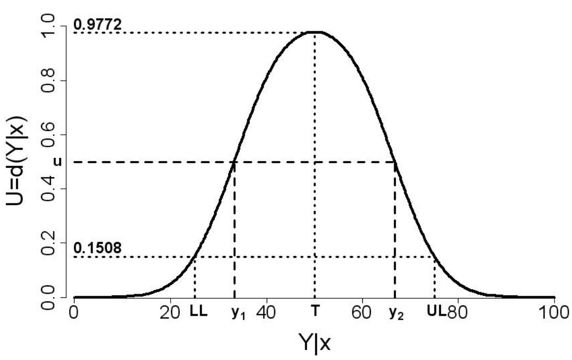

Figure 4: DFs based on the standard Normal *cdf* in the case of a targeted property. $y_1 \in (-\infty, T]$ is such that $d(y_1) = u$ and $y_2 \in [T, \infty)$ is such that $d(y_2) = u$.

Proposition 3 (DF - target)**.** *The cdf and the pdf of* $U \equiv d(Y|\mathbf{x}) = \sqrt{\left[\Phi\left(\frac{Y|\mathbf{x}-a_1}{b_1}\right)\right] \cdot \left[1-\Phi\left(\frac{Y|\mathbf{x}-a_2}{b_2}\right)\right]}$ *are respectively given by*

$$F_U(u) = \begin{cases} 0 & \text{if u} \leq 0 \\ \Phi_{\mu_Y;\sigma_\epsilon^2}(y_1) + 1 - \Phi_{\mu_Y;\sigma_\epsilon^2}(y_2) & \text{if } 0 < \text{u} < \text{d(T)} = 0.9772 \\ 1 & \text{if u} \geq \text{d(T)} = 0.9772 \end{cases} \tag{13}$$

and

$$\begin{aligned} f_U(u) \quad &= \phi_{\mu_Y;\sigma_\epsilon^2}(y_1) \cdot \left[0.5 \cdot \frac{\phi\left(\frac{y_1-a_1}{b_1}\right)\cdot\frac{1}{b_1}}{\sqrt{\Phi\left(\frac{y_1-a_1}{b_1}\right)}} \cdot \sqrt{1-\Phi\left(\frac{y_1-a_2}{b_2}\right)} - 0.5 \cdot \frac{\phi\left(\frac{y_1-a_2}{b_2}\right)\cdot\frac{1}{b_2}}{\sqrt{1-\Phi\left(\frac{y_1-a_2}{b_2}\right)}} \cdot \sqrt{\Phi\left(\frac{y_1-a_1}{b_1}\right)}\right]^{-1} \\ &-\phi_{\mu_Y;\sigma_\epsilon^2}(y_2) \cdot \left[0.5 \cdot \frac{\phi\left(\frac{y_2-a_1}{b_1}\right)\cdot\frac{1}{b_1}}{\sqrt{\Phi\left(\frac{y_2-a_1}{b_1}\right)}} \cdot \sqrt{1-\Phi\left(\frac{y_2-a_2}{b_2}\right)} - 0.5 \cdot \frac{\phi\left(\frac{y_2-a_2}{b_2}\right)\cdot\frac{1}{b_2}}{\sqrt{1-\Phi\left(\frac{y_2-a_2}{b_2}\right)}} \cdot \sqrt{\Phi\left(\frac{y_2-a_1}{b_1}\right)}\right]^{-1} \\ &\text{if } 0 < \text{u} < \text{d(T)} = 0.9772\,; \quad 0 \quad \text{otherwise} \end{aligned} \tag{14}$$

where $y_1 \in (-\infty, T]$ *and* $y_2 \in [T, \infty)$ *depend on u through the equations* $\sqrt{\Phi\left(\frac{y_1-a_1}{b_1}\right) \cdot \left[1-\Phi\left(\frac{y_1-a_2}{b_2}\right)\right]} = u$ *and* $\sqrt{\Phi\left(\frac{y_2-a_1}{b_1}\right) \cdot \left[1-\Phi\left(\frac{y_2-a_2}{b_2}\right)\right]} = u$.

It has to be noted that the two roots y_1 and y_2 such that $d(y_1) = d(y_2) = u$ can not analytically be calculated. An adequate algorithm, such as the one of Newton-Raphson, provides easily the two values.

The shapes of this density vary with parameters μ_Y, a_1 and a_2, b_1 and b_2, and σ_ϵ as presented in Figure **??**.

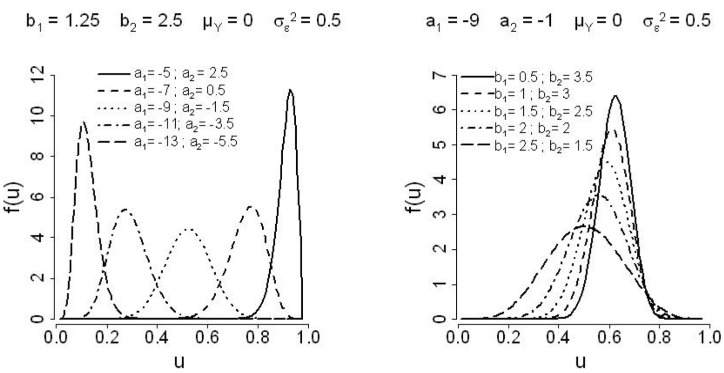

Figure 5: Different shapes for the *pdf* of DFs based on the standard Normal *cdf* in the case of targeted property. The graphs show how the shape varies with a_1 and a_2 (inverse behaviour with μ_Y) and with b_1 and b_2 (inverse behaviour with σ_ϵ).

3.2 *Pdf* of the weighted desirability function $(d(Y|\mathbf{x}))^w$

Propositions **??**, **??** and **??** provide the *pdf* of DFs for the three optimization cases, $d(Y|\mathbf{x}) = \Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)$, $d(Y|\mathbf{x}) = 1 - \Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)$ and $d(Y|\mathbf{x}) = \sqrt{\Phi\left(\frac{Y|\mathbf{x}-a_1}{b_1}\right) \cdot \left[1-\Phi\left(\frac{Y|\mathbf{x}-a_2}{b_2}\right)\right]}$. The DI based on the geometric mean (equation (**??**)) makes use of such functions weighted by a certain exponent $w \in (0, 1)$. The following question arises then: what's the *pdf* of a weighted DF $(d(Y|\mathbf{x}))^w$? The density transformation theorem (Theorem **??**) provides the following results as $(d(Y|\mathbf{x}))^w$ is an increasing function of $d(Y|\mathbf{x})$:

Proposition 4 (weighted DF - maximization)**.** *The pdf of* $V \equiv (d(Y|\mathbf{x}))^w = \left[\Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)\right]^w$ *is given by*

$$f_V(v) = \begin{cases} \frac{\phi_{\frac{\mu_Y - a}{b};\frac{\sigma_\epsilon^2}{b^2}}\left(\Phi^{-1}(v^{1/w})\right)}{\phi\left(\Phi^{-1}(v^{1/w})\right)} \cdot \left(\frac{1}{w} v^{\frac{1}{w}-1}\right) & \text{if } 0 < \text{v} < 1 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

Proposition 5 (weighted DF - minimization)**.** *The pdf of* $V \equiv (d(Y|\mathbf{x}))^w = \left[1 - \Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)\right]^w$ *is given by*

$$f_V(v) = \begin{cases} \frac{\phi_{\frac{\mu_Y - a}{b};\frac{\sigma_\epsilon^2}{b^2}}\left(\Phi^{-1}(1-v^{1/w})\right)}{\phi\left(\Phi^{-1}(1-v^{1/w})\right)} \cdot \left(\frac{1}{w} v^{\frac{1}{w}-1}\right) & \text{if } 0 < \text{v} < 1 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

For both maximization and minimization cases, the *pdf*'s are the ratio of two Normal *pdf*'s times a simple factor depending on the realisation v and the weight w. Those two *pdf*'s are still easy to compute with any statistical software. The shapes of the densities for both cases are similar and vary in the same way with parameters μ_Y, a, b, σ_ϵ and w as presented in Figure **??**.

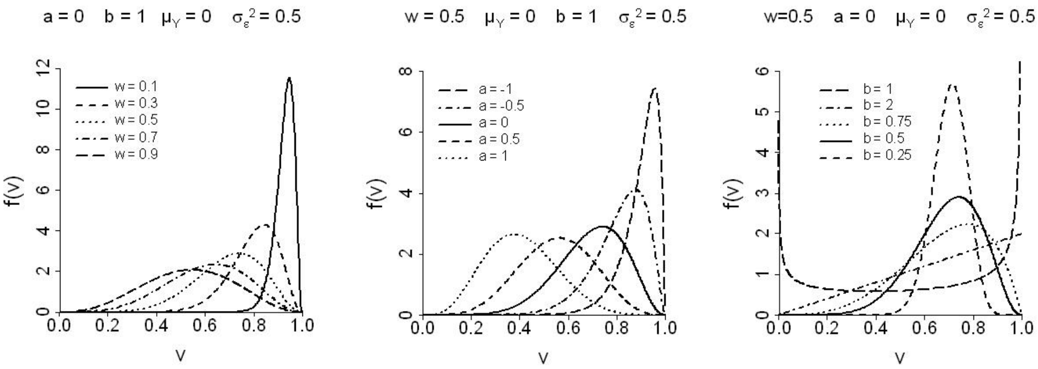

Figure 6: Different shapes for the *pdf* of weighted DFs based on the standard Normal *cdf* in the case of maximized property. The three graphs show how the shape varies with w, a (inverse behaviour with μ_Y) and b (inverse behaviour with σ_ϵ) respectively.

The *cdf* and the *pdf* of the weighted DF in the case of a targeted property can be obtained in the same way as proposition **??**, using the particular increasing-decreasing shape of $(d(Y|\mathbf{x}))^w$ like $d(Y|\mathbf{x})$ (Figure **??**).

Proposition 6 (weighted DF - target)**.** *The cdf and the pdf of*
$V \equiv (d(Y|\mathbf{x}))^w = \left(\sqrt{\left[\Phi\left(\frac{Y|\mathbf{x}-a_1}{b_1}\right)\right] \cdot \left[1 - \Phi\left(\frac{Y|\mathbf{x}-a_2}{b_2}\right)\right]}\right)^w$ *are respectively given by*

$$F_V(v) = \begin{cases} 0 & \text{if v} \le 0 \\ \Phi_{\mu_Y;\sigma_\epsilon^2}(y_1) + 1 - \Phi_{\mu_Y;\sigma_\epsilon^2}(y_2) & \text{if } 0 < \text{v} < \text{d(T)}^\text{w} = 0.9772^\text{w} \\ 1 & \text{if v} \ge \text{d(T)} = 0.9772^\text{w} \end{cases} \tag{17}$$

and

$$f_V(v) = \phi_{\mu_Y;\sigma^2_\epsilon}(y_1)\cdot\left[\frac{w}{2}\cdot\frac{\phi\left(\frac{y_1-a_1}{b_1}\right)\cdot\frac{1}{b_1}}{\left[\Phi\left(\frac{y_1-a_1}{b_1}\right)\right]^{1-\frac{w}{2}}}\cdot\left[1-\Phi\left(\frac{y_1-a_2}{b_2}\right)\right]^{\frac{w}{2}} - \frac{w}{2}\cdot\frac{\phi\left(\frac{y_1-a_2}{b_2}\right)\cdot\frac{1}{b_2}}{\left[1-\Phi\left(\frac{y_1-a_2}{b_2}\right)\right]^{1-\frac{w}{2}}}\cdot\left[\Phi\left(\frac{y_1-a_1}{b_1}\right)\right]^{\frac{w}{2}}\right]^{-1}$$
$$-\phi_{\mu_Y;\sigma^2_\epsilon}(y_2)\cdot\left[\frac{w}{2}\cdot\frac{\phi\left(\frac{y_2-a_1}{b_1}\right)\cdot\frac{1}{b_1}}{\left[\Phi\left(\frac{y_2-a_1}{b_1}\right)\right]^{1-\frac{w}{2}}}\cdot\left[1-\Phi\left(\frac{y_2-a_2}{b_2}\right)\right]^{\frac{w}{2}} - \frac{w}{2}\cdot\frac{\phi\left(\frac{y_2-a_2}{b_2}\right)\cdot\frac{1}{b_2}}{\left[1-\Phi\left(\frac{y_2-a_2}{b_2}\right)\right]^{1-\frac{w}{2}}}\cdot\left[\Phi\left(\frac{y_2-a_1}{b_1}\right)\right]^{\frac{w}{2}}\right]^{-1}$$
$$\text{if } 0 < \text{v} < \text{d(T)}^{\text{w}} = 0.9772^{\text{w}};\quad 0 \quad \text{otherwise} \tag{18}$$

where $y_1 \in (-\infty, T]$ *and* $y_2 \in [T,\infty)$ *depend on* v *by the equations* $\left(\sqrt{\left[\Phi\left(\frac{y_1-a_1}{b_1}\right)\right]\cdot\left[1-\Phi\left(\frac{y_1-a_2}{b_2}\right)\right]}\right)^w =$ v *and* $\left(\sqrt{\left[\Phi\left(\frac{y_2-a_1}{b_1}\right)\right]\cdot\left[1-\Phi\left(\frac{y_2-a_2}{b_2}\right)\right]}\right)^w = v.$

3.3 *Pdf* of the weighted desirability function $w \cdot d(Y|\mathbf{x})$

In the DI based on the arithmetic mean (equation (**??**)), DFs can be multiplied by a constant $w \in (0,1)$ to take into account the relative importance of each response. The *pdf* of $w \cdot d(Y|\mathbf{x})$ can be easily derived by applying once again the density transformation theorem for the maximization and the minimization cases where $d(Y|\mathbf{x}) = \Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)$ and $d(Y|\mathbf{x}) = 1-\Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)$. For the targeted case, the double monotonic shape of $d(Y|\mathbf{x}) = \sqrt{\Phi\left(\frac{Y|\mathbf{x}-a_1}{b_1}\right)\cdot\left[1-\Phi\left(\frac{Y|\mathbf{x}-a_2}{b_2}\right)\right]}$ allows to derive the *pdf* of $w \cdot d(Y|\mathbf{x})$.

Proposition 7 (weighted DF - maximization)**.** *The pdf of* $V \equiv w\cdot d(Y|\mathbf{x}) = w\cdot\Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)$ *is given by*

$$f_V(v) = \begin{cases} \frac{\phi_{\frac{\mu_Y-a}{b};\frac{\sigma^2_\epsilon}{b^2}}\left(\Phi^{-1}\left(\frac{v}{w}\right)\right)}{\phi\left(\Phi^{-1}\left(\frac{v}{w}\right)\right)}\cdot\frac{1}{w} & \text{if } 0<\text{v}<\text{w} \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

Proposition 8 (weighted DF - minimization)**.** *The pdf of* $V \equiv w\cdot d(Y|\mathbf{x}) = w\cdot\left[1-\Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)\right]$ *is given by*

$$f_V(v) = \begin{cases} \frac{\phi_{\frac{\mu_Y-a}{b};\frac{\sigma^2_\epsilon}{b^2}}\left(\Phi^{-1}\left(1-\frac{v}{w}\right)\right)}{\phi\left(\Phi^{-1}\left(1-\frac{v}{w}\right)\right)}\cdot\frac{1}{w} & \text{if } 0<\text{v}<\text{w} \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

Proposition 9 (weighted DF - target)**.** *The cdf and the pdf of* $V \equiv w\cdot d(Y|\mathbf{x}) = w\cdot\left[\Phi\left(\frac{Y|\mathbf{x}-a_1}{b_1}\right)\right]^{0.5}\left[1-\Phi\left(\frac{Y|\mathbf{x}-a_2}{b_2}\right)\right]^{0.5}$ *are respectively given by*

$$F_V(v) = \begin{cases} 0 & \text{if v} \le 0 \\ \Phi_{\mu_Y;\sigma^2_\epsilon}(y_1) + 1 - \Phi_{\mu_Y;\sigma^2_\epsilon}(y_2) & \text{if } 0<\text{v}<\text{w}\cdot\text{d(T)} = \text{w}\cdot 0.9772 \\ 1 & \text{if v} \ge \text{w}\cdot\text{d(T)} = \text{w}\cdot 0.9772 \end{cases} \tag{21}$$

and

$$f_V(v) = \phi_{\mu_Y;\sigma^2_\epsilon}(y_1)\cdot\left[w\cdot\left(0.5\cdot\frac{\phi\left(\frac{y_1-a_1}{b_1}\right)\cdot\frac{1}{b_1}}{\sqrt{\Phi\left(\frac{y_1-a_1}{b_1}\right)}}\cdot\sqrt{1-\Phi\left(\frac{y_1-a_2}{b_2}\right)} - 0.5\cdot\frac{\phi\left(\frac{y_1-a_2}{b_2}\right)\cdot\frac{1}{b_2}}{\sqrt{1-\Phi\left(\frac{y_1-a_2}{b_2}\right)}}\cdot\sqrt{\Phi\left(\frac{y_1-a_1}{b_1}\right)}\right)\right]^{-1}$$
$$-\phi_{\mu_Y;\sigma^2_\epsilon}(y_2)\cdot\left[w\cdot\left(0.5\cdot\frac{\phi\left(\frac{y_2-a_1}{b_1}\right)\cdot\frac{1}{b_1}}{\sqrt{\Phi\left(\frac{y_2-a_1}{b_1}\right)}}\cdot\sqrt{1-\Phi\left(\frac{y_2-a_2}{b_2}\right)} - 0.5\cdot\frac{\phi\left(\frac{y_2-a_2}{b_2}\right)\cdot\frac{1}{b_2}}{\sqrt{1-\Phi\left(\frac{y_2-a_2}{b_2}\right)}}\cdot\sqrt{\Phi\left(\frac{y_2-a_1}{b_1}\right)}\right)\right]^{-1}$$
$$\text{if } 0<\text{v}<\text{w}\cdot\text{d(T)} = \text{w}\cdot 0.9772;\quad 0\quad \text{otherwise} \tag{22}$$

where $y_1 \in (-\infty, T]$ *and* $y_2 \in [T, \infty)$ *depend on* v *by the equations* $w \cdot \sqrt{\Phi\left(\frac{y_1-a_1}{b_1}\right) \cdot \left[1 - \Phi\left(\frac{y_1-a_2}{b_2}\right)\right]} = v$ *and* $w \cdot \sqrt{\Phi\left(\frac{y_2-a_1}{b_1}\right) \cdot \left[1 - \Phi\left(\frac{y_2-a_2}{b_2}\right)\right]} = v$.

Those expressions are similar to the *pdf* of the unweighted cases of section 3.1 but evaluated in $\frac{v}{w}$ and multiplied by $\frac{1}{w}$.

4 Distribution of desirability indexes

Different formulas can be used to aggregate the p desirability functionss into an unique desirability index. This section analyze the *pdf*'s of the three most often used indexes presented in equations (**??**). The DFs $d_i(Y_i|\mathbf{x})$ considered are still the ones based on the *cdf* of the standard Normal.

4.1 Distribution of the weighted geometric mean of DFs

The weighted geometric mean is defined as the product of the weighted individual DFs: $D(\mathbf{Y}|\mathbf{x}) = \prod_{i=1}^{p} \left(d_i\left(Y_i|\mathbf{x}\right)\right)^{w_i}$ with $w_i \in (0,1)$ and $\sum_{i=1}^{p} w_i = 1$. This DI is a random variable, abbreviated D, constructed by the product of p continuous random variables $V_i \equiv d_i(Y_i|\mathbf{x})^{w_i}$: $D = \prod_{i=1}^{p} V_i$. The *pdf* of a product of such p random variables can be generalized from the following theorem that provides the *pdf* of the product of two random variables. A demonstration can be found in (**?**).

Theorem 2 (Density of the product of two random variables)**.** *Let* (V_1, V_2) *be a vector of two continuous random variables with known joint distribution* $f_{V_1,V_2}(\cdot,\cdot)$ *then* $W \equiv V_1 \cdot V_2$ *is a continuous random variable with pdf given by.*

$$f_W(w) = \int_{-\infty}^{\infty} f_{V_1,V_2}(t, \frac{w}{t}) \frac{1}{|t|} dt = \int_{-\infty}^{\infty} f_{V_1,V_2}(\frac{w}{t}, t) \frac{1}{|t|} dt \tag{23}$$

The density of the product $V_1 \cdot V_2$ depends on the joint density of the vector (V_1, V_2). If the realistic assumption is made that, conditionally to $\mathbf{x}$, the two responses are independent, *i.e.* $(Y_1|\mathbf{x}) \amalg (Y_2|\mathbf{x})$, then $V_1 \equiv d_1(Y_1|\mathbf{x})^{w_1} \amalg V_2 \equiv d_2(Y_2|\mathbf{x})^{w_2}$ and the joint density of (V_1, V_2) is simply the product of the two known marginal densities: $f_{V_1,V_2}(v_1, v_2) = f_{V_1}(v_1) \cdot f_{V_2}(v_2)$. Generalizing that, by recurrence, to the product of p independent random variables leads to the new proposition:

Proposition 10 (*Pdf* of the weighted geometric mean)**.** *Under the assumption that* $(Y_i|\mathbf{x}) \amalg (Y_j|\mathbf{x})\ \forall i \neq j$, *the pdf of* $D \equiv D(\mathbf{Y}|\mathbf{x}) = \prod_{i=1}^{p} \left(d_i\left(Y_i|\mathbf{x}\right)\right)^{w_i}$ *is given by*

$$f_D(\mathrm{d}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{V_1}(t_1) \cdot f_{V_2}\left(\frac{t_2}{t_1}\right) \cdots f_{V_{p-1}}\left(\frac{t_{p-1}}{t_{p-2}}\right) \cdot f_{V_p}\left(\frac{\mathrm{d}}{t_{p-1}}\right) \frac{1}{|t_1 \cdot t_2 \cdots t_{p-1}|} dt_1 dt_2 \cdots dt_{p-1} \tag{24}$$

where $f_{V_i}(\cdot)$ *is the pdf of the random variable* $V_i \equiv d_i(Y_i|\mathbf{x})^{w_i}$ *for* $i = 1, 2, \cdots, p$.

This results can not be extented to the case of non-independent random variables. It has also to be noted that the range of each integral can be simplified has the marginal densities $f_{V_i}(v_i)$ are positive for $0 < v_i < 1$ if the i^{th} response must be maximized or minimized and for $0 < v_i < d(T)^w = 0.9772^w$ if the i^{th} response must reach some optimal target value T (see propositions **??**-**??**-**??**). According to those domains to have $f_{V_i}(v_i) > 0$, the density $f_D(\mathrm{d})$ is non zero for $\mathrm{d} \in (0,1)$ without any targeted property but could be more narrow otherwise.

The *pdf* of $D(\mathbf{Y}|\mathbf{x})$ (**??**) is difficult to handle analytically as the multiple integral to solve makes use of complex marginal *pdf*'s $f_{V_i}(\cdot)$. The quantity of interest $E[D(\mathbf{Y}|\mathbf{x})]$ can, on the other hand, be computed without using the *pdf* of $D(\mathbf{Y}|\mathbf{x})$ as the expectation of a product of independent random variables is the product of the marginal expectations.

Proposition 11 (Expectation of the weighted geometric mean). *If* $(Y_i|\mathbf{x}) \amalg (Y_j|\mathbf{x})\ \forall i \neq j$*, then*

$$E\left[D(\mathbf{Y}|\mathbf{x})\right] = E\left[\prod_{i=1}^{p} \left(d_i\left(Y_i|\mathbf{x}\right)\right)^{w_i}\right] = \prod_{i=1}^{p} E\left[\left(d_i\left(Y_i|\mathbf{x}\right)\right)^{w_i}\right] \tag{25}$$

where each individual expectation $E\left[\left(d_i\left(Y_i|\mathbf{x}\right)\right)^{w_i}\right]$*,* $i = 1, 2, \cdots, p$*, can be computed using the pdf of* $\left(d_i\left(Y_i|\mathbf{x}\right)\right)^{w_i}$ *presented in proposition* **??***,* **??** *or* **??***.*

Proposition **??** can be used to associate to any combination of factors levels $\mathbf{x}$ the expected desirability of the resulting product $E[D(Y|\mathbf{x})]$ and find the corresponding optimum. This approach is different from the classical use of DI, $\prod_{i=1}^{p}\left(d_i\left(E\left[Y_i|\mathbf{x}\right]\right)\right)^{w_i}$, as $\left(d_i\left(\cdot\right)\right)^{w_i}$ is a non linear function.

4.2 Distribution of the weighted arithmetic mean of DFs

The weighted arithmetic mean is constructed by summing the individual DFs multiplied by a constant weight: $D(\mathbf{Y}|\mathbf{x}) = \sum_{i=1}^{p} w_i \cdot d_i\left(Y_i|\mathbf{x}\right)$ with $w_i \in (0,1)$ and $\sum_{i=1}^{p} w_i = 1$. This DI is a random variable, abbreviated D, constructed by the sum of p continuous random variables $V_i \equiv w_i \cdot d_i\left(Y_i|\mathbf{x}\right)$: $D = \sum_{i=1}^{p} V_i$. There is a well-known result (**?**) for the case of the sum of two random variables (**?**).

Theorem 3 (Density of the sum of two random variables). *Let* (V_1, V_2) *be a vector of two continuous random variables with known joint distribution* $f_{V_1,V_2}(\cdot,\cdot)$ *then* $W \equiv V_1 + V_2$ *is a continuous random variable with pdf given by*

$$f_W(w) = \int_{t=-\infty}^{\infty} f_{V_1,V_2}(t, w-t)dt = \int_{t=-\infty}^{\infty} f_{V_1,V_2}(w-t, t)dt \tag{26}$$

This theorem is easely proved by derivating the *cdf* of $W \equiv V_1 + V_2$. Unfortunatly theorem **??** can not be generalized to obtain the *pdf* of the sum of p continuous random variables. The only result about the *pdf* of the sum of p continuous random variables necessitates the independence assumption. This leads to the next corollary:

Corollary 1 (Density of the sum of two independent random variables). *Let* (V_1, V_2) *be a vector of two continuous independent random variables with known marginal distributions* $f_{V_1}(\cdot)$ *and* $f_{V_2}(\cdot)$ *then* $W \equiv V_1 + V_2$ *is a continuous random variable with pdf given by the convolution of* $f_{V_1}(\cdot)$ *and* $f_{V_2}(\cdot)$*:*

$$f_W(w) = f_{V_1} * f_{V_2}(x) = \int_{t=-\infty}^{\infty} f_{V_1}(t) \cdot f_{V_2}(w-t)dt = f_{V_2} * f_{V_1}(x) = \int_{t=-\infty}^{\infty} f_{V_1}(w-t) \cdot f_{V_2}(t)dt \tag{27}$$

This corollary can be extended, by recurrence, to any number of independent random variables. This way, the *pdf* of the weighted arithmetic mean can be obtained only under the independence assumption.

Proposition 12 (weighted arithmetic mean). *Under the assumption that* $(Y_i|\mathbf{x}) \amalg (Y_j|\mathbf{x})\ \forall i \neq j$*, the pdf of* $D \equiv D(\mathbf{Y}|\mathbf{x}) = \sum_{i=1}^{p} w_i \cdot d_i\left(Y_i|\mathbf{x}\right)$ *is given by*

$$f_D(\mathrm{d}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{V_1}\left(v_1\right) \cdot f_{V_2}\left(v_2\right) \cdots f_{V_{p-1}}\left(v_{p-1}\right) \cdot f_{V_p}\left(\mathrm{d} - v_{p-1} - \cdots - v_2 - v_1\right) dv_1 dv_2 \cdots dv_{p-1} \tag{28}$$

where $f_{V_i}(\cdot)$ *is the pdf of the random variable* $V_i \equiv w_i \cdot d_i\left(Y_i|\mathbf{x}\right)$ *for* $i = 1, 2, \cdots, p$*.*

The range of each integral can be simplified according to the values of v_i such that $f_{V_i}(v_i) > 0$ (see propositions **??**-**??**-**??**). According to those domains to have $f_{V_i}(v_i) > 0$, the density $f_D(\mathrm{d})$ is non zero for $\mathrm{d} \in (0,1)$ without any targeted property but could be more narrow otherwise.

The *pdf* of $D(\mathbf{Y}|\mathbf{x})$ is difficult to handle analytically as the multiple integral to solve makes use of complex marginal *pdf*'s $f_{V_i}(\cdot)$. The quantity of interest $E[D(\mathbf{Y}|\mathbf{x})]$ can, on the other hand, be computed without using the *pdf* of $D(\mathbf{Y}|\mathbf{x})$ as the expectation of a sum is the sum of the expectations for independent or non-independent random variables.

Proposition 13 (Expectation of the weighted arithmetic mean)**.**

$$E\left[D(\mathbf{Y}|\mathbf{x})\right] = E\left[\sum_{i=1}^{p} w_i \cdot d_i\left(Y_i|\mathbf{x}\right)\right] = \sum_{i=1}^{p} E\left[w_i \cdot d_i\left(Y_i|\mathbf{x}\right)\right] = \sum_{i=1}^{p} w_i \cdot E\left[d_i\left(Y_i|\mathbf{x}\right)\right] \tag{29}$$

where each individual expectation $E\left[d_i\left(Y_i|\mathbf{x}\right)\right]$*,* $i = 1, 2, \cdots, p$*, can be computed using the pdf of* $d_i\left(Y_i|\mathbf{x}\right)$ *presented in propositions* **??***,* **??** *and* **??***.*

This proposition can be used to associate to any combination of factors levels $\mathbf{x}$ the expected desirability of the resulting responses $E[D(Y|\mathbf{x})]$ and to find the corresponding optimum. Like the geometric mean DI, $E\left[\sum_{i=1}^{p} w_i \cdot d_i\left(Y_i|\mathbf{x}\right)\right]$ is different from the classical use of DI $\sum_{i=1}^{p} w_i \cdot d_i\left(E\left[Y_i|\mathbf{x}\right]\right)$ as $d_i\left(\cdot\right)$ is a non linear function.

4.3 Distribution of the minimum of DFs

A third often used way to summarize the individual DFs is to take their minimum: $D(\mathbf{Y}|\mathbf{x}) = \min_{i=1,2,\cdots,p} d_i\left(\mathbf{Y_i}|\mathbf{x}\right)$. This desirability index is a random variable, abbreviated D, constructed as the min of p continuous random variables $V_i \equiv d_i\left(Y_i|\mathbf{x}\right)$: $D = \min_{i=1,2,\cdots,p} V_i$. In most cases, nothing can be said about the density of the minimum of random variables without assuming their independence (**?**).

Theorem 4 (Density of the minimum of p random variables)**.** *If* $V_1, V_2, \cdots, V_p$ *are independent random variables with pdf* $f_{V_1}(\cdot), f_{V_2}(\cdot), \cdots, f_{V_p}(\cdot)$ *and cdf* $F_{V_1}(\cdot), F_{V_2}(\cdot), \cdots, F_{V_p}(\cdot)$*, then the pdf of* $D = \min_{i=1,2,\cdots,p} V_i$ *is given by*

$$f_D(\mathrm{d}) = \sum_{i=1}^{p}\left[f_{V_i}(\mathrm{d}) \cdot \prod_{\substack{j=1\\ j\neq i}}^{p}\left(1 - F_{V_j}(\mathrm{d})\right)\right] \tag{30}$$

Applying this result to the minimum of DFs leads to the following proposition:

Proposition 14 (*Pdf* of the minimum)**.** *Under the assumption that* $(Y_i|\mathbf{x}) \perp\!\!\!\perp (Y_j|\mathbf{x})\ \forall i \neq j$*, the pdf of* $D \equiv D(\mathbf{Y}|\mathbf{x}) = \min_{i=1,2,\cdots,p} d_i\left(\mathbf{Y_i}|\mathbf{x}\right)$ *is given by*

$$f_D(\mathrm{d}) = \sum_{i=1}^{p}\left[f_{V_i}(\mathrm{d}) \cdot \prod_{\substack{j=1\\ j\neq i}}^{p}\left(1 - F_{V_j}(\mathrm{d})\right)\right] \tag{31}$$

where $f_{V_i}(\cdot)$ *and* $F_{V_i}(\cdot)$ *are respectively the pdf's and the cdf's of the random variables* $V_i \equiv d_i(Y_i|\mathbf{x})$ *for* $i = 1, 2, \cdots, p$*.*

The values of d for which $f_D(\mathrm{d})$ is non zero depends on the types of individual DFs $d_i\left(\mathbf{Y_i}|\mathbf{x}\right)$. Without any targeted response, $\mathrm{d} \in (0, 1)$ but this could be more narrow otherwise.

The expected minimum desirability can then be computed by intergration of its density for any combination of factors levels $\mathbf{x}$ and the optimum can be found using an adequate optimization algorithm.

5 Distribution of DI when models are estimated

All the densities presented in the preceding sections depend on the quantities $E[Y_i|x]$ and $\sigma^2_{\epsilon_i}$ $(i = 1, 2, \ldots, p)$, that are unknown in practice. Indeed, the models given by equation (**??**) are not known and have to be estimated. Using powerful tools of experimental design, a set of n experiences within the domain of interest

χ can be adequately chosen and performed to collect data on which the assumed models can be fitted. For each property, the estimated model is denoted by

$$\hat{Y}_i = \hat{f}_i\left(x, \hat{\boldsymbol{\beta}}_{\boldsymbol{i}}\right). \tag{32}$$

Thanks to equation (**??**), the responses can be predicted for any point x in the experimental domain χ, and the unknown quantities, $E[Y_i|x]$ and $\sigma^2_{\epsilon_i}$ $(i = 1, 2, \ldots, p)$, can be estimated by

$$\hat{E}[Y_i|x] = \hat{Y}_i = \hat{f}_i\left(x, \hat{\boldsymbol{\beta}}_{\boldsymbol{i}}\right) \quad \text{and} \quad \hat{\sigma}^2_{\epsilon_i} = \frac{1}{n - q_i}\sum_{j=1}^{n}\left(y_{ij} - \hat{y}_{ij}\right)^2 \tag{33}$$

where n is the number of observations, q_i is the number of model parameters, y_{ij} $(j = 1, 2, \cdots, n)$ are the observed values for the i^{th} property and $\hat{y}_{ij}$ are the corresponding predicted values.

Before using the estimated models for prediction, it's crucial to analyze their adequacy for the training data (using R^2, adjusted R^2, goodness-of-fit tests,...) and their performance for predicting responses of new points x (using a test set, cross-validation, bootstrap,...). Nevertheless, even if the conclusions of these analysis are good, it is well known that all models are wrong and that $\hat{E}[Y_i|x]$ and $\hat{\sigma}^2_{\epsilon_i}$ are only uncertain estimation of the true parameters $E[Y_i|x]$ and $\sigma^2_{\epsilon_i}$, leading to uncertain estimation of the densities of DFs and DI, as well as uncertain estimation of the expectation of interest: $E[D(Y|x)]$.

There is no method available to quantify the propagation of the uncertainty of the estimated parameters $E[Y_i|x]$ and $\sigma^2_{\epsilon_i}$ on the estimated expected desirability index $E[D(Y|x)]$. This is a subject for futur research. If the uncertainty is small, *i.e.* $E[Y_i|x]$ and $\sigma^2_{\epsilon_i}$ are precisely estimated by $\hat{E}[Y_i|x]$ and $\hat{\sigma}^2_{\epsilon_i}$, using the estimated densities provides quite accurate estimate of $E[D(Y|x)]$.

6 Conclusion

New DFs based on the standard Normal distribution are presented for cases of maximized, minimized or targeted responses. The parameters of those functions are easy to choose in practice with or without the knowledge of a production specialist.

Due to the error terms of the assumed models linking the responses to the factors, DFs and DIes are random variables and the optimization consists of maximizing the expected desirabity index to take the random errors into account, as suggested by **?**.

The densities of the Normal based DFs are derived analytically using the univariate density transformation theorem under the assumption that the error terms are Normally distributed.

Under the assumption that conditionally to the factors levels, the responses are independent, the densities of the geometric mean and the arithmetic mean DIes are provided by applying two theorems of the statistical literature about the density of the product of random variables and the sum of random variables (*convolution*). Those densities can be used to compute some kind of prediction interval for the DI of a new experiment. But they are difficult to handle when integrating to obtain the quantity of interest, the expected DI. Under the same independence assumption, the expected DI can be computed on the basis of the expected weighted DFs for which densities are also known and more easy to handle. The density of the minimum of DFs is also derived and can be integrated to obtain the expected minimum.

The developed densities assume that the models parameters as well as the errors variances are known. Nevertheless, in practice, only estimates of those quantities are available. If those estimates are quite accurate, the computation of the expected DI on the basis of estimated densities is of reasonnable precision. Otherwise, the estimated expected DI is suiled with error and the propagation of the models prediction error has to be taken into account. A way to quantify this uncertainty propagation is through the use of the Delta method theorem (**?**).

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Distribution of Desirability Index in Multicriteria Optimization using Desirability Functions based on the Cumulative Distribution Function of the Standard Normal

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1 Introduction

Multiresponse optimization is a common problem in industries. Each response defining the quality of the product, Y_i $(i = 1, 2, \ldots, p)$, is assumed to be related to the same set of varying factors, x_j's $(j = 1, 2, \ldots, k)$. The objective is to find the factors settings $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ that simultaneously optimize the p responses $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_k)$.

The quality of a compromise between the responses can be measured by the *desirability* concept (**?**). Using *desirability functions*, the adequacy of each of the p responses, Y_i, are first quantified by a value between 0 and 1 (the higher, the better). Those p values are then aggregated in a *desirability index* providing, for any combination of factors levels, a value between 0 and 1 measuring the desirability of the resulting product quality.

As industries can not face testing all possible combinations of factors levels and measure the quality of resulting products, a model capturing the relationship between each response and factors is assumed over the domain of interest, denoted χ, through an equation of the form

$$Y_i = f_i\left(\mathbf{x}, \boldsymbol{\beta_i}\right) + \epsilon_i \quad \text{with} \quad \epsilon_i \sim N(0, \sigma^2_{\epsilon_i}). \tag{1}$$

We assume that the link function f_i and the model parameters $\boldsymbol{\beta_i}$ are known as well as the error term variance $\sigma^2_{\epsilon_i}$.

The most well-known class of models is the multiple linear regression. The link between the i^{th} response Y_i and transformed factors $\mathbf{z} = g(\mathbf{x})$ is assumed to follow an equation of the form

$$Y_i = \mathbf{z}'\boldsymbol{\beta_i} + \epsilon_i \quad with \quad \epsilon_i \sim N(0, \sigma^2_{\epsilon_i}). \tag{2}$$

For given factors settings $\mathbf{x}$, each response $Y_i|\mathbf{x}$ is supposed to be a random variable with known distribution:

$$Y_i|\mathbf{x} \sim N(E\left[Y_i|\mathbf{x}\right], \sigma^2_{\epsilon_i}). \tag{3}$$

The usual way to deal with those p random variables is to optimize the desirability of the expected quality responses, $E[Y_i|\mathbf{x}]$ (**??**). An other treatment of randomness could be, as proposed by **?**, to optimize the expected desirability of responses considering the desirability index as a random variable. This is possible if

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the distribution of the desirability index is known.

This paper discusses the derivation of the desirability index distribution using the density transformation theorem (**?**). First, desirability functions and desirability indexes are reviewed. A focus is made on desirability functions based on the standard Normal distribution function. Then, the desirability functions distribution is studied and formulas derived in the case of desirability functions based on the standard Normal distribution function. Finally, the question of desirability index distribution is treated. The practical case of unknown models and parameters estimation is also discussed.

2 Desirability functions and desirability index

The concept of *desirability* was introduced by **?** to provide a solution to multiresponse optimization problems. It allows to balance the optimized properties, Y_i's, one against the other, taking into account their target value, their relative importance and their scale.

Harrington proceeds in two steps. First, each response Y_i is transformed to the same scale using a *desirability function*, denoted by d_i, such that $d_i(Y_i) \in [0,1]$. If $d_i(Y_i) = 0$, the product is not at all acceptable according to the specifications of the i^{th} property and if $d_i(Y_i) = 1$, the product fullfields them perfectly. Secondly, the properties transformed by desirability functions are aggregated in a single value still in the $[0,1]$ interval, the *desirability index*, representing the overall desirability of the product. The weighted geometric mean or the weighted arithmetic mean of the desirability functions, as well as their minimum are the three most often used desirability index, denoted by D:

$$D(\mathbf{Y}) = \prod_{i=1}^{p} \left(d_i(Y_i)\right)^{w_i}, \quad D(\mathbf{Y}) = \sum_{i=1}^{p} w_i \cdot d_i(Y_i) \quad \text{or} \quad D(\mathbf{Y}) = \min d_i(Y_i) \quad \text{with} \sum_{i=1}^{p} w_i = 1 \tag{4}$$

The most well-known desirability functions are the Harrington's ones (1965) based on the exponential function of a linear transformation of the Y_i's and the Derringer and Suich's ones (1980) based on a power of a linear transformation of the Y_i's. **?** proposed also smoother and differentiable desirability functions using the logit function. These three types of desirability functions are presented in Table 1 for the cases where the response Y (the sub-index i has been removed to simplify notations) must be maximized, minimized or reach a target value. Desirability functions depend on parameters (a, b, s or T) that have to be fixed by a specialist of the product to define which responses values are desirable. As it can be seen on Figure **??**, the three desirability functions may provide similar desirability curves.

	Maximum	Minimum	Target Value
Harrington (1965)	$exp(-exp(-a-bY))$	$1-exp(-exp(-a-bY))$	$exp(-\|\frac{Y-T}{b}\|^n)$
Derringer and Suich (1980)	$\begin{cases} 0 & \text{if } Y<a \\ (\frac{Y-a}{b-a})^s & \text{if } a\leq Y\leq b \\ 1 & \text{if } Y>b \end{cases}$	$\begin{cases} 1 & \text{if } Y<b \\ (\frac{a-Y}{a-b})^s & \text{if } b\leq Y\leq a \\ 0 & \text{if } Y>a \end{cases}$	$\begin{cases} 0 & \text{if } Y<a_1 \\ (\frac{Y-a_1}{T-a_1})^{s_1} & \text{if } a_1\leq Y\leq T \\ (\frac{a_2-Y}{a_2-T})^{s_2} & \text{if } T\leq Y\leq a_2 \\ 0 & \text{if } Y>a_2 \end{cases}$
Gibb *et al* (2001)	$(1+exp(-\frac{Y-a}{b}))^{-1}$	$(1-exp(-\frac{Y-a}{b}))^{-1}$	$exp(-\frac{1}{2}(\frac{Y-T}{b})^2)$

Table 1: Examples of desirability functions. Y is a response; the target value T and the parameters a, b and s have to be adjusted according to the specifications.

We propose a new classe of desirability functions to transform a quality property Y in the $(0,1)$ interval. It is based on the cumulative distribution function, abbreviated *cdf*, of the standard Normal:

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp\left(-\frac{t^2}{2}\right) dt$$

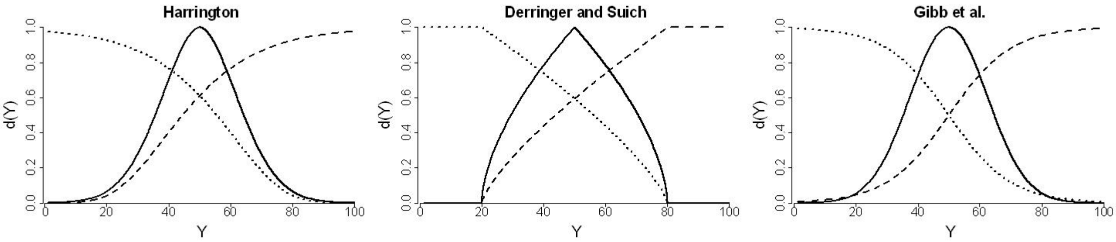

Figure 1: Different desirability functions. The continuous lines represent cases of a targeted property and the dotted and dashed lines represent cases of a minimized and a maximized property respectively.

This function is continuous and differentiable, which will be usefull to apply the density transformation theorem further. According to the desirable values of the property Y, we will use the following transformations as depicted on Figure ??:

$$d(Y) = \Phi\left(\frac{Y-a}{b}\right) \quad \text{if } Y \text{ has to be maximized,} \tag{5}$$

$$d(Y) = 1-\Phi\left(\frac{Y-a}{b}\right) \quad \text{if } Y \text{ has to be minimized,} \tag{6}$$

$$d(Y) = \sqrt{\Phi\left(\frac{Y-a_1}{b_1}\right)\cdot\left[1-\Phi\left(\frac{Y-a_2}{b_2}\right)\right]} \quad \text{if } Y \text{ has to reach some optimal value.} \tag{7}$$

The parameters a and b have to be chosen according to the knowledge of both the statistician and the specialist of the product. They can be fixed using one of the two simple following rules, :

- If the specialist notifies that under the value LL (Lower Limit) and over the value UL (Upper Limit) changes for the property Y have no additional interest, the parameters a and b can be chosen as a function of those limits according to the properties of the standard Normal distribution:

  - If Y has to be maximized, take $a = \frac{LL+UL}{2}$ and $b = \frac{UL-\frac{LL+UL}{2}}{2}$ such that $d(LL) \approx 0.023$ and $d(UL) \approx 0.977$

  - If Y has to be minimized, take $a = \frac{LL+UL}{2}$ and $b = \frac{UL-\frac{LL+UL}{2}}{2}$ such that $d(LL) \approx 0.977$ and $d(UL) \approx 0.023$

  - If Y has to reach some optimal value T, take $a_1 = \frac{LL+T}{2}$, $b_1 = \frac{T-\frac{LL+T}{2}}{2}$, $a_2 = \frac{T+UL}{2}$ and $b_2 = \frac{UL-\frac{T+UL}{2}}{2}$ such that $d(LL) = d(UL) \approx 0.151$ and $d(T) = 0.977$

- If the specialist does not notify any limit for the property Y and just desires to maximize or minimize it, or to reach a target value T, the two parameters a and b can be chosen according to the values of the response Y observed on a small sample of the experimental domain.

  - If Y has to be maximized or minimized, take a and b respectively as the sample average and the sample standard deviation.

  - If Y has to reach some optimal value T, divide the sample into a first subset with smaller values than T and a second subset with higher value than T. Take a_1 and b_1, as the arithmetic mean and the standard deviation of the first subset and a_2 and b_2, as the arithmetic mean and the standard deviation of the second subset respectively.

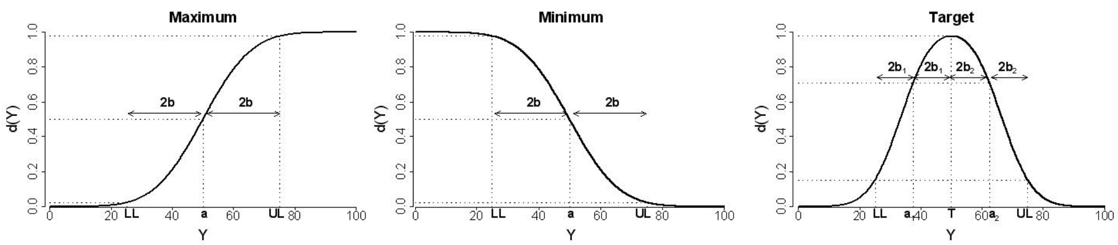

Figure 2: Desirability functions based on the standard Normal *cdf*. The three graphs correspond to the case of maximized property (left), minimized property (center) and targeted property (right). The dashed lines correspond to the limits (LL and USL). Decreasing or increasing parameter a shifts the desirability curves respectively to the left or to the right and decreasing or increasing parameter b makes the curves respectively more or less stiff.

If statistical models of the form (**??**) are assumed, there are two possibilities to associate a desirability value to any factors settings x: applying desirability functions on expected responses or taking the expectation of the desirability index. The first case is the classical use of desirability, $D^C(x)$, as Harrington and Derringer and Suich proposed in their papers and the second case is the new concept introduced by **?**, $D^N(x)$.

If for instance the weighted geometric mean is used as desirability index (**??**) the optimization problem can be formalised as below:

$$\max_{\mathbf{x}\in\chi} D^C(x) = \max_{\mathbf{x}\in\chi} D\left(E\left[\mathbf{Y}|\mathbf{x}\right]\right) = \max_{\mathbf{x}\in\chi} \prod_{i=1}^{p} \left[d_i\left(E[Y_i|\mathbf{x}]\right)\right]^{w_i} = \max_{\mathbf{x}\in\chi} \prod_{i=1}^{p} \left[d_i\left(f_i\left(\mathbf{x}, \boldsymbol{\beta_i}\right)\right)\right]^{w_i} \quad \text{or} \tag{8}$$

$$\max_{\mathbf{x}\in\chi} D^N(x) = \max_{\mathbf{x}\in\chi} E\left[D(\mathbf{Y}|\mathbf{x})\right] = \max_{\mathbf{x}\in\chi} E\left[\prod_{i=1}^{p} \left[d_i\left(Y_i|\mathbf{x}\right)\right]^{w_i}\right] = \max_{\mathbf{x}\in\chi} E\left[\prod_{i=1}^{p} \left[d_i\left(f_i\left(\mathbf{x}, \boldsymbol{\beta_i}\right) + \epsilon_i\right)\right]^{w_i}\right] \tag{9}$$

As the expectation of a random variable function is the function of the random variable expectation if and only if the transformation is linear, most of the time, $D\left(E\left[\mathbf{Y}|\mathbf{x}\right]\right) \neq E\left[D(\mathbf{Y}|\mathbf{x})\right]$ and the corresponding optima, $\mathbf{x}^C_{opt}$ and $\mathbf{x}^N_{opt}$, are different.

The idea of maximizing the expected DI instead of the DI of expected responses is the same as in the utility theory field (**?**). The expected utility function, u, is maximized instead of the utility function of expected results, R, to take risk factors into account: $\max E\left[u(R)\right]$ instead of $\max u\left(E[R]\right)$. Similarly, as it takes into account the propagation of the uncertainty of the response ϵ_i on the desirability index, Steuer proposes to associate to each factors setting $\mathbf{x}$ the average desirability of the resulting product quality, $E\left[D\left(\mathbf{Y}|\mathbf{x}\right)\right]$.

Steuer approximates for each design point $\mathbf{x} \in \chi$ the distribution of $D(\mathbf{Y}|\mathbf{x})$ by Monte-Carlo simulations on the basis of the model error distribution often assumed to be Normal. This is an heavy procedure, especially if there is more than two optimized properties and a huge experimental domain χ to explore.

To avoid intensive use of Monte-Carlo simulations, **?**, suggest to derive analytically the distribution of $D(\mathbf{Y}|\mathbf{x})$ on the basis of the model error distribution using the density transformation theorem. If the probability density function (abbreviated *pdf*) of $D(\mathbf{Y}|\mathbf{x})$ is known, its expectation $E\left[D(\mathbf{Y}|\mathbf{x})\right]$ can be computed by analytical or numerical integration. They derived the analytical expression of the Harrington DFs distribution and deduced, for special cases, the DI distribution. In this paper we derive analytically the distribution of DFs based on the Normal *cdf*.

3 Distribution of desirability functions

In this section, the *pdf* of the DFs $d_i(Y_i|\mathbf{x})$ and of the weighted DFs $[d_i(Y_i|\mathbf{x})]^{w_i}$ are derived analytically in the case of maximized, minimized and targeted properties. For simplicity we remove the i indice standing for the i^{th} property Y_i.

3.1 *Pdf* of the desirability function $d(Y|\mathbf{x})$

Most of the following results are based on the univariate density transformation theorem.

Theorem 1 (Density transformation). *Let Z have a pdf $f_Z(z)$. If $h(z)$ is either increasing or decreasing for all z such that $f_Z(z) > 0$, then $U = h(Z)$ has a pdf given by*

$$f_U(u) = f_Z\big(h^{-1}(u)\big) \cdot \left|\frac{d\left(h^{-1}(u)\right)}{du}\right| \tag{10}$$

Given the assumption that the error term ϵ for each model (**??**) follows a $N(0, \sigma_\epsilon^2)$ distribution, we know that $Y|\mathbf{x} \sim N(E[Y|\mathbf{x}], \sigma_\epsilon^2)$ and $\frac{Y|\mathbf{x}-a}{b} \sim N\left(\frac{E[Y|\mathbf{x}]-a}{b}, \frac{\sigma_\epsilon^2}{b^2}\right)$. As the DFs based on the Normal *cdf* for the maximization and the minimization cases (see equations (**??**-**??**)) are respectively increasing and decreasing functions of $\frac{Y|\mathbf{x}-a}{b}$, we can derive their *pdf*'s using the density transformation theorem.

In the following $E[Y|\mathbf{x}]$ is abbreviated μ_Y, $\phi_{M;S^2}$ and $\Phi_{M;S^2}$ denote respectively the *pdf* and the *cdf* of the Normal with expectation M and variance S^2 and ϕ and Φ without any indice denote respectively the *pdf* and the *cdf* of the standard Normal.

Proposition 1 (DF - maximization). *The pdf of $U \equiv d(Y|\mathbf{x}) = \Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)$ is given by*

$$f_U(u) = \begin{cases} \frac{\phi_{\frac{\mu_Y-a}{b};\frac{\sigma_\epsilon^2}{b^2}}\left(\Phi^{-1}(u)\right)}{\phi(\Phi^{-1}(u))} & \text{if } 0 < \text{u} < 1 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Proposition 2 (DF - minimization). *The pdf of $U \equiv d(Y|\mathbf{x}) = 1 - \Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)$ is given by*

$$f_U(u) = \begin{cases} \frac{\phi_{\frac{\mu_Y-a}{b};\frac{\sigma_\epsilon^2}{b^2}}\left(\Phi^{-1}(1-u)\right)}{\phi(\Phi^{-1}(1-u))} & \text{if } 0 < \text{u} < 1 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

For both maximization and minimization cases, the DFs *pdf*'s are the ratio of two Normal *pdf*'s and are then easy to compute in practice with any statistical software. The shapes of those two densities are similar and vary in the same way with parameters μ_Y, a, b and σ_ϵ^2 as presented in Figure **??**.

The *pdf* of the DF in the case of a targeted property can not be derived directly using the density transformation theorem (Theorem **??**) as $U \equiv d(Y|\mathbf{x}) = \sqrt{\Phi\left(\frac{Y|\mathbf{x}-a_1}{b_1}\right) \cdot \left[1 - \Phi\left(\frac{Y|\mathbf{x}-a_2}{b_2}\right)\right]}$ does not satisfy the monotonicity assumption. Nevertheless, $d(Y|\mathbf{x})$ is strictly increasing in $Y|\mathbf{x}$ below the value target T and strictly decreasing above the value target T. Thanks to this particular shape, as represented in Figure **??**, realizations of U smaller than a given u correspond either to realizations of $Y|\mathbf{x}$ smaller than a certain $y_1 \in (-\infty, T]$ such that $d(y_1) = u$, or realizations of $Y|\mathbf{x}$ higher than a certain $y_2 \in [T, \infty)$ such that $d(y_2) = u$. This allows to compute first the *cdf* of $d(Y|\mathbf{x})$ and then derive this analytical expression to obtain its *pdf*.

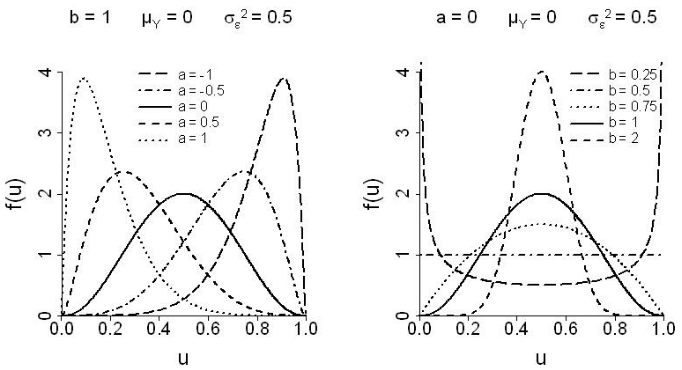

Figure 3: Different shapes for the *pdf* of DFs based on the standard Normal *cdf* in the case of maximized property. The first graphs shows how the shape varies with a (inverse behaviour with μ_Y) and the second graph shows how the shape varies with b (inverse behaviour with σ_ϵ^2).

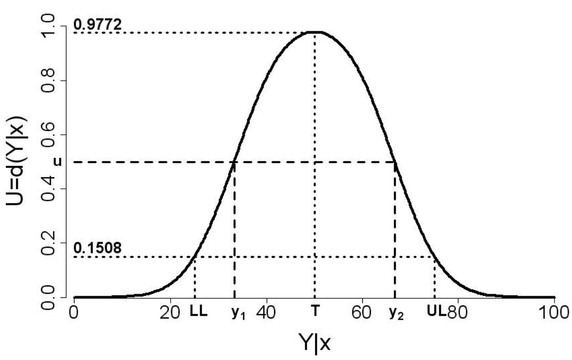

Figure 4: DFs based on the standard Normal *cdf* in the case of a targeted property. $y_1 \in (-\infty, T]$ is such that $d(y_1) = u$ and $y_2 \in [T, \infty)$ is such that $d(y_2) = u$.

Proposition 3 (DF - target)**.** *The cdf and the pdf of* $U \equiv d(Y|\mathbf{x}) = \sqrt{\left[\Phi\left(\frac{Y|\mathbf{x}-a_1}{b_1}\right)\right]\cdot\left[1-\Phi\left(\frac{Y|\mathbf{x}-a_2}{b_2}\right)\right]}$ *are respectively given by*

$$F_U(u) = \begin{cases} 0 & \text{if u} \leq 0 \\ \Phi_{\mu_Y;\sigma^2_\epsilon}(y_1) + 1 - \Phi_{\mu_Y;\sigma^2_\epsilon}(y_2) & \text{if } 0 < \text{u} < \text{d(T)} = 0.9772 \\ 1 & \text{if u} \geq \text{d(T)} = 0.9772 \end{cases} \tag{13}$$

and

$$\begin{aligned} f_U(u) &= \phi_{\mu_Y;\sigma^2_\epsilon}(y_1)\cdot\left[0.5\cdot\frac{\phi\left(\frac{y_1-a_1}{b_1}\right)\cdot\frac{1}{b_1}}{\sqrt{\Phi\left(\frac{y_1-a_1}{b_1}\right)}}\cdot\sqrt{1-\Phi\left(\frac{y_1-a_2}{b_2}\right)} - 0.5\cdot\frac{\phi\left(\frac{y_1-a_2}{b_2}\right)\cdot\frac{1}{b_2}}{\sqrt{1-\Phi\left(\frac{y_1-a_2}{b_2}\right)}}\cdot\sqrt{\Phi\left(\frac{y_1-a_1}{b_1}\right)}\right]^{-1} \\ &\quad -\phi_{\mu_Y;\sigma^2_\epsilon}(y_2)\cdot\left[0.5\cdot\frac{\phi\left(\frac{y_2-a_1}{b_1}\right)\cdot\frac{1}{b_1}}{\sqrt{\Phi\left(\frac{y_2-a_1}{b_1}\right)}}\cdot\sqrt{1-\Phi\left(\frac{y_2-a_2}{b_2}\right)} - 0.5\cdot\frac{\phi\left(\frac{y_2-a_2}{b_2}\right)\cdot\frac{1}{b_2}}{\sqrt{1-\Phi\left(\frac{y_2-a_2}{b_2}\right)}}\cdot\sqrt{\Phi\left(\frac{y_2-a_1}{b_1}\right)}\right]^{-1} \\ &\quad \text{if } 0 < \text{u} < \text{d(T)} = 0.9772\ ; \quad 0 \quad \text{otherwise} \end{aligned} \tag{14}$$

where $y_1 \in (-\infty, T]$ *and* $y_2 \in [T, \infty)$ *depend on* u *through the equations* $\sqrt{\Phi\left(\frac{y_1-a_1}{b_1}\right)\cdot\left[1-\Phi\left(\frac{y_1-a_2}{b_2}\right)\right]} = u$ *and* $\sqrt{\Phi\left(\frac{y_2-a_1}{b_1}\right)\cdot\left[1-\Phi\left(\frac{y_2-a_2}{b_2}\right)\right]} = u$.

It has to be noted that the two roots y_1 and y_2 such that $d(y_1) = d(y_2) = u$ can not analytically be calculated. An adequate algorithm, such as the one of Newton-Raphson, provides easily the two values.

The shapes of this density vary with parameters μ_Y, a_1 and a_2, b_1 and b_2, and σ_ϵ as presented in Figure **??**.

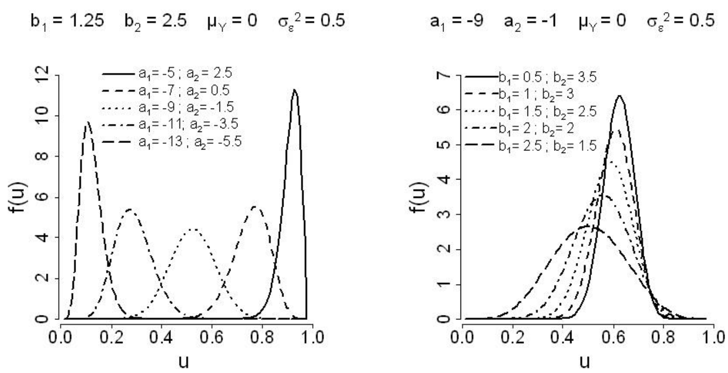

Figure 5: Different shapes for the *pdf* of DFs based on the standard Normal *cdf* in the case of targeted property. The graphs show how the shape varies with a_1 and a_2 (inverse behaviour with μ_Y) and with b_1 and b_2 (inverse behaviour with σ_ϵ).

3.2 *Pdf* of the weighted desirability function $(d(Y|\mathbf{x}))^w$

Propositions **??**, **??** and **??** provide the *pdf* of DFs for the three optimization cases, $d(Y|\mathbf{x}) = \Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)$, $d(Y|\mathbf{x}) = 1-\Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)$ and $d(Y|\mathbf{x}) = \sqrt{\Phi\left(\frac{Y|\mathbf{x}-a_1}{b_1}\right)\cdot\left[1-\Phi\left(\frac{Y|\mathbf{x}-a_2}{b_2}\right)\right]}$. The DI based on the geometric mean (equation (**??**)) makes use of such functions weighted by a certain exponent $w \in (0,1)$. The following question arises then: what's the *pdf* of a weighted DF $(d(Y|\mathbf{x}))^w$? The density transformation theorem (Theorem **??**) provides the following results as $(d(Y|\mathbf{x}))^w$ is an increasing function of $d(Y|\mathbf{x})$:

Proposition 4 (weighted DF - maximization)**.** *The pdf of* $V \equiv (d(Y|\mathbf{x}))^w = \left[\Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)\right]^w$ *is given by*

$$f_V(v) = \begin{cases} \dfrac{\phi_{\frac{\mu_Y-a}{b};\frac{\sigma_\epsilon^2}{b^2}}\left(\Phi^{-1}(v^{1/w})\right)}{\phi\left(\Phi^{-1}(v^{1/w})\right)} \cdot \left(\frac{1}{w}v^{\frac{1}{w}-1}\right) & \text{if } 0 < \text{v} < 1 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

Proposition 5 (weighted DF - minimization)**.** *The pdf of* $V \equiv (d(Y|\mathbf{x}))^w = \left[1 - \Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)\right]^w$ *is given by*

$$f_V(v) = \begin{cases} \dfrac{\phi_{\frac{\mu_Y-a}{b};\frac{\sigma_\epsilon^2}{b^2}}\left(\Phi^{-1}(1-v^{1/w})\right)}{\phi\left(\Phi^{-1}(1-v^{1/w})\right)} \cdot \left(\frac{1}{w}v^{\frac{1}{w}-1}\right) & \text{if } 0 < \text{v} < 1 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

For both maximization and minimization cases, the *pdf*'s are the ratio of two Normal *pdf*'s times a simple factor depending on the realisation v and the weight w. Those two *pdf*'s are still easy to compute with any statistical software. The shapes of the densities for both cases are similar and vary in the same way with parameters μ_Y, a, b, σ_ϵ and w as presented in Figure **??**.

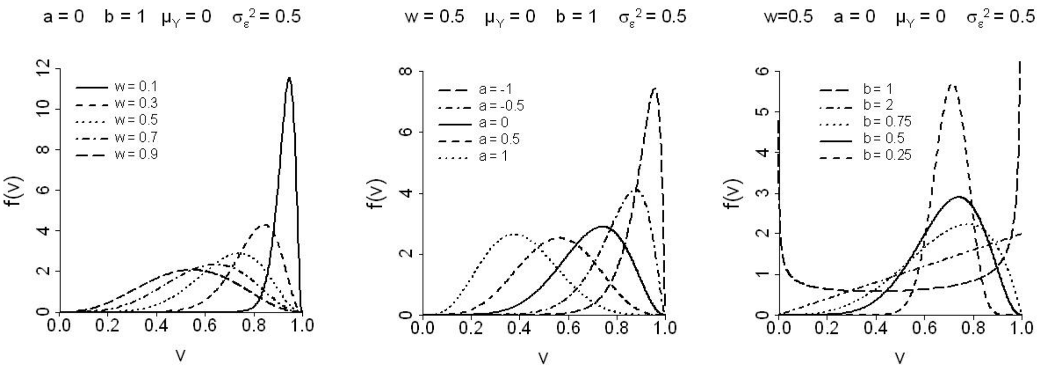

Figure 6: Different shapes for the *pdf* of weighted DFs based on the standard Normal *cdf* in the case of maximized property. The three graphs show how the shape varies with w, a (inverse behaviour with μ_Y) and b (inverse behaviour with σ_ϵ) respectively.

The *cdf* and the *pdf* of the weighted DF in the case of a targeted property can be obtained in the same way as proposition **??**, using the particular increasing-decreasing shape of $(d(Y|\mathbf{x}))^w$ like $d(Y|\mathbf{x})$ (Figure **??**).

Proposition 6 (weighted DF - target)**.** *The cdf and the pdf of*
$V \equiv (d(Y|\mathbf{x}))^w = \left(\sqrt{\left[\Phi\left(\frac{Y|\mathbf{x}-a_1}{b_1}\right)\right] \cdot \left[1 - \Phi\left(\frac{Y|\mathbf{x}-a_2}{b_2}\right)\right]}\right)^w$ *are respectively given by*

$$F_V(v) = \begin{cases} 0 & \text{if v} \leq 0 \\ \Phi_{\mu_Y;\sigma_\epsilon^2}(y_1) + 1 - \Phi_{\mu_Y;\sigma_\epsilon^2}(y_2) & \text{if } 0 < \text{v} < \text{d(T)}^\text{w} = 0.9772^\text{w} \\ 1 & \text{if v} \geq \text{d(T)} = 0.9772^\text{w} \end{cases} \tag{17}$$

and

$$f_V(v) \quad = \phi_{\mu_Y;\sigma^2_\epsilon}(y_1)\cdot\left[\frac{w}{2}\cdot\frac{\phi\left(\frac{y_1-a_1}{b_1}\right)\cdot\frac{1}{b_1}}{\left[\Phi\left(\frac{y_1-a_1}{b_1}\right)\right]^{1-\frac{w}{2}}}\cdot\left[1-\Phi\left(\frac{y_1-a_2}{b_2}\right)\right]^{\frac{w}{2}}-\frac{w}{2}\cdot\frac{\phi\left(\frac{y_1-a_2}{b_2}\right)\cdot\frac{1}{b_2}}{\left[1-\Phi\left(\frac{y_1-a_2}{b_2}\right)\right]^{1-\frac{w}{2}}}\cdot\left[\Phi\left(\frac{y_1-a_1}{b_1}\right)\right]^{\frac{w}{2}}\right]^{-1}$$
$$-\phi_{\mu_Y;\sigma^2_\epsilon}(y_2)\cdot\left[\frac{w}{2}\cdot\frac{\phi\left(\frac{y_2-a_1}{b_1}\right)\cdot\frac{1}{b_1}}{\left[\Phi\left(\frac{y_2-a_1}{b_1}\right)\right]^{1-\frac{w}{2}}}\cdot\left[1-\Phi\left(\frac{y_2-a_2}{b_2}\right)\right]^{\frac{w}{2}}-\frac{w}{2}\cdot\frac{\phi\left(\frac{y_2-a_2}{b_2}\right)\cdot\frac{1}{b_2}}{\left[1-\Phi\left(\frac{y_2-a_2}{b_2}\right)\right]^{1-\frac{w}{2}}}\cdot\left[\Phi\left(\frac{y_2-a_1}{b_1}\right)\right]^{\frac{w}{2}}\right]^{-1}$$
$$\text{if } 0<\mathrm{v}<\mathrm{d(T)^w}=0.9772^{\mathrm{w}};\quad 0 \quad \text{otherwise} \tag{18}$$

where $y_1 \in (-\infty, T]$ *and* $y_2 \in [T, \infty)$ *depend on* v *by the equations* $\left(\sqrt{\left[\Phi\left(\frac{y_1-a_1}{b_1}\right)\right]\cdot\left[1-\Phi\left(\frac{y_1-a_2}{b_2}\right)\right]}\right)^w =$ v *and* $\left(\sqrt{\left[\Phi\left(\frac{y_2-a_1}{b_1}\right)\right]\cdot\left[1-\Phi\left(\frac{y_2-a_2}{b_2}\right)\right]}\right)^w = v$.

3.3 *Pdf* of the weighted desirability function $w \cdot d(Y|\mathbf{x})$

In the DI based on the arithmetic mean (equation (**??**)), DFs can be multiplied by a constant $w \in (0,1)$ to take into account the relative importance of each response. The *pdf* of $w \cdot d(Y|\mathbf{x})$ can be easily derived by applying once again the density transformation theorem for the maximization and the minimization cases where $d(Y|\mathbf{x}) = \Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)$ and $d(Y|\mathbf{x}) = 1-\Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)$. For the targeted case, the double monotonic shape of $d(Y|\mathbf{x}) = \sqrt{\Phi\left(\frac{Y|\mathbf{x}-a_1}{b_1}\right)\cdot\left[1-\Phi\left(\frac{Y|\mathbf{x}-a_2}{b_2}\right)\right]}$ allows to derive the *pdf* of $w \cdot d(Y|\mathbf{x})$.

Proposition 7 (weighted DF - maximization). *The pdf of* $V \equiv w \cdot d(Y|\mathbf{x}) = w \cdot \Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)$ *is given by*

$$f_V(v) = \begin{cases} \frac{\phi_{\frac{\mu_Y-a}{b};\frac{\sigma^2_\epsilon}{b^2}}\left(\Phi^{-1}\left(\frac{v}{w}\right)\right)}{\phi\left(\Phi^{-1}\left(\frac{v}{w}\right)\right)}\cdot\frac{1}{w} & \text{if } 0<\mathrm{v}<\mathrm{w} \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

Proposition 8 (weighted DF - minimization). *The pdf of* $V \equiv w \cdot d(Y|\mathbf{x}) = w \cdot \left[1-\Phi\left(\frac{Y|\mathbf{x}-a}{b}\right)\right]$ *is given by*

$$f_V(v) = \begin{cases} \frac{\phi_{\frac{\mu_Y-a}{b};\frac{\sigma^2_\epsilon}{b^2}}\left(\Phi^{-1}\left(1-\frac{v}{w}\right)\right)}{\phi\left(\Phi^{-1}\left(1-\frac{v}{w}\right)\right)}\cdot\frac{1}{w} & \text{if } 0<\mathrm{v}<\mathrm{w} \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

Proposition 9 (weighted DF - target). *The cdf and the pdf of* $V \equiv w \cdot d(Y|\mathbf{x}) = w \cdot \left[\Phi\left(\frac{Y|\mathbf{x}-a_1}{b_1}\right)\right]^{0.5}\left[1-\Phi\left(\frac{Y|\mathbf{x}-a_2}{b_2}\right)\right]^{0.5}$ *are respectively given by*

$$F_V(v) = \begin{cases} 0 & \text{if } \mathrm{v} \le 0 \\ \Phi_{\mu_Y;\sigma^2_\epsilon}(y_1) + 1 - \Phi_{\mu_Y;\sigma^2_\epsilon}(y_2) & \text{if } 0<\mathrm{v}<\mathrm{w}\cdot\mathrm{d(T)}=\mathrm{w}\cdot 0.9772 \\ 1 & \text{if } \mathrm{v}\ge\mathrm{w}\cdot\mathrm{d(T)}=\mathrm{w}\cdot 0.9772 \end{cases} \tag{21}$$

and

$$f_V(v) \quad = \phi_{\mu_Y;\sigma^2_\epsilon}(y_1)\cdot\left[w\cdot\left(0.5\cdot\frac{\phi\left(\frac{y_1-a_1}{b_1}\right)\cdot\frac{1}{b_1}}{\sqrt{\Phi\left(\frac{y_1-a_1}{b_1}\right)}}\cdot\sqrt{1-\Phi\left(\frac{y_1-a_2}{b_2}\right)}-0.5\cdot\frac{\phi\left(\frac{y_1-a_2}{b_2}\right)\cdot\frac{1}{b_2}}{\sqrt{1-\Phi\left(\frac{y_1-a_2}{b_2}\right)}}\cdot\sqrt{\Phi\left(\frac{y_1-a_1}{b_1}\right)}\right)\right]^{-1}$$
$$-\phi_{\mu_Y;\sigma^2_\epsilon}(y_2)\cdot\left[w\cdot\left(0.5\cdot\frac{\phi\left(\frac{y_2-a_1}{b_1}\right)\cdot\frac{1}{b_1}}{\sqrt{\Phi\left(\frac{y_2-a_1}{b_1}\right)}}\cdot\sqrt{1-\Phi\left(\frac{y_2-a_2}{b_2}\right)}-0.5\cdot\frac{\phi\left(\frac{y_2-a_2}{b_2}\right)\cdot\frac{1}{b_2}}{\sqrt{1-\Phi\left(\frac{y_2-a_2}{b_2}\right)}}\cdot\sqrt{\Phi\left(\frac{y_2-a_1}{b_1}\right)}\right)\right]^{-1}$$
$$\text{if } 0<\mathrm{v}<\mathrm{w}\cdot\mathrm{d(T)}=\mathrm{w}\cdot 0.9772;\quad 0 \quad \text{otherwise} \tag{22}$$

where $y_1 \in (-\infty, T]$ *and* $y_2 \in [T, \infty)$ *depend on* v *by the equations* $w \cdot \sqrt{\Phi\left(\frac{y_1-a_1}{b_1}\right) \cdot \left[1 - \Phi\left(\frac{y_1-a_2}{b_2}\right)\right]} = v$ *and* $w \cdot \sqrt{\Phi\left(\frac{y_2-a_1}{b_1}\right) \cdot \left[1 - \Phi\left(\frac{y_2-a_2}{b_2}\right)\right]} = v$.

Those expressions are similar to the *pdf* of the unweighted cases of section 3.1 but evaluated in $\frac{v}{w}$ and multiplied by $\frac{1}{w}$.

4 Distribution of desirability indexes

Different formulas can be used to aggregate the p desirability functionss into an unique desirability index. This section analyze the *pdf*'s of the three most often used indexes presented in equations (**??**). The DFs $d_i(Y_i|\mathbf{x})$ considered are still the ones based on the *cdf* of the standard Normal.

4.1 Distribution of the weighted geometric mean of DFs

The weighted geometric mean is defined as the product of the weighted individual DFs: $D(\mathbf{Y}|\mathbf{x}) = \prod_{i=1}^{p} \left(d_i\left(Y_i|\mathbf{x}\right)\right)^{w_i}$ with $w_i \in (0,1)$ and $\sum_{i=1}^{p} w_i = 1$. This DI is a random variable, abbreviated D, constructed by the product of p continuous random variables $V_i \equiv d_i(Y_i|\mathbf{x})^{w_i}$: $D = \prod_{i=1}^{p} V_i$. The *pdf* of a product of such p random variables can be generalized from the following theorem that provides the *pdf* of the product of two random variables. A demonstration can be found in (**?**).

Theorem 2 (Density of the product of two random variables)**.** *Let* (V_1, V_2) *be a vector of two continuous random variables with known joint distribution* $f_{V_1,V_2}(\cdot,\cdot)$ *then* $W \equiv V_1 \cdot V_2$ *is a continuous random variable with pdf given by.*

$$f_W(w) = \int_{-\infty}^{\infty} f_{V_1,V_2}(t, \frac{w}{t}) \frac{1}{|t|} dt = \int_{-\infty}^{\infty} f_{V_1,V_2}(\frac{w}{t}, t) \frac{1}{|t|} dt \tag{23}$$

The density of the product $V_1 \cdot V_2$ depends on the joint density of the vector (V_1, V_2). If the realistic assumption is made that, conditionally to $\mathbf{x}$, the two responses are independent, *i.e.* $(Y_1|\mathbf{x}) \amalg (Y_2|\mathbf{x})$, then $V_1 \equiv d_1(Y_1|\mathbf{x})^{w_1} \amalg V_2 \equiv d_2(Y_2|\mathbf{x})^{w_2}$ and the joint density of (V_1, V_2) is simply the product of the two known marginal densities: $f_{V_1,V_2}(v_1, v_2) = f_{V_1}(v_1) \cdot f_{V_2}(v_2)$. Generalizing that, by recurrence, to the product of p independent random variables leads to the new proposition:

Proposition 10 (*Pdf* of the weighted geometric mean)**.** *Under the assumption that* $(Y_i|\mathbf{x}) \amalg (Y_j|\mathbf{x})\ \forall i \neq j$, *the pdf of* $D \equiv D(\mathbf{Y}|\mathbf{x}) = \prod_{i=1}^{p} \left(d_i\left(Y_i|\mathbf{x}\right)\right)^{w_i}$ *is given by*

$$f_D(\mathrm{d}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{V_1}(t_1) \cdot f_{V_2}\left(\frac{t_2}{t_1}\right) \cdots f_{V_{p-1}}\left(\frac{t_{p-1}}{t_{p-2}}\right) \cdot f_{V_p}\left(\frac{\mathrm{d}}{t_{p-1}}\right) \frac{1}{|t_1 \cdot t_2 \cdots t_{p-1}|} dt_1 dt_2 \cdots dt_{p-1} \tag{24}$$

where $f_{V_i}(\cdot)$ *is the pdf of the random variable* $V_i \equiv d_i(Y_i|\mathbf{x})^{w_i}$ *for* $i = 1, 2, \cdots, p$.

This results can not be extented to the case of non-independent random variables. It has also to be noted that the range of each integral can be simplified has the marginal densities $f_{V_i}(v_i)$ are positive for $0 < v_i < 1$ if the i^{th} response must be maximized or minimized and for $0 < v_i < d(T)^w = 0.9772^w$ if the i^{th} response must reach some optimal target value T (see propositions **??**-**??**-**??**). According to those domains to have $f_{V_i}(v_i) > 0$, the density $f_D(\mathrm{d})$ is non zero for $\mathrm{d} \in (0,1)$ without any targeted property but could be more narrow otherwise.

The *pdf* of $D(\mathbf{Y}|\mathbf{x})$ (**??**) is difficult to handle analytically as the multiple integral to solve makes use of complex marginal *pdf*'s $f_{V_i}(\cdot)$. The quantity of interest $E[D(\mathbf{Y}|\mathbf{x})]$ can, on the other hand, be computed without using the *pdf* of $D(\mathbf{Y}|\mathbf{x})$ as the expectation of a product of independent random variables is the product of the marginal expectations.

Proposition 11 (Expectation of the weighted geometric mean). *If* $(Y_i|\mathbf{x}) \amalg (Y_j|\mathbf{x})\ \forall i \neq j$*, then*

$$E[D(\mathbf{Y}|\mathbf{x})] = E\left[\prod_{i=1}^{p} (d_i(Y_i|\mathbf{x}))^{w_i}\right] = \prod_{i=1}^{p} E\left[(d_i(Y_i|\mathbf{x}))^{w_i}\right] \tag{25}$$

where each individual expectation $E\left[(d_i(Y_i|\mathbf{x}))^{w_i}\right]$*,* $i = 1, 2, \cdots, p$*, can be computed using the pdf of* $(d_i(Y_i|\mathbf{x}))^{w_i}$ *presented in proposition* **??**, **??** *or* **??**.

Proposition **??** can be used to associate to any combination of factors levels $\mathbf{x}$ the expected desirability of the resulting product $E[D(Y|\mathbf{x})]$ and find the corresponding optimum. This approach is different from the classical use of DI, $\prod_{i=1}^{p} (d_i(E[Y_i|\mathbf{x}]))^{w_i}$, as $(d_i(\cdot))^{w_i}$ is a non linear function.

4.2 Distribution of the weighted arithmetic mean of DFs

The weighted arithmetic mean is constructed by summing the individual DFs multiplied by a constant weight: $D(\mathbf{Y}|\mathbf{x}) = \sum_{i=1}^{p} w_i \cdot d_i(Y_i|\mathbf{x})$ with $w_i \in (0,1)$ and $\sum_{i=1}^{p} w_i = 1$. This DI is a random variable, abbreviated D, constructed by the sum of p continuous random variables $V_i \equiv w_i \cdot d_i(Y_i|\mathbf{x})$: $D = \sum_{i=1}^{p} V_i$. There is a well-known result (**?**) for the case of the sum of two random variables (**?**).

Theorem 3 (Density of the sum of two random variables). *Let* (V_1, V_2) *be a vector of two continuous random variables with known joint distribution* $f_{V_1,V_2}(\cdot,\cdot)$ *then* $W \equiv V_1 + V_2$ *is a continuous random variable with pdf given by*

$$f_W(w) = \int_{t=-\infty}^{\infty} f_{V_1,V_2}(t, w-t)dt = \int_{t=-\infty}^{\infty} f_{V_1,V_2}(w-t, t)dt \tag{26}$$

This theorem is easely proved by derivating the *cdf* of $W \equiv V_1 + V_2$. Unfortunatly theorem **??** can not be generalized to obtain the *pdf* of the sum of p continuous random variables. The only result about the *pdf* of the sum of p continuous random variables necessitates the independence assumption. This leads to the next corollary:

Corollary 1 (Density of the sum of two independent random variables). *Let* (V_1, V_2) *be a vector of two continuous independent random variables with known marginal distributions* $f_{V_1}(\cdot)$ *and* $f_{V_2}(\cdot)$ *then* $W \equiv V_1 + V_2$ *is a continuous random variable with pdf given by the convolution of* $f_{V_1}(\cdot)$ *and* $f_{V_2}(\cdot)$*:*

$$f_W(w) = f_{V_1} * f_{V_2}(x) = \int_{t=-\infty}^{\infty} f_{V_1}(t) \cdot f_{V_2}(w-t)dt = f_{V_2} * f_{V_1}(x) = \int_{t=-\infty}^{\infty} f_{V_1}(w-t) \cdot f_{V_2}(t)dt \tag{27}$$

This corollary can be extended, by recurrence, to any number of independent random variables. This way, the *pdf* of the weighted arithmetic mean can be obtained only under the independence assumption.

Proposition 12 (weighted arithmetic mean). *Under the assumption that* $(Y_i|\mathbf{x}) \amalg (Y_j|\mathbf{x})\ \forall i \neq j$*, the pdf of* $D \equiv D(\mathbf{Y}|\mathbf{x}) = \sum_{i=1}^{p} w_i \cdot d_i(Y_i|\mathbf{x})$ *is given by*

$$f_D(\mathrm{d}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{V_1}(v_1) \cdot f_{V_2}(v_2) \cdots f_{V_{p-1}}(v_{p-1}) \cdot f_{V_p}(\mathrm{d} - v_{p-1} - \cdots - v_2 - v_1)\, dv_1 dv_2 \cdots dv_{p-1} \tag{28}$$

where $f_{V_i}(\cdot)$ *is the pdf of the random variable* $V_i \equiv w_i \cdot d_i(Y_i|\mathbf{x})$ *for* $i = 1, 2, \cdots, p$.

The range of each integral can be simplified according to the values of v_i such that $f_{V_i}(v_i) > 0$ (see propositions **??**-**??**-**??**). According to those domains to have $f_{V_i}(v_i) > 0$, the density $f_D(\mathrm{d})$ is non zero for $\mathrm{d} \in (0,1)$ without any targeted property but could be more narrow otherwise.

The *pdf* of $D(\mathbf{Y}|\mathbf{x})$ is difficult to handle analytically as the multiple integral to solve makes use of complex marginal *pdf*'s $f_{V_i}(\cdot)$. The quantity of interest $E[D(\mathbf{Y}|\mathbf{x})]$ can, on the other hand, be computed without using the *pdf* of $D(\mathbf{Y}|\mathbf{x})$ as the expectation of a sum is the sum of the expectations for independent or non-independent random variables.

Proposition 13 (Expectation of the weighted arithmetic mean)**.**

$$E\left[D(\mathbf{Y}|\mathbf{x})\right] = E\left[\sum_{i=1}^{p} w_i \cdot d_i\left(Y_i|\mathbf{x}\right)\right] = \sum_{i=1}^{p} E\left[w_i \cdot d_i\left(Y_i|\mathbf{x}\right)\right] = \sum_{i=1}^{p} w_i \cdot E\left[d_i\left(Y_i|\mathbf{x}\right)\right] \tag{29}$$

where each individual expectation $E\left[d_i\left(Y_i|\mathbf{x}\right)\right]$*,* $i = 1, 2, \cdots, p$*, can be computed using the pdf of* $d_i\left(Y_i|\mathbf{x}\right)$ *presented in propositions* **??***,* **??** *and* **??***.*

This proposition can be used to associate to any combination of factors levels $\mathbf{x}$ the expected desirability of the resulting responses $E[D(Y|\mathbf{x})]$ and to find the corresponding optimum. Like the geometric mean DI, $E\left[\sum_{i=1}^{p} w_i \cdot d_i\left(Y_i|\mathbf{x}\right)\right]$ is different from the classical use of DI $\sum_{i=1}^{p} w_i \cdot d_i\left(E\left[Y_i|\mathbf{x}\right]\right)$ as $d_i(\cdot)$ is a non linear function.

4.3 Distribution of the minimum of DFs

A third often used way to summarize the individual DFs is to take their minimum: $D(\mathbf{Y}|\mathbf{x}) = \min_{i=1,2,\cdots,p} d_i\left(\mathbf{Y_i}|\mathbf{x}\right)$. This desirability index is a random variable, abbreviated D, constructed as the min of p continuous random variables $V_i \equiv d_i\left(Y_i|\mathbf{x}\right)$: $D = \min_{i=1,2,\cdots,p} V_i$. In most cases, nothing can be said about the density of the minimum of random variables without assuming their independence (**?**).

Theorem 4 (Density of the minimum of p random variables)**.** *If* $V_1, V_2, \cdots, V_p$ *are independent random variables with pdf* $f_{V_1}(\cdot), f_{V_2}(\cdot), \cdots, f_{V_p}(\cdot)$ *and cdf* $F_{V_1}(\cdot), F_{V_2}(\cdot), \cdots, F_{V_p}(\cdot)$*, then the pdf of* $D = \min_{i=1,2,\cdots,p} V_i$ *is given by*

$$f_D(\mathrm{d}) = \sum_{i=1}^{p}\left[f_{V_i}(\mathrm{d}) \cdot \prod_{\substack{j=1\\ j\neq i}}^{p}\left(1 - F_{V_j}(\mathrm{d})\right)\right] \tag{30}$$

Applying this result to the minimum of DFs leads to the following proposition:

Proposition 14 (*Pdf* of the minimum)**.** *Under the assumption that* $(Y_i|\mathbf{x}) \amalg (Y_j|\mathbf{x})\ \forall i \neq j$*, the pdf of* $D \equiv D(\mathbf{Y}|\mathbf{x}) = \min_{i=1,2,\cdots,p} d_i\left(\mathbf{Y_i}|\mathbf{x}\right)$ *is given by*

$$f_D(\mathrm{d}) = \sum_{i=1}^{p}\left[f_{V_i}(\mathrm{d}) \cdot \prod_{\substack{j=1\\ j\neq i}}^{p}\left(1 - F_{V_j}(\mathrm{d})\right)\right] \tag{31}$$

where $f_{V_i}(\cdot)$ *and* $F_{V_i}(\cdot)$ *are respectively the pdf's and the cdf's of the random variables* $V_i \equiv d_i(Y_i|\mathbf{x})$ *for* $i = 1, 2, \cdots, p$*.*

The values of d for which $f_D(\mathrm{d})$ is non zero depends on the types of individual DFs $d_i\left(\mathbf{Y_i}|\mathbf{x}\right)$. Without any targeted response, $\mathrm{d} \in (0, 1)$ but this could be more narrow otherwise.

The expected minimum desirability can then be computed by intergration of its density for any combination of factors levels $\mathbf{x}$ and the optimum can be found using an adequate optimization algorithm.

5 Distribution of DI when models are estimated

All the densities presented in the preceding sections depend on the quantities $E[Y_i|x]$ and $\sigma^2_{\epsilon_i}$ $(i = 1, 2, \ldots, p)$, that are unknown in practice. Indeed, the models given by equation (**??**) are not known and have to be estimated. Using powerful tools of experimental design, a set of n experiences within the domain of interest

χ can be adequately chosen and performed to collect data on which the assumed models can be fitted. For each property, the estimated model is denoted by

$$\hat{Y}_i = \hat{f}_i\left(x, \hat{\boldsymbol{\beta}}_{\boldsymbol{i}}\right). \tag{32}$$

Thanks to equation (**??**), the responses can be predicted for any point x in the experimental domain χ, and the unknown quantities, $E[Y_i|x]$ and $\sigma^2_{\epsilon_i}$ $(i = 1, 2, \dots, p)$, can be estimated by

$$\hat{E}[Y_i|x] = \hat{Y}_i = \hat{f}_i\left(x, \hat{\boldsymbol{\beta}}_{\boldsymbol{i}}\right) \quad \text{and} \quad \hat{\sigma}^2_{\epsilon_i} = \frac{1}{n - q_i}\sum_{j=1}^{n}(y_{ij} - \hat{y}_{ij})^2 \tag{33}$$

where n is the number of observations, q_i is the number of model parameters, y_{ij} $(j = 1, 2, \cdots, n)$ are the observed values for the i^{th} property and $\hat{y}_{ij}$ are the corresponding predicted values.

Before using the estimated models for prediction, it's crucial to analyze their adequacy for the training data (using R^2, adjusted R^2, goodness-of-fit tests,...) and their performance for predicting responses of new points x (using a test set, cross-validation, bootstrap,...). Nevertheless, even if the conclusions of these analysis are good, it is well known that all models are wrong and that $\hat{E}[Y_i|x]$ and $\hat{\sigma}^2_{\epsilon_i}$ are only uncertain estimation of the true parameters $E[Y_i|x]$ and $\sigma^2_{\epsilon_i}$, leading to uncertain estimation of the densities of DFs and DI, as well as uncertain estimation of the expectation of interest: $E[D(Y|x)]$.

There is no method available to quantify the propagation of the uncertainty of the estimated parameters $E[Y_i|x]$ and $\sigma^2_{\epsilon_i}$ on the estimated expected desirability index $E[D(Y|x)]$. This is a subject for futur research. If the uncertainty is small, *i.e.* $E[Y_i|x]$ and $\sigma^2_{\epsilon_i}$ are precisely estimated by $\hat{E}[Y_i|x]$ and $\hat{\sigma}^2_{\epsilon_i}$, using the estimated densities provides quite accurate estimate of $E[D(Y|x)]$.

6 Conclusion

New DFs based on the standard Normal distribution are presented for cases of maximized, minimized or targeted responses. The parameters of those functions are easy to choose in practice with or without the knowledge of a production specialist.

Due to the error terms of the assumed models linking the responses to the factors, DFs and DIes are random variables and the optimization consists of maximizing the expected desirabity index to take the random errors into account, as suggested by **?**.

The densities of the Normal based DFs are derived analytically using the univariate density transformation theorem under the assumption that the error terms are Normally distributed.

Under the assumption that conditionally to the factors levels, the responses are independent, the densities of the geometric mean and the arithmetic mean DIes are provided by applying two theorems of the statistical literature about the density of the product of random variables and the sum of random variables (*convolution*). Those densities can be used to compute some kind of prediction interval for the DI of a new experiment. But they are difficult to handle when integrating to obtain the quantity of interest, the expected DI. Under the same independence assumption, the expected DI can be computed on the basis of the expected weighted DFs for which densities are also known and more easy to handle. The density of the minimum of DFs is also derived and can be integrated to obtain the expected minimum.

The developed densities assume that the models parameters as well as the errors variances are known. Nevertheless, in practice, only estimates of those quantities are available. If those estimates are quite accurate, the computation of the expected DI on the basis of estimated densities is of reasonnable precision. Otherwise, the estimated expected DI is suiled with error and the propagation of the models prediction error has to be taken into account. A way to quantify this uncertainty propagation is through the use of the Delta method theorem (**?**).

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