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FIRST EXIT FROM AN INTERVAL AND TOTAL DURATION OF STAY INSIDE AN INTERVAL BY A POISSON PROCESS WITH A NEGATIVE EXPONENTIAL COMPONENT

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First exit from an interval and total duration of stay inside an interval by a Poisson process with a negative exponential component

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Abstract

Several two-boundary problems for a Poisson process with an exponential component are solved in the present article. We obtain the integral transforms of the joint distribution of the first exit time from a fixed interval and the value of the overshoot through boundaries at the epoch of the exit. The Laplace transform is found of the total duration of the process's stay inside the interval.

1 Introduction

Many interesting problems in applied probability (inventory theory, option pricing, dam theory, ruin problems, etc.) are related to the joint distribution of the first exit time and the value of the process at the epoch of exit. In the present paper we study a Poisson process with a negative exponential component, i.e. a compound Poisson process with arbitrary positive jumps and exponentially distributed negative jumps (for a more rigorous definition of the process see section 2). First, a two-sided exit problem is solved, which consists of specifying the joint distribution of the first exit time from a fixed interval and the value of the overshoot at the epoch of the exit. The second problem solved in this framework is determining of the distribution of the duration of stay of the Poisson process with a negative exponential component inside the interval. We derive integral equations (systems of integral equations) for the integral transforms using probabilistic methods, space homogeneity and the strong Markov property of the process. To solve these equations (systems), we use the method of successive iterations.

Exit problems for different types of Lévy processes (and for compound Poisson process in particular) have been considered by many authors (see for instance [9], [13]). Compound Poisson process with linear deterministic decrease between positive and negative jumps was studied by D. Perry, W. Stadje, and S. Zacks in [16]. In [15] these authors consider the lower boundary crossing problem for the difference of two independent compound Poisson processes. A martingale approach for solving exit problems has been applied in [14]. The integral transforms of the duration of stay of the process inside the interval have been obtained in [5] for a semi-continuous process with independent increments. The distribution of the duration of stay of the Wiener process inside the interval has been found in the same paper.

The paper is organized as follows. First we state auxiliary results and then introduce the process which we are going to study. In Section 2 we derive and solve a system of linear integral equations for the integral transforms of the joint distribution of the first exit time from a fixed interval and the value of the overshoot at the epoch of the exit. The corresponding results for a Poisson process with a negative exponential component are proved in Section 3. The distribution os duration of stay of the Poisson process with a negative exponential component inside the fixed interval is determined in Section 4.

2 Main definitions and auxiliary results

To derive the joint distribution distribution of the first exit time and the value of the overshoot at the epoch of the exit for a Poisson process with a negative exponential component we apply a general theorem for Lévy processes which has been proved in [9]. Before stating the theorem we introduce a Lévy process and auxiliary functionals which we will use while proving the theorem. Let $\{\xi(t);\, t\geq 0\}$ be a real-valued Lévy process, i.e. homogeneous process with independent increments ([18], p.110) with the Laplace exponent

$$k(p)=\frac{1}{2}p^2\sigma^2-\alpha p+\int_{-\infty}^{\infty}\left(e^{-px}-1+\frac{px}{1+x^2}\right)\Pi(dx),\qquad \mathrm{Re}\,p=0. \tag{1}$$

Here and in the sequel the paths of the process are supposed to be right-continuous and $\xi(0)=0$. Note, that the introduced process is a strong Markov process [17]. For $x\geq 0$ introduce the random variables

$$\tau^x=\inf\{\,t:\ \xi(t)>x\,\},\quad T^x=\xi(\tau^x)-x,\qquad \tau_x=\inf\{\,t:\ \xi(t)<-x\,\},\quad T_x=-\xi(\tau_x)-x$$

the first passage time of the level x and the value of the overshoot through this level at the epoch of the passage; the first passage time of the level $-x$ and the value of the overshoot through this level at the epoch of the passage. Integral transforms of the joint distributions $\{\,\tau^x,T^x\,\}$, $\{\,\tau_x,T_x\,\}$, satisfy the following equalities

$$E\,[\,e^{-s\tau^x-pT^x}\,]=\left(E\,[e^{-p\xi^+(\nu_s)}]\right)^{-1}E\,[\,e^{-p(\xi^+(\nu_s)-x)};\ \xi^+(\nu_s)>x\,],\qquad \mathrm{Re}\,p\geq 0,$$
$$E\,[\,e^{-s\tau_x-pT_x}\,]=\left(E\,[e^{p\xi^-(\nu_s)}]\right)^{-1}E\,[\,e^{p(\xi^-(\nu_s)+x)};\ -\xi^-(\nu_s)>x\,],\qquad \mathrm{Re}\,p\geq 0, \tag{2}$$

where $\xi^+(t)=\sup_{u\leq t}\xi(u)$, $\xi^-(t)=\inf_{u\leq t}\xi(u)$, ν_s is an exponential variable with parameter $s>0$, independent of the process $\{\xi(t),\ t\geq 0\}$, $P[\,\nu_s>t\,]=e^{-st}$, and

$$E[e^{-p\,\xi^{\pm}(\nu_s)}\,]=\exp\left\{\int_0^{\infty}\frac{1}{t}\,e^{-st}\,E\,[\,e^{-p\xi(t)}-1;\ \pm\xi(t)>0\,]\,dt\right\},\qquad \pm\mathrm{Re}\,p\geq 0.$$

The formulae (2) were obtained by Pecherskii E. and Rogozin B. in [11]. A simple proof of these equalities is given in [9]. Let us give a strict definition of the first exit time from the interval. Let $B>0$ be fixed, $y\in[0,B]$, $x=B-y$, $\xi(0)=0$ and define

$$\chi(y)=\inf\{\,t>0:\ y+\xi(t)\notin[0,B]\,\}$$

the instant of the first exit by the process $y+\xi(t)$ from the interval $[0,B]$. The random variable $\chi(y)$ is a Markov time [4] and $P\,[\,\chi(y)<\infty\,]=1$. Observe that the exit from the interval $[0,B]$ can take place either through the upper level B, or through the lower level 0. Introduce events
$A^B=\{\,\xi(\chi(y))>B\,\}$ — the exit from the interval by the process takes place through the upper level;
$A_0=\{\,\xi(\chi(y))<0\,\}$ — the exit from the interval by the process takes place through the lower level. Denote by

$$X(y)=(\xi(\chi(y))-B)\,I_{A^B}+(-\xi(\chi(y)))\,I_{A_0},\qquad P\,[\,A^B+A_0\,]=1,$$

the value of the overshoot through a boundary at the epoch of the exit from the interval by the given process. Here $I_A = I_A(\omega)$ is an indicator of the event A.

The following theorem is true.

Theorem 1 ([9],[10]). *Let $\{\xi(t);\ t \geq 0\}$ be a real valued Lévy process increments with the Laplace exponent (1), $B > 0$ is fixed, $y \in [0, B]$, $x = B - y$, $\xi(0) = 0$, and*

$$\chi(y) = \inf\{\, t > 0 :\ y + \xi(t) \notin [0, B]\,\}, \qquad X(y) = (\xi(\chi(y)) - B)\, I_{A^B} + (-\xi(\chi(y)))\, I_{A_0}$$

the moment of the first exit by the process $y+\xi(t)$ from the interval $[0, B]$ and the value of the overshoot through a boundary at the epoch of the exit from the interval by the given process. The Laplace transforms of the joint distribution of the variables $\{\,\chi(y), X(y)\,\}$ for $s > 0$ satisfy the following formulae

$$\begin{aligned} E\,[\,e^{-s\chi(y)};\ X(y) \in du,\ A^B\,] &= f_+^s(x, du) + \int_0^\infty f_+^s(x, dv)\, K_+^s(v, du), \\ E\,[\,e^{-s\chi(y)};\ X(y) \in du,\ A_0\,] &= f_-^s(y, du) + \int_0^\infty f_-^s(y, dv)\, K_-^s(v, du), \end{aligned} \tag{3}$$

where

$$\begin{aligned} f_+^s(x, du) &= E\,[\,e^{-s\tau^x};\ T^x \in du\,] - \int_0^\infty E\,[\,e^{-s\tau_y};\ T_y \in dv\,]\, E\,[\,e^{-s\tau^{v+B}};\ T^{v+B} \in du\,], \\ f_-^s(y, du) &= E\,[\,e^{-s\tau_y};\ T_y \in du\,] - \int_0^\infty E\,[\,e^{-s\tau^x};\ T^x \in dv\,]\, E\,[\,e^{-s\tau_{v+B}};\ T_{v+B} \in du\,]; \end{aligned}$$

and $K_\pm^s(v, du) = \sum\limits_{n=1}^{\infty} K_\pm^{(n)}(v, du, s)$, $v \geq 0$ are the series of successive iterations, $n \in \mathbb{N}$; which are given by

$$K_\pm^{(1)}(v, du, s) = K_\pm(v, du, s), \quad K_\pm^{(n+1)}(v, du, s) = \int_0^\infty K_\pm^{(n)}(v, dl, s)\, K_\pm(l, du, s) \tag{4}$$

where the kernels $K_\pm(v, du, s)$, are determined by the defining equalities

$$\begin{aligned} K_+(v, du, s) &= \int_0^\infty E\,[\,e^{-s\tau_{v+B}};\ T_{v+B} \in dl\,]\, E\,[\,e^{-s\tau^{l+B}};\ T^{l+B} \in du\,], \\ K_-(v, du, s) &= \int_0^\infty E\,[\,e^{-s\tau^{v+B}};\ T^{v+B} \in dl\,]\, E\,[\,e^{-s\tau_{l+B}};\ T_{l+B} \in du\,]. \end{aligned} \tag{5}$$

To summarize, we have obtained the integral transforms of the joint distribution of the first exit time and the value of the overshoot at the epoch of the exit. These transforms are given in terms of integral transforms of one-boundary functionals which are well known. We apply now the formulae of Theorem 1 for the case when the underlying process is the Poisson process with an exponentially distributed negative component. Let us explain what do we mean by this. Let $\eta \in (0, \infty)$ be a positive random variable, and γ be an exponential variable with the parameter $\lambda > 0$: $P\,[\,\gamma > x\,] = e^{-\lambda x}$, $x \geq 0$. Introduce the random variable $\xi \in \mathbb{R}$ by its distribution function

$$F(x) = a\, e^{x\lambda}\, I\{x \leq 0\} + (a + (1 - a)\, P\,[\,\eta \leq x\,])\, I\{x > 0\}, \qquad a \in (0, 1), \quad \lambda > 0.$$

Consider a right-continuous Poisson process $\{\xi(t);\ t \geq 0\}$ with the Laplace exponent

$$k(p) = c \int_{-\infty}^{\infty} (e^{-xp} - 1)\, dF(x) = a_1 \frac{p}{\lambda - p} + a_2(E[\,e^{-p\eta}] - 1), \qquad c > 0, \quad \mathrm{Re}\, p = 0, \tag{6}$$

where $a_1 = ac$, $a_2 = (1 - a)c$. Here and in the sequel we will call such process the Poisson process with a negative exponential component. Note, that jumps of the process $\{\xi(t);\ t \geq 0\}$

occur at the time epochs that are exponentially distributed with parameter c. With the probability $1-a$ there occur positive jumps with value distributed as η, and with the probability a there occur negative jumps (jumps, which value is γ that is exponentially distributed with the parameter λ). The first term of (6) is the simplest case of a rational function, while the second term is nothing but the Laplace exponent of a monotone Poisson process with positive jumps of value η. It is well known fact (see for instance [1] p.65, [2]), that in this case the equation $k(p)-s=0,\quad s>0$ has a unique root $c(s)\in(0,\lambda)$, in the semi-plane $\mathrm{Re}\,p>0$. Moreover, for the integral transforms of the random variables $\xi^+(\nu_s),\quad \xi^-(\nu_s)$ the following formulae hold

$$
\begin{aligned}
E[\,e^{-p\xi^-(\nu_s)}] &= \frac{c(s)}{\lambda}\,\frac{\lambda-p}{c(s)-p}, \qquad \mathrm{Re}\,p\le 0,\\
E[\,e^{-p\xi^+(\nu_s)}] &= \frac{s\lambda}{c(s)}\,(p-c(s))\,R(p,s), \qquad \mathrm{Re}\,p\ge 0,
\end{aligned}
\tag{7}
$$

where

$$
R(p,s)=\big(a_1p+(p-\lambda)[s-a_2(\,E[\,e^{-p\eta}]-1)]\big)^{-1}, \qquad \mathrm{Re}\,p\ge 0,\quad p\ne c(s) \tag{8}
$$

It follows from (2) and (7) after some calculations that the integral transforms of the joint distributions $\{\tau_x,\,T_x\},\quad \{\tau^x,\,T^x\}$ of the Poisson process with a negative exponential component satisfy the equalities

$$
\begin{aligned}
&E[\,e^{-s\tau_x};\,T_x\in du\,]=(\lambda-c(s))\,e^{-xc(s)}\,e^{-\lambda u}\,du=E[\,e^{-s\tau_x}]\,P\,[\,\gamma\in du\,], \qquad (9)\\
&\int_0^\infty e^{-px}\,E[\,e^{-s\tau^x-z\xi(\tau^x)}]\,dx=\frac{1}{p}\left(1-\frac{p+z-c(s)}{z-c(s)}\,\frac{R(p+z,s)}{R(z,s)}\right), \quad \mathrm{Re}\,p>0,\ \mathrm{Re}\,z\ge 0.
\end{aligned}
$$

The first equality of (9) yields that τ_x and T_x are independent, moreover, for all $x\ge 0$ the value of the overshoot through the lower level T_x is exponentially distributed with the parameter λ. This fact serves as a characterizing feature of the Poisson process with a negative exponential component. Observe, that the function $R(p,s)$ is analytic in the semi-plane $\mathrm{Re}\,p>c(s)$, and $\lim\limits_{p\to\infty}R(p,s)=0$. Therefore, it allows the representation in the form of an absolutely convergent Laplace integral ([3])

$$
R(p,s)=\int_0^\infty e^{-px}R_x(s)\,dx, \qquad \mathrm{Re}\,p>c(s). \tag{10}
$$

We will call the function $R_x(s),\quad x\ge 0$ the resolvent of the Poisson process with a negative exponential component. We assume that $R_x(s)=0$, for $x<0$. Notice, that $R_0(s)=\lim\limits_{p\to\infty}p\,R(p,s)=(c+s)^{-1}$, and the equalities (7) imply that

$$
P\,[\,\xi^-(\nu_s)=0\,]=\frac{c(s)}{\lambda}, \qquad P\,[\,\xi^+(\nu_s)=0\,]=\frac{\lambda}{c(s)}\,\frac{s}{s+c}.
$$

The second formula of (7) yields

$$
R(p,s)=\frac{c(s)}{s\lambda}\,\frac{1}{p-c(s)}\,E[\,e^{-p\xi^+(\nu_s)}], \qquad \mathrm{Re}\,p>c(s). \tag{11}
$$

The functions

$$
\frac{1}{p-c(s)}=\int_0^\infty e^{-u(p-c(s))}du,\ \mathrm{Re}\,p>c(s), \quad E[\,e^{-p\xi^+(\nu_s)}]=\int_0^\infty e^{-up}d\,P\,[\,\xi^+(\nu_s)<u\,],\ \mathrm{Re}\,p\ge 0,
$$

which enter the right-hand side of (11), are the Laplace transforms for $\mathrm{Re}\,p>c(s)$. Therefore, the original functions of the left-hand side and the right-hand side of (11) coincide, and

$$
R_x(s)=\frac{c(s)}{s\lambda}\int_{-0}^{x}e^{c(s)(x-u)}d\,P\,[\,\xi^+(\nu_s)<u\,], \qquad x\ge 0. \tag{12}
$$

The latter formula is another representation for the resolvent of the Poisson process with a negative exponential component. Note, that the representation for the resolvent similar to (12) was obtained by V. Shurenkov and V. Suprun in [20, 19]. The representation (12) implies that $R_x(s), \quad x \geq 0$ is positive, monotone, continuous, increasing function and $R_x(s) < A(s)\exp\{xc(s)\}, \quad 0 < A(s) < \infty$. Consequently

$$\int_0^\infty R_x(s) e^{-\alpha x} dx < \infty, \qquad \alpha > c(s).$$

Moreover, in the neighborhood of any $x \geq 0$ the function $R_x(s)$ has bounded variation. Hence, the inversion formula [3, p. 68] is valid

$$R_x(s) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} e^{xp} R(p,s)\, dp, \qquad \alpha > c(s). \tag{13}$$

The latter equality together with (10) determines the resolvent of the Poisson process with a negative exponential component. Now we state the main results.

3 Exit from the interval by the Poisson process with the negative exponential component

In this section the integral transforms of the joint distributions of the instant of the first exit from the interval by the Poisson process with a negative exponential component and the value of the overshoot through a boundary are determined. The following corollary from Theorem 1 holds.

Corollary 1. *Let* $\{\xi(t);\, t \geq 0\}$ *be a Poisson process with a negative exponential component as specified in the previous section,* $B \geq 0, \quad y \in [0, B], \quad x = B - y, \quad \xi(0) = 0,$ *and*

$$\chi(y) = \inf\{\, t > 0 :\ y + \xi(t) \notin [0,B]\,\}, \qquad X(y) = (\xi(\chi(y)) - B)\, I_{A^B} + (-\xi(\chi(y)))\, I_{A_0}$$

the moment of the first exit from the interval and the value of the of the overshoot through one of the boundaries. Then for $s > 0$,

1) the integral transforms of the joint distribution $\{\, \chi(y),\ X(y)\,\}$ *satisfy the following equalities*

$$E[\, e^{-s\chi(y)};\ X(y) \in du,\ A_0] = e^{-\lambda u}(\lambda - c(s))\, e^{-yc(s)} \left(1 - E[\, e^{-s\tau^x - c(s)\xi(\tau^x)}\,]\right) K(s)^{-1} du, \tag{14}$$

$$E[\, e^{-s\chi(y)};\ X(y) \in du,\ A^B\,] = E[\, e^{-s\tau^x};\ T^x \in du\,] - E\,[\, e^{-s\chi(y)};\ A_0]\ E\,[\, e^{-s\tau^{\gamma+B}};\ T^{\gamma+B} \in du\,],$$

where

$$K(s) = 1 - E[\, e^{-s\tau_B}]\ E[\, e^{-s\tau^{\gamma+B} - c(s)T^{\gamma+B}}\,],$$

$$E[\, e^{-s\tau^{\gamma+B} - c(s)T^{\gamma+B}}\,] = \lambda \int_0^\infty e^{-\lambda u} E\,[\, e^{-s\tau^{u+B} - c(s)T^{u+B}}\,]\, du,$$

in particular

$$E[\, e^{-s\chi(y)};\ A_0\,] = \left(1 - \frac{c(s)}{\lambda}\right) e^{-yc(s)} \left(1 - E[\, e^{-s\tau^x} e^{-c(s)\xi(\tau^x)}\,]\right) K(s)^{-1}, \tag{15}$$

$$E[\, e^{-s\chi(y)};\ A^B\,] = E[\, e^{-s\tau^x}] - E[\, e^{-s\chi(y)};\ A_0\,]\ E[\, e^{-s\tau^{\gamma+B}}];$$

2) for the Laplace transforms of the random variable $\chi(y)$ *the following representations hold*

$$E[\,e^{-s\chi(y)};\ X(y)\in du,\ A_0\,] = e^{-\lambda(u+B)}\ \frac{R_x(s)}{\hat R_B(\lambda,s)}\,du, \qquad E[\,e^{-s\chi(y)};\ A_0\,] = \frac{1}{\lambda}\,e^{-\lambda B}\ \frac{R_x(s)}{\hat R_B(\lambda,s)},$$

$$E[\,e^{-s\chi(y)};\ A^{\,B}\,] = 1 - \frac{R_x(s)}{\hat R_B(\lambda,s)}\ \left[\frac{1}{\lambda}\,e^{-\lambda B} + s\lambda\,\hat S_B(\lambda,s)\right] + s\lambda\,S_x(s),$$

$$\int_0^\infty e^{-st}P\,[\,\chi(y)>t\,]\,dt = \lambda\ \frac{R_x(s)}{\hat R_B(\lambda,s)}\ \hat S_B(\lambda,s) - \lambda\,S_x(s), \tag{16}$$

where $R_x(s),\quad x\ge 0$ *the resolvent of the process, defined by (10), (13);*

$$S_x(s)=\int_0^x R_u(s)\,du, \qquad \hat R_B(\lambda,s)=\int_B^\infty e^{-\lambda u}R_u(s)\,du, \qquad \hat S_B(\lambda,s)=\int_B^\infty e^{-\lambda u}S_u(s)\,du.$$

Proof. We apply the results of Theorem 1 for the Poisson process with a negative exponential component, which in this case take a simplified form. Using the equality (9) and the defining formulae (5) of the kernels $K_\pm(v,du,s)$, yields

$$K_+(v,du,s) = \left(1-\frac{c(s)}{\lambda}\right)\ e^{-c(s)(v+B)}\,E\,[\,e^{-s\tau^{\gamma+B}};\ T^{\gamma+B}\in\ du\,],$$

$$K_-(v,du,s) = e^{-\lambda u}(\lambda-c(s))\,e^{-c(s)B}\,E\,[\,e^{-s\tau^{v+B}-c(s)T^{v+B}}\,]\,du,$$

where γ is an exponentially distributed random variable with the parameter λ. The latter equalities imply the following form of the successive iterations $K_\pm^{(n)}(v,du,s),\quad n\in N$ of the kernels $K_\pm(v,du,s)$:

$$K_-^{(n)}(v,du,s) = E[\,e^{-s\tau^{v+B}-c(s)T^{v+B}}\,]\ \left(E[\,e^{-s\tau_B}]\right)^n\ \left(E[\,e^{-s\tau^{\gamma+B}-c(s)T^{\gamma+B}}\,]\right)^{n-1}\ \lambda e^{-\lambda u}\,du,$$

$$K_+^{(n)}(v,du,s) = e^{-c(s)v}\ \left(E[\,e^{-s\tau_B}]\right)^n\ \left(E[\,e^{-s\tau^{\gamma+B}-c(s)T^{\gamma+B}}\,]\right)^{n-1}\ E[\,e^{-s\tau^{\gamma+B}};\ T^{\gamma+B}\in\ du\,].$$

Then

$$K_-^s(v,du) = \sum_{n=1}^\infty K_-^{(n)}(v,du,s) = E[\,e^{-s\tau^{v+B}}\ e^{-c(s)T^{v+B}}\,]\ \ E[\,e^{-s\tau_B}]\ K(s)^{-1}\lambda\,e^{-\lambda u}\,du,$$

$$K_+^s(v,du) = \sum_{n=1}^\infty K_+^{(n)}(v,du,s) = e^{-c(s)v}\ E[\,e^{-s\tau_B}]\ E[\,e^{-s\tau^{\gamma+B}};\ T^{\gamma+B}\in\ du\,]\ K(s)^{-1},$$

where

$$K(s) = 1 - E[\,e^{-s\tau_B}]\ E[\,e^{-s\tau^{\gamma+B}-c(s)T^{\gamma+B}}\,].$$

Substituting the obtained expressions for $K_\pm^s(v,du)$ into the equalities (3) of Theorem 1, we get the formulae (14) of Corollary 1. Integrating (14) with respect to $u\in\mathbb{R}_+$, yields the equalities (15) of Corollary 1. Further, utilizing the representation (10), (13) for the resolvent and the formulae (9), we obtain the resolvent representations for the functions $E\,[\,e^{-s\tau^{\,x}-c(s)\xi(\tau^x)}\,]$, $E[\,e^{-s\tau^x}\,]$:

$$E\,[\,e^{-s\tau^{\,x}}e^{-c(s)\xi(\tau^x)}\,] = 1 - e^{-c(s)x}\ R_x(s)\ r(c(s),s),$$

$$E[\,e^{-s\tau^x}\,] = 1 - \frac{s\lambda}{c(s)}\ R_x(s) + s\lambda\,S_x(s),$$

where

$$S_x(s)=\int_0^x\ R_u(s)\,du, \qquad r(c(s),s) = \left.\frac{d}{dp}R(p,s)^{-1}\right|_{p=c(s)}.$$

Substituting now the latter equalities into (14), (15) yields the formulae (16) of the corollary. Note, that for a random walk with a geometrically distributed negative component the resolvent representations similar to (16) have been obtained in [7]. □

Remark 1. *The formulae of Corollary 1 can be also obtained after solving the following system of the integral equations*

$$\begin{aligned} E\,[\,e^{-s\tau^x};\ T^x\in du\,] =& E\,[\,e^{-s\chi(y)};\ X(y)\in du,\ A^B\,] \\ &+\int_0^\infty E\,[\,e^{-s\chi(y)};\ X(y)\in dv,\ A_0\,]\,E\,[\,e^{-s\tau^{v+B}};\ T^{v+B}\in du\,], \\ E\,[\,e^{-s\tau_y};\ T_y\in du\,] =& E\,[\,e^{-s\chi(y)};\ X(y)\in du,\ A_0\,] \\ &+\int_0^\infty E\,[\,e^{-s\chi(y)};\ X(y)\in dv,\ A^B\,]\,E\,[\,e^{-s\tau_{v+B}};\ T_{v+B}\in du\,]. \end{aligned} \tag{17}$$

which takes a simple form for the Poisson process with a negative exponential component. This system has been solved in [9] for general Lévy processes (see also Lemma 1 in [10]).

4 Duration of stay inside the interval by the Poisson process with a negative exponential component

In this section we obtain the Laplace transform of the total duration of stay of the process inside the interval. Let $\{\xi(t);\ t\geq 0\}$, $\xi(0)=0$ be a real-valued Poisson process with a negative exponential component. Denote by

$$\sigma_y(t)=\int_0^t I\,\{\,y+\xi(u)\in[0,B]\,\}\,du,\qquad y\in\mathbb{R}$$

the total duration of stay of the process $y+\xi(\cdot)$ inside of the interval $[0,B]$ up to the moment t. We will determine

$$C_a^s(y)=E[e^{-a\sigma_y(\nu_s)}],\qquad y\in\mathbb{R},\quad a\geq 0$$

the Laplace transform of the total duration of stay of the process $y+\xi(\cdot)$ inside the interval $[0,B]$ on the exponentially distributed time interval $[0,\nu_s]$. We require several auxiliary functions to solve the stated problem. Let us introduce these functions. Let $y\geq 0$, $\xi(0)=0$, and $\tau_y=\inf\{\,t:\ y+\xi(t)<0\,\}$, $\sigma_y=\sigma_y(\tau_y)$ be the instant of the first exit from the upper semi-plane by the process $y+\xi(\cdot)$ and the total duration of stay of the process inside $[0,B]$ on the time interval $[0,\tau_y]$. On the event $\{\,\tau_y=\infty\,\}$ we set per definition $\sigma_y=\infty$. The following statement is true.

Lemma 1. *Let $\{\xi(t);\ t\geq 0\}$, $\xi(0)=0$ be a Poisson process with a negative exponential component as specified above. Then for the integral transform*

$$D_a^s(y)=E[e^{-s\tau_y-a\sigma_y}],\qquad y\geq 0,\quad a\geq 0,\ s>0$$

of the joint distribution $\{\,\tau_y,\sigma_y\,\}$ the following equality holds

$$D_a^s(y)=\Big[V_a^s(B-y)-a\,R_{B-y}(s+a)\,e^{-c(s)(B-y)}\Big]\,V_a^s(B)^{-1}E[\,e^{-s\tau_y}],\qquad y\geq 0, \tag{18}$$

where $R_x(s)=0$ for $x<0$, and the function $V_a^s(x)$, $x\in\mathbb{R}$ is given by

$$V_a^s(x)=1+a(\lambda-c(s))\int_0^x e^{-uc(s)}R_u(s+a)\,du,\quad x\geq 0,\qquad V_a^s(x)=1,\quad x<0. \tag{19}$$

Proof. For the functions $D_a^s(y)$, $y\geq 0$ according to the total probability law and the strong Markov property of the underlying process we can write the system

$$\begin{aligned} D_a^s(y)=&\,E\,[e^{-(s+a)\chi(y)};\ A_0\,]+\int_0^\infty E\,[e^{-(s+a)\chi(y)};\ X(y)\in dv,A^B\,]\,E\,[e^{-s\tau_v};\ T_v>B\,] \\ &+\int_0^\infty E\,[e^{-(s+a)\chi(y)};\ X(y)\in dv,A^B\,]\int_0^B E\,[e^{-s\tau_v};\ T_v\in du\,]\,D_a^s(B-u),\quad y\in[0,B], \\ D_a^s(y)=&\,E\,[e^{-s\tau_{y-B}};\ T_{y-B}>B\,]+\int_0^B E\,[e^{-s\tau_{y-B}};\ T_{y-B}\in du\,]\,D_a^s(B-u),\qquad y>B. \end{aligned}$$

The first equation of this system represents the fact that the total duration of stay of the process $y+\xi(\cdot)$, $y \in [0,B]$ inside $[0,B]$ on the time interval $[0,\tau_y]$ is realized either on sample paths which do not intersect the upper level (the first term of the right-hand side of the equation), or on the paths which do intersect the upper level and then overleap the interval $[0,B]$ (the second term of the right-hand side of the equation), or on the paths which intersect the upper level and then return to the interval $[0,B]$ (the third term of the right-hand side of the equation).The second equation is written analogously. Now, using the first equality of (9) and the equalities (16) of Corollary 1, we get after some simplifications

$$D_a^s(y) = \frac{1}{\lambda}e^{-\lambda B}\frac{R_{B-y}(s+a)}{\hat{R}_B(\lambda,s+a)} + (\lambda - c(s))\, E_y^{s+a}(c(s))\left(\frac{1}{\lambda}e^{-\lambda B} + \tilde{D}_a^s(\lambda)\right), \qquad y \in [0,B], \quad (20)$$

$$D_a^s(y) = \left(1 - \frac{c(s)}{\lambda}\right) e^{-c(s)(y-B)}e^{-\lambda B} + (\lambda - c(s))\, e^{-c(s)(y-B)}\tilde{D}_a^s(\lambda), \qquad y > B,$$

where

$$\tilde{D}_a^s(\lambda) = \int_0^B e^{-\lambda u} D_a^s(B-u)\, du, \qquad E_y^{s+a}(c(s)) = E\,[\, e^{-(s+a)\chi(y)}e^{-c(s)X(y)};\ A^B\,].$$

The only unknown function in (20) is $\tilde{D}_a^s(\lambda)$. This function can be determined from the first equation of (20). For this we first make auxiliary calculations. Denote

$$T_x^s(z) = E\,[e^{-s\tau^x - zT^x}\,], \qquad x \ge 0, \quad \mathrm{Re}\, z \ge 0.$$

Utilizing the second equality of (9) and the defining formulae (10), (13) yields

$$T_x^{s+a}(c(s)) = e^{xc(s)} + a\frac{\lambda - c(s)}{c(s) - c(s+a)}R_x(s+a)$$
$$+\, a(\lambda - c(s))\int_0^x e^{-c(s)(u-x)}R_u(s+a)\, du, \qquad x \ge 0.$$

Using the latter equality and (14), (16) of Corollary 1, we find

$$E_y^{s+a}(c(s)) = e^{xc(s)}V_a^s(B-y)$$
$$- \frac{e^{-B(\lambda - c(s))}}{\lambda - c(s)}\,\frac{R_{B-y}(s+a)}{\hat{R}_B(\lambda,s+a)}V_a^s(B) - aR_{B-y}(s+a), \qquad y \in [0,B]. \quad (21)$$

Multiplying this equality by $e^{-\lambda(B-y)}$ and integrating with respect to $y \in [0,B]$, implies

$$(\lambda - c(s))\int_0^B e^{-\lambda(B-y)}E_y^{s+a}(c(s))\, dy = 1 - V_a^s(B)\left[1 + \frac{\check{R}_B(\lambda,s+a)}{\hat{R}_B(\lambda,s+a)}\right]e^{-B(\lambda - c(s))} \quad (22)$$

where $\check{R}_B(\lambda,s+a) = \int_0^B e^{-\lambda u}R_u(s+a)\, du$. Further, multiplying the first equation of (20) by $e^{-\lambda(B-y)}$ and integrating it with respect to $y \in [0,B]$, we get

$$\tilde{D}_a^s(\lambda) = \frac{1}{\lambda}e^{-\lambda B}\frac{\check{R}_B(\lambda,s+a)}{\hat{R}_B(\lambda,s+a)} + \left(\frac{1}{\lambda}e^{-\lambda B} + \tilde{D}_a^s(\lambda)\right)\left(1 - V_a^s(B)\left[1 + \frac{\check{R}_B(\lambda,s+a)}{\hat{R}_B(\lambda,s+a)}\right]e^{-B(\lambda - c(s))}\right)$$

which is a linear equation with respect to the function $\tilde{D}_a^s(\lambda)$. It yields

$$\tilde{D}_a^s(\lambda) + \frac{1}{\lambda}e^{-\lambda B} = \frac{1}{\lambda}e^{-c(s)B}V_a^s(B)^{-1}$$

Substituting this expression for the function $\tilde{D}_a^s(\lambda)$ into the equalities (20), and taking into account the fact, that $R_x(s) = 0$ for $x < 0$ and $V_a^s(x) = 1,\ x < 0$, implies the equality of the lemma. Thus, we have proved Lemma 1. □

Now for $y \ge 0$ we determine another auxiliary function

$$Q_a^s(y) = E\,[\,e^{-a\sigma_y(\nu_s)};\ \tau_y > \nu_s\,], \qquad y \ge 0, \quad a \ge 0$$

the Laplace transform of the total duration of stay of the process $y + \xi(\cdot)$ inside $[0, B]$ on the time interval $[0, \nu_s]$, on the event $\{\tau_y > \nu_s\}$, i.e. on the event that there is no exit from the upper semi-plane by the process $y + \xi(t)$ up to the epoch ν_s. The following statement is true.

Lemma 2. *Let $\{\xi(t);\ t \ge 0\}$, $\xi(0) = 0$ be a real-valued Poisson process with a negative exponential component with the Laplace exponent (6). Then the functions $Q_a^s(y)$, $y \ge 0$, for $s > 0$ satisfy the equality*

$$Q_a^s(y) = v_a^s(B - y) - aR_{B-y}(s + a) - v_a^s(B)\, D_a^s(y), \qquad y \ge 0, \tag{23}$$

where $R_x(s) = 0,\ x < 0$, and $D_a^s(y)$, $y \ge 0$ is given by (18), $v_a^s(x)$, $x \in \mathbb{R}$ is given by the following formula

$$v_a^s(x) = 1 + a\lambda \int_0^x R_u(s + a)\, du, \quad x \ge 0, \qquad v_a^s(x) = 1, \quad x < 0. \tag{24}$$

Proof. In accordance with the total probability law and due to the fact that $\chi(y)$, τ_y are the Markov times, for the functions $Q_a^s(y)$, $y \ge 0$ the following system of equations is valid

$$\begin{aligned}
Q_a^s(y) &= \frac{s}{s+a}\left(1 - E[\,e^{-(s+a)\chi(y)}]\right) + \int_0^\infty E\,[e^{-(s+a)\chi(y)};\ X(y) \in dv, A^B\,]\,(1 - E[\,e^{-s\tau_v}])\\
&\quad + \int_0^\infty E\,[e^{-(s+a)\chi(y)};\ X(y) \in dv, A^B\,] \int_0^B E\,[e^{-s\tau_v};\ T_v \in du\,]\, Q_a^s(B-u), \quad y \in [0, B],\\
Q_a^s(y) &= 1 - E[\,e^{-s\tau_{y-B}}] + \int_0^B E\,[e^{-s\tau_{y-B}};\ T_{y-B} \in du\,]\, Q_a^s(B-u), \qquad y > B.
\end{aligned}$$

Let us interpret the obtained equations. For the first equation we have that the duration of stay inside $[0, B]$ on the event $\{\tau_y > \nu_s\}$ can be realized on the following disjunct events: 1) sample paths of the process do not leave the interval $[0, B]$ (the first term of the right-hand side); 2) the paths intersect the upper level B and do not return to the interval (the second term); 3) the paths leave the interval $[0, B]$ through the upper level and then return to the interval (the third term of the equation). The second equation is set up analogously. Using the first equality of (9) and the equalities (16) it follows from the latter system that

$$\begin{aligned}
Q_a^s(y) &= 1 - \frac{1}{\lambda} e^{-\lambda B} \frac{R_{B-y}(s+a)}{\hat{R}_B(\lambda, s+a)} - a\lambda \frac{R_{B-y}(s+a)}{\hat{R}_B(\lambda, s+a)} \hat{S}_B(\lambda, s+a)\\
&\quad + a\lambda S_{B-y}(s+a) + (\lambda - c(s))\, E_y^{s+a}(c(s)) \left(\tilde{Q}_a^s(\lambda) - \frac{1}{\lambda}\right), \qquad y \in [0, B],\\
Q_a^s(y) &= 1 - \left(1 - \frac{c(s)}{\lambda}\right) e^{-c(s)(y-B)} + (\lambda - c(s))\, e^{-c(s)(y-B)} \tilde{Q}_a^s(\lambda), \qquad y > B,
\end{aligned} \tag{25}$$

where the function $E_y^{s+a}(c(s)) = E\,[\,e^{-(s+a)\chi(y) - c(s)X(y)};\ A^B\,]$ is determined by (21), and

$$\tilde{Q}_a^s(\lambda) = \int_0^B e^{-\lambda u} Q_a^s(B-u)\, du, \qquad \hat{S}_B(\lambda, s+a) = \int_B^\infty e^{-\lambda u} S_u(s+a)\, du.$$

The function $\tilde{Q}_a^s(\lambda)$ can be determined straightforwardly from the first equation of (25), since we have already given the auxiliary calculations in the previous lemma. Multiplying the first equation of (25) by $e^{-\lambda(B-y)}$ and integrating it with respect to $y \in [0, B]$, implies

$$\tilde{Q}_a^s(\lambda) - \frac{1}{\lambda} = -\frac{1}{\lambda} e^{-\lambda B} \left[1 + \frac{R_{B-y}(s+a)}{\hat{R}_B(\lambda, s+a)}\right] v_a^s(B) + (\lambda - c(s))\, \tilde{E}^{s+a}(c(s)) \left(\tilde{Q}_a^s(\lambda) - \frac{1}{\lambda}\right), \tag{26}$$

which is a linear equation for the function $\tilde{Q}_a^s(\lambda)$ and the function $\tilde{E}^{s+a}(c(s)) = \int_0^B e^{-\lambda(B-y)} E_y^{s+a}(c(s))\, dy$ is determined by (22), $v_a^s(x)$, $x \in \mathbb{R}$ is given by the equality (24) of Lemma 2. To obtain this equation we have also used the obvious identities

$$\lambda \hat{S}_B(\lambda, s+a) = \hat{R}_B(\lambda, s+a) + S_B(s+a)\, e^{-\lambda B},$$
$$\lambda \int_0^B e^{-\lambda u} S_u(s+a)\, du = \int_0^B e^{-\lambda u} R_u(s+a)\, du - S_B(s+a)\, e^{-\lambda B}.$$

Utilizing the equality (22), we find from (26)

$$\tilde{Q}_a^s(\lambda) - \frac{1}{\lambda} = -\frac{v_a^s(B)}{V_a^s(B)} \frac{1}{\lambda}\, e^{-c(s)B}.$$

Substituting the latter expression for the function $\tilde{Q}_a^s(\lambda)$ into (25), and taking into account the fact, that $v_a^s(x) = V_a^s(x) = 1,\ x < 0,$ we obtain the equality of Lemma 2. □

The auxiliary functions $D_a^s(y)$, $Q_a^s(y)$, $y \geq 0$ are determined by the equalities (18), (23), hence now we can obtain the integral transforms of the distribution of the duration of stay of the process inside the interval. The following statement holds.

Theorem 2. *Let $\{\xi(t);\ t \geq 0\}$, $\xi(0) = 0$ be a Poisson process with a negative exponential component with the Laplace exponent (6), $B > 0$, $a \geq 0$ and*

$$\sigma_y(t) = \int_0^t I\{\, y + \xi(u) \in [0, B]\,\}\, du, \quad C_a^s(y) = s \int_0^\infty e^{-st} E[e^{-a\sigma_y(t)}]\, dt, \qquad y \in \mathbb{R}$$

be the total duration of stay of the process $y + \xi(\cdot)$ inside the interval $[0, B]$ up to the instant t and the integral transform of the distribution of $\sigma_y(t)$. Then for the function $C_a^s(y)$, $y \in \mathbb{R}$ for $s > 0$ the following equalities hold

$$C_a^s(y) = v_a^s(B-y) - a\, R_{B-y}(s+a) + D_a^s(y)\, C^*(B), \qquad y \geq 0, \tag{27}$$
$$C_a^s(-y) = 1 - E[\, e^{-s\tau_y}] + \int_0^\infty E\,[\, e^{-s\tau_y};\ T^y \in du\,]\, C_a^s(u), \qquad y > 0,$$

where

$$C^*(B) = \frac{\frac{a\lambda}{c(s)} \left(v_a^s(B) - V_a^s(B) e^{c(s)B}\right) V_a^s(B)}{r(c(s), s) + a(\lambda - c(s)) \int\limits_0^B \left(V_a^s(x) - a\, e^{-xc(s)} R_x(s+a)\right)\, dx},$$

$$v_a^s(x) = 1 + a\lambda \int_0^x R_u(s+a)\, du, \quad x \geq 0, \qquad v_a^s(x) = 1, \quad x < 0,$$
$$V_a^s(x) = 1 + a(\lambda - c(s)) \int_0^x e^{-uc(s)} R_u(s+a)\, du, \quad x \geq 0, \qquad V_a^s(x) = 1, \quad x < 0.$$

Proof. The negative jumps of the underlying process being exponentially distributed, the random variables τ_x, T_x are independent (see the first formula of (9)) and for all $x \geq 0$ the value of the overshoot through the lower level T_x is exponentially distributed with the parameter λ. Hence,

$$E\,[\, e^{-s\tau_y - a\sigma_y};\ T_x \in du\,] = D_a^s(y) \lambda e^{-\lambda u} du.$$

Then, according to the total probability law and the strong Markov property of the process for the function $C_a^s(y)$, $y \geq 0$, we can write the system of equations

$$C_a^s(y) = Q_a^s(y) + D_a^s(y) \tilde{C}_a^s(\lambda), \qquad y \geq 0,$$
$$\tilde{C}_a^s(\lambda) = 1 - m_\gamma^s + \int_0^\infty m_\gamma^s(dy)\, C_a^s(y), \tag{28}$$

where $\tilde{C}_a^s(\lambda) = \lambda \int_0^\infty e^{-\lambda x} C_a^s(-x)\, dx$ is an unknown function, which we will determine later, and

$$m_\gamma^s = \lambda \int_0^\infty e^{-\lambda x} E\, e^{-s\tau^x}\, dx, \qquad m_\gamma^s(dy) = \lambda \int_0^\infty e^{-\lambda x} E\, [\, e^{-s\tau^x};\ T^x \in dy\,]\, dx.$$

Substituting the right-hand side of the first equation into the second equation of the system, implies

$$\tilde{C}_a^s(\lambda) = 1 - m_\gamma^s + \int_0^\infty m_\gamma^s(dy)\, Q_a^s(y) + \tilde{C}_a^s(\lambda) \int_0^\infty m_\gamma^s(dy)\, D_a^s(y)$$

which is a linear equation with respect to the function $\tilde{C}_a^s(\lambda)$. Using the expressions (18), (23) for the functions $D_a^s(y)$, $Q_a^s(y)$, we find from the latter equation that

$$\tilde{C}_a^s(\lambda) = v_a^s(B) + \qquad (29)$$

$$\frac{a\lambda}{1 - \int\limits_0^\infty m_\gamma^s(dy)\, D_a^s(y)} \left(\int\limits_0^B m_\gamma^s(dy) \int\limits_0^{B-y} R_u(s+a)\, du - \frac{1}{\lambda} \int\limits_0^B m_\gamma^s(dy)\, R_{B-y}(s+a) - S_B(s+a) \right),$$

where $S_B(s+a) = \int_0^B R_u(s+a)\, du$. Substituting the latter expression for $\tilde{C}_a^s(\lambda)$ into the first equation of the system (28) yields

$$C_a^s(y) = v_a^s(B-y) - a\, R_{B-y}(s+a) + D_a^s(y)\, C^*(B), \qquad y \ge 0$$

which is the first equality of Theorem 2. Now we need to determine the constant $C^*(B) = \tilde{C}_a^s(\lambda) - v_a^s(B)$. To do this, we require the following formulae

$$\int_0^B m_\gamma^s(dy)\, R_{B-y}(s+a) = \lambda \hat{R}_B(\lambda, s+a) e^{\lambda B} - \lambda R(\lambda, s) V_a^s(B) e^{c(s)B},$$

$$\int_0^B m_\gamma^s(dy) \int_0^{B-y} R_u(s+a)\, du = S_B(s+a)$$

$$+ \hat{R}_B(\lambda, s+a)\, e^{\lambda B} + \frac{R(\lambda, s)}{c(s)} \left((\lambda - c(s)) v_a^s(B) - \lambda V_a^s(B) e^{c(s)B} \right), \qquad (30)$$

$$\int_0^B m_\gamma^s(dy) e^{-yc(s)} \int_0^{B-y} e^{-uc(s)} R_u(s+a)\, du = \frac{\lambda R(\lambda, s)}{\lambda - c(s)} \left(e^{B(\lambda - c(s))} - 1 \right)$$

$$+ \frac{\lambda}{\lambda - c(s)} \left(\int_0^B e^{-uc(s)} R_u(s+a)\, du - \check{R}_B(\lambda, s+a)\, e^{(\lambda - c(s))B} \right) - \lambda R(\lambda, s) \int_0^B V_a^s(x)\, dx,$$

where

$$\check{R}_B(\lambda, s+a) = \int_0^B e^{-\lambda u} R_u(s+a)\, du, \qquad \hat{R}_B(\lambda, s+a) = \int_B^\infty e^{-\lambda u} R_u(s+a)\, du.$$

Integrals which enter the left-hand sides of the latter formulae are the convolutions of known functions and are easy to calculate by utilizing the following equalities

$$\int\limits_0^\infty e^{-yz} m_\gamma^s(dy) = \frac{\lambda}{\lambda - z} \left(1 - \frac{\lambda - c(s)}{z - c(s)}\, \frac{R(\lambda, s)}{R(z, s)} \right), \qquad R(z, s)^{-1} = R(z, s+a)^{-1} + a(\lambda - z).$$

The first equality follows from (9) when $p = \lambda - z$, and the second equality follows from the defining formula (8). Substituting the formulae (30) into (29), we obtain $C^*(B)$:

$$C^*(B) = \frac{\frac{a\lambda}{c(s)} \left(v_a^s(B) - V_a^s(B) e^{c(s)B} \right) V_a^s(B)}{r(c(s), s) + a(\lambda - c(s)) \int\limits_0^B \left(V_a^s(x) - a\, e^{-xc(s)} R_x(s+a) \right) dx},$$

where $r(c(s), s) = \frac{d}{dp} R(p, s)^{-1} \Big|_{p=c(s)}$. The second formula of Theorem 2 follows form the total probability law and the fact that τ^y is a Markov time. □

Therefore, we have obtained the Laplace transforms of the distribution of the duration of stay of the process inside the fixed interval. Note, that for a semi-continuous process with independent increments the integral transforms of the duration of stay of the process inside the interval were obtained in [5]. For a symmetric Wiener process the distribution of the duration of stay of the process inside the interval was obtained in [5].

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