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A COPULA MODEL FOR RESIDUAL DEPENDENCY IN IRT MODELS

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A copula model for residual dependency in IRT models.

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Abstract

Most item response theory models are not robust to violations of local independence. However, several modeling approaches (e.g., conditioning on other responses, additional random effects) exist that try to incorporate local item dependencies, but they have some drawbacks (e.g., non-reproducibility of marginal probabilities, interpretation problems). A new class of models that makes use of copulas to deal with local item dependencies is introduced. These models belong to the bigger class of marginal models in which marginals and association structure are modeled separately. It is shown how this approach overcomes some of the problems associated with other local item dependency models.

Keywords:

item response theory, local item dependency, copula

Introduction

A well-known measurement model used in different areas of research in psychology and educational measurement is the Rasch model (Rasch, 1960). For a person p (p=1, …, P) and item i (i=1,…,I), a binary random variable Y_{pi} is defined and the probability of a realization y_{pi} equals:

$$\Pr\left(Y_{pi} = y_{pi} \mid \theta_p\right) = \frac{\exp\left(y_{pi}\left(\theta_p - \beta_i\right)\right)}{1+\exp\left(\theta_p - \beta_i\right)},$$

where θ_p is the propensity of person p and β_i is the difficulty of the item i. For the moment we will assume that Y_{pi} = 1 corresponds to a correct response and Y_{pi} = 0 to an incorrect one (but see the second application in the Application section).

For the remainder of the paper, it will be convenient to define the Rasch model as a latent threshold model in which an underlying (latent) continuous variable X_{pi} is logistically distributed with mean $\theta_p - \beta_i$, scale parameter equal to one and a threshold parameter set at 0 (see e.g. Lord & Novick, 1968). More formally:

$$X_{pi} = \theta_p - \beta_i + \varepsilon_{pi} \quad \text{with} \quad \varepsilon_{pi} \sim \text{Logistic}\left(0,1\right) \tag{1}$$

such that

$$\Pr(Y_{pi} = 1 \mid \theta_p) = \Pr(X_{pi} > 0 \mid \theta_p) = \Pr(\varepsilon_{pi} > -\theta_p + \beta_i \mid \theta_p)$$
$$= 1 - \frac{\exp\left(-\theta_p + \beta_i\right)}{1+\exp\left(-\theta_p + \beta_i\right)} = \frac{\exp\left(\theta_p - \beta_i\right)}{1+\exp\left(\theta_p - \beta_i\right)}.$$

This is illustrated in Figure 1. The random variable ε_{pi} can be called a latent residual for person p and item i.

INSERT FIGURE 1 ABOUT HERE

A basic assumption of the Rasch model is local independence:

$$\Pr\left(\boldsymbol{Y}_p = \boldsymbol{y}_p \mid \theta_p\right) = \prod_{i}^{I} \Pr\left(Y_{pi} = y_{pi} \mid \theta_p\right),$$

where $\boldsymbol{Y}_p$ is the random vector of responses for person p on the set of I items (and $\boldsymbol{y}_p$ is the vector of corresponding realizations). Note that in the latent threshold model formulation above, the local independence assumption is equivalent to uncorrelated identically distributed logistic error terms ε_{pi} across items.

The Rasch model, and most item response theory models in general, are not robust to violations of local independence. Local item dependencies can affect the estimation and the reliability of the model parameters (see e.g., Ackerman, 1987; Chen & Thissen, 1997; Sireci, Thissen, & Wainer, 1991; Yen, 1984; 1993). Consider the extreme case wherein several slightly differently phrased questions are used in one test; this is almost equivalent to asking a single question. This redundancy situation will lead to an inflation in information, and consequently, to an underestimation of the standard error of a person's propensity parameter θ_p (see e.g., Junker, 1991). Moreover and in general, when an item response model that assumes local independency is used for a test that suffers from local item dependencies, this can result in biased estimates for both person and item parameters (e.g., for the effect on item discrimination parameters, see Masters, 1988).

Various types of tools have been developed to detect local item or residual dependencies (see e.g., Chen & Thissen, 1997; Stout, 1990; Holland & Rosenbaum, 1986; Rosenbaum, 1984; Yen, 1984; for a comparison see Tate, 2003). Once local item dependency problems have been detected, there are several possible approaches to model them; for an overview see Tuerlinckx and De Boeck (2004): Two typical problems that are associated with some of the most popular models for local item dependencies are non-reproducibility and the impossibility of interpreting

β_i as the difficulty of item *i*. To illustrate these problems, take the constant combination interaction model of Hoskens and De Boeck (1997) as a starting point (for simplicity, consider an interaction between only two items 1 and 2). The joint probability of response (y_{p1},y_{p2}) then equals:

$$\Pr\left(Y_{p1}=y_{p1},Y_{p2}=y_{p2}\mid\theta_p\right)=\frac{\exp\left(y_{p1}\left(\theta_p-\beta_1\right)+y_{p2}\left(\theta_p-\beta_2\right)+y_{p1}y_{p2}\delta\right)}{1+\exp\left(\theta_p-\beta_1\right)+\exp\left(\theta_p-\beta_2\right)+\exp\left(2\theta_p-\beta_1-\beta_2+\delta\right)}.$$

The δ parameter equals the logodds ratio (conditional on θ_p). If we now calculate the probability of answering correctly on the first item:

$$\begin{aligned}\Pr\left(Y_{p1}=1\mid\theta_p\right)&=\sum_{y_{p2}=0}^{1}\Pr\left(Y_{p1}=1,Y_{p2}=y_{p2}\mid\theta_p\right)\\&=\sum_{y_{p2}=0}^{1}\frac{\exp\left(\left(\theta_p-\beta_1\right)+y_{p2}\left(\theta_p-\beta_2\right)+y_{p2}\delta\right)}{1+\exp\left(\theta_p-\beta_1\right)+\exp\left(\theta_p-\beta_2\right)+\exp\left(2\theta_p-\beta_1-\beta_2+\delta\right)}\\&=\frac{\exp\left(\theta_p-\beta_1\right)+\exp\left(2\theta_p-\beta_1-\beta_2+\delta\right)}{1+\exp\left(\theta_p-\beta_1\right)+\exp\left(\theta_p-\beta_2\right)+\exp\left(2\theta_p-\beta_1-\beta_2+\delta\right)}.\end{aligned}$$

It can be seen that the marginal probability of responding correctly to item 1 is not a Rasch model anymore. Therefore, it is said that the marginals are not reproducible[1]. Moreover, the parameter β_1 cannot be interpreted as the difficulty parameter of item 1 because it is not simply the location on the latent trait for which the probability of responding correctly is 0.5. It can be seen that the probability of responding correctly to item 1 also depends the association parameter δ . The same

[1] Note that we employ the concept of reproducibility here in an intuitive fashion. More formal definitions can be found in Ip (2002) and Fitzmaurice, Laird, and Rotnitzky (1993).

problems also occur with random-effect models (Bradlow et al., 1999) and other types of conditional models (e.g., Verhelst & Glas, 1993).

A class of models that do not suffer from the aforementioned problems are the so-called marginal models. In these models, the univariate marginals and the dependence structure are modeled separately. In consequence, these models will not suffer of the problem of not-reproducible marginals. Examples proposed in the literature are the Bahadur-Ip model (Bahadur, 1961; Ip, 2000; 2001) and the likelihood-based variant of the generalized estimating equation method (see e.g., Ip, 2002; Liang & Zeger, 1986; Fitzmaurice, Laird, & Rotnitzky, 1993).

An interesting, but often neglected, type of marginal model is the multivariate probit random effect model (Ashford & Sowden, 1970). Instead of assuming a latent logistic distribution random variable underlying the response process, one could also use a normal distribution leading to the probit model. For simplicity, suppose a test consists of two binary items. The probit model for these two items is then defined as:

$$\begin{pmatrix} X_{p1} \\ X_{p2} \end{pmatrix} = \begin{pmatrix} \theta_p - \beta_1 \\ \theta_p - \beta_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_{p1} \\ \varepsilon_{p2} \end{pmatrix}$$

with

$$\begin{pmatrix} \varepsilon_{p1} \\ \varepsilon_{p2} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

The probability of responding correctly to item i (i=1, 2) now equals:

$$\Pr(Y_{pi} = 1) = \Phi(\theta_p - \beta_i). \tag{2}$$

Taking into account residual dependencies can be done by allowing the latent random variables (ε_{p1}, ε_{p2}) to be correlated:

$$\begin{pmatrix} \varepsilon_{p1} \\ \varepsilon_{p2} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right).$$

Because the marginal distributions of the ε_{pi}'s are not affected, the marginal probability of responding correctly to the item will be the same as in Equation 2. Hence, the multivariate probit model is reproducible and the item parameter can be interpreted as a difficutly parameter.

Thus, the multivariate probit model has all the nice properties of a marginal model, but there are also some disadvantages. First, the probit link function is used instead of the logit link, and in consequence the nice interpretation of the parameters in terms of odds ratios is not available anymore. This is especially relevant if covariates are added to the model (see De Boeck & Wilson, 2004). Second, the multivariate probit model is computationally very demanding because to compute the joint probability for a certain response pattern, a multivariate normal cumulative distribution function has to be computed. For the two-item case this gives:

$$\Pr(Y_{p1} = 1, Y_{p2} = 1 \mid \theta_p) = \int_0^{+\infty} \int_0^{+\infty} \phi\left(\begin{pmatrix} x_{p1} \\ x_{p2} \end{pmatrix}; \begin{pmatrix} \theta_p - \beta_1 \\ \theta_p - \beta_2 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right) dx_{p1} dx_{p2}$$

where $\phi(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{R})$ denotes here the bivariate normal density function and ρ refers to the correlation between X_{p1} and X_{p2}. This integral is intractable, and it can only be approximated very well in low dimensions through standard numerical integration techniques. For this reason, Chib and Greenberg (1998) propose a Bayesian approach for parameter estimation.

If we want to make use of logistic marginals, as is the case for the Rasch model, a similar approach as outlined above for the multivariate probit model can be taken. However, there is no single multivariate logistic distribution (analogous to the multivariate normal) that can be used as a starting point for introducing dependencies between the latent residuals. Convenient tools to study and construct multivariate distributions are copula functions which will be the main focus of this paper.

The remainder of the paper is organized as follows. First, some basic formal theory with respect to copulas will be discussed. Subsequently, a marginal residual dependency model with Rasch marginals will be defined. Next, the setup and results of a simulation study to investigate how well the model can be estimated will be described. Then the copula model will be applied to two data sets containing residual dependencies. Finally, the paper is closed with a discussion.

An overview of copula theory

In this section copula functions will be introduced as a mathematical concept and some necessary theory will be given. A more thorough overview can be found in the reference work by Joe (1997) or by Nelsen (1998). In mathematics a copula (Latin for link or tie) defines a function that relates a multivariate uniform cumulative distribution function (CDF) to its univariate margins that have uniform CDFs. A copula function enables the separate and independent modeling of the marginals and the association structure. Some more formal theory will make this clear.

Definition. An R-dimensional copula is a function $C:[0,1]^R \rightarrow [0,1]$, which has the following properties:

1. C($\boldsymbol{u}$) is increasing in each component u_r with $r \in \{1,2,...,R\}$.
2. For every vector $\boldsymbol{u} \in [0,1]^R$, $C(\boldsymbol{u}) = 0$ if at least one coordinate of the vector is 0 and $C(\boldsymbol{u}) = u_r$ if all the coordinates of the vector are equal to one except the r-th one.
3. For every $\boldsymbol{a}, \boldsymbol{b} \in [0,1]^R$ with $a_r \leq b_r \ \forall \ r \in [1, R]$, given a hypercube

$\mathbf{B} = [\boldsymbol{a}, \boldsymbol{b}] = [a_1, b_1] x [a_2, b_2] ... x [a_R, b_R]$ whose vertices lie in the domain of *C*, $V_C(\mathbf{B}) \geq 0$.

The volume $V_C(\mathbf{B})$ is defined as:

$$V_C(\mathbf{B}) = \sum_d \operatorname{sgn}(\mathbf{d}) C(\mathbf{d})$$
$$= \sum_{k_1=1}^{2} \sum_{k_2=1}^{2} ... \sum_{k_R=1}^{2} (-1)^{k_1+k_2...+k_R} C\left(d_{1k_1}, d_{2k_2}, ..., d_{Rk_R}\right).$$

This definition shows that the copula function *C* is a multivariate CDF with uniformly distributed margins.

Sklar's Theorem (Sklar, 1959)**.** Let $F(\boldsymbol{Y})$ be an *R*-dimensional distribution function with univariate margins $F(Y_1), F(Y_2), ..., F(Y_R)$. Then there exists a copula *C* for $\boldsymbol{Y} \in \mathbb{R}^R$ such that $F(\boldsymbol{Y}) = C\left[F(Y_1), F(Y_2), ..., F(Y_R)\right]$. The statement holds also conversely. Moreover, if the margins are continuous, C is unique, else C is uniquely determined on the Range of $F(\boldsymbol{Y})$.

In words, the first part of Sklar's theorem means that every joint distribution can be expressed as a copula function of its marginals. Thus, we can rephrase any known joint CDF by means of a copula function of its marginals. The second part is more of interest to us since it states conversely that, based upon the univariate marginals, a joint (i.e., multivariate) distribution can be constructed by means of a copula function.

There is a wide variety of possible copula functions. However, our modeling approach will focus on the class of Archimedean copulas (see e.g., Genest & MacKay, 1986; Nelsen, 1998). Archimedean copulas have a simple structure and can be written as:

$$C(u_1, ..., u_R) = \phi^{-1}\left[\phi(u_1) + ... + \phi(u_R)\right]$$

where $\phi:[0,1]\to[0,\infty]$ is a generator function satisfying,

$$\phi(1)=0,$$
$$\text{for all } t\in[0,1],\ \phi'(t)<0:\ \phi \text{ is decreasing,}$$
$$\text{for all } t\in[0,1],\ \phi''(t)\geq 0:\ \phi \text{ is convex.}$$

This class of copulas has also some nice properties like for instance permutation symmetry, $C(u_1,u_2)=C(u_2,u_1)$, and associativity, $C(u_1,u_2,u_3)=C(u_1,C(u_2,u_3))=C(C(u_1,u_2),u_3)$.

These symmetry properties lead to the conclusion that the association parameter for Archimedean copulas has an interpretation of an association parameter between any pair of variables within this copula function. Since the association parameter is common to all variables in the copula, Archimedean copulas are best suited for data that are symmetrically dependent, as is often the case when there are residual dependencies.

In this paper, we will work with three instantiations of the class of Archimedean copulas (1) the independence or product copula :

$$F(\boldsymbol{Y})=C\left[F(Y_1),F(Y_2),...,F(Y_R)\right]=\exp\left(-\sum_r^R\left[-\log\left(F(Y_r)\right)\right]\right)=\prod_r^R F(Y_r);$$

(2) Frank's Copula (Frank, 1979):

$$F(\boldsymbol{Y})=C\left[F(Y_1),...,F_R(Y_R)\right]=\frac{-1}{\alpha}\log\left\{1-\left(1-\exp(-\alpha)\right)\prod_{r=1}^R\left[\frac{1-\exp\left(-\alpha F(Y_r)\right)}{1-\exp(-\alpha)}\right]\right\},$$

with $\lim_{\alpha\to 0}\left\{C\left[F(Y_1),...,F(Y_R)\right]\right\}=\prod_{r=1}^R F(Y_r)$;

Frank's copula contains only one parameter α that is able to capture the whole range of dependency, from negative dependency ($\alpha < 0$) to positive dependency ($\alpha > 0$), with the independence case in the interior of the parameter space ($\alpha \to 0$). If there are only two variables in the copula, the parameter α can take any real value, with the values of $-\infty$ en ∞ respectively corresponding to the limiting Fréchet-Hoeffding lower and upper bound. However, for the case that more than 2 variables appear in the copula, Meester (1991) showed that the lower bound needs to be adapted in function of the number of variables in the copula. For any number of items, this adapted lower-bound is always strictly less than zero, thus not restricting the positive association range and leaving the independence point in the interior of the parameter space.

(3) Clayton's copula (Clayton, 1978; Cook & Johnson, 1981):

$$F(\boldsymbol{Y}) = C\left[F(Y_1),\dots,F_R(Y_R)\right] = \left[1+\sum_r^R \left(F(Y_r)-1\right)^{-\alpha}\right]^{-1/\alpha} ,$$

$$\text{with } \lim_{\alpha\to 0}\left\{C\left[F(Y_1),\dots,F(Y_R)\right]\right\} = \prod_{r=1}^R F(Y_r) .$$

Clayton's copula contains only one parameter α that is able to capture the whole range of positive dependency. The parameter α can take any positive real value, with α equal to 0 being the independence case and α equal to ∞ being the case of absolute positive dependence.

Copula model for residual dependencies

In order to construct a residual dependency model with Rasch margins, we will start again from the latent underlying continuous variables as defined in Equation 1. For simplicity, assume for the moment that a test consists of two items and these are suspected to show local item dependencies. The extension to more than two items is explained below.

The random variables ε_{p1} and ε_{p2} refer to the latent random variables. Instead of assuming two independent logistic distributions for both random variables, they could as well be modeled jointly such that their univariate CDFs are logistic distributions. Thus the items 1 and 2 are jointly modeled as

$$\begin{pmatrix} X_{p1} \\ X_{p2} \end{pmatrix} = \begin{pmatrix} \theta_p - \beta_1 \\ \theta_p - \beta_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_{p1} \\ \varepsilon_{p2} \end{pmatrix}$$

with

$$F(\boldsymbol{\varepsilon}) = F\left(\begin{pmatrix} \varepsilon_{p1} \\ \varepsilon_{p2} \end{pmatrix}\right) = C(F(\varepsilon_{p1}), F(\varepsilon_{p2})),$$

where

$$F(\varepsilon_{pi}) = \frac{\exp\left(\varepsilon_{pi}\right)}{1 + \exp\left(\varepsilon_{pi}\right)}.$$

The probability of responding correctly to item i (i=1, 2) now equals:

$$\Pr(Y_{pi} = 1) = \Pr\left(X_{pi} > 0\right) = \Pr(\varepsilon_{pi} > -\theta_p + \beta_i) = \frac{\exp\left(\theta_p - \beta_i\right)}{1 + \exp\left(\theta_p - \beta_i\right)}.$$

The residual dependency is taken into account by allowing the latent random variables (ε_{p1}, ε_{p2}) to be dependent; instead of the multivariate normal CDF with specified correlation matrix as used in the probit random effect model, a multivariate logistic CDF is created by means of a copula.

Thus, roughly said, the logistic distribution and the copula function replace the normal distribution and the specified correlation matrix, respectively.

A residual dependency model with logistic margins at the latent level is possible by choosing appropriate margins and an appropriate copula function. Because a copula model is a marginal model different association structures (by means of different copulas) for the residual dependency can be compared without changing something essential to the base model of the marginal probabilities. Figure 2 shows samples of the joint distributions of the latent logistically distributed random variables (ε_{p1}, ε_{p2}) constructed by means of the aforementioned three instantiations of Archimedean copulas with varying levels of residual dependence. Notice that the copulas differ in the type of dependence they induce; for instance Clayton's copula has a prominent lower tail (i.e. more formally, $C(u,\ldots,u)/u$ converges to a constant c in [0,1] as $u \to 0$).

Insert Figure 3 around here.

Since the continuous latent variables will be locally dependent, the discrete responses Y_{p1} and Y_{p2} will be too. In order to calculate the joint probabilities from the joint CDFs of the latent random variables, the volume under the density for the corresponding quadrant is calculated (see e.g., Mood, Graybill, & Boes, 1974):

$$\begin{aligned}
&\Pr\left(Y_{p1}=0,Y_{p2}=0\right)=C\left(F\left(X_{p1}=0\right),F\left(X_{p2}=0\right)\right)\\
&\Pr\left(Y_{p1}=1,Y_{p2}=1\right)=1-F\left(X_{p1}=0\right)-F\left(X_{p2}=0\right)+C\left(F\left(X_{p1}=0\right),F\left(X_{p2}=0\right)\right)\\
&\Pr\left(Y_{p1}=1,Y_{p2}=0\right)=F\left(X_{p2}=0\right)-C\left(F\left(X_{p1}=0\right),F\left(X_{p2}=0\right)\right),\\
&\Pr\left(Y_{p1}=0,Y_{p2}=1\right)=F\left(X_{p1}=0\right)-C\left(F\left(X_{p1}=0\right),F\left(X_{p2}=0\right)\right).
\end{aligned}$$

Figure 3, presenting a bivariate density with quadrants made up by the solid lines (the dashed lines indicating the marginal means), offers an intuïtive insight in these calculations.

Insert Figure 3 around here.

Besides making use of latent responses (the *X*'s), one could also define the copula directly on the discrete responses (the *Y*'s). It can be shown that this leads to the same model as the one above (see Appendix I for a formal prove).

In order to illustrate that the introduction of the copula can take residual dependency into account, the odds ratio for items 1 and 2 involved in the copula (conditional on θ_p) can be computed as follows:

$$OR(\theta_p) = \frac{\Pr(Y_{p1}=1, Y_{p2}=1 \mid \theta_p)\Pr(Y_{p1}=0, Y_{p2}=0 \mid \theta_p)}{\Pr(Y_{p1}=1, Y_{p2}=0 \mid \theta_p)\Pr(Y_{p1}=0, Y_{p2}=1 \mid \theta_p)}$$

$$= \frac{\left(1 - F(Y_{p1}=0 \mid \theta_p) - F(Y_{p2}=0 \mid \theta_p) + C\right)C}{\left(F(Y_{p2}=0 \mid \theta_p) - C\right)\left(F(Y_{p1}=0 \mid \theta_p) - C\right)},$$

with $C = C\left[F(Y_{p1}=0 \mid \theta_p), F(Y_{p2}=0 \mid \theta_p)\right].$

Using Frank's copula, the value of the log odds ratio is then computed for several values of α and for θ_p ranging from –4 to +4; the result is shown in Figure 4a. For Clayton's copula, the same procedure is followed and this result is shown in Figure 4b. For ease of demonstration the two marginals were set equal to one another, with difficulty parameters β_1 and β_2 equal to zero, so that the log odds ratio (conditional on θ_p) was only a function of the copula's association parameter and the marginal probabilities determined by θ_p. From both figures it can be seen that there is a weak dependency between the (log) odds ratio and the value of the latent trait. The difference between the figures of the two copula functions can be ascribed to the lower tail dependence of Clayton's copula. As shown by Trégouet et al. (1999), the (log) odds ratio is dependent of the association parameter and the marginal probabilities (which are, in their turn, influenced by θ_p). Hence, no perfect separation between the marginal and the association structure is possible. However, for low to moderate values of the association parameter α, the dependency is negligible. Furthermore, for the Frank's copula the odds ratio can reach the limiting cases of 0 and infinity (corresponding with absolute negative and positive dependency respectively) without any restriction; for Clayton's copula, that only captures positive dependency, the odds ratio can reach the limiting case of infinity (i.e., absolute positive dependence) without any restriction. These results indicate that the existing dependency between the marginals and α is weak; this was also confirmed for real data by observations of Meester and MacKay (1994) and by our own simulation study (results presented further on in the corresponding section).

If R (R>2) items are showing residual dependencies, the copula approach can be easily extended (see also copula definition, property 3):

$$\Pr\left(Y_{p1}=y_1,\dots,Y_{pr}=y_r,\dots,Y_{pR}=y_R \mid \theta_p\right)=\Pr\left(d_{11}<Y_{p1}\le d_{12},\dots,d_{r1}<Y_{pr}\le d_{r2},\dots,d_{R1}<Y_{pR}\le d_{R2}\mid \theta_p\right)$$

$$=\sum_{k_1=1}^{2}\sum_{k_2=1}^{2}\dots\sum_{k_R=1}^{2}(-1)^{k_1+k_2+\dots+k_R}\,F\left(d_{1k_1},\dots,d_{rk_r},\dots,d_{Rk_R}\right)$$

with $\forall\, r\in[1,R]$:

$$\text{if } Y_{pr}=1,\ d_{r1}=0, d_{r2}=\infty; \qquad (3)$$

$$\text{if } Y_{pr}=0,\ d_{r1}=-\infty, d_{r2}=0.$$

Note that this can be easily extended beyond binary variables to polytomous variables, the number of possible interval points $d_{r.}$ in the CDF just increases with increasing categories; with the limit being the continuous case. Note that the formulation of the conditional odds ratio remains the same given the symmetry properties of Archimedean copulas (i.e., the association created by the copula is exactly the same for each item pair in the copula).

Thus, copula functions can be easily incorporated into the regular Rasch model. The items with a common local item dependency issue are put into a copula function, resulting in an adapted joint probability of the set of items in the copula that takes into account the detected (or theoretical) residual dependence. For the moment let us assume that there is only one group of locally dependent items, than the extension of the Rasch model we propose is formulated in a more traditional way as:

$$\Pr\left(\boldsymbol{Y}_p = \boldsymbol{y}_p \mid \theta_p\right) = \Pr\left(\boldsymbol{Y}_{pC} = \boldsymbol{y}_{pC} \mid \theta_p, \boldsymbol{\beta}_C, \alpha\right) \prod_{i=R+1}^{I} \Pr\left(Y_{pi} = y_{pi} \mid \theta_p\right),$$

with

$Y_{pC} = y_{p1}, \ldots, y_{pR}$ a vector of responses on the items in copula C,

$\boldsymbol{\beta}_C = \beta_1, \ldots, \beta_R$ a vector consisting of the difficulty parameters of the items in copula C,

α the parameter(s) of the copula function C.

Thus the joint probability of a person's vector of responses $\boldsymbol{Y}_p$ is calculated as the product of the joint probability of the person's response vector on the items within the copula and each of the separate individual probabilities of the locally independent items.

Statistical inference in the copula model

The estimation of the model's parameter can be done through marginal maximum likelihood (MML) estimation. In this article it is assumed that the θ_p's stem from a normal population distribution with mean zero and an unknown variance. Compared to a regular Rasch model, the only additional parameter is the association parameter in the copula.

The major difference with models assuming conditional independence is that each time the likelihood has to be computed, the probabilities of the response patterns of the items involved in the copula have to be calculated from the joint CDF (see Equation 3). Because terms of approximately the same order of magnitude may get subtracted, this step in the estimation process might cause numerical instability. Usually it has no consequences for finding the mode of the loglikelihood (i.e., the parameter estimates), but our experience learns that the numerical approximation to the Hessian matrix may suffer from it, mainly when a moderate to a large number of items are involved in the copula (e.g., 4 or more). Given these computational problems, a parametric bootstrap approach is adopted here to estimate standard errors (see Efron & Tibshirani, 1993). We also recommend using appropriate starting values to let the MML algorithm run as smooth as possible.

With respect to model checking and selection, the tools usually applied in nonlinear mixed models are available (Wald, score and likelihood ratio tests). Special attention has to be given to model selection with respect to the copula function. Frank's and Clayton's copula do not have a nested relation, the results should be compared with methods such as the AIC or BIC (Schwarz, 1978). The independence model, however, is nested within the two models. Therefore, a test of the null hypothesis that the association parameter equals a particular value (0 for both

Frank's and Clayton's copula) is a comparison between the Rasch model and a residual dependency model.

Simulation study

A simulation study was carried out for two reasons. First, we wanted to investigate the goodness-of-recovery of the copula model. Second, we have shown that the (log) odds ratio for two items depends not only on the association parameter α but also on the univariate marginal probabilities (hence, on the θ_p and β_i). Therefore, our goal is to find out whether the association parameter α can be estimated independently of the item parameters.

The setup of the simulation study was as follows: Data sets with the binary responses of 500 fictive persons on 10 items were simulated (person and item parameters were randomly drawn from a standard normal distribution) The responses on the first two items showed residual dependencies by means of Frank's copula (we only used Frank's copula here, but the results with Clayton's copula were similar). To study in a systematical way how the item parameters of the copula items (β_1 and β_2) influence the association parameter estimate, these item parameters were manipulated following the scheme in Table 1.

Insert Table 1 about here.

The copula parameter α was varied systematically from 0 (being the local item independency case) to 4 with increments of 0.5, resulting in 9 possible values. In total, there are 54 conditions (6 item parameter combinations for the two copula items and 9 assocation parameters). In each cell of the study, there are 50 replications.

To evaluate the goodness-of-recovery, three measures were used (see also Maris, 1999): the Root Mean Square Deviation (RMSD), the Monte Carlo Standard Error (MCSE) and bias (Note that $RMSD^2 = MCSE^2 + BIAS^2$).

To determine the degree of dependence between the estimate of the association parameter α and the marginals (the second reason for the simulation study), ideally one should evaluate the expected value of the second derivative of the loglikelihood with respect to the assocation parameter and the item parameters involved in the copula and this expectation should be zero:

$$E\left[\frac{\partial^2 \ell(\alpha,\beta_i)}{\partial\alpha\,\partial\beta_i}\right]=0,$$

for i=1, 2 (the two copula items). Because there is no closed-form solution to this expectation, we will use the correlation (computed over replications within a cell) between the estimate of the association parameter and the estimate of the copula item parameters as an indication of the dependence between the estimates of $\alpha \,\&\, \beta_i$.

In each condition a relative goodness-of-fit measure was constructed by taking the proportion of replications within a cell where the copula model was chosen above the regular Rasch model based upon the likelihood ratio test.

The results of the simulation study are presented in Table 2a. For each of the 54 conditions the RMSE of the model parameters, the empirical correlation between the association parameter and the estimates of respectively, the copula item parameters, one examplary item parameter outside the copula, and the estimated standard deviation of the proficiency in the population, is given, as well as the relative fit of the copula model with Rasch margins. A summary of the results can be found in Table 2b. where the same measures are presented but averaged over conditions, offering an indication of the main effects of the different manipulations in the simulation study (i.e., degree of residual dependence and the location of the copula item parameters).

The goodness-of-recovery of the copula model for the association parameter decreases with increasing α. In the conditions where the copula item parameters have opposite sign, the copula model with Rasch margins seem to have problems in estimating the true association parameter α, but leaving the marginal parameters largely unaffected. This will be probably caused by the limited availability of informative response patterns for the estimation of the association parameter (i.e., $(1,1)$ and $(0,0)$ patterns), because the copula items, each one located around an opposite side of the latent continuum (i.e. -2 and 2 on the logit scale), will largely generate $(1,0)$ response patterns and almost no other response patterns. The RMSE of the item parameters is quite low. When α equals 0 (i.e., the independence case) the copula item parameters are as well estimated as the parameters of the items outside the copula; as α rises above 0 the copula item parameters are slightly less exact recovered, but still quite accurate and not influencing the items outside the copula (0.26 VS 0.12 RMSE). The standard deviation of the proficiency in the population is not affected by the association parameter α and the recovery of the true parameter is quite good. The RMSE for this parameter is largest when the two copula item parameters are of equal sign. This is quite natural because the two items are of exact equal difficulty and thus are in a sense redundant while giving the same information about the proficiencies among the test population.

The simulation study gives no evidence to reject the approximate independence relation between the association parameter α and the marginals. The empirical correlations show no clear pattern over the conditions and remain within a relatively small range (between -0.3 and 0.3).

As could be expected the copula model with Rasch margins shows better fit than the regular Rasch model with increasing α, regardless of the marginal parameters. Note that the relative goodness-of-fit of the copula model is somehow slightly less in the extreme conditions (i.e., where the copula item parameters have opposite sign) compared to the other conditions. However the copula model remains supreme with increasing α.

Applications

In this section, we will apply the copula model to two data sets. Both Frank's and Clayton's copula will be fitted to the data. The marginal maximum likelihood method is implemented using a Gauss-Hermite quadrature with 21 nodes and a quasi-Newton optimization technique. The standard errors of the parameter estimates were approximated using a parametric bootstrap with 100 replications.

Application 1: Small reading test

A group of high school students interested in studying law in college (P = 441) answered six multiple choice questions about a text on the president and the separation of powers in the United States of America. The answers were binary recoded in right and false for ease of demonstration and scored 1 and 0, respectively. In previous analyses (Tuerlinckx & De Boeck, 2001) it is shown that two pairs of items showed residual dependencies: items 1 and 6, and items 4 and 5. A series of seven models was fitted (the Rasch model and the different possible residual dependency models using Frank's and Clayton's copula) and compared using likelihood ratio tests and/or the AIC.

Insert Table 3 about here

From the results in Table 3, it can be seen that the most appropriate model is that copula model with a separate Frank's copula for each item pair. The model has a significant better fit than the regular Rasch model (*LRT*=49, *df*=2, p<0.0001). Both association parameters differ significantly from zero (p<0.0001 for $\alpha(1,6)$ and p<0.003 for $\alpha(4,5)$). Looking at the Mantel-Haenszel test for equal conditional odds ratio as a diagnostic tool (Ip, 2001), one notices a strong reduction in unmodeled local residual dependency for the copula modeled item pairs: for the item pair 1 and 6, from 7.01 (p<0.0001) under the Rasch model to 0.37 (n.s.) under the chosen model, for the item pair 4 and 5 from 2.6 (p<0.02) to -0.61 (n.s.). Thus the copula model succeeds in its primary objective, namely modeling the local item dependencies. Notice that the standard errors of the item parameters and the standard deviation of the propensity distribution in the population are of comparable size between the Rasch model and the copula model with Rasch margins (see Table 4).

Insert Table 4 about here

Application 2: Verbal Aggression data

As a second example, we analyze the data from a behavioral questionnaire (Vansteelandt, 2000; see also De Boeck & Wilson, 2004). The questionnaire consists of 24 items that each refer to verbally aggressive reactions in a frustrating situation, and it was administered from 316 persons. Four different situations were used to construct the 24 items: 'A bus fails to stop for me', 'I miss a train because a clerk gave me faulty information', 'The grocery store closes just as I am about to enter', and 'The operator disconnects me when I had used up my last 10 cents for a call'.

The description of the situation is followed by a statement about the behavioral mode of a verbal aggressive reaction ('I would want to …' or 'I would …'). Three verbal aggressive reactions are studied: cursing, scolding, and shouting (the questionnaire was translated from Dutch, and shouting refers to an expressive aggressive reaction in Dutch). The original items had three response categories ('yes', 'perhaps', and 'no') but we dichotomized them ('yes' and 'perhaps' were scored 1 and 'no' was scored 0).

Because the 24 items are clustered in four groups of six items due to the common situations, we may expect residual dependencies. Therefore, a model with four copulas (each containing six items) and Rasch marginals is fitted.

The results are presented in Table 5. The common Rasch model was fitted, a copula model with four copulas and also a model with copula for items 3, 6, 9 and 12. These four items were diagnosed by means of the Q3-index (Yen, 1984) with showing large residual dependencies and are all items with the question whether the person wants to shout in that situation; thus this residual dependency can be reasonably attributed to this item characteristic.

From the results in Table 5, one can see that the copula models with univariate Rasch margins outperforms the regular Rasch model. Not only the interpretation of the individual parameters remains the same, but notice that also their standard errors are largely comparable to the standard errors of the Rasch model (this also holds for the previous application and the simulation study).

Discussion

In this paper we have introduced the use of copulas for modeling residual dependencies. The model has a simple form and allows easy and flexible model construction by modeling the marginals and the dependence structure separately and independent of each other. Extensions to other more complicated models can be easily implemented and it is not restricted to binary response variables.

The Rasch Copula model does not suffer from the difficulties and flaws of the other existing models that try to incorporate local item dependencies. The model has the property of reproducibility, such that the univariate marginals are still Rasch models and the item parameters can be interpreted as difficulty parameters.

In the case of Frank's copula, the association parameter α can cover the range of dependencies, from negative to positive association, with the independence case in the interior of the parameter space. For Clayton's copula, only positive dependencies can be taken into account, but those are the most frequent.

The general class of Archimedean copulas (with Frank and Clayton as special cases) is limited to the case in which the dependency is symmetric. Therefore, copulas are less suited to model learning phenomena.

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Appendix I : Equivalence of the Rasch Copula model defined at the latent level or at the discrete level.

Latent Level:

$$X_{pi} = \theta_p - \beta_i + \varepsilon_{pi} \quad \text{with} \quad \varepsilon_{pi} \sim \text{Logistic}(0,1)$$

such that

$$\begin{aligned}\Pr(Y_{pi} = 1 \mid \theta_p) &= \Pr(X_{pi} > 0 \mid \theta_p) = \Pr(\varepsilon_{pi} > -\theta_p + \beta_i \mid \theta_p) \\ &= 1 - \frac{\exp(-\theta_p + \beta_i)}{1 + \exp(-\theta_p + \beta_i)} = \frac{\exp(\theta_p - \beta_i)}{1 + \exp(\theta_p - \beta_i)}.\end{aligned}$$

$$\begin{aligned}&\Pr(Y_{pi} = 0, Y_{pj} = 0 \mid \theta_p) = C\left(F(X_{pi} = 0 \mid \theta_p), F(X_{pj} = 0 \mid \theta_p)\right)\\ &\Pr(Y_{pi} = 1, Y_{pj} = 1 \mid \theta_p) = C\left(1 - F(X_{pi} = 0 \mid \theta_p), 1 - F(X_{pj} = 0 \mid \theta_p)\right),\\ &\Pr(Y_{pi} = 1, Y_{pj} = 0 \mid \theta_p) = F(X_{pj} = 0 \mid \theta_p) - C\left(F(X_{pi} = 0 \mid \theta_p), F(X_{pj} = 0 \mid \theta_p)\right),\\ &\Pr(Y_{pi} = 0, Y_{pj} = 1 \mid \theta_p) = F(X_{pi} = 0 \mid \theta_p) - C\left(F(X_{pi} = 0 \mid \theta_p), F(X_{pj} = 0 \mid \theta_p)\right).\end{aligned}$$

Discrete Level:

Since a copula is defined as a cumulative distribution function, a transformation to probabilities is required to make proper use of copula functions in probabilistic measurement models like the Rasch and other item response theory models. One can easily use property 3 in the definition of a copula. For continuous variables one would apply the Probability Integral Transform, for discrete variables as in our case one uses the discrete equivalent.

The cumulative distribution function of a single response in the Rasch model is defined as:

$$F(Y_{pi} = y_{pi}) = \begin{cases} 0 & \text{for } y_{pi} = -1 \\ \Pr(Y_{pi} = 0) = \dfrac{1}{1 + \exp(\theta_p - \beta_i)} & \text{for } y_{pi} = 0 \\ \Pr(Y_{pi} = 0) + \Pr(Y_{pi} = 1) = 1 & \text{for } y_{pi} = 1 \end{cases}.$$

Thus, one has to evaluate the cumulative distribution functions using these three points to obtain the joint probability as defined by the copula function C. This means that for instance the joint probability of answering correctly on item 1 and 2 equals:

$$
\begin{aligned}
C\left[P\left(Y_{p1}=1,Y_{p2}=1\,|\,\theta_p\right)\right] = & \, C\left[F\left(Y_{p1}=1\,|\,\theta_p\right),F\left(Y_{p2}=1\,|\,\theta_p\right)\right] \\
& -C\left[F\left(Y_{p1}=1\,|\,\theta_p\right),F\left(Y_{p2}=0\,|\,\theta_p\right)\right] \\
& -C\left[F\left(Y_{p1}=0\,|\,\theta_p\right),F\left(Y_{p2}=1\,|\,\theta_p\right)\right] \\
& +C\left[F\left(Y_{p1}=0\,|\,\theta_p\right),F\left(Y_{p2}=0\,|\,\theta_p\right)\right].
\end{aligned}
$$

Leading to:

$$
\Pr\left(Y_{pi}\,|\,\theta_p\right)=\frac{\exp\left(y_{pi}\left(\theta_p-\beta_i\right)\right)}{1+\exp\left(\theta_p-\beta_i\right)}
$$

$$
\begin{aligned}
F\left(Y_{pi}=0\,|\,\theta_p\right) &= \Pr\left(Y_{pi}=0\,|\,\theta_p\right) \\
F\left(Y_{pi}=1\,|\,\theta_p\right) &= \Pr\left(Y_{pi}=0\,|\,\theta_p\right)+\Pr\left(Y_{pi}=1\,|\,\theta_p\right)=1
\end{aligned}
$$

$$
\begin{aligned}
& C\left(F\left(Y_{pi}=1\,|\,\theta_p\right),F\left(Y_{pj}=1\,|\,\theta_p\right)\right)=1, \\
& C\left(F\left(Y_{pi}=1\,|\,\theta_p\right),F\left(Y_{pj}=0\,|\,\theta_p\right)\right)=F\left(Y_{pj}=0\,|\,\theta_p\right), \\
& C\left(F\left(Y_{pi}=0\,|\,\theta_p\right),F\left(Y_{pj}=1\,|\,\theta_p\right)\right)=F\left(Y_{pi}=0\,|\,\theta_p\right), \\
& C\left(F\left(Y_{pi}=0\,|\,\theta_p\right),F\left(Y_{pj}=0\,|\,\theta_p\right)\right)= \\
& \qquad \frac{-1}{\alpha}\log\left\{1-\frac{\left(1-\exp\left(-\alpha F\left(Y_{pi}=0\,|\,\theta_p\right)\right)\right)\left(1-\exp\left(-\alpha F\left(Y_{pj}=0\,|\,\theta_p\right)\right)\right)}{1-\exp\left(-\alpha\right)}\right\};
\end{aligned}
$$

$$
\begin{aligned}
& \Pr\left(Y_{pi}=1,Y_{pj}=1\,|\,\theta_p\right)=1+C\left(F\left(Y_{pi}=0\,|\,\theta_p\right),F\left(Y_{pj}=0\,|\,\theta_p\right)\right)-F\left(Y_{pi}=0\,|\,\theta_p\right)-F\left(Y_{pj}=0\,|\,\theta_p\right), \\
& \Pr\left(Y_{pi}=1,Y_{pj}=0\,|\,\theta_p\right)=F\left(Y_{pj}=0\,|\,\theta_p\right)-C\left(F\left(Y_{pi}=0\,|\,\theta_p\right),F\left(Y_{pj}=0\,|\,\theta_p\right)\right), \\
& \Pr\left(Y_{pi}=0,Y_{pj}=1\,|\,\theta_p\right)=F\left(Y_{pi}=0\,|\,\theta_p\right)-C\left(F\left(Y_{pi}=0\,|\,\theta_p\right),F\left(Y_{pj}=0\,|\,\theta_p\right)\right), \\
& \Pr\left(Y_{pi}=0,Y_{pj}=0\,|\,\theta_p\right)=C\left(F\left(Y_i=0\,|\,\theta_p\right),F\left(Y_j=0\,|\,\theta_p\right)\right).
\end{aligned}
$$

Proven equivalence since $F\left(Y_{pi}=0\,|\,\theta_p\right)=\Pr\left(Y_{pi}=0\,|\,\theta_p\right)=F\left(X_{pi}=0\,|\,\theta_p\right)$.

Tables

Table 1: Item parameters of the residual dependency items in the simulation study.

condition	1	2	3	4	5	6
β_1	-2	-2	0	+2	+2	-2
β_2	-2	0	0	0	+2	+2

Table 2a. *Results of the simulation study for each of the conditions.*

True parameters			RMSE					Correlation with α				Relative fit
α	b1	b2	α	b1	b2	b3	sd(θ)	b1	b2	b3	sd(θ)	%Copula
0	-2	-2	0.74	0.14	0.17	0.10	0.58	-0.01	0.27	-0.04	0.06	0.2
0.5	-2	-2	0.96	0.33	0.34	0.10	0.65	0.20	-0.10	-0.18	0.02	0.48
1	-2	-2	1.01	0.40	0.39	0.12	0.63	0.16	0.52	-0.09	-0.23	0.68
1.5	-2	-2	0.98	0.41	0.41	0.13	0.64	0.23	0.17	0.13	0.03	0.72
2	-2	-2	0.99	0.34	0.36	0.13	0.63	0.25	0.04	0.16	0.09	0.94
2.5	-2	-2	1.12	0.34	0.37	0.12	0.70	0.00	-0.18	-0.05	-0.08	0.94
3	-2	-2	1.11	0.32	0.34	0.11	0.56	-0.21	-0.10	0.00	-0.02	1
3.5	-2	-2	1.14	0.32	0.31	0.13	0.58	0.05	-0.01	0.04	-0.08	1
4	-2	-2	1.14	0.30	0.27	0.13	0.48	0.11	0.14	0.01	-0.14	1
0	-2	0	0.63	0.16	0.11	0.12	0.27	0.11	-0.08	-0.12	-0.17	0.12
0.5	-2	0	0.66	0.37	0.30	0.10	0.29	0.10	-0.17	-0.04	0.06	0.36
1	-2	0	0.72	0.41	0.31	0.13	0.06	-0.14	-0.10	-0.17	0.18	0.7
1.5	-2	0	0.80	0.37	0.26	0.12	0.06	-0.07	-0.22	0.32	0.18	0.88
2	-2	0	0.80	0.38	0.24	0.09	0.29	0.25	0.10	0.17	0.00	0.94
2.5	-2	0	0.82	0.32	0.22	0.13	0.06	-0.25	-0.33	0.11	-0.05	0.98
3	-2	0	0.80	0.35	0.16	0.13	0.29	0.15	0.33	-0.07	0.03	1
3.5	-2	0	0.80	0.30	0.18	0.11	0.07	0.00	0.05	0.03	-0.02	1
4	-2	0	1.00	0.33	0.17	0.11	0.06	0.15	0.08	0.18	-0.11	1
0	0	0	0.48	0.11	0.10	0.13	0.05	0.00	0.03	0.11	0.26	0.16
0.5	0	0	0.57	0.26	0.28	0.12	0.05	-0.17	-0.09	0.15	0.00	0.58
1	0	0	0.56	0.27	0.27	0.12	0.05	-0.22	-0.19	-0.02	0.11	0.88
1.5	0	0	0.46	0.26	0.25	0.12	0.06	-0.15	0.09	-0.19	-0.29	1
2	0	0	0.52	0.21	0.21	0.13	0.06	-0.27	-0.14	-0.04	-0.10	1
2.5	0	0	0.57	0.21	0.20	0.12	0.06	0.17	0.02	0.02	-0.01	1
3	0	0	0.48	0.19	0.20	0.12	0.05	-0.22	0.13	0.05	0.02	1
3.5	0	0	0.57	0.16	0.17	0.11	0.06	0.10	0.09	0.17	0.08	1
4	0	0	0.65	0.17	0.17	0.12	0.06	-0.06	0.09	0.16	-0.21	1
0	2	0	0.52	0.12	0.11	0.11	0.06	-0.11	-0.15	-0.28	0.11	0.1
0.5	2	0	0.52	0.25	0.28	0.11	0.05	0.09	0.19	0.07	-0.06	0.36
1	2	0	0.57	0.22	0.29	0.13	0.05	0.21	0.08	0.09	0.13	0.56

1.5	2	0	0.67	0.21	0.28	0.13	0.06	-0.18	0.22	0.12	-0.37	0.86
2	2	0	0.68	0.19	0.22	0.12	0.28	-0.12	0.18	0.26	-0.10	0.96
2.5	2	0	0.68	0.20	0.22	0.13	0.06	-0.11	-0.21	0.01	-0.04	0.98
3	2	0	0.79	0.16	0.21	0.11	0.06	0.19	-0.19	-0.05	0.24	1
3.5	2	0	0.90	0.15	0.17	0.11	0.29	-0.10	-0.06	0.20	-0.14	1
4	2	0	1.00	0.15	0.17	0.11	0.38	0.10	0.10	-0.09	0.02	1
0	2	2	0.77	0.13	0.13	0.12	0.69	-0.19	-0.14	-0.10	0.02	0.16
0.5	2	2	0.71	0.26	0.29	0.12	0.56	0.00	-0.06	-0.12	0.04	0.24
1	2	2	0.69	0.26	0.25	0.11	0.49	0.13	0.08	-0.39	0.30	0.54
1.5	2	2	0.68	0.21	0.24	0.12	0.50	0.06	-0.10	0.01	0.06	0.7
2	2	2	0.77	0.17	0.18	0.12	0.59	0.07	-0.08	0.05	0.00	0.82
2.5	2	2	0.81	0.18	0.16	0.15	0.69	-0.07	-0.20	0.24	0.06	0.96
3	2	2	0.86	0.16	0.18	0.12	0.57	-0.05	-0.17	-0.08	0.01	1
3.5	2	2	0.89	0.16	0.16	0.08	0.57	0.22	0.29	0.02	-0.06	1
4	2	2	0.86	0.15	0.13	0.12	0.71	-0.10	-0.28	0.24	-0.15	1
0	-2	2	0.90	0.14	0.13	0.11	0.05	0.29	0.03	-0.05	0.13	0.08
0.5	-2	2	1.07	0.34	0.29	0.11	0.06	-0.01	0.33	0.07	0.15	0.22
1	-2	2	3.19	0.42	0.25	0.16	0.06	0.15	0.16	0.08	0.19	0.38
1.5	-2	2	4.88	0.39	0.22	0.12	0.08	-0.14	0.05	-0.07	0.19	0.52
2	-2	2	6.15	0.32	0.18	0.11	0.06	0.09	-0.05	-0.14	0.16	0.6
2.5	-2	2	6.12	0.32	0.15	0.12	0.07	-0.02	0.39	0.05	0.03	0.84
3	-2	2	8.49	0.33	0.19	0.11	0.06	-0.14	0.20	0.13	-0.02	0.78
3.5	-2	2	8.59	0.32	0.16	0.12	0.05	-0.07	0.20	0.15	-0.02	0.86
4	-2	2	10.73	0.28	0.16	0.10	0.06	0.05	0.08	0.22	0.09	0.96

Table 2b. *Results of the simulation study averaged over conditions.*

True parameters			**RMSE**					**Correlation with α**				**Relative fit**
α	b1	b2	α	b1	b2	b3	sd(θ)	b1	b2	b3	sd(θ)	%Copula Rasch
0	averaged		0.67	0.13	0.12	0.11	0.28	0.02	0.00	-0.08	0.07	0.14
0.5	averaged		0.75	0.30	0.30	0.11	0.28	0.04	0.02	-0.01	0.04	0.37
1	averaged		1.12	0.33	0.29	0.13	0.22	0.05	0.09	-0.08	0.11	0.62
1.5	averaged		1.41	0.31	0.28	0.12	0.23	-0.04	0.03	0.05	-0.03	0.78
2	averaged		1.65	0.27	0.23	0.12	0.32	0.05	0.01	0.07	0.01	0.88
2.5	averaged		1.69	0.26	0.22	0.13	0.27	-0.05	-0.09	0.06	-0.02	0.95
3	averaged		2.09	0.25	0.21	0.12	0.27	-0.05	0.03	0.00	0.04	0.96
3.5	averaged		2.15	0.23	0.19	0.11	0.27	0.03	0.09	0.10	-0.04	0.98
4	averaged		2.56	0.23	0.18	0.12	0.29	0.04	0.03	0.12	-0.08	0.99
averaged	-2	-2	1.02	0.33	0.33	0.12	0.61	0.09	0.08	0.00	-0.04	0.77
averaged	-2	0	0.78	0.33	0.22	0.12	0.16	0.04	-0.04	0.04	0.01	0.78
averaged	0	0	0.54	0.20	0.21	0.12	0.06	-0.09	0.00	0.05	-0.02	0.85
averaged	2	0	0.70	0.18	0.22	0.12	0.14	0.00	0.02	0.04	-0.02	0.76
averaged	2	2	0.78	0.19	0.19	0.12	0.60	0.01	-0.07	-0.02	0.03	0.71
averaged	-2	2	5.57	0.32	0.19	0.12	0.06	0.02	0.15	0.05	0.10	0.58

overall mean	1.57	0.26	0.23	0.12	0.27	0.01	0.03	0.03	0.01	0.74
overall std	2.25	0.09	0.08	0.01	0.25	0.15	0.18	0.14	0.13	0.31

Table 3. *Rasch and Rasch Copula models for the small reading test.*

Model	**AIC**	$\alpha_{1,6}$	$\alpha_{4,5}$	$\alpha_{1,6,4,5}$
Rasch	3157	.	.	.
$C_{Frank}(1,6)C_{Frank}(4,5)$	3108	3.56	2.53	.
$C_{Frank}(1,6)C_{Clayton}(4,5)$	3113	3.57	0.39	.
$C_{Clayton}(1,6)C_{Frank}(4,5)$	3111	1.00	2.50	.
$C_{Clayton}(1,6)C_{Clayton}(4,5)$	3116	1.00	0.38	.
$C_{Frank}(1,6,4,5)$	3140	.	.	1.40
$C_{Clayton}(1,6,4,5)$	3148	.	.	0.25

Table 4. *Parameter estimates for the Rasch model and the copula model with Rasch margins.*

Model	**Rasch**	$C_{Frank}(1,6)C_{Frank}(4,5)$
Parameter	**Estimate**	**Estimate**
b1	-0,36 (0.12)	-0,34 (0.13)
b2	0,99 (0.15)	0,94 (0.12)
b3	0,01 (0.11)	0,00 (0.12)
b4	-1,38 (0.14)	-1,32 (0.16)
b5	-1,50 (0.13)	-1,44 (0.16)
b6	-0,02 (0.13)	-0,03 (0.13)
sd(θ)	1,28 (0.09)	1,12 (0.10)
$\alpha(1,6)$	.	3,56 (0.61)
$\alpha(4,5)$	.	2,53 (0.84)
AIC	3157	3108

Table 5. *Rasch and Rasch Copula models for the verbal aggression data.*

Model	**Rasch**	$C_{Frank}\left(situation\right)$	$C_{Frank}\left(3,6,9,12\right)$	$C_{Clayton}\left(3,6,9,12\right)$
Parameter	**Estimate**	**Estimate**	**Estimate**	**Estimate**
b1	-1,22	-1.20 (0.20)	-1.23 (0.16)	-1.22 (0.17)
b2	-0,56	-0.57 (0.16)	-0.57 (0.17)	-0.56 (0.15)
b3	-0,08	-0.08 (0.18)	-0.06 (0.18)	-0.10 (0.17)
b4	-1,74	-1.72 (0.19)	-1.76 (0.22)	-1.75 (0.17)
b5	-0,70	-0.71 (0.17)	-0.71 (0.17)	-0.71 (0.14)
b6	-0,01	-0.00 (0.18)	-0.02 (0.17)	-0.04 (0.16)
b7	-0,52	-0.47 (0.15)	-0.54 (0.17)	-0.53 (0.16)
b8	0,69	0.63 (0.17)	0.68 (0.15)	0.69 (0.16)
b9	1,53	1.50 (0.15)	1.51 (0.18)	1.51 (0.17)
b10	-1,08	-0.98 (0.16)	-1.09 (0.16)	-1.08 (0.14)
b11	0,35	0.34 (0.18)	0.35 (0.16)	0.35 (0.16)
b12	1,05	1.06 (0.19)	1.04 (0.17)	1.03 (0.17)
b13	-1,22	-1.19 (0.18)	-1.23 (0.16)	-1.22 (0.16)
b14	-0,38	-0.40 (0.18)	-0.40 (0.16)	-0.39 (0.18)
b15	0,88	0.85 (0.13)	0.87 (0.18)	0.87 (0.16)
b16	-0,87	-0.85 (0.15)	-0.88 (0.16)	-0.87 (0.16)
b17	0,06	0.03 (0.14)	0.05 (0.17)	0.06 (0.16)
b18	1,49	1.42 (0.17)	1.48 (0.18)	1.48 (0.17)
b19	0,22	0.23 (0.15)	0.21 (0.17)	0.21 (0.15)
b20	1,51	1.43 (0.17)	1.50 (0.16)	1.50 (0.19)
b21	2,98	2.86 (0.26)	2.96 (0.24)	2.95 (0.24)
b22	-0,70	-0.67 (0.14)	-0.71 (0.18)	-0.71 (0.15)
b23	0,39	0.38 (0.18)	0.38 (0.19)	0.38 (0.16)
b24	2,00	1.94 (0.21)	1.99 (0.19)	1.99 (0.20)
sd(θ)	1,37	1.28 (0.06)	1.37 (0.08)	1.36 (0.07)
α(sit1)	.	1.00 (0.27)	.	.
α(sit2)	.	1.22 (0.53)	.	.
α(sit3)	.	1.68 (0.36)	.	.
α(sit4)	.	1.92 (0.31)	.	.
α(3,6,9,12)	.		2.97 (0.42)	1.02 (0.16)
AIC	8125	7951	8043	8045

Figures

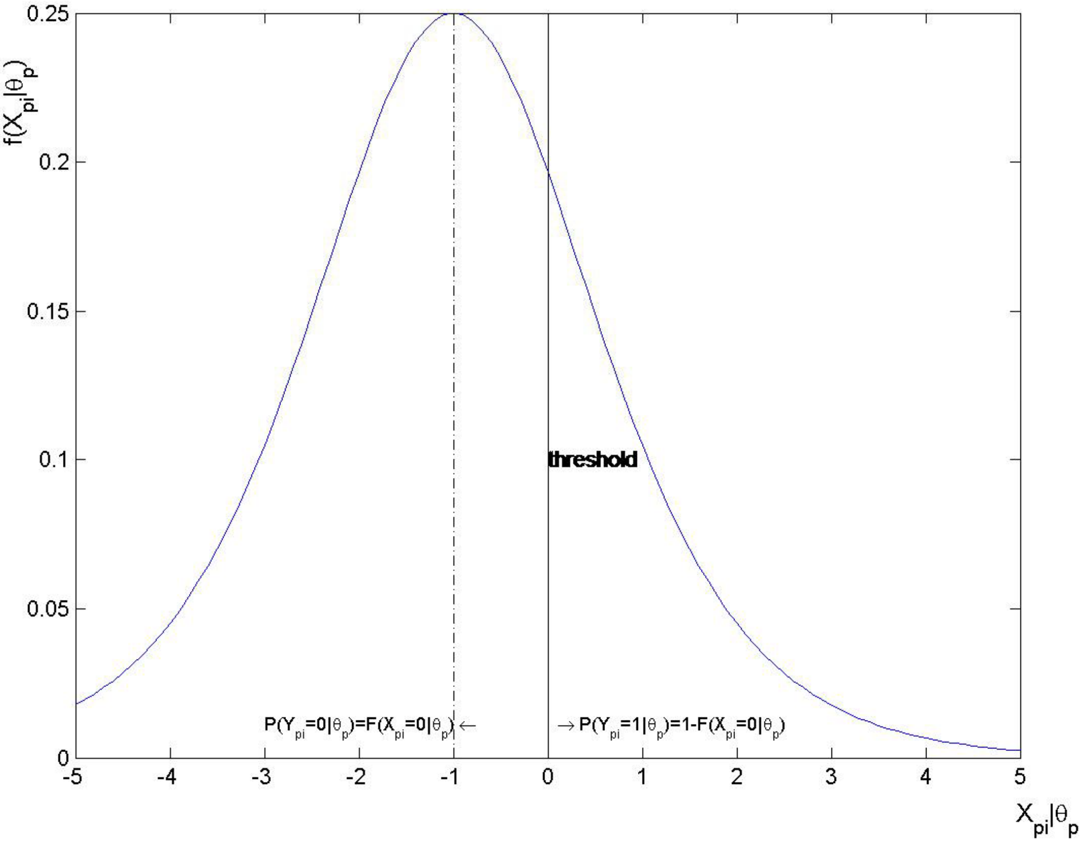

Figure 1. Latent threshold formulation of the Rasch model.

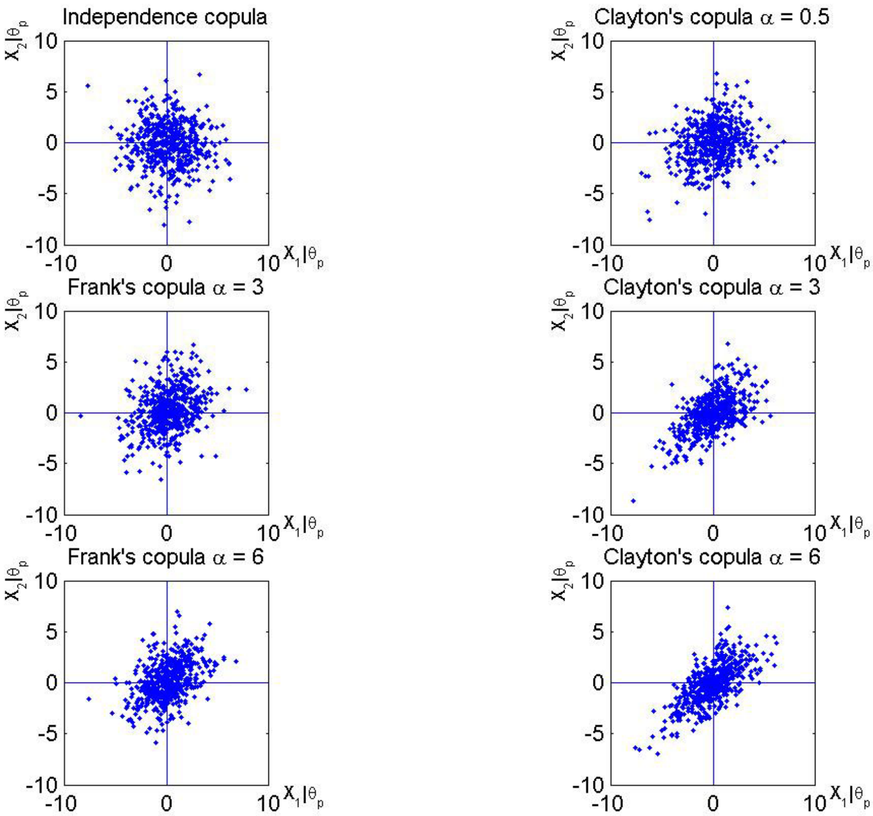

Figure 2. Sample data of multivariate latent logistic distributions (with means zero and scale parameters equal to one) for varying levels of dependence and using different copulas.

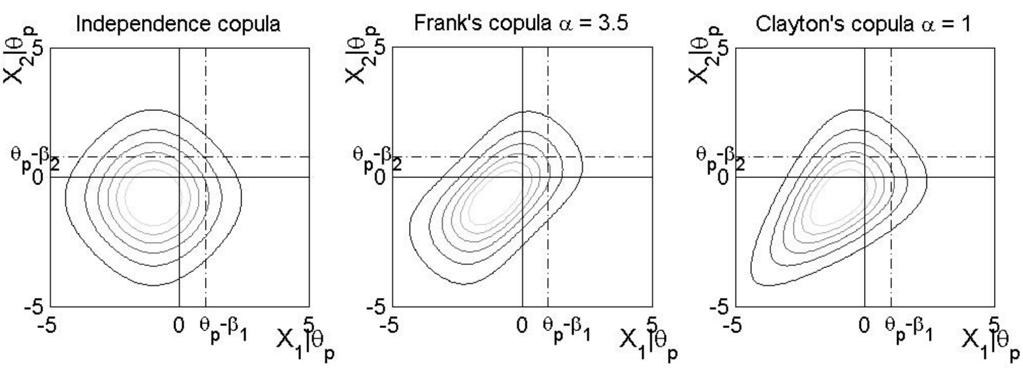

Figure 3. Bivariate density for different logistic copula models.

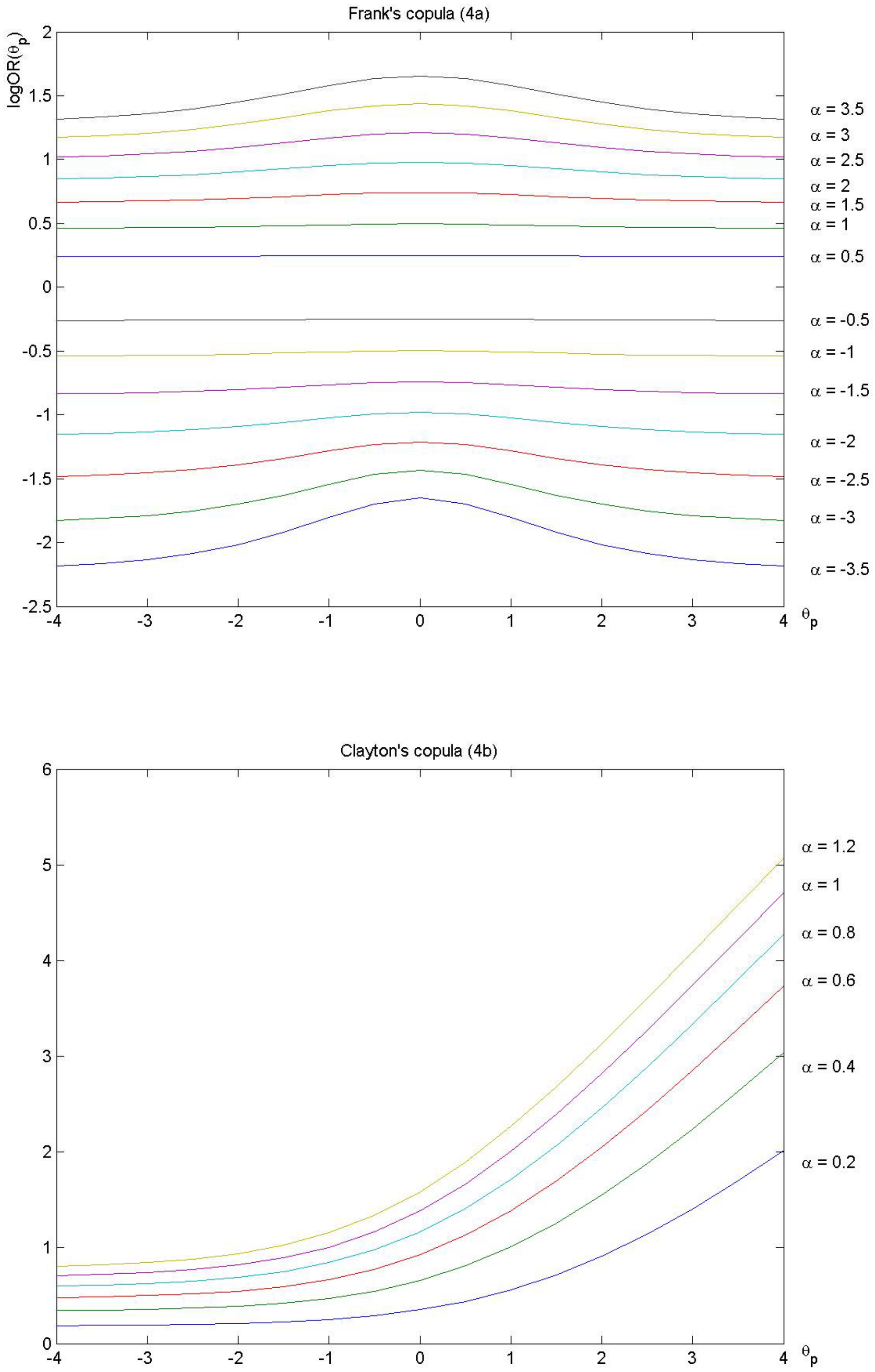

Figure 4. The conditional log odds ratio for different values of the copula parameter.