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TWO-SIDED EXIT PROBLEM FOR GENERAL LEVY PROCESSES THEORY AND APPLICATION

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Two-sided exit problem for general Lévy processes. Theory and application.

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Abstract

In this paper we will determine the joint distribution of the first exit time from a fixed interval by a Lévy process with arbitrary jumps and the overshoot values across the boundaries. The joint distribution of the infimum, the supremum and the value of the process are obtained. Possible applications in mathematical finance are discussed.

Keywords: Lévy process; first exit time; Markov property; Wiener-Hopf factorization.

AMS Subject Classification: 60G51, 62P05.

1 Introduction

Lévy processes have been considered in applied probability, in connection with theories of dams, queues, insurance risks, etc. A further motivation for their study has been provided more recently by the link with financial mathematics. One of the most interesting aspects of the theory of Lévy processes concerns the so-called two-sided exit problem or two-boundary problem, which consists of specifying the distribution of certain random variables related to the first exit time form an interval. Many academics such as Emery [13], Pecherskii [22], Suprun, Shurenkov [34], Doney [10] have enormously contributed to the development of this field. Spectrally one-sided Lévy processes have been studied by many researches (Pistorius [24], Avram et al. [1], Doney [9], Rogers [27], Mordecki [21] and many others). Boyarchenko en Levendorskii [7], Alili and Kyprianou [3], Geman [11], Asmussen et al. [2], Madan et al. [19] and many others studied Lévy processes in context of mathematical finance. In the present paper we will determine the joint distribution of the first exit time from a fixed interval and the value of the overshoot across the boundaries by a general Lévy process. This distribution will be obtained in terms of the integral transforms of the one-boundary characteristics of the process. We will show how the results obtained in the general framework simplify for special classes of Lévy processes (spectrally positive Lévy processes, and for the Wiener process, in particular). The results obtained can also be applied straightforwardly for the discrete version of Lévy processes, i.e. random walks. The distribution of the first exit time from a fixed interval by a Lévy process plays a crucial role in solving other two-boundary problems. In this article we will also obtain the joint distribution of the infimum, the supremum and the value of the process. The rest of the paper is organized as follows. First, we will introduce a general Lévy process and state auxiliary results. The main results with proofs are given in Section 3. In Section 4 we prove corresponding results for spectrally positive Lévy processes and for the Wiener process in particular. The joint distribution of the supremum, the infimum and the value of the Lévy processwill be determined in Section 5.

2 Definition of the process and auxiliary results

Let $(\Omega,\ \mathfrak{F}, \{\mathfrak{F}_t\}, P)$ be a filtered probability space, where the filtration $\mathfrak{F} = \{\mathfrak{F}_t\}$ satisfies the usual conditions of right-continuity and completion. We assume that all random variables and stochastic processes are defined on this probability space. A Lévy process is an $\mathfrak{F}$-adapted stochastic process $\{\xi(t);\, t \geq 0\}$ which has independent and stationary increments and its paths are right-continuous with left limits [6]. In assumption that $\xi(0) = 0$ the Laplace transform of the process $\{\xi(t);\, t \geq 0\}$ has the form $E[e^{-p\xi(t)}] = e^{t\,k(p)}$, $Re p = 0$, where the function $k(p)$ is called Laplace exponent and is given by the formula

$$k(p) = \frac{1}{2}\,p^2\sigma^2 - \alpha p + \int_{-\infty}^{\infty}\left(e^{-px} - 1 + \frac{px}{1+x^2}\right)\Pi(dx), \qquad Re\,p = 0, \tag{1}$$

where $\alpha \in \mathbb{R}$ is the drift, $\sigma^2 > 0$ is the diffusion and $\Pi(\cdot)$ is a measure on $\mathbb{R}$, called the Lévy measure. Note, that (1) is one of the possible representations for the Laplace exponent. For the equivalent formulae and restrictions on the Lévy measure we refer to [12]. Let us fix $B > 0$ and assume $x \in (0, B),\quad y = B - x$, and define the random variable

$$\chi = \inf\{\, t > 0 :\ \xi(t) \notin (-y, x)\,\},$$

the first exit time from the interval $(-y, x)$ by the process $\xi(t)$. Observe that χ is a Markov time ([12], p. 52) and $P\,[\,\chi < \infty\,] = 1$. It is clear that the exit from the interval can occur either through the upper boundary x or through the lower boundary $-y$. Introduce events:

$A^x = \{\, \xi(\chi) \geq x \,\}$, i.e. the exit from the interval occurs through the upper boundary;

$A_y = \{\, \xi(\chi) \leq -y \,\}$, i.e. the process exits the interval through the lower boundary.

Introduce the random variable

$$X = (\xi(\chi) - x)I_{A^x} + (-\xi(\chi) - y)I_{A_y},$$

i.e. the value of the overshoot through a boundary at the instant of the first exit from the interval by the given process. Here $I_A = I_A(w)$ is an indicator of the event A; $P[A^x \cup A_y] = 1$, $P[A^x \cap A_y] = 0$. The aim of this section is to determine the integral transforms $E\,[\,e^{-s\chi};\ X \in du]$ of the joint distributions of $\{\,\chi, X\}$. To solve this problem we require the distributions of the one-boundary functionals $\{\,\tau^x, T^x\,\}$, $\{\,\tau_y, T_y\,\}$, where

$$\tau^x = \inf\{\, t > 0 :\ \xi(t) \geq x \,\}, \qquad T^x = \xi(\tau^x) - x \in \mathbb{R}^+, \qquad x > 0$$

the first passage time of the upper level by the process $\xi(t)$, $t \geq 0$ and the value of the overshoot across this level at the first passage time of the upper level;

$$\tau_y = \inf\{\, t > 0 :\ \xi(t) \leq -y \,\}, \qquad T_y = -\xi(\tau_y) - y \in \mathbb{R}^+, \qquad y > 0$$

the first passage time of the lower level by the process $\xi(t)$ and value of the overshoot across the lower level at this instant. The following lemma is true.

Lemma 1 *The integral transforms of $\{\,\tau^x, T^x\,\}, \{\,\tau_y, T_y\,\}$, $\{\,\xi(\nu_s),\ \xi^+(\nu_s)\,\}$ satisfy the following identities*

$$\begin{aligned}
&E\,[\,e^{-s\tau^x - pT^x}\,] = \left(E[e^{-p\xi^+(\nu_s)}]\right)^{-1} E\,[\,e^{-p(\xi^+(\nu_s)-x)};\ \xi^+(\nu_s) \geq x\,], && \mathrm{Re}\, p \geq 0,\\
&E\,[\,e^{-s\tau_y - pT_y)}\,] = \left(E[\,e^{p\xi^-(\nu_s)}]\right)^{-1} E\,[\,e^{p(\xi^-(\nu_s)+y)};\ -\xi^-(\nu_s) \geq y\,], && \mathrm{Re}\, p \geq 0,\\
&E\,[\,e^{-p\xi(\nu_s)};\ \xi^+(\nu_s) < x\,] = E[\,e^{-p\xi^-(\nu_s)}]E\,[\,e^{-p\xi^+(\nu_s)};\ \xi^+(\nu_s) < x\,], && \mathrm{Re}\, p \leq 0,\\
&E\,[\,e^{-p\xi(\nu_s)};\ \xi^-(\nu_s) > -x\,] = E[\,e^{-p\xi^+(\nu_s)}]E\,[\,e^{-p\xi^-(\nu_s)};\ \xi^-(\nu_s) > -x\,], && \mathrm{Re}\, p \geq 0.
\end{aligned} \tag{2}$$

where

$$\xi^+(t) = \sup_{u \leq t} \xi(u), \qquad \xi^-(t) = \inf_{u \leq t} \xi(u),$$

$$E\,[e^{-p\xi^{\pm}(\nu_s)}] = \exp\left\{\int_0^\infty \frac{1}{t}\, e^{-st}\, E\,[\, e^{-p\xi(t)} - 1;\ \pm\xi(t) > 0\,]\, dt\right\}, \qquad \pm Re\, p \geq 0,$$

ν_s is an exponentially distributed random variable with parameter s, i.e. $P[\,\nu_s > t\,] = e^{-st}$. It is supposed to be independent of the underlying process.

Proof. The integral transforms of the distributions in Lemma are well known for random walks and for processes with independent increments as well (see [22]). We sketch a brief proof of the equalities (2), based on the probabilistic reasoning and on the Wiener-Hopf factorization:

$$E[e^{-p\,\xi(\nu_s)}] = \frac{s}{s-k(p)} = E[e^{-p\,\xi^{+}(\nu_s)}]\; E[e^{-p\,\xi^{-}(\nu_s)}], \qquad Re\, p = 0$$

According to the total probability formula and the strong Markov property of the process, the Laplace transform of the process's increments satisfies the following equation:

$$E\,[e^{-p\xi(\nu_s)}] = E\,[\,e^{-p\xi(\nu_s)};\ \xi^{+}(\nu_s) < x] + E\,[\,e^{-s\tau^x - p\xi(\tau^x)}\,]E[\,e^{-p\xi(\nu_s)}], \qquad Re\, p = 0. \quad (3)$$

This equation represents the fact that the increments of $\{\xi(t);\ t \geq 0\}$ on the time-interval $[0,\ \nu_s]$ can occur either on the sample paths which do not cross the level x (the first term in the right-hand side), or on paths which cross the level x and where the continuation of the process is a probabilistic replica of the entire process on the exponentially distributed interval $[0, \nu_s]$ (the second term in the right-hand side). Let us give more thorough explanation. In accordance with the total probability formula the following equality holds

$$\begin{aligned} E[\,e^{-p\xi(t)}] &= E\,[\,e^{-p\xi(t)};\ \tau^x > t\,] + E\,[\,e^{-p\xi(t)};\ \tau^x \leq t\,] \\ &= E\,[\,e^{-p\xi(t)};\ \xi^{+}(t) < x\,] + E\,[\,e^{-p\xi(\tau^x)}\, e^{-p\,\theta_{\tau^x}\xi(t-\tau^x)};\ \tau^x \leq t\,], \qquad \mathrm{Re}\, p = 0, \end{aligned}$$

where θ_t is a shift operator ([12],p. 85). The random variable τ^x being a Markov time, the increments of the process $\theta_{\tau^x}\xi(t-\tau^x)$ do not depend on the sigma-algebra $\mathfrak{F}_{\tau^x}$, generated by events $\{\,\xi(u) < v\,\} \cap \{\,\tau^x > u\,\}$ for all u, v. Hence,

$$E\,[\,e^{-p\xi(\tau^x)}\, e^{-p\,\theta_{\tau^x}\xi(t-\tau^x)};\ \tau^x \leq t\,] = \int_0^t E\,[\,e^{-p\xi(u)};\ \tau^x \in du\,]\, E\,[\,e^{-p\xi(t-u)}\,].$$

Substituting the right-hand side of this formula into the previous equality yields

$$E[\,e^{-p\xi(t)}] = E\,[\,e^{-p\xi(t)};\ \xi^{+}(t) < x\,] + \int_0^t E\,[\,e^{-p\xi(u)};\ \tau^x \in du\,]\, E\,[\,e^{-p\xi(t-u)}\,].$$

Now multiplying both sides of the last equality by the density $s\,e^{-st}$ of the random variable ν_s, and integrating over $t \geq 0$, implies (3). The next step is to partition the functions which enter the equation (3). Using the Wiener-Hopf factorization, we rewrite this equation in the form:

$$\left(E\,[e^{-p\xi^-(\nu_s)}]\right)^{-1} E\,[\,e^{-p(\xi(\nu_s)-x)};\ \xi^+(\nu_s) < x\,] - E\,[\,e^{-p(\xi^+(\nu_s)-x)};\ \xi^+(\nu_s) < x\,] =$$

$$E\,[\,e^{-p(\xi^+(\nu_s)-x)};\ \xi^+(\nu_s) \geq x\,] - E\,[e^{-p\xi^+(\nu_s)}]E\,[\,e^{-s\tau^x - pT^x}\,], \qquad Re\,p = 0.$$

Observe that the function in the left-hand side of the equality is analytical in the left semi-plane $Re\,p < 0$, bounded and continuous including the boundary $Re\,p = 0$. By means of the latter equality it can be extended to the function being analytical in the right semi-plane, which enters the right-hand side of the equality. Therefore, this function is analytic and bounded on the entire complex plane. But then in view of Liouville's theorem [8] the function in the left hand side of the equality is a constant with respect to p. We denote it by $C(s)$. To find this constant let $p \to \infty$, which implies that $C(s) = 0$. These factorization arguments [4] yield the following formulae

$$E\,[\,e^{-s\tau^x - pT^x)}\,] = \left(E\,[e^{-p\xi^+(\nu_s)}]\right)^{-1} E\,[\,e^{-p(\xi^+(\nu_s)-x)};\ \xi^+(\nu_s) \geq x\,], \qquad Re\,p \geq 0$$

$$E\,[\,e^{-p\xi(\nu_s)};\ \xi^+(\nu_s) < x\,] = E\,[e^{-p\xi^-(\nu_s)}]E\,[\,e^{-p\xi^+(\nu_s)};\ \xi^+(\nu_s) < x\,], \qquad Re\,p \leq 0$$

Applying now the obtained formulae for the process $-\xi(t)$ yields the last two equalities (2) of Lemma. Thus the lemma is proved. We now introduce some notation. Denote $(v > 0)$

$$K_+(v, du, s) = \int_0^\infty E[e^{-s\tau_{v+B}};\ T_{v+B} \in dv_1]\, E[e^{-s\tau^{v_1+B}};\ T^{v_1+B} \in du],$$

$$K_-(v, du, s) = \int_0^\infty E[e^{-s\tau^{v+B}};\ T^{v+B} \in dv_1]\, E[e^{-s\tau_{v_1+B}};\ T_{v_1+B} \in du].$$

We introduce the sequences $K_\pm^{(n)}(v, du, s),\ n \in \mathbb{N}$ by means of recurrence relation

$$K_\pm^{(1)}(v, du, s) = K_\pm(v, du, s), \qquad K_\pm^{(n+1)}(v, du, s) = \int_0^\infty K_\pm^{(n)}(v, dv_1, s) K_\pm(v_1, du, s). \quad (4)$$

We now formulate the main results.

3 Main results

Theorem 1 *Let* $\{\xi(t);\ t \geq 0\}$, $\xi(0) = 0$ *be a general Lévy process as specified above,* $B > 0$, $x \in (0, B)$, $\quad y = B - x$, *and*

$$\chi = \inf\{\,t > 0 :\ \xi(t) \notin (-y, x)\,\}, \qquad X = (\xi(\chi) - x)\, I_{A^x} + (-\xi(\chi) - y)\, I_{A_y}$$

be the instant of the first exit from the interval $(-y,x)$ *and the value of the overshoot across one of the boundaries. Then the Laplace transforms of the joint distributions of the random variables* $\{\chi, X\}$ *satisfy the following equalities*

$$\begin{aligned} E\,[\,e^{-s\chi};\ X\in du,\ A^x] &= f_+^s(x,du) + \int_0^\infty f_+^s(x,dv)\,K_+^s(v,du), \\ E\,[\,e^{-s\chi};\ X\in du,\ A_y] &= f_-^s(y,du) + \int_0^\infty f_-^s(y,dv)\,K_-^s(v,du), \end{aligned} \tag{5}$$

where

$$f_+^s(x,du) = E\,[\,e^{-s\tau^x};\ T^x\in du\,] - \int_0^\infty E\,[\,e^{-s\tau_y};\ T_y\in dv\,]\,E\,[\,e^{-s\tau^{v+B}};\ T^{v+B}\in du\,];$$

$$f_-^s(y,du) = E\,[\,e^{-s\tau_y};\ T_y\in du\,] - \int_0^\infty E\,[\,e^{-s\tau^x};\ T^x\in dv\,]\,E\,[\,e^{-s\tau_{v+B}};\ T_{v+B}\in du\,];$$

$K_\pm^s(v,du) = \sum_{n=1}^{\infty} K_\pm^{(n)}(v,du,s)$, $v\ge 0$ *are the series of the iterations of the* $K_\pm^{(n)}(v,du,s)$ *defined by (4).*

Proof of Theorem 1 According to the total probability formula and the strong Markov property of the process the functions
$E\,[\,e^{-s\chi};\ X\in du\,A^x]$, $E\,[\,e^{-s\chi};\ X\in du\,A_y]$ satisfy the following system of equations:

$$\begin{aligned} E\,[\,e^{-s\tau^x};\ T^x\in du\,] =& E\,[\,e^{-s\chi};\ X\in du,\ A^x\,] + \\ & \int_0^\infty E\,[\,e^{-s\chi};\ X\in dv,\ A_y\,]\,E\,[\,e^{-s\tau^{v+B}};\ T^{v+B}\in du\,], \\ E\,[\,e^{-s\tau_y};\ T_y\in du\,] =& E\,[\,e^{-s\chi};\ X\in du,\ A_y\,] + \\ & \int_0^\infty E\,[\,e^{-s\chi};\ X\in dv,\ A^x\,]\,E\,[\,e^{-s\tau_{v+B}};\ T_{v+B}\in du\,]. \end{aligned} \tag{6}$$

The first equation represents the fact that the first passage of the upper level x by the process $\{\xi(t);\ t\ge 0\}$ (the left-hand side of the equation) can be realized either on paths which do not intersect the lower level $-y$ (the first term of the right-hand side) or on the paths which do intersect the level $-y$ and then pass the level x (the second term in the left-hand side). The second equation is written analogously. Let us explain more carefully. It is clear that

$$E\,[\,e^{-s\tau^x};\ T^x\in du,\ \tau^x<\tau_y\,] = E\,[\,e^{-s\chi};\ X\in du,\ A^x\,],$$

$$E\,[\,e^{-s\tau_y};\ T_y\in du,\ \tau_y<\tau^x\,] = E\,[\,e^{-s\chi};\ X\in du,\ A_y\,].$$

Then, by the total probability formula the chain of equalities is valid:

$$\begin{aligned} E\,[\,e^{-s\tau^x};\ T^x\in du\,] &= E\,[\,e^{-s\tau^x};\ T^x\in du,\ \tau^x<\tau_y\,] + E\,[\,e^{-s\tau^x};\ T^x\in du,\ \tau_y<\tau^x\,] \\ &= E\,[\,e^{-s\chi};\ X\in du,\ A^x\,] + E\,[\,e^{-s\chi}\,e^{-s\tau^{B+X}};\ T^{B+X}\in du, A_y\,]. \end{aligned} \tag{7}$$

Since χ is a Markov time, the random variables τ^{B+X}, T^{B+X} do not depend on sigma-algebra F_χ, and therefore

$$E\,[\,e^{-s\chi}\,e^{-s\tau^{B+X}};\;T^{B+X}\in du, A_y\,]=\int_0^\infty E\,[\,e^{-s\chi};\;X\in dv,\;A_y\,]\,E\,[\,e^{-s\tau^{v+B}};\;T^{v+B}\in du\,].$$

Substituting the right-hand side of this equality into (7) we obtain the first equality of (6). The second equality of the system can be verified analogously. Let us solve the system of integral equations (6). It is a linear system with two unknown functions, which is very similar to a linear system with two variables. Substituting the expression for the function $E\,[\,e^{-s\chi};\;X\in du\,A_y]$ from the second equation into the first one, implies

$$\begin{aligned}E\,[\,e^{-s\chi};\;X\in du,\;A^x\,]&=f_+^s(x,du)\\&+\int_{l=0}^\infty\int_{v=0}^\infty E\,[\,e^{-s\chi};\;X\in dv,\;A^x\,]\,E\,[\,e^{-s\tau_{v+B}};\;T_{v+B}\in dl\,]\,E\,[\,e^{-s\tau^{l+B}};\;T^{l+B}\in du\,],\end{aligned}\tag{8}$$

where

$$f_+^s(x,du)=E\,[\,e^{-s\tau^x};\;T^x\in du\,]-\int_0^\infty E\,[\,e^{-s\tau_y};\;T_y\in dv\,]\,E\,[\,e^{-s\tau^{v+B}};\;T^{v+B}\in du\,].$$

Changing the order of integration in the second term of the right-hand side of (8) leads to

$$E\,[\,e^{-s\chi};\;X\in du,\;A^x\,]=f_+^s(x,du)+\int_0^\infty E\,[\,e^{-s\chi};\;X\in dv,\;A^x\,]\,K_+(v,du,s)\tag{9}$$

which is a linear integral equation with respect to $E\,[\,e^{-s\chi};\;X\in du\,A^x]$. The kernel $K_+(v,du,s)$ of this equation enjoys the following property for all $v,u\in\mathbb{R}^+$, $\quad s>s_0>0$

$$\begin{aligned}K_+(v,du,s)&=\int_0^\infty E\,[\,e^{-s\tau_{v+B}};\;T_{v+B}\in dv_1\,]\,E\,[\,e^{-s\tau^{v_1+B}};\;T^{v_1+B}\in du\,]\\&\le\int_0^\infty E\,[\,e^{-s\tau_{v+B}};\;T_{v+B}\in dv_1\,]\,E\,[e^{-s\tau^{v_1+B}}]\\&\le E\,[e^{-s\tau^B}]\int_0^\infty E\,[\,e^{-s\tau_{v+B}};\;T_{v+B}\in dv_1\,]\\&\le E[\,e^{-s\tau^B}]\,E\,[e^{-s\tau_B}]\le\lambda<1,\qquad\qquad s>s_0>0,\end{aligned}$$

where

$$\lambda=E\,[e^{-s_0\tau^B}]\,E\,[e^{-s_0\tau_B}]<1,\qquad s_0>0.$$

This chain of inequalities can be found in [12, p. 308]. We also used the fact that: $E\,[e^{-s\tau^{v+B}}]\le E\,[e^{-s\tau^B}]$, $\quad v\ge 0$, which follows immediately from

$$\begin{aligned}E\,[e^{-s\tau^{v+B}}]=&E\,[\,e^{-s\tau^B};\;T^B>v\,]+\int_0^v E\,[\,e^{-s\tau^B};\;T^B\in du\,]\,E\,[e^{-s\tau^{v-u}}]\\\le&E\,[\,e^{-s\tau^B};\;T^B>v\,]+E\,[\,e^{-s\tau^B};\;T^B\le v\,]=E\,[e^{-s\tau^B}].\end{aligned}$$

It is clear, that the inequality $E[e^{-s\tau_{v+B}}] \le E\,[e^{-s\tau_B}], \quad v \ge 0$ also holds. Now, considering the sequence of $n-th$ iterations of the kernels $K_+(v, du, s)$,

$$K_+^{(1)}(v, du, s) = K_+(v, du, s), \qquad K_+^{(n+1)}(v, du, s) = \int_0^\infty K_+^{(n)}(v, dv_1, s)\, K_+(v_1, du, s),$$

one can establish by means of induction that for all $v,\ u \in \mathbb{R}^+, \quad s > s_0 > 0$

$$K_+^{(n)}(v, du, s) < \lambda^n, \qquad\qquad n \in \mathbb{N}.$$

Therefore, the series of successive iterations

$$K_+^s(v, du) = \sum_{n=1}^\infty K_+^{(n)}(v, du, s) < \frac{\lambda}{1-\lambda}$$

converges uniformly with respect to $v, u \in \mathbb{R}^+, \quad s > s_0 > 0$. Therefore, we can apply the method of successive iterations [25] to solve the integral equation (8). Using this method yields

$$E\,[\,e^{-s\chi};\ X \in du, A^x] = f_+^s(x, du) + \int_0^\infty f_+^s(x, dv)\, K_+^s(v, du)$$

which is the first equality of Theorem 1. We complete the proof by noticing that the second equality can be verified analogously. In the following section we present some applications of the results obtained.

4 Spectrally positive Lévy process

Lévy processes with no positive (no negative jumps) form a remarkable class of real-valued Lévy processes, which were first studied in applied probability as models for queuing, insurance risk, dam theory. For a related literature we refer to Prabhu [26], Borovkov [4], Pistorius [24], Doney [9]. We now apply the results obtained in the previous section for this class of Lévy processes. Let $\{\xi(t); t \ge 0\}, \xi(0) = 0$ be a spectrally positive Lévy process. This process has only positive jumps, i.e. the Lévy measure in the representation (1) of the Laplace exponent ([12], p. 311) assigns no mass to $(-\infty, 0)$:

$$k(p) = \frac{1}{2}\,p^2\sigma^2 - \alpha p + \int_0^\infty \left(e^{-px} - 1 + \frac{px}{1+x^2}\right)\Pi(dx), \qquad Re\,p \ge 0.$$

In this case the lower level is reached continuously, i.e. $P[\,T_x = 0\,] = 1$ and the integral transforms of $\tau_x, \quad \xi^-(\nu_s)$ are given by

$$E\,[e^{-s\tau_x}] = e^{-xc(s)}, \qquad\qquad E\,[e^{-p\xi^-(\nu_s)}] = \frac{c(s)}{c(s)-p}, \qquad Re\,p \le 0,$$

where $c(s) > 0$ is a unique root of the equation $k(p) - s = 0$ in semi-plane $Re\, p > 0$ ([12], p. 312.) Using the Wiener-Hopf factorization and the first equality of (2) implies

$$\int_0^\infty e^{-px} E\,[\,e^{-s\tau^x - \lambda\xi(\tau^x)}]\,dx = \frac{1}{p}\left(1 - \frac{p+\lambda-c(s)}{k(p+\lambda)-s}\;\frac{k(\lambda)-s}{\lambda-c(s)}\right), \tag{10}$$

Let $x \in (0,\ B),\ y = B - x.\xi(0) = 0$ and

$$\chi = \inf\{\,t > 0:\ \xi(t) \notin (-y, x)\,\}, \qquad X = (\xi(\chi) - x)\,I_{A^x} + (-\xi(\chi) - y)\,I_{A_y}$$

be the first exit time from $(-y, x)$ by the process $\{\xi(t);\ t \geq 0\}$ and the value of the overshoot through one of the boundaries at the instant of the first exit, where
$A^x = \{\,\xi(\chi) \geq x\,\},\ A_y = \{\,\xi(\chi) = -y\,\}$.

Proposition 1 *Let* $\{\xi(t);\ t \geq 0\},\ \xi(0) = 0$ *be a spectrally positive Lévy process as specified above,* $x \in (0,\ B),\ y = B - x$.

Then

1) the integral transforms of the joint distribution of $\chi,\ \{\,\chi, X\}$ *satisfy the formulae:*

$$\begin{aligned} &E\,[e^{-s\chi};\ A_y] = \frac{1 - G_x^s(c(s))}{1 - G_B^s(c(s))}\; e^{-yc(s)}, \\ &E\,[\,e^{-s\chi};\ X \in du, A^x] = E\,[\,e^{-s\tau^x};\ T^x \in du\,] - E\,[\,e^{-s\chi};\ A_y]\;E\,[\,e^{-s\tau^B};\ T^B \in du\,], \end{aligned} \tag{11}$$

where the function

$$G_x^s(\lambda) = E[e^{-s\tau^x - \lambda\xi(\tau^x)}], \qquad x > 0$$

is given by (10);

2) the integral transforms of the first exit time from the interval admit the following resolvent representation

$$\begin{aligned} &E\,[e^{-s\chi};\ A_y] = \frac{R_x(s)}{R_B(s)}, \\ &E\,[e^{-s\chi};\ A^x] = 1 - \frac{R_x(s)}{R_B(s)} - s\frac{R_x(s)}{R_B(s)}\int_0^B R_u(s)\,du + s\int_0^x R_u(s)\,du, \end{aligned} \tag{12}$$

where

$$R_x(s) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{xp}\,\frac{1}{k(p) - s}\,dp, \qquad \gamma > c(s) \tag{13}$$

is the resolvent of a spectrally positive Lévy process (see [5]).

Remark 1 *This function often appears in literature as a scale function ([6]). The scale function naturally appears when computing various transient characteristics of the spectrally positive (negative) Lévy process. Its properties have been studied in [5], [34], [35], [?], and more recently in [18].*

Remark 2 *Let us mention that Emery [13] and Pecherskii [23] were the first authors who obtained the integral transforms of the distributions of the first exit time from the interval by a spectrally one-sided Lévy process. The equalities (12) were obtained by means of resolvent methods in [34, 33]. The equalities (11) which contain the distribution of the value of the overshoot across the upper boundary at the instant of the first exit were derived in [16].*

Proof of Proposition 1 Let us verify the equalities (11), using the equalities of Theorem 1 which substantially simplify for the spectrally positive Lévy process. In this case the functions $f_-^s(y,du)$, $K_-^{(n)}(v,du,s)$, $\quad n\in\mathbb{N}$ and $K_-^s(v,du)$ are easily calculated. In view of the formulae

$$E[e^{-s\tau^x - c(s)T^x}] = e^{xc(s)}E[e^{-s\tau^x - c(s)\xi(\tau^x)}] = e^{xc(s)}G_x^s(c(s)),$$

$$E[e^{-s\tau_x},\, T_x\in du] = e^{-x\, c(s)}\delta(u)du,$$

we have

$$\begin{aligned} f_-^s(y,du) &= e^{-yc(s)}\,(1-G_x^s(c(s)))\,\delta(u)\,du,\\ K_-^{(n)}(v,du,s) &= e^{vc(s)}G_{v+B}^s(c(s))\ \left(G_B^s(c(s))\right)^{n-1}\ \delta(u)\,du, \qquad\qquad n\in\mathbb{N},\\ K_-^s(v,du) &= \sum_{n=1}^{\infty} K_-^{(n)}(v,du,s) = e^{vc(s)}\ \frac{G_{v+B}^s(c(s))}{1-G_B^s(c(s))}\ \delta(u)\,du, \end{aligned}$$

Substituting the expressions for $f_-^s(y,du)$ and $K_-^s(v,du)$ into the second formula of Theorem 1 yields

$$E\,[\,e^{-s\chi};\ X\in du, A_y] = e^{-yc(s)}\ \frac{1-G_x^s(c(s))}{1-G_B^s(c(s))}\ \delta(u)\,du. \tag{14}$$

Integrating (14) with respect to $u\in\mathbb{R}^+$, implies the first equality of (11). Now, observe that

$$\int_0^\infty E[e^{-s\tau_y};\ T_y\in dv]\,E[e^{-s\tau^{v+B}};\ T^{v+B}\in du] = e^{-c(s)y}E[e^{-s\tau^B};\ T^B\in du]$$

Taking into account the latter expressions we derive the functions $f_+^s(x,du)$, $K_+^{(n)}(v,du,s)$ and $K_+^s(v,du)$ for the spectrally positive Lévy process:

$$\begin{aligned} f_+^s(x,du) &= E\,[\,e^{-s\tau^x};\ T^x\in du\,] - e^{-yc(s)}\,E\,[\,e^{-s\tau^B};\ T^B\in du\,],\\ K_+^{(n)}(v,du,s) &= e^{-c(s)(v+B)}\ \left(G_B^s(c(s))\right)^{n-1}\ E\,[\,e^{-s\tau^B};\ T^B\in du\,], \qquad n\in\mathbb{N}; \end{aligned}$$

$$K_{+}^{s}(v,du)=\sum_{n=1}^{\infty}K_{+}^{(n)}(v,du,s)=\frac{e^{-c(s)(v+B)}}{1-G_{B}^{s}(c(s))}\;E\,[\,e^{-s\tau^{B}};\;T^{B}\in du\,].$$

Substituting the expressions for $f_{+}^{s}(x,du)$, $K_{+}^{s}(v,du)$ into the first formula of Theorem 1, we get:

$$E\,[\,e^{-s\chi};\;X\in du,A^{x}]=E\,[\,e^{-s\tau^{x}};\;T^{x}\in du\,]-e^{-yc(s)}\;\frac{1-G_{x}^{s}(c(s))}{1-G_{B}^{s}(c(s))}\;E\,[\,e^{-s\tau^{B}};\;T^{B}\in du\,] \tag{15}$$

which is the second equality of (11). Integrating the equalities (14), (15) all over $u\in\mathbb{R}^{+}$, yields

$$\begin{aligned}E\,[e^{-s\chi};\;A_{y}]&=e^{-yc(s)}\;\frac{1-G_{x}^{s}(c(s))}{1-G_{B}^{s}(c(s))},\\ E\,[\,e^{-s\chi};\;A^{x}]&=E[e^{-s\tau^{x}}]-e^{-yc(s)}\;\frac{1-G_{x}^{s}(c(s))}{1-G_{B}^{s}(c(s))}\;E[e^{-s\tau^{B}}\;].\end{aligned}$$

Employing the defining formula of the resolvent (13), and also the integral transforms of the joint distribution of $\{\,\tau^{x},T^{x}\,\}$ in (10), we obtain the expressions of the functions $G_{x}^{s}(c(s))$, $E[e^{-s\tau^{x}}]$ in terms of the resolvent function

$$G_{x}^{s}(c(s))=1-k'(c(s))\,e^{-xc(s)}\;R_{x}(s),\qquad E[e^{-s\tau^{x}}]=1-\frac{s}{c(s)}\;R_{x}(s)+s\int_{0}^{x}R_{u}(s)\,du.$$

Substituting these resolvent representations in the previous equalities, implies

$$\begin{aligned}E\,[e^{-s\chi};\;A_{y}]&=\frac{R_{x}(s)}{R_{B}(s)},\\ E\,[e^{-s\chi};\;A^{x}]&=1-\frac{R_{x}(s)}{R_{B}(s)}-s\frac{R_{x}(s)}{R_{B}(s)}\int_{0}^{B}R_{u}(s)\;du+s\int_{0}^{x}R_{u}(s)\;du\end{aligned}$$

which are the formulae (12) of Corollary 1. This completes the proof. The results of Theorem 1 become more clear for a Wiener process.

4.1 Wiener process

Let $\{w(t);\;t\geq 0\}$ be a Wiener process with the Laplace exponent of the form $k(p)=\frac{1}{2}p^{2}$. In this case

$$P[\,T^{x}=T_{x}=0\,]=1,\qquad\qquad E[e^{-s\tau^{x}}]=e^{-x\sqrt{2s}}=E[e^{-s\tau_{x}}].$$

Proposition 2 *Let $\{w(t);\,t\geq 0\}$ be a Wiener process as specified above, $x\in(0,\,B), y=B-x$ and*

$$\chi=\inf\{\,t>0:\;w(t)\notin(-y,x)\,\},\quad A^{x}=\{w(\chi)=x\},\quad A_{y}=\{w(\chi)=-y\}$$

be the first exit time from the interval $(-y, x)$, $\quad x, y > 0,\ x + y = B$. Then

1) the integral transforms of the random variable χ are determined by

$$\begin{aligned}
E\,[\,e^{-s\chi};\ A^x\,] =& \frac{e^{y\sqrt{2s}} - e^{-y\sqrt{2s}}}{e^{B\sqrt{2s}} - e^{-B\sqrt{2s}}} = \frac{\operatorname{sh}\left(y\sqrt{2s}\right)}{\operatorname{sh}\left(B\sqrt{2s}\right)},\\
E\,[\,e^{-s\chi};\ A_y\,] =& \frac{e^{x\sqrt{2s}} - e^{-x\sqrt{2s}}}{e^{B\sqrt{2s}} - e^{-B\sqrt{2s}}} = \frac{\operatorname{sh}\left(x\sqrt{2s}\right)}{\operatorname{sh}\left(B\sqrt{2s}\right)},\\
E\,[\,e^{-s\chi}\,] =& \operatorname{ch}\left(\frac{x-y}{2}\sqrt{2s}\right)\Big/\operatorname{ch}\left(\frac{B}{2}\sqrt{2s}\right);
\end{aligned}\tag{16}$$

2) the distribution of χ satisfies the formulae

$$\begin{aligned}
P\,[\,\chi > t;\ A^x\,] =& \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\exp\left(-\frac{t}{2}\,(k\pi/B)^2\right)\sin\left(\frac{x}{B}k\pi\right),\\
P\,[\,\chi > t;\ A_y\,] =& \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\exp\left(-\frac{t}{2}\,(k\pi/B)^2\right)\sin\left(\frac{y}{B}k\pi\right),\\
P\,[\,\chi > t\,] =& \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left(-\frac{t}{2}\,((2k+1)\pi/B)^2\right)\cos\left(\frac{x-y}{2B}(2k+1)\pi\right);
\end{aligned}\tag{17}$$

3) the moments of χ are of the form

$$E[\chi^n] = \frac{1}{(2n-1)!!}\left(\frac{B}{2}\right)^{2n}\sum_{k=0}^{n}(-1)^k\left(\frac{x-y}{B}\right)^{2k}\binom{2n}{2k}E_{n-k}, \qquad n > 0, \tag{18}$$

in particular

$$E[\chi] = xy, \qquad E[\chi^2] = \frac{1}{3}xy(x^2 + 3xy + y^2),$$

and setting $x = y = \frac{B}{2}$ yields

$$E[\chi^n] = \frac{1}{(2n-1)!!}\left(\frac{B}{2}\right)^{2n}E_n, \qquad n > 0,$$

where $E_1 = 1,\ E_2 = 5, \ldots$ are the Euler numbers.

Proof of Proposition 2 The series of kernels take the simple form:

$$\begin{aligned}
K_-(v, du, s) &= e^{-v\sqrt{2s}}e^{-2B\sqrt{2s}}\,\delta(u)\,du = K_+(v, du, s),\\
K_-^s(v, du) &= e^{-v\sqrt{2s}}\,\frac{e^{-2B\sqrt{2s}}}{1 - e^{-2B\sqrt{2s}}}\,\delta(u)\,du = K_+^s(v, du).
\end{aligned}$$

Substituting these expressions into the equalities of Theorem 1 we obtain the formulae (16). These formulae are also obtained in the monograph [14, p.47]. To determine the distribution of the first exit time on the event that the exit occurs through the upper boundary, we need to calculate the following contour integral

$$P\left[\,\chi < t;\ A^x\,\right] = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma+i\infty} e^{st}\ \frac{1}{s}\ \frac{e^{y\sqrt{2s}} - e^{-y\sqrt{2s}}}{e^{B\sqrt{2s}} - e^{-B\sqrt{2s}}}\ ds, \qquad \gamma > 0.$$

Observe that the integrand has simple poles (see [20], p 206) in $s_k = -\frac{1}{2}\left(\frac{k\pi}{B}\right)^2, \quad k \geq 0.$ Choosing the appropriate contour and utilizing the Cauchy's residue theorem (see [20], p. 217) yields the first formula of (16). Other equalities of (16) follow from the first equality.

One can notice that the distributions (16) are the limit distributions for corresponding homogeneous processes with independent increments and for random walks. Moreover, the formulae (16) are also the asymptotic expansions for the probabilities which enter the left-hand sides of the formulae (16). Formula (17) follows from the series expansion of the functions $\mathrm{ch}x,\ \mathrm{sech}x = (\mathrm{ch}x)^{-1}.$

It is worth of noticing that in case when one of the Wiener-Hopf factors is a rational function, the sums of the series $K_-^s(v,du),\ K_+^s(v,du)$ have an explicit form which is not the case in general.

Remark 3 *As one of the possible applications we give here the formula for obtaining the series* $(x, y > 0, \quad x + y = B)$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \cos\left(\frac{x-y}{2B}(2k+1)\pi\right) = \frac{\pi^{2n+1}}{2^{2n+2}} \sum_{k=0}^{n} (-1)^k \left(\frac{x-y}{B}\right)^{2k} \frac{E_{n-k}}{(2k)!(2n-2k)!},$$

which follows from (17) and the third equality of (16). In particular, when $x = y$ *this formula becomes*

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2}}\frac{1}{(2n)!}\,E_n \qquad n \geq 0, \quad E_0 = 1$$

which is well-known from mathematical analysis.

5 Joint distribution of the supremum, the infimum and the value of the Lévy process

Now we are able to determine the joint distribution of the supremum, the infimum and the value of the process and the function

$$Q^s(p) = E[\,e^{-p\,\xi(\nu_s)};\ \chi > \nu_s\,] = \int_{-y}^{x} e^{-up}\,P[\,-y < \xi^-(\nu_s),\ \xi(\nu_s) \in du,\ \xi^+(\nu_s) < x\,].$$

Let us mention that the joint distribution of $\{\,\xi^-(\nu_s),\ \xi(\nu_s),\ \xi^+(\nu_s)\,\}$ has been obtained in the monograph of Gihman and Skorohod [12, pp. 306-310], in terms of the distributions of $\xi(t),\ \{\,\tau^x, T^x\,\},\ \{\,\tau_y, T_y\,\}$. The authors used probability-combinatorics methods, based on the inclusion-exclusion formula to set up equations. The method of successive iterations was applied to solve the integral equations. In the proposition below we derive the formulae for the integral transform of the joint distribution of $\{\,\xi^-(\nu_s),\ \xi(\nu_s),\ \xi^+(\nu_s)\,\}$ in terms of the joint distribution of $\{\,\chi,\ X\,\}$, and of the distribution of $\{\,\xi(t), \xi^+(t)\,\}$, given by (2).

Proposition 3 *The integral transform of the joint distribution of* $\{\,\xi^-(\nu_s),\ \xi(\nu_s),\ \xi^+(\nu_s)\,\}$ *satisfies the formula*

$$Q^s(p) = U^x(s,p) - e^{yp}\int_0^{\infty} E[\,e^{-s\chi};\ X \in dv, A_y\,]\,e^{vp}\,U^{v+B}(s,p), \tag{19}$$

where

$$U^x(s,p) = E[\,e^{-p\,\xi(\nu_s)};\ \xi^+(\nu_s) < x\,] = E[e^{-p\xi^-(\nu_s)}]\,E[\,e^{-p\,\xi^+(\nu_s)};\ \xi^+(\nu_s) < x\,].$$

In particular,

$$P[\,\chi > \nu_s\,] = P[\,\xi^+(\nu_s) < x\,] - \int_0^{\infty} E[\,e^{-s\chi};\ X \in dv,\ A_y]\ P[\,\xi^+(\nu_s) < v + B\,],$$

Proof of Proposition 3 To derive the solving equation we observe first, that the event $\{\xi^+(t) < x\}$ is equivalent to the event $\{\tau^x > t\}$. Then by the total probability formula we write

$$E\,[\,e^{-p\,\xi(t)};\ \xi^+(t) < x\,] = E\,[\,e^{-p\,\xi(t)};\ \xi^+(t) < x,\ \tau_y > t\,] + E\,[\,e^{-p\,\xi(t)};\ \tau^x > t,\ \tau_y \le t\,] =$$
$$E\,[\,e^{-p\,\xi(t)};\ \xi^-(t) > -y,\ \xi^+(t) < x\,] + E\,[\,e^{p(y+X)}\,e^{-p\,\theta_\chi\,\xi(t-\chi)};\ \chi \le t,\ \tau^{\,B+X} > t-\chi,\ A_y\,], \tag{20}$$

where θ_t is a shift operator. Since χ is a Markov time, then the increments of the process $\theta_\chi\,\xi(t-\chi)$ and τ^{B+X} do not depend on the sigma-algebra F_χ, and hence

$$E\,[\,e^{p(y+X)}\,e^{-p\,\theta_\chi\,\xi(t-\chi)};\;\chi\le t,\,\tau^{\,B+X}>t-\chi,\,A_y\,]=$$
$$e^{py}\int_0^t\int_0^\infty e^{pv}\,E\,[\,\chi\in du,\,X\in dv,\,A_y\,]\,E\,[\,e^{-p\,\xi(t-u)};\;\tau^{\,v+B}>t-u\,].$$

Substituting the right-hand side of this equality into (20) implies

$$E\,[\,e^{-p\,\xi(t)};\;\xi^+(t)<x\,]=E\,[\,e^{-p\,\xi(t)};\;\xi^-(t)>-y,\,\xi^+(t)<x\,]$$
$$+\,e^{py}\int_0^t\int_0^\infty e^{pv}\,P\,[\,\chi\in du,\,X\in dv,\,A_y\,]\,E\,[\,e^{-p\,\xi(t-u)};\;\xi^+(t-u)<v+B\,].$$

Multiplying this equality by the density $s\,e^{-st}$ of the random variable ν_s and integrating all over $t\ge 0$, yields the following equality

$$E\,[\,e^{-p\,\xi(\nu_s)};\;\xi^+(\nu_s)<x\,]=E\,[\,e^{-p\,\xi(\nu_s)};\;\chi>\nu_s\,]$$
$$+\int_0^\infty E\,[\,e^{-s\chi};\;X\in dv,A_y\,]\,e^{(v+y)p}\,E\,[\,e^{-p\,\xi(\nu_s)};\;\xi^+(\nu_s)<v+B\,],\qquad Re\,p\le 0.\quad(21)$$

This equation means that the increments of the process $\xi(t)$ on the sample paths which do not intersect the level x on an exponentially distributed time interval $[0,\nu_s]$, can be realized either on the paths which do not intersect the lower level $-y$ (the first term of the right-hand side), or on the paths which do intersect the level $-y$ and further do not intersect the level x on the time interval $[0,\nu_s]$ (the second term). It follows from (21) that

$$Q^s(p)=U^x(s,p)-e^{yp}\int_0^\infty E[\,e^{-s\chi};\;X\in dv,\,A_y]\,e^{vp}\,U^{v+B}(s,p),$$

which is the formula (19) of Proposition 2. The second formula of the proposition can be obtained after setting $p=0$ in (19). The function $Q^s(p)$ can be determined also from the equation

$$E\,[e^{-p\,\xi(\nu_s)}]=E\,[\,e^{-p\,\xi(\nu_s)};\;\chi>\nu_s\,]+E\,[\,e^{-s\chi}\,e^{-p\xi(\chi)};\;\xi(\chi)>x]\,E\,[e^{-p\,\xi(\nu_s)}]$$
$$+\,E\,[\,e^{-s\chi}\,e^{-p\xi(\chi)};\;A_y]\,E\,[e^{-p\,\xi(\nu_s)}],\qquad Re\,p\le 0.$$

This equation means that the increments of the process $\xi(t)$ on the exponentially distributed time interval $[0,\nu_s]$ can be realized either on sample paths which do not leave the interval $(-y,x)$, or on the paths which leave the interval through the lower boundary. Let us obtain the equalities of Proposition 2 using the latter equation. Using the Wiener-Hopf factorization we rewrite it in the following way:

$$E\,[e^{-p\,\xi^+(\nu_s)}] = Q^s(p)\,\left(E\,[e^{-p\,\xi^-(\nu_s)}]\right)^{-1} + e^{-px}\,E\,[\,e^{-s\chi}\,e^{-pX};\,A^x]\,E\,[e^{-p\,\xi^+(\nu_s)}]$$
$$+e^{py}\,E\,[\,e^{-s\chi}\,e^{-pX};\,A_y]\,E\,[e^{-p\,\xi^+(\nu_s)}], \qquad Re\,p \le 0.$$

Consider a class of functions $A(u), \quad u \in \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} e^{-pu}|A(u)|\,du < \infty, \quad Re\,p = 0 \tag{22}$$

and introduce an operator (the so-called projector) which acts on these functions via the formula

$$\mathbf{I}^x\left(\int_{-\infty}^{\infty} e^{-pu}A(u)\,du\right) = \int_{-\infty}^{x} e^{-pu}A(u)\,du, \qquad Re\,p = 0.$$

Note that the functions in the equation considered satisfy the condition (20), we can apply $\mathbf{I}^x(\cdot)$ to both sides, which implies

$$E[\,e^{-p\,\xi^+(\nu_s)};\,\xi^+(\nu_s) < x\,] = Q^s(p)\,\left(E[e^{-p\,\xi^-(\nu_s)}]\right)^{-1}$$
$$+\int_0^{\infty} E[\,e^{-s\chi};\,X \in dv,\,A_y]\,e^{(v+y)p}\,E[\,e^{-p\,\xi^+(\nu_s)};\,\xi^+(\nu_s) < v + B\,].$$

From this equality we find

$$\begin{aligned} Q^s(p) =& E[e^{-p\,\xi^-(\nu_s)}]\,E[\,e^{-p\,\xi^+(\nu_s)};\,\xi^+(\nu_s) < x\,] \\ &-e^{py}\int_0^{\infty} E[\,e^{-s\chi};\,X \in dv,\,A_y]\,e^{vp}\,E[e^{-p\,\xi^-(\nu_s)}]\,E[\,e^{-p\,\xi^+(\nu_s)};\,\xi^+(\nu_s) < v + B\,] \\ =& U^x(s,p) - e^{yp}\int_0^{\infty} E[\,e^{-s\chi};\,X \in dv, A_y\,]\,e^{vp}\,U^{v+B}(s,p) \end{aligned}$$

and therefore we obtained the first formula of Proposition 3. When $\{\xi(t);\,t \ge 0\}$ is a spectrally positive Lévy process, the equalities of Proposition 1 simplify. In this case we have

$$Q^s(p) = U^x(s,p) - e^{yp}\,\frac{R_x(s)}{R_B(s)}\,U^B(s,p), \qquad Re\,p \le 0, \tag{23}$$

where

$$U^x(s,p) = \frac{c(s)}{c(s) - p}\,E\,[\,e^{-p\,\xi^+(\nu_s)};\,\xi^+(\nu_s) < x\,], \qquad Re\,p \le 0.$$

The formula (23) was obtained in the paper [15] from the equation (21), for spectrally positive Lévy processes. The author has found the resolvent representation for the joint distribution of $\{\,\xi^-(\nu_s),\,\xi(\nu_s),\,\xi^+(\nu_s)\,\}$

$$P\,[\,-y < \inf_{u \le \nu_s}\xi(u), \quad \xi(\nu_s) \in (\alpha,\,\beta), \quad \sup_{u \le \nu_s}\xi(u) < x\,] =$$
$$s\,\frac{R_x(s)}{R_B(s)}\int_{\alpha}^{\beta} R_{y+u}(s)\,du - s\int_{\max\{0,\alpha\}}^{\max\{0,\beta\}} R_u(s)\,du, \qquad -y \le \alpha < \beta \le x$$

where $R_x(s)$ is the resolvent of a spectrally positive Lévy process defined in (13). Further, in the same paper for the Wiener process $\{w(t), \quad t \geq 0\}$ with Laplace exponent $k(p) = \frac{1}{2}\sigma^2 p^2$ the joint distribution of $\{\, \xi^-(t),\ \xi(t),\ \xi^+(t)\,\}$ was determined:

$$P\,[\,-y < \inf_{u \leq t} w(u), \quad w(t) \in (\alpha,\, \beta), \quad \sup_{u \leq t} w(u) < x\,] =$$
$$\frac{4}{\pi} \sum_{\nu=1}^{\infty} \frac{1}{\nu}\, \exp\left(-\frac{t}{2}(\pi\nu\sigma/B)^2\right)\, \sin\left(\frac{x}{B}\pi\nu\right)\, \sin\left(\frac{2x-\alpha-\beta}{2B}\pi\nu\right)\, \sin\left(\frac{\beta-\alpha}{2B}\pi\nu\right).$$

This distribution can be treated as the limit distribution for the corresponding distributions of the processes with independent increments and random walks. Moreover, the right-hand side of this equality serves as an asymptotic expansion of the probability which enters the left-hand side of the equality.

6 Possible applications in finance

In this section we will be concerned with a financial model of type

$$S(t) = \exp\{\xi(t)\}$$

where $S(t)$ is an asset price (price of a stock, options, an index, an exchange rate); $\xi(t)$ is a Lévy process that satisfies some integrability conditions. Such models are called exponential Lévy models. In recent years these models have become very popular since they are able to take into account different important features of financial time series. The exponential Lévy models offer analytically tractable examples of positive jump processes. For an overview of Lévy processes in finance we refer to [30] or [31]. In literature several choices for Lévy processes have been studied. The most common are as follows: the Variance Gamma process, the generalized Hyperbolic model, the Normal inverse Gaussian motion and the Meixner process.

Let us consider a double barrier option. In order for the investor to receive a payout, one of two situations must occur: the price must reach one of the range boundaries or the price must avoid touching either boundary. Thus, for pricing double-barrier options we require the distribution of the first exit time from the interval. Nowadays numerical methods are widely used to price such options. However obtaining exact formulae for pricing of the double-barrier options is of great interest. This forms the subject of ongoing research.

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