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REGRESSION QUANTILES UNDER DEPENDENT CENSORING

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REGRESSION QUANTILES UNDER DEPENDENT CENSORING

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ABSTRACT. In a recent paper, Braekers and Veraverbeke [1] obtained asymptotic results for an estimator of the conditional distribution function of the lifetime, in a fixed design regression model with dependence between lifetime and censoring time. We complement these findings by corresponding results for the conditional quantile estimator. Our main results are uniform strong consistency and a Bahadur-type representation. The latter is an invaluable tool for deriving further asymptotic results, such as the weak convergence of the quantile process.

Key words: Archimedean copula, Bahadur representation, dependent censoring, fixed design regression, quantiles, strong consistency, weak convergence.

MSC2000 Classification: Primary 62N01, Secondary 62N02, 62G08

1. INTRODUCTION

We consider the estimation of quantiles of the conditional lifetime distribution in a fixed design regression model in which the lifetime Y_x at a fixed covariate value $x \in [0,1]$ is subject to right censoring by a non-negative censoring variable C_x. We do not necessarily assume the independence of Y_x and C_x, but allow dependence via an Archimedean copula model for the joint survival function of Y_x and C_x.

In many practical situations, the independence of Y_x and C_x is doubtful. For example, in a clinical trial, one might have that patients withdraw (are censored) because they are doing poorly with the new treatment. Or in engineering, one might have that a piece of equipment is replaced (is censored) because it has given some sign of future failure.

Copula models for censored observations, in the absence of covariates, have been studied by Zheng and Klein [8] and Rivest and Wells [4]. For Archimedean copula models with covariate information, Braekers and Veraverbeke [1] studied the asymptotic properties of an estimator $F_{xh}(t)$ for the conditional distribution function $F_x(t) = P(Y_x \leq t)$.

In this paper we focus on the estimation of quantiles of the distribution function $F_x(t)$. For $0 < p < 1$, the p-th quantile is defined as $F_x^{-1}(p) = \inf\{t : F_x(t) \geq p\}$ and the corresponding estimator is given by $F_{xh}^{-1}(p) = \inf\{t : F_{xh}(t) \geq p\}$. Our results are the rate of uniform strong consistency (Section 3), a Bahadur-type asymptotic representation (Section 4) and the weak convergence result for the quantile process (Section 5). Two basic lemmas on the almost sure (a.s.) behaviour of $F_{xh}(t)$ are proved in Section 6. We begin with preliminaries on the copula model and the estimator.

2. THE COPULA MODEL AND THE DISTRIBUTION FUNCTION ESTIMATOR

We assume that, at fixed design points $0 \leq x_1 \leq \ldots \leq x_n \leq 1$, we have responses $Y_1, \ldots, Y_n$ and censoring times $C_1, \ldots, C_n$. The observed random variables are (Z_i, δ_i), where for each $i = 1, \ldots, n : Z_i = \min(Y_i, C_i)$ and $\delta_i = I(Y_i \leq C_i)$. At a given design point value $x \in [0,1]$, we write F_x, G_x, H_x for the distribution function of the lifetime Y_x, the censoring time C_x and the observed variable $Z_x = \min(Y_x, C_x)$. We will also write $\delta_x = I(Y_x \leq C_x)$. (Note that at the n design variables x_i, we write $Y_i, C_i, Z_i, F_i, \ldots$ instead of $Y_{x_i}, C_{x_i}, Z_{x_i}, F_{x_i}, \ldots$)

Our Archimedean copula model assumes that the joint survival function of Y_x and

C_x at x can be written as

$$S_x(t_1, t_2) = P(Y_x > t_1, C_x > t_2) = \varphi_x^{[-1]}(\varphi_x(\overline{F}_x(t_1)) + \varphi_x(\overline{G}_x(t_2))).$$

Here φ_x is a known generator function depending in a general way on the covariate value x and $\overline{F}_x$ (respectively $\overline{G}_x$) is the survival function of Y_x (respectively C_x) at x. We refer to Nelsen [3] or Genest and MacKay [2] for definitions and properties of Archimedean copulas and their generators. We have that, for each x, $\varphi_x : [0,1] \to [0,\infty]$ is a known, continuous, convex, strictly decreasing function with $\varphi_x(0) = 0$. $\varphi_x^{[-1]}$ is the pseudo-inverse of φ_x given by $\varphi_x^{-1}(s)$ if $0 \le s \le \varphi_x(0)$ and zero if $\varphi_x(0) \le s \le +\infty$.
The estimator $F_{xh}(t)$ for $F_x(t)$ in Braekers and Veraverbeke [1] is given by

$$\begin{aligned} &\overline{F}_{xh}(t) \\ &= \varphi_x^{-1}\left(\sum_{Z_i \le t, \delta_i = 1} \left[\varphi_x(\overline{H}_{xh}(Z_i-) - \varphi_x(\overline{H}_{xh}(Z_i-) - w_{ni}(x;h_n))\right]\right) I(t \le Z_{(n)}). \end{aligned} \tag{2.1}$$

Here $Z_{(1)} \le \ldots \le Z_{(n)}$ are the ordered $Z_1, \ldots, Z_n$ and

$$H_{xh}(t) = \sum_{i=1}^{n} w_{ni}(x;h_n) I(Z_i \le t). \tag{2.2}$$

The weights $\{w_{ni}(x;h_n)\}$ in (2.1) and (2.2) are smoothing weights which give larger weight to observations at design points close to x. Here we take the Gasser-Müller weights, which is the natural choice in the fixed design situation. They are given by

$$\begin{aligned} w_{ni}(x;h_n) &= \frac{1}{c_n(x;h_n)} \int_{x_{i-1}}^{x_i} \frac{1}{h_n} K\left(\frac{x-z}{h_n}\right) dz \\ c_n(x;h_n) &= \int_0^{x_n} \frac{1}{h_n} K\left(\frac{x-z}{h_n}\right) dz \end{aligned}$$

where $x_0 = 0$, K is a known probability density function (kernel) and $h = \{h_n\}$ is a sequence of positive constants, tending to 0 as $n \to \infty$ (bandwidth sequence). Note that for the independence copula ($\varphi_x(t) = -\log t$), the estimator in (2.2) becomes the classical Beran generalization of the Kaplan-Meier estimator, studied by Van Keilegom and Veraverbeke [6], [7].
For the design points $x_1, \ldots, x_n$, we write $\underline{\Delta}_n = \min(x_i - x_{i-1})$ and $\overline{\Delta}_n = \max(x_i - x_{i-1})$. The notations $\|K\|_\infty = \sup_{u \in I\!R} K(u)$, $\|K\|_2^2 = \int_{-\infty}^{\infty} K^2(u)du$, $\mu_1^K = \int_{-\infty}^{\infty} uK(u)du$,

$\mu_2^K = \int\limits_{-\infty}^{\infty} u^2 K(u) du$ will be used for the kernel K. We use the following assumptions on the design and on the kernel

(C1) $x_n \to 1$, $\overline{\Delta}_n = O(n^{-1})$, $\overline{\Delta}_n - \underline{\Delta}_n = o(n^{-1})$

(C2) K is a probability density function with finite support $[-M, M]$, for some $M > 0$, $\mu_1^K = 0$ and K is Lipschitz of order 1.

Note that (C1) implies that $c_n(x; h_n) = 1$ for n sufficiently large. Therefore we may take $c_n(x; h_n) = 1$ in all proofs of asymptotic results.
If L is any (sub)distribution, then $T_L = \inf\{t : L(t) = L(+\infty)\}$ denotes the right endpoint of its support. Here we have that $T_{H_x} \leq \min(T_{F_x}, T_{G_x})$, with equality if $\varphi_x(0) = +\infty$. In our results we will need typical types of smoothness conditions on functions like $H_x(t) = P(Z_x \leq t)$ and $H_x^u(t) = P(Z_x \leq t, \delta_x = 1)$. We formulate them here for a general (sub)distribution function $L_x(t)$, $t \in \mathbb{R}$, $0 \leq x \leq 1$ and for a fixed $T > 0$

(C3) $\dot{L}_x(t) = \frac{\partial}{\partial x} L_x(t)$ exists and is continuous in $(x, t) \in [0, 1] \times [0, T]$

(C4) $L'_x(t) = \frac{\partial}{\partial t} L_x(t)$ exists and is continuous in $(x, t) \in [0, 1] \times [0, T]$

(C5) $\ddot{L}_x(t) = \frac{\partial^2}{\partial x^2} L_x(t)$ exists and is continuous in $(x, t) \in [0, 1] \times [0, T]$

(C6) $L''_x(t) = \frac{\partial^2}{\partial t^2} L_x(t)$ exists and is continuous in $(x, t) \in [0, 1] \times [0, T]$

(C7) $\dot{L}'_x(t) = \frac{\partial^2}{\partial x \partial t} L_x(t)$ exists and is continuous in $(x, t) \in [0, 1] \times [0, T]$

Note that (C3) implies that $L_x(t)$ is Lipschitz, in the sense that for all $0 \leq x,\, x' \leq 1$,

$$\sup_{0 \leq t \leq T} |L_x(t) - L_{x'}(t)| \leq \|\dot{L}\| |x - x'|$$

where $\|M\| = \sup\limits_{0 \leq x \leq 1} \sup\limits_{0 \leq t \leq T} |M_x(t)|$ for any function $M_x(t)$ on $[0, 1] \times [0, T]$. Similarly, (C4) implies that for all $0 \leq t,\, t' \leq T$

$$\sup_{0 \leq x \leq 1} |L_x(t) - L_x(t')| \leq \|L'\| |t - t'|.$$

The generator φ_x of the Archimedean copula needs to satisfy the following properties

(C8) $\varphi'_x(v) = \dfrac{\partial}{\partial x}\varphi_x(v)$ and $\varphi''_x(v) = \dfrac{\partial^2}{\partial x^2}\varphi_x(v)$ are Lipschitz in the x-direction with a bounded Lipschitz constant, and $\varphi'''_x(v) = \dfrac{\partial^3}{\partial x^3}\varphi_x(v) \leq 0$ exists and is continuous in $(x, v) \in [0, 1]\times]0, 1]$.

These assumptions and the fact that φ_x is the generator of an Archimedean copula, give that φ'_x is monotone increasing with $\varphi'_x < 0$ and φ''_x is monotone decreasing with $\varphi''_x \geq 0$.

3. STRONG CONSISTENCY OF THE QUANTILE ESTIMATOR

Our first result is the uniform strong consistency of $F_{xh}^{-1}(p)$ as estimator for $F_x^{-1}(p)$. This will follow as a consequence of a stronger result which gives an exponential bound for $P\left(\sup_{\varepsilon_0 \leq p \leq p_0} \mid F_{xh}^{-1}(p) - F_x^{-1}(p) \mid > \varepsilon\right)$, where $0 < \varepsilon_0 < p_0 < 1$.

Theorem 1.
Assume (C1), (C2), $H_x(t)$ and $H_x^u(t)$ satisfy (C3) and (C4) in $[0, T]$ with $T < T_{H_x}$, $h_n \to 0$, $\dfrac{\log n}{nh_n} \to 0$.
Assume that φ_x satisfies (C8) and also that $\varphi'_x(1) < 0$.
Let $0 < \varepsilon_0 < p_0 < 1$ be such that $F_x^{-1}(p_0) < T$ and $\inf_{\varepsilon_0 \leq p \leq p_0} f_x(F_x^{-1}(p)) = \lambda > 0$, where $f_x = F'_x$.

(a) For $\varepsilon > 0$ such that $F_x^{-1}(\varepsilon_0) - \varepsilon \geq 0$, $F_x^{-1}(p_0) + \varepsilon \leq T$ and

$$\inf_{F_x^{-1}(\varepsilon_0) - \varepsilon \leq y \leq F_x^{-1}(p_0) + \varepsilon} f_x(y) \geq \frac{\lambda}{2},$$

for n sufficiently large and

$$\varepsilon \geq \frac{2}{\lambda} A \max\left(\sqrt{6}\|K\|_2 \frac{1}{(nh_n)^{1/2}},\ 4(\|\dot{H}\| \vee \|\dot{H}^u\|)\left(\int |u|K(u)du\right) h_n\right)$$

with

$$A = \frac{3}{2}\left(-\frac{1}{\varphi'_x(1)}\right)\left(|\varphi'_x(\overline{H}_x(T)) \mid + \varphi''_x(\overline{H}_x(T))\right) \tag{3.1}$$

we have

$$P\left(\sup_{\varepsilon_0 \leq p \leq p_0} \mid F_{xh}^{-1}(p) - F_x^{-1}(p) \mid > \varepsilon\right) \leq \frac{1}{4A} d_0\ \lambda\varepsilon\ nh_n\ e^{-d_1 nh_n \frac{\lambda^2\varepsilon^2}{256A^2}}$$

with

$$d_0 = \frac{8e^2}{\|K\|_2^2}, \qquad d_1 = \frac{4}{3\|K\|_2^2}. \tag{3.2}$$

(b) If $\dfrac{nh_n^5}{\log n} = O(1)$, then, as $n \to \infty$,

$$\sup_{\varepsilon_0 \le p \le p_0} | F_{xh}^{-1}(p) - F_x^{-1}(p) | = O((nh_n)^{-1/2}(\log n)^{1/2}) \qquad \text{a.s.}$$

Proof. (a) We have that

$$\begin{aligned} & P\left(\sup_{\varepsilon_0 \le p \le p_0} | F_{xh}^{-1}(p) - F_x^{-1}(p) | > \varepsilon \right) \\ \le \ & P\left(F_{xh}^{-1}(p) > F_x^{-1}(p) + \varepsilon \text{ for some } \varepsilon_0 \le p \le p_0\right) \\ + \ & P\left(F_{xh}^{-1}(p) < F_x^{-1}(p) - \varepsilon \text{ for some } \varepsilon_0 \le p \le p_0\right) \end{aligned} \tag{3.3}$$

Now, for the first term in (3.3), we have

$$\begin{aligned} & P\left(F_{xh}^{-1}(p) > F_x^{-1}(p) + \varepsilon \text{ for some } \varepsilon_0 \le p \le p_0\right) \\ \le \ & P\left(\sup_{0 \le t \le T} | F_{xh}(t) - F_x(t) | > \inf_{\varepsilon_0 \le p \le p_0} (F_x(F_x^{-1}(p) + \varepsilon) - p) \right) \end{aligned} \tag{3.4}$$

Since $\inf_{\varepsilon_0 \le p \le p_0} (F_x(F_x^{-1}(p) + \varepsilon) - p) \ge \inf_{F_x^{-1}(\varepsilon_0) \le y \le F_x^{-1}(p_0) + \varepsilon} f_x(y)\varepsilon \ge \frac{1}{2}\lambda\varepsilon > 0$ we can apply Lemma 1 to (3.4).

The second term in (3.3) can be bounded in the same way.

(b) This follows from the Borel-Cantelli lemma.

4. ALMOST SURE ASYMPTOTIC REPRESENTATION FOR THE QUANTILE ESTIMATOR

Theorem 2.

Assume (C1), (C2), $H_x(t)$ and $H_x^u(t)$ satisfy (C5) - (C7) in $[0, T]$ with $T < T_{H_x}$, $h_n \to 0$, $\dfrac{\log n}{nh_n} \to 0$, $\dfrac{nh_n^5}{\log n} = O(1)$.

Assume that φ_x satisfies (C8) and also that $\varphi_x'(1) < 0$. Let $0 < \varepsilon_0 < p_0 < 1$ be such that $F_x^{-1}(p) < T$ and $\inf_{\varepsilon_0 \le p \le p_0} f_x(F_x^{-1}(p)) = \lambda < 0$. Then, for $\varepsilon_0 \le p \le p_0$, we have

$$F_{xh}^{-1}(p) - F_x^{-1}(p) = \frac{p - F_{xh}(F_x^{-1}(p))}{f_x(F_x^{-1}(p))} + r_n(x, p)$$

where

$$\sup_{\varepsilon_0 \le p \le p_0} | r_n(x,p) | = O((nh_n)^{-3/4}(\log n)^{3/4}) \text{ a.s.}$$

as $n \to \infty$.

Proof. Using Lemma 2 we have that

$$\sup_{\varepsilon_0 \le p \le p_0} | F_{xh}(F_{xh}^{-1}(p)) - p | \le \sup_{\varepsilon_0 \le p \le p_0} | F_{xh}(F_{xh}^{-1}(p)) - F_{xh}(F_{xh}^{-1}(p)-) |$$

$$\le \sup_{\substack{0 \le s,t \le T \\ |t-s| \le c(nh_n)^{-1/2}(\log n)^{1/2}}} | F_{xh}(t) - F_{xh}(s) - F_x(t) + F_x(s) |$$

$$= O((nh_n)^{-3/4}(\log n)^{3/4}) \text{ a.s.}$$

Hence,

$$\sup_{\varepsilon_0 \le p \le p_0} \left| F_{xh}^{-1}(p) - F_x^{-1}(p) - \frac{p - F_{xh}(F_x^{-1}(p))}{f_x(F_x^{-1}(p))} \right|$$

$$\le \frac{1}{\lambda} \sup_{\varepsilon_0 \le p \le p_0} \left| f_x(F_x^{-1}(p))(F_{xh}^{-1}(p) - F_x^{-1}(p)) - (F_{xh}(F_{xh}^{-1}(p)) - F_{xh}(F_x^{-1}(p)) \right|$$

$$+O((nh_n)^{-3/4}(\log n)^{3/4}) \text{ a.s.}$$

$$\le \frac{1}{\lambda} \sup_{\varepsilon_0 \le p \le p_0} \left| f_x(F_x^{-1}(p))(F_{xh}^{-1}(p) - F_x^{-1}(p)) - F_x(F_{xh}^{-1}(p)) - F_x(F_x^{-1}(p)) \right|$$

$$+O((nh_n)^{-3/4}(\log n)^{3/4}) \text{ a.s.}$$

where the last inequality follows from Theorem 1 and Lemma 2. By Taylor expansion, this can be rewritten as

$$\frac{1}{2\lambda} \sup_{\varepsilon_0 \le p \le p_0} | f_x'(\theta) | (F_{xh}^{-1}(p) - F_x^{-1}(p))^2 + O((nh_n)^{-3/4}(\log n)^{3/4}) \text{ a.s.}$$

where θ is between $F_{xh}^{-1}(p)$ abnd $F_x^{-1}(p)$. From (C6), f_x' is bounded on $[0,T]$. The theorem is proved after a further application of Theorem 1.

5. WEAK CONVERGENCE OF THE QUANTILE PROCESS

An important consequence of the result in the previous section is the weak convergence of the quantile process $\{(nh_n)^{1/2}(F_{xh}^{-1}(p) - F_x^{-1}(p))\}$ in the space $\ell^\infty[\varepsilon_0, p_0]$ of uniformly bounded, real functions, endowed with the uniform topology. Combining our Theorem 2 with Theorem 3 in Braekers and Veraverbeke [1] we also obtain the following weak convergence result.

Theorem 3.
Assume (C1), (C2), $H_x(t)$ and $H_x^u(t)$ satisfy (C5) - (C7) in $[0,T]$ with $T < T_{H_x}$. Assume that φ_x satisfies (C8) and also that $\varphi_x'(1) < 0$. Let $0 < \varepsilon_0 < p_0 < 1$ be such that $F_x^{-1}(p) < T$ and $\inf_{\varepsilon_0 \le p \le p_0} f_x(F_x^{-1}(p)) = \lambda > 0$.

(a) If $nh_n^5 \to 0$ and $\dfrac{(\log n)^3}{nh_n} \to 0$, then, as $n \to \infty$,

$$(nh_n)^{1/2}(F_{xh}^{-1}(\cdot) - F_x^{-1}(\cdot)) \to \frac{W_x(F_x^{-1}(\cdot))}{f_x(F_x^{-1}(\cdot))} \text{ in } \ell^\infty[\varepsilon_0, p_0]$$

(b) if $h_n = Cn^{-1/5}$, for some $c > 0$, then, as $n \to \infty$,

$$(nh_n)^{1/2}(F_{xh}^{-1}(\cdot) - F_x^{-1}(\cdot)) \to \frac{\widetilde{W}_x(F_x^{-1}(\cdot))}{f_x(F_x^{-1}(\cdot))} \text{ in } \ell^\infty[\varepsilon_0, p_0]$$

where $W_x(\cdot)$ and $\widetilde{W}_x(\cdot)$ are Gaussian processes with covariance function given in Theorem 3 of Braekers and Veraverbeke [1].

6. TWO LEMMAS

In this section we prove two crucial lemmas that have been used in the previous sections. Lemma 1 provides an exponential probability bound for $\sup_{0 \le t \le T} \mid F_{xh}(t) - F_x(t) \mid$. Lemma 2 describes the a.s. behaviour of the modulus of continuity of the estimator $F_{xh}(t)$.

Lemma 1.
Assume (C1), (C2), $H_x(t)$ and $H_x^u(t)$ satisfy (C3) and (C4) in $[0,T]$ with $T < T_{H_x}$, $h_n \to 0$, $\dfrac{\log n}{nh_n} \to 0$.
Assume that φ_x satisfies (C8) and also that $\varphi_x'(1) < 0$. For n sufficiently large and

$$\varepsilon \ge 4A \ \max(\sqrt{6}\|K\|_2 \frac{1}{(nh_n)^{1/2}}, 4(\|\dot{H}\| \vee \|\dot{H}^u\|)(\textstyle\int |u|K(u)du)h_n)$$

with A as in (3.1), we have

$$P\left(\sup_{0\le t\le T} \mid F_{xh}(t) - F_x(t) \mid > \varepsilon\right) \le \frac{1}{4A} d_0 \; \varepsilon \; n h_n e^{-d_1 n h_n \frac{\varepsilon^2}{64A^2}}$$

with d_0 and d_1 as in (3.2).

Proof. From Lemma 1 in Braekers and Veraverbeke [1] we have that

$$\overline{F}_x(t) = \varphi_x^{-1}\left(-\int_0^t \varphi_x'(\overline{H}_x(s))dH_x^u(s)\right) \tag{6.1}$$

With $H_{xh}(t)$ as in (2.2) and $H_{xh}^u(t) = \sum_{i=1}^n w_{ni}(x;h_n)I(Z_i \le t, \delta_i = 1)$, we have from (2.1) and (6.1) and an application of the mean value theorem that

$$\begin{aligned}
& F_{xh}(t) - F_x(t) \\
= \; & \Bigg\{-\varphi_x^{-1}\left(-\sum_{Z_i\le t,\delta_i=1}[\varphi_x(\overline{H}_{xh}(Z_i-)) - \varphi_x(\overline{H}_{xh}(Z_i-) - w_{ni}(x;h_n))]\right) \\
& +\varphi_x^{-1}\left(-\sum_{Z_i\le t,\delta_i=1}\varphi_x'(\overline{H}_{xh}(Z_i)w_{ni}(x;h_n)\right)\Bigg\} \\
& -\left\{\varphi_x^{-1}\left(-\int_0^t \varphi_x'\left(\overline{H}_{xh}(s)\right)dH_{xh}^u(s)\right) - \varphi_x^{-1}\left(-\int_0^t \varphi_x'\left(\overline{H}_x(s)\right)dH_x^u(s)\right)\right\} \\
= \; & \frac{-1}{\varphi_x'(\varphi_x^{-1}(\theta_1))}\Bigg\{\sum_{Z_i\le t,\delta_i=1}\varphi_x'(\overline{H}_{xh}(Z_i))w_{ni}(x;h_n) \\
& -\sum_{Z_i\le t,\delta_i=1}\left[\varphi_x(\overline{H}_{xh}(Z_i-)) - \varphi_x(\overline{H}_{xh}(Z_i-) - w_{ni}(x;h_n))\right]\Bigg\} \\
& -\frac{1}{\varphi_x'(\varphi_x^{-1}(\theta_2))}\left\{\int_0^t \varphi_x'(\overline{H}_x(s))dH_x^u(s) - \int_0^t \varphi_x'(\overline{H}_{xh}(s))dH_{xh}^u(s)\right\}
\end{aligned} \tag{6.2}$$

with θ_1 between $-\sum_{Z_i\le t,\delta_i=1}[\varphi_x(\overline{H}_{xh}(Z_i-)) - \varphi_x(\overline{H}_{xh}(Z_i-) - w_{ni}(x;h_n))]$ and $-\sum_{Z_i\le t,\delta_i=1}\varphi_x'(\overline{H}_{xh}(Z_i))w_{ni}(x;h_n)$ and θ_2 between $-\int_0^t \varphi_x'(\overline{H}_{xh}(s))dH_{xh}^u(s)$ and $-\int_0^t \varphi_x'(\overline{H}_x(s))dH_x^u(s)$.

To bound the two terms on the right hand side of (6.2) we first note that $\frac{-1}{\varphi_x'(\varphi_x^{-1}(\theta_1))}$ converges a.s. to $\frac{-1}{\varphi_x'(\overline{F}_x(t))}$. Since $\overline{F}_x(t) \leq 1$ and φ_x' is monotone increasing, we have that $-\varphi_x'(\overline{F}_x(t)) \geq -\varphi_x'(1)$ and that $\frac{-1}{\varphi_x'(\varphi_x^{-1}(\theta_1))} \leq \frac{3}{2}\left(\frac{-1}{\varphi_x'(1)}\right)$ a.s., for n sufficiently large. The same bound holds for $\frac{-1}{\varphi_x'(\varphi_x^{-1}(\theta_2))}$.
Using Lemma 3 in Braekers and Veraverbeke [1], we obtain that the first term in the right hand side of (6.2) is a.s. bounded above by

$$\frac{9}{4}\left(\frac{-1}{\varphi_x'(1)}\right)\varphi_x''(\overline{H}_x(T))\sum_{i=1}^{n} w_{ni}^2(x;h_n) \leq \frac{9}{4}\left(\frac{-1}{\varphi_x'(1)}\right)\varphi_x''(\overline{H}_x(T))\|K\|_\infty \frac{\overline{\Delta}_n}{h_n}.$$

Using integration by parts on the second term in the right hand side of (6.2), we obtain the a.s. bound

$$\frac{3}{2}\left(\frac{-1}{\varphi_x'(1)}\right)\left\{\left[\mid \varphi_x'(\overline{H}_x(T)) \mid + \varphi_x''(\overline{H}_x(T))\right] \sup_{0\leq t\leq T} \mid H_{xh}^u(t) - H_x^u(t) \mid \right.$$

$$\left. + \mid \varphi_x'(\overline{H}_x(T)) \mid \sup_{0\leq t\leq T} \mid H_{xh}(t) - H_x(t) \mid \right\}$$

$$\leq \; A\left\{\sup_{0\leq t\leq T} \mid H_{xh}^u(t) - H_x^u(t) \mid + \sup_{0\leq t\leq T} \mid H_{xh}(t) - H_x(t) \mid\right\}$$

with A as in (3.1).
With $\varepsilon > 0$ and n sufficiently large, we have that

$$P\left(\sup_{0\leq t\leq T} \mid F_{xh}(t) - F_x(t) \mid > \varepsilon\right) \leq P\left(\sup_{0\leq t\leq T} \mid H_{xh}^u(t) - H_x^u(t) \mid > \frac{\varepsilon}{4A}\right)$$

$$+P\left(\sup_{0\leq t\leq T} \mid H_{xh}(t) - H_x(t) \mid > \frac{\varepsilon}{4A}\right).$$

For each of the two terms on the right hand side we have an exponential inequality by applying Lemma A3 in Van Keilegom and Veraverbeke [7].

Lemma 2.
Assume (C1), (C2), $H_x(t)$ and $H_x^u(t)$ satisfy (C3), (C6), (C7) in $[0,T]$ with $T < T_{H_x}$, $1 - H_x(T) > \delta > 0$, $h_n \to 0$, $nh_n \to \infty$.
Assume that φ_x satisfies (C8) and also that $\varphi_x'(1) < 0$.

(a) Let $\{a_n\}$ be a sequence of positive constants, tending to 0 as $n \to \infty$, and satisfying $na_n \to \infty$.
Then, for n sufficiently large and

$$\varepsilon \leq 12 \left(-\frac{1}{\varphi_x'(1)} \right) \mid \varphi_x'(\delta) \mid \left\{ \|\dot{H}^u\| \overline{\Delta}_n + M \|\dot{H}^{u'}\| a_n h_n + \|H^{u''}\| a_n^2 \right\}$$
$$+6 \frac{\varphi_x''(\delta)}{(\varphi_x'(1))^2} \|F'\| \mid \varphi_x'(\delta) \mid \|H^{u'}\| a_n^2 \tag{6.3}$$

and

$$\varepsilon \leq 12 \left(-\frac{1}{\varphi_x'(1)} \right) \varphi_x''(\delta) \|H^{u'}\| a_n$$
$$\max(\sqrt{6} \|K\|_2 (nh_n)^{-1/2} + 2\|\dot{H}\| \overline{\Delta}_n + 2\mu_2^K \|\ddot{H}\| h_n^2) \tag{6.4}$$

we have

$$P \left(\sup_{\substack{0 \leq s,t \leq T \\ |t-s| \leq a_n}} \mid F_{xh}(t) - F_{xh}(s) - F_x(t) + F_x(s) \mid > \varepsilon \right)$$

$$\leq C_1 \frac{a_n}{\varepsilon^2} \exp \left(-\frac{C_2 nh_n \varepsilon^2}{C_3 a_n + \varepsilon} \right) + C_4 \frac{nh_n \varepsilon}{a_n} exp \left(-C_5 \frac{nh_n \varepsilon^2}{a_n^2} \right)$$

$$+C_6 nh_n \exp(-C_7 nh_n) + C_8 \frac{1}{a_n} \exp(-C_9 nh_n a_n)$$

where $C_1, \ldots, C_9$ are positive constants.

(b) If $\dfrac{nh_n^5}{\log n} = O(1)$ and $\{a_n\}$ is a sequence of positive constants of the form $a_n = c_1 (nh_n)^{-1/2} (\log n)^{1/2}$ for some $c_1 > 0$, then, as $n \to \infty$,

$$\sup_{\substack{0 \leq s,t \leq T \\ |t-s| \leq a_n}} \mid F_{xh}(t) - F_{xh}(s) - F_x(t) + F_x(s) \mid = O((nh_n)^{-3/4} (\log n)^{3/4}) \text{ a.s.}$$

Proof. (a) Defining

$$\widetilde{F}_{xh}(t) = \varphi_x^{-1} \left(- \int_0^t \varphi_x'(\overline{H}_x(y)) dH_{xh}^u(y) \right) \tag{6.5}$$

we can write

$$
\begin{aligned}
& F_{xh}(t) - F_{xh}(s) - F_x(t) + F_x(s) \\
= & \left\{\widetilde{F}_{xh}(t) - \widetilde{F}_{xh}(s) - F_x(t) + F_x(s)\right\} \\
+ & \left\{F_{xh}(t) - F_{xh}(s) - \widetilde{F}_{xh}(t) + \widetilde{F}_{xh}(s)\right\}
\end{aligned} \tag{6.6}
$$

From (6.1), (6.5) and the mean value theorem, we can rewrite the first term in the right hand side of (6.6) as

$$
\begin{aligned}
& \frac{1}{\varphi_x'(\varphi_x^{-1}(\theta_1))} \sum_{i=1}^n w_{ni}(x;h_n) \left\{\varphi_x'(\overline{H}_x(Z_i)) I(s \le Z_i \le t, \delta_i = 1)\right. \\
& \left. -E\left[\varphi_x'(\overline{H}_x(Z_i) I(s \le Z_i \le t, \delta_i = 1)\right]\right\} \\
+ & \frac{1}{\varphi_x'(\varphi_x^{-1}(\theta_1))} \sum_{i=1}^n w_{ni}(x;h_n) \int_s^t \varphi_x'(\overline{H}_x(y)) d(H_{x_i}(y) - H_x^u(y)) \\
+ & \left\{\frac{1}{\varphi_x'(\varphi_x^{-1}(\theta_1))} - \frac{1}{\varphi_x'(\varphi_x^{-1}(\theta_2))}\right\} \int_s^t \varphi_x'(\overline{H}_x(y)) dH_x^u(y) \\
= & T_1 + T_2 + T_3,
\end{aligned}
$$

where θ_1 is between $-\int_0^t \varphi_x'(\overline{H}_x(y)) dH_{xh}^u(y)$ and $-\int_0^s \varphi_x'(\overline{H}_x(y)) dH_{xh}^u(y)$ and θ_2 is between $-\int_0^t \varphi_x'(\overline{H}_x(y)) dH_x^u(y)$ and $-\int_0^s \varphi_x'(\overline{H}_x(y)) dH_x^u(y)$.

As in the proof of Lemma 1 it follows that $\left|\frac{1}{\varphi_x'(\varphi_x^{-1}(\theta_1))}\right|$ and $\left|\frac{1}{\varphi_x'(\varphi_x^{-1}(\theta_2))}\right|$ are both bounded above by $\frac{3}{2}\left(\frac{-1}{\varphi_x'(1)}\right)$ a.s.. Also, by the Lipschitz property of $\frac{1}{\varphi_x'}$, we have that $\left|\frac{1}{\varphi_x'(\varphi_x^{-1}(\theta_1))} - \frac{1}{\varphi_x'(\varphi_x^{-1}(\theta_2))}\right| \le \frac{3}{2}\frac{\varphi_x''(\delta)}{(\varphi_x'(1))^2}\|F'\||t-s|$.

We will now bound the terms T_1, T_2 and T_3. For T_3 we have

$$
\sup_{\substack{0 \le s,t \le T \\ |t-s| \le a_n}} |T_3| \le \frac{3}{2}\frac{\varphi_x''(\delta)}{(\varphi_x'(1))^2}\|F'\||\varphi_x'(\delta)\mid \|H^{u'}\| a_n^2. \tag{6.7}
$$

For T_2, we obtain, after integration by parts,

$$\begin{aligned}
T_2 &= \frac{1}{\varphi'_x(\varphi_x^{-1}(\theta_1))}\{\varphi'_x(\overline{H}_x(t))[EH^u_{xh}(t) - EH^u_{xh}(s) - H^u_x(t) + H^u_x(s)] \\
&+[EH_{xh}(s) - H^u_x(s)][\varphi'_x(\overline{H}_x(t)) - \varphi'_x(\overline{H}_x(s))] \\
&+\int_s^t [EH^u_{xh}(y) - H^u_x(y)]\varphi''_x(\overline{H}_x(y))dH_x(y)\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sup_{\substack{0\le s,t\le T\\ |t-s|\le a_n}} |T_2| \\
&\le \frac{3}{2}\left(\frac{-1}{\varphi'_x(1)}\right)\{|\varphi'_x(\delta)| \sup_{\substack{0\le s,t\le T\\ |t-s|\le a_n}} |EH^u_{xh}(t) - EH^u_{xh}(s) - H^u_x(t) - H^u_x(s)| \\
&+|\varphi'_x(\delta)|\|H'\|a_n \sup_{0\le s\le T}|EH^u_{xh}(s) - H^u_x(s)| \\
&+\varphi''_x(\delta)\|H'\|a_n \sup_{0\le s\le T}|EH^u_{xh}(s) - H^u_x(s)|\} \\
&\le 3\left(\frac{-1}{\varphi'_x(1)}\right)|\varphi'_x(\delta)|\{\|\dot{H}^u\|\overline{\Delta}_n + M\|\dot{H}^{u'}\|a_n h_n + \|H^{u''}\|a_n^2\}. \qquad (6.8)
\end{aligned}$$

In the last inequality we used Lemma A.1(a) and (the proof of) Lemma A.5(b) in Van Keilegom and Veraverbeke [7].
From (6.7) and (6.8) it follows that

$$\sup_{\substack{0\le s,t\le T\\ |t-s|\le a_n}} |T_2 + T_3| \le \frac{\varepsilon}{4}$$

because of our imposed condition (6.3) on ε.
Therefore,

$$\begin{aligned}
& P\left(\widetilde{F}_{xh}(t) - F_{xh}(s) - F_x(t) + F_x(s)| > \frac{\varepsilon}{2}\right) \\
\le\ & P\left(|T_1| > \frac{\varepsilon}{4}\right) \\
=\ & P\Bigg(|\sum_{i=1}^n w_{ni}(x;h_n)\{\varphi'_x(\overline{H}_x(Z_i))I(s\le Z_i\le t, \delta_i = 1) \qquad (6.9)\\
& -E[\varphi'_x(\overline{H}_x(Z_i))I(s\le Z_i\le t,\delta_i=1)]\}| > \frac{\varepsilon}{6}(-\varphi'_x(1))\Bigg).
\end{aligned}$$

In order to obtain an upper bound for the probabililty in (6.9), we apply Bernstein's inequality (Serfling [5]). The variance of the sum above is bounded by

$$\sum_{i=1}^{n} w_{ni}^2(x;h_n) \int_s^t \varphi_x'^2(\overline{H}_x(y))dH_{x_i}^u(y)$$

$$\leq \;\; (\varphi_x'(\delta))^2 \|H^{u'}\| a_n \sum_{i=1}^{n} w_{ni}^2(x;h_n) \leq (\varphi_x'(\delta))^2 \|H^{u'}\| \|K\|_\infty \frac{\overline{\Delta}_n}{h_n} a_n.$$

Hence, Berstein's inequality gives that

$$P\left(|\widetilde{F}_{xh}(t) - \widetilde{F}_{xh}(s) - F_x(t) + F_x(s) \,| > \frac{\varepsilon}{2} \right) \leq 2\exp\left(-\frac{C_2 n h_n \varepsilon^2}{C_3 a_n + \varepsilon} \right)$$

for some constants $C_2 > 0$ and $C_3 > 0$.
By a classical argument, based on partitioning the interval $[0,T]$, we can replace $\sup\limits_{0\leq t\leq T}$ and $\sup\limits_{0\leq s\leq T}$ maxima and obtain that, for some constant $C_1 < 0$,

$$P\left(\sup_{\substack{0\leq s,t\leq T\\ |t-s|\leq a_n}} |\widetilde{F}_{xh}(t) - \widetilde{F}_{xh}(s) - F_x(t) + F_x(s)| > \frac{\varepsilon}{2} \right) \leq C_1 \frac{a_n}{\varepsilon^2} \exp\left(-\frac{C_2 n h_n \varepsilon^2}{C_3 a_n + \varepsilon} \right).$$

For the second term in the right hand side of (6.6), we first note that, using Lemma 3 in Braekers and Veraverbeke [1], we have that

$$\sup_{0\leq t\leq T} \left| \overline{F}_{xh}(t) - \varphi_x^{-1}\left(-\int_0^t (\overline{H}_{xh}(t)) dH_{xh}^u(y) \right) \right| \leq \frac{9}{4}\left(\frac{-1}{\varphi_x'(1)} \right) \varphi_x''(\delta) \|K\|_\infty \frac{\overline{\Delta}_n}{h_n}.$$

Hence, by adding and subtracting terms, we obtain that

$$\sup_{\substack{0\leq s,t\leq T\\ |t-s|\leq a_n}} |F_{xh}(t) - F_{xh}(s) - \widetilde{F}_{xh}(t) + \widetilde{F}_{xh}(s)|$$

$$\leq \frac{3}{2}\left(\frac{-1}{\varphi_x'(1)} \right) \varphi_x''(\delta)| \sup_{0\leq t\leq T} |H_{xh}(t) - H_x(t)| \sup_{\substack{0\leq s,t\leq T\\ |t-s|\leq a_n}} |H_{xh}^u(t) - H_{xh}^u(s)|$$

$$+2\frac{\varphi_x''(\delta)}{(\varphi_x'(1))^2} \|F'\| |\varphi_x'(\delta)| + \frac{9}{2}\left(\frac{-1}{\varphi_x'(1)} \right) \varphi_x''(\delta) \|K\|_\infty \frac{\overline{\Delta}_n}{h_n}.$$

For n sufficiently large, the sum of the last two terms is less than $\frac{\varepsilon}{4}$, and hence

$$P\left(\sup_{\substack{0\le s,t\le T\\|t-s|\le a_n}} |F_{xh}(t)-F_{xh}(s)-\widetilde{F}_{xh}(t)+\widetilde{F}_{xh}(s)|>\frac{\varepsilon}{2}\right)$$

$$\le\ P\left(\sup_{0\le t\le T}|H_{xh}(t)-H_x(t)|\cdot\sup_{\substack{0\le s,t\le T\\|t-s|\le a_n}}|H^u_{xh}(t)-H^u_{xh}(s)|>\frac{\varepsilon}{4a}\right) \quad (6.10)$$

where $a=\frac{3}{2}\left(\frac{-1}{\varphi'_x(1)}\right)\varphi''_x(\delta)$.

To deal with the right hand side of (6.10) we call the event Z and use that $P(Z)\le P(Z\cap(A_n\cap B_n))+P(A_n^c)+P(B_n^c)$ where $A_n=\{1-H_x(t)\ge\frac{\delta}{2}$ for all $0\le t\le T\}$ and $B_n=\{|H^u_{xh}(t)-H^u_{xh}(s)|\le 2\|H^{u'}\|a_n$ for all $0\le s,t\le T,|t-s|\le a_n\}$.
Then, since $1-H_x(t)>\delta$ for all $0\le t\le T$, we have that

$$P(A_n^c)\le P\left(\sup_{0\le t\le T}|H_{xh}(t)-H_x(t)|>\frac{\delta}{2}\right)$$

and also

$$P(B_n^c)\le P\left(\sup_{\substack{0\le s,t\le T\\|t-s|\le a_n}}|H^u_{xh}(t)-H^u_{xh}(s)-H^u_x(t)+H^u_x(s)|>\|H^{u'}\|a_n\right)$$

Hence,

$$P\left(\sup_{\substack{0\le s,t\le T\\|t-s|\le a_n}} |F_{xh}(t)-F_{xh}(s)-\widetilde{F}_{xh}(t)+\widetilde{F}_{xh}(s)|>\frac{\varepsilon}{2}\right)$$

$$\le\ P\left(\sup_{0\le t\le T}|H_{xh}(t)-H_x(t)|>\frac{\varepsilon}{4a}\,\frac{1}{2\|H^{u'}\|a_n}\right)+P\left(\sup_{0\le t\le T}|H_{xh}(t)-H_x(t)|>\frac{\delta}{2}\right)$$

$$+\ P\left(\sup_{\substack{0\le s,t\le T\\|t-s|\le a_n}}|H^u_{xh}(t)-H^u_{xh}(s)-H^u_x(t)+H^u_x(s)|\ge\|H^{u'}\|a_n\right).$$

We can now apply the exponential inequalities in Lemma A.3 in Van Keilegom and Veraverbeke [7] and in Theorem 4 in Van Keilegom and Veraverbeke [6]. This requires the restriction (6.4) on ε.

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