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BAYESIAN HIERARCHICAL CLASSES ANALYSIS

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Abstract

Hierarchical classes models are models for n-way n-mode data that represent the association among the n modes and simultaneously yield, for each mode, a hierarchical classification of its elements. In this paper, we present a stochastic extension of the hierarchical classes model for two-way two-mode binary data. The proposed extension includes an additional (pair of) probability parameter(s) to account for a predicted cell entry deviating from the corresponding observed data entry. In line with the original model, the new probabilistic extension still represents both the association among the two modes and the hierarchical classifications. Methods for model fitting and model checking are presented within a Bayesian framework using Markov Chain Monte Carlo simulation techniques; in particular, some tests for specific model assumptions using posterior predictive checks are discussed, exemplifying how the latter technique can be applied to test virtually any underlying assumption. The most important advantages of the stochastic extension over the original deterministic approach are: (1) conceptually, the extended model explicitly includes an account of its relation to the observed data; (2) uncertainty with regard to the parameter estimations is formalized; and (3) model checks are available. We illustrate these advantages with an application in the domain of the psychology of choice.

1 Introduction

Hierarchical classes models (De Boeck and Rosenberg, 1988; Van Mechelen, De Boeck, and Rosenberg, 1995; Leenen, Van Mechelen, De Boeck, and Rosenberg, 1999; Ceulemans, Van Mechelen, and Leenen, 2003), dubbed HICLAS, are a family of deterministic models for n-way n-mode data. In this paper, we will focus on hierarchical classes models for two-way two-mode binary data (i.e., data that can be represented in a 0/1 matrix), although extensions to the more general case can be considered. For examples of this type of data, one may think of person by item success/failure data, object by attribute presence/abscence data, person by opinion agreement/disagreement data, etc.. HICLAS models for two-way two-mode data imply the representation of three types of relations in the data: (a) an association relation that links the two modes together; (b) an equivalence relation defined on each mode, yielding a two-sided classification; and (c) a partial order relation defined on both classifications —the latter can be interpreted as an *if-then* type of relation (Van Mechelen, Rosenberg, and De Boeck, 1997) and yields a hierarchical organization of the classes (and the elements) in each mode—. Several types and variants of hierarchical classes models have been presented, including the original disjunctive model (De Boeck and Rosenberg, 1988) and the conjunctive model (Van Mechelen et al., 1995); they differ in the way the three types of relations are represented. The HICLAS approach has been succesfully applied in various substantive domains including person perception (Gara and Rosenberg, 1979; Cheng, 1999), developmental psychology (Elbogen, Carlo, and Spaulding, 2001), personality psychology (Vansteelandt and Van Mechelen, 1998; ten Berge and de Raad, 2001), psychiatric diagnosis (Gara, Silver, Escobar, Holman, and Waitzkin, 1998), the psychology of learning (Luyten, Lowyck, and Tuerlinckx, 2001), and the psychology of choice (Van Mechelen and Van Damme, 1994). Overviews can be found in De Boeck, Rosenberg, and Van Mechelen (1993) and Rosenberg, Van Mechelen, and De Boeck (1996).

The HICLAS model, as a deterministic model for binary data, predicts the value for each cell in a given matrix exactly to be either 0 or 1. Strictly speaking, this implies that one should reject the model as soon as a single discrepancy (i.e., a cell value that is mispredicted) is found. In analyses of data, however, discrepancies are allowed for: Algorithms were developed (De Boeck and Rosenberg, 1988; Leenen and Van Mechelen, 2001) to find a hierarchical classes model (of a prespecified complexity) that has a minimal number of discrepancies with a given data set. Although this deterministic approach has proven to yield satisfactory results in many practical applications, some drawbacks are to be considered. Firstly, the deterministic model is incomplete (and, as a result, has a somewhat unclear conceptual status), since the relation with the data is not specified in the model. Secondly, the proposed minimization algorithms yield a single solution (which in the best case is a global optimum), whereas (a) several solutions may exist that fit the data equally well and (b) many other solutions can be considered that are not but slightly worse compared to the best solution and that may be interesting from a substantive point of view. Thirdly, as a consequence of the model's incompleteness, no statistical testing tools for model checking can be developed (apart from the fact that any discrepancy, in principle, implies a rejection of the model as a whole).

Maris, De Boeck, and Van Mechelen (1996) proposed a probabilistic variant of the HICLAS family, called probability matrix decomposition models, which to some extent meet the objections above. However, in this probabilistic variant, the representation of the classifications and hierarchical relations is lost. The classifications and/or hierarchy relations in binary data arrays have often been of important substantive interest, though, for example, in concept analysis (Ganter and Wille, 1996, pp. 1–15), knowledge space theory (Falmagne, Koppen, Vilano, Doignon, and Johannesen, 1990), and person perception (Gara and Rosenberg, 1979).

This paper introduces an alternative stochastic extension of the HICLAS model that does retain the hierarchical classification feature. The extension is based on a generic Bayesian methodology proposed by Gelman, Leenen, Van Mechelen, De Boeck, and Poblome (2003). Within the proposed framework, a stochastic component is added to a deterministic model while fully retaining its deterministic core. As a result, many estimation and checking tools that are common for stochastic models become available for deterministic models as well. An application illustrating this methodology in the context of Boolean regression can be found in Leenen, Van Mechelen, and Gelman (2000).

For brevity's sake, we will limit the exposition of the stochastic extension to the conjunctive HICLAS model, which previously appeared most relevant within the context of the illustrative application under study. The stochastic extension of the other HICLAS models, including De Boeck and Rosenberg's (1988) original disjunctive model, is straightforward and fully analogous to that of the conjunctive model. In the remainder of this paper, we will first recapitulate the deterministic HICLAS model (Section 2); next, we introduce the new stochastic extension (Section 3) and we illustrate with an application from the domain of the psychology of choice (Section 4). Finally, we present some concluding remarks and a discussion on possible further extensions of the proposed model (Section 5).

2 The deterministic hierarchical classes model

2.1 Model

Consider a binary data matrix $Y = (y_{11}, y_{12}, \ldots, y_{mn})$. We will index the rows and columns of Y with i ($i = 1, \ldots, m$) and j ($j = 1, \ldots, n$), respectively. A hierarchical classes model for Y implies the representation of three types of relations defined on Y. We will first define these three types of relations and subsequently discuss their representation in a HICLAS model. As a guiding example in this section, we will use the hypothetical child by item data matrix in Table 1. The items are part of a test on addition of fractions and child i succeeds in item j (denoted as $y_{ij} = 1$) if and only if (s)he returns the correct solution in its simplest form (i.e., with the possible integer part separated and the fraction part reduced as far as possible).

The *association relation* is defined as the binary relation between the row elements and column elements as defined by the 1-entries in Y. From Table 1, for example, we can read that $\left(\text{John}, \frac{5}{9} + \frac{7}{9}\right)$ is an element of the relation, while $\left(\text{John}, \frac{2}{3} + \frac{1}{2}\right)$ is not.

Two *equivalence relations* are defined, one on the row elements and one on the column elements of Y. Two row [resp. column] elements i and i' [resp. j and j'] are equivalent (denoted as $i \sim_{\text{Row}} i'$ [resp. $j \sim_{\text{Col}} j'$]) if and only if they are associated with the same column [resp. row] elements.

In Table 1, for example, John and Dave are equivalent because the sets of items they succeed in are identical. Likewise, $(\frac{6}{7}+\frac{4}{7})$ and $(\frac{3}{5}+\frac{4}{5})$ are equivalent: They are solved by the same persons. The equivalence relations induce a partitioning of the row and column elements into a number of partition classes. For example, $\left\{\frac{2}{3}+\frac{1}{2}, \frac{3}{4}+\frac{5}{6}, \frac{2}{5}+\frac{3}{4}\right\}$ constitute an item class for the data in Table 1.

The *hierarchical relations* are defined on the rows and on the columns as well. A row i [resp. column j] is considered hierarchically below row i' [resp. column j'] (denoted as $i \prec_{\text{Row}} i'$ [resp. $j \prec_{\text{Col}} j'$]) if and only if the set of column [resp. row] elements that i [resp. j] is associated with constitutes a proper subset of the set of column [resp. row] elements that i' [resp. j'] is associated with. For example, in Table 1 Patrick is hierarchically below Susan as Patrick succeeds in only a subset of the items that Susan succeeds in; likewise, $(\frac{5}{9}+\frac{7}{9})$ is hierarchically below $(\frac{3}{5}+\frac{4}{5})$ as all children that succeed in the former item succeed in the latter item as well. Obviously, the hierarchical relation on the row [resp. column] elements directly implies a hierarchical relation on the row [resp. column] classes: For example, the item class $\left\{\frac{2}{3}+\frac{1}{2}, \frac{3}{4}+\frac{5}{6}, \frac{2}{5}+\frac{3}{4}\right\}$ is hierarchically below $\left\{\frac{6}{7}+\frac{4}{7}, \frac{3}{5}+\frac{4}{5}\right\}$. The equivalence and hierarchy relations are called set-theoretical relations or implication relations and we denote $i \preceq_{\text{Row}} i'$ [resp. $j \preceq_{\text{Col}} j'$] if either $i \sim_{\text{Row}} i'$ [resp. $j \sim_{\text{Col}} j'$] or $i \prec_{\text{Row}} i'$ [resp. $j \prec_{\text{Row}} j'$].

A hierarchical classes model implies a decomposition of Y into two binary[1] matrices, one for each mode, which represent the three above-mentioned types of relations. Those matrices are called bundle matrices and are denoted by $S = (s_{11}, \ldots, s_{mr})$ and $P = (p_{11}, \ldots, p_{nr})$ for the rows and the columns, respectively, with entries $s_{ik}, p_{jk} \in \{0, 1\}$ $(k = 1, \ldots, r)$; the integer r is called the rank of the model and the r columns of S and P are called bundles. The set of bundles with $s_{ik} = 1$ [resp. $p_{jk} = 1$] $(k = 1, \ldots, r)$ will be referred to as the bundle pattern of row i [resp. column j].

We now explain how the bundle matrices represent the association, equivalence, and hierarchy relations in Van Mechelen et al.'s (1995) conjunctive HICLAS model. A correct representation of the *association relation* implies for any i and j:

$$y_{ij} = \begin{cases} 1 & \text{if } \forall k(1 \leq k \leq r) : s_{ik} \geq p_{jk} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Equation (1) means that a row is associated with a column if and only if the column's bundle pattern is a (proper or improper) subset of the row's bundle pattern. Table 2 shows the bundle matrices S and P of a conjunctive model in rank 3 for the hypothetical matrix in Table 1. We find that, for example, John's bundle pattern ($\{\mathrm{II}, \mathrm{III}\}$) includes all the bundles in the pattern of item ($\frac{6}{7} + \frac{4}{7}$) ($\{\mathrm{II}\}$) while Table 1 shows that John succeeds in the item; similarly, John does not solve ($\frac{1}{4} + \frac{2}{3}$) as his bundle pattern lacks Bundle I, which is included in the item's bundle pattern.

Optionally one may look for a substantive interpretation of the bundles and the association rule in the example: The three bundles may be regarded as underlying abilities that children either have ($s_{ik} = 1$) or do not have ($s_{ik} = 0$), and that are either required ($p_{jk} = 1$) or not required ($p_{jk} = 0$) for solving the item. In particular, Bundle I can be interpreted as the ability of finding the lowest common multiple of two integers (needed to reduce two fractions to a common denominator), Bundle II can be interpreted as the ability of dealing with fractions where the denominator exceeds the nominator, and Bundle III as the ability of finding the greatest common divisor of two integers (needed to reduce a fraction as far as possible). Then, Eq. (1) can be read as: A child succeeds in an item if and only if (s)he has all the abilities required by the item.

The conjunctive hierarchical classes model further adds the following restriction on S and P to represent the *set-theoretical relations*:

$$\forall i, i' : i \preceq_{\text{Row}} i' \quad \text{iff} \quad \forall k(1 \leq k \leq r) : s_{ik} \leq s_{i'k} \tag{2a}$$

$$\forall j, j' : j \preceq_{\text{Col}} j' \quad \text{iff} \quad \forall k(1 \leq k \leq r) : p_{jk} \geq p_{j'k}. \tag{2b}$$

A pair of bundle matrices (S, P) for which (2a) and (2b) hold are called set-theoretically consistent. Eq. (2a) [resp. Eq. (2b)] implies that equivalent rows [resp. columns] have identical bundle patterns. In the example of Table 2, John and Dave, who are equivalent in Table 1, have an identical set of abilities. Eq. (2a) further implies that, if row i is hierarchically below row i', then the bundle pattern of i is a *proper subset* of the bundle pattern of i'. In Table 2, for example, Mary's abilities include as a proper subset those of John and Dave, who are hierarchically below Mary. On the other hand, Eq. (2b) implies that if column j is hierarchally below column j' then the bundle pattern of j is a *proper superset* of the bundle pattern of j'. The items in class $\{\frac{2}{3}+\frac{1}{2}, \frac{3}{4}+\frac{5}{6}, \frac{2}{5}+\frac{3}{4}\}$, for example, which is hierarchically below the item class $\{\frac{6}{7}+\frac{4}{7}, \frac{3}{5}+\frac{4}{5}\}$, require a proper superset of the abilities required by the latter items ($\{\mathrm{I}, \mathrm{II}\} \supset \{\mathrm{II}\}$). Within an ability context the inverse representation of the item hierarchy naturally follows from the fact that the more abilities are required by an item, the less children dominate it.

2.2 Graphical representation

Van Mechelen et al. (1995) proposed a graphical representation that gives a full account of the relations represented by the conjunctive hierarchical classes model. A graphical representation of the model in Table 2 is found in Figure 1. In the graphical representation, the child and item classes appear as paired boxes, the upper box of each pair being a child class and the lower box an item class. The lines connecting the classes represent the hierarchical relations, with the item hierarchy to be read upside down. The association relation can be read from the graph as a dominance relation: A child succeeds in all items below him/her and an item is solved by all children above it.

2.3 Data analysis

Van Mechelen et al. (1995) (relying on a proposition by De Boeck and Rosenberg, 1988) showed that a perfectly fitting hierarchical classes model can be found for any matrix Y. However, in most realistic applications, models of high rank are needed for an exact decomposition of Y, such that the resulting model is often difficult to interpret. Therefore, in practice one usually searches for an approximate reconstructed data matrix $\widehat{Y}$ that can be represented by a low rank HICLAS model. In particular, in a typical HICLAS analysis (De Boeck and Rosenberg, 1988; Leenen and Van Mechelen, 2001), a matrix $\widehat{Y}$ for which a hierarchical classes model in some prespecified rank

r holds is looked for such that the loss function

$$D(\widehat{Y}, Y) = \sum_{i,j} (\hat{y}_{ij} - y_{ij})^2 \tag{3}$$

has minimal value. Note that, since Y and $\widehat{Y}$ only contain 0/1 values, the least squares loss function in Eq. (3) comes down to a least absolute deviations loss function (Chaturvedi and Carroll, 1997), and equals the number of discrepancies between Y and $\widehat{Y}$.

2.4 Model checking and drawing inferences

Although yielding quite satisfactory results in general, the widespread use of finding a best-fitting HICLAS model $\widehat{Y}$ for given data Y is conceptually unclear. For, as in all deterministic models, in an approximate HICLAS model, the relation to the data is not specified (i.e., it is not specified how and why predictions by the model may differ from the observed data), and hence, the slightest deviation between model and data strictly speaking implies a rejection of the model. Furthermore, because the model either fully holds or is rejected as a whole, it does not make sense to develop checks for specific model assumptions.

Furthermore, in most practical applications, the basis for pre-specifying the rank r of an approximating HICLAS model for $\widehat{Y}$ is often unclear. As a way out, a number of heuristics have been developed to infer a rank from the data, which in most cases is based on comparing the goodness-of-fit values for models of successive ranks. Examples of such heuristics include variants of the scree test and Leenen and Van Mechelen's (2001) pseudo-binomial test. Although these heuristics may yield satisfactory results, in some cases the selection of a particular rank seems arbitrary and susceptible to chance.

3 Stochastic extension within a Bayesian framework

3.1 Model

The standard HICLAS approach with its search for an approximate solution that has an optimal fit to a given data set reveals the assumption of an underlying stochastic process that may cause the predicted value to be different from the observed value. This implicit assumption can be formalized by introducing a parameter π which is defined (regardless of i and j) as:

$$\Pr\left[y_{ij} = \hat{y}_{ij}(S,P) \mid S,P,\pi\right] = 1-\pi, \tag{4}$$

where $\hat{y}_{ij}(S,P)$ is the general notation for the predicted value in cell entry (i,j) obtained by combining the bundle matrices S and P by Equation (1). Clearly, the interpretation for π is the expected proportion of discrepancies in the model.

As an alternative, a slightly more general model may be considered, which allows predicted values 0 and 1 to have different error probabilities:

$$\Pr\left[y_{ij} = \hat{y}_{ij}(S,P) \mid S,P,\pi_0,\pi_1\right] = \begin{cases} 1-\pi_0 & \text{if } \hat{y}_{ij}(S,P)=0 \\ 1-\pi_1 & \text{if } \hat{y}_{ij}(S,P)=1 \end{cases} \tag{5}$$

In both Equation (4) and (5), it is additionally assumed that the realizations y_{ij} are independent. The two models will be refered to as the one- and the two-error probability model, respectively.

3.2 Estimation

The likelihood of the data Y under the one-error probability model can readily be shown to be:

$$p(Y|s,p,\pi) = \pi^{D(\widehat{Y}(S,P),Y)}(1-\pi)^{C(\widehat{Y}(S,P),Y)} \tag{6}$$

with $D(\widehat{Y}(S,P),Y)$ the number of discrepancies [as defined in (3)] and $C(\hat{Y}(S,P),Y)$ the number of concordances [i.e., $mn - D(\hat{Y}(S,P),Y)$]. Equation (6) may be used to obtain maximum likelihood estimates for (S,P,π). Note in this respect that $(\widehat{S},\widehat{P},\hat{\pi})$ is a maximum likelihood estimate if and only if the loss-function $D(\widehat{Y}(\widehat{S},\widehat{P}),Y)$ in (3) has minimal value and $\hat{\pi} = D(\widehat{Y}(\widehat{S},\widehat{P}),Y)/mn$. One may wish, however, to go beyond finding the parameter vector (S,P,π) that maximizes the likelihood function for two reasons. Firstly, apart from a trivial kind of nonidentifiability due to a joint permutability of the columns of S and P, in some cases different pairs of bundle matrices (S,P) may combine to the same $\widehat{Y}$. Secondly, and more importantly, different matrices $\widehat{Y}(S,P)$ may exist with (almost) identical minimal values on the loss-function (3) and, hence, with (almost) maximal likelihood values. Therefore, we propose to inspect the full posterior distribution of the parameters given the data:

$$p(S, P, \pi|Y) \propto p(Y|S, P, \pi)\, p(S, P, \pi) \tag{7}$$

We assume a uniform prior, although other prior distributions can be considered as well (depending on additional information on the parameters or if restricted HICLAS models are called for, see also Section 5). A uniform prior in (7) implies that (S, P, π) is a posterior mode if and only if it maximizes the likelihood function.

For the two-error probability model, the likelihood of the data Y is given by:

$$p(Y|S, P, \pi_0, \pi_1) = \pi_0^{n_{10}(Y,\widehat{Y}(S,P))}(1-\pi_0)^{n_{00}(Y,\widehat{Y}(S,P))}\pi_1^{n_{01}(Y,\widehat{Y}(S,P))}(1-\pi_1)^{n_{11}(Y,\widehat{Y}(S,P))}, \tag{8}$$

where $n_{hh'}(Y, \widehat{Y}(S, P)) = \#\{(i, j) \mid y_{ij} = h \text{ and } \hat{y}_{ij}(S, P) = h'\}$ $(h, h' \in \{0, 1\})$. Again, the posterior distribution $p(S, P, \pi_0, \pi_1|Y)$ will be considered and a uniform prior will be assumed. Note, however, that the two-error probability model no longer implies that a pair of bundle matrices (S, P) with minimal value on the loss-function $D(\widehat{Y}(S, P), Y)$ maximizes the likelihood or, equivalently, is a posterior mode. This can be easily seen by considering the case where π_0 and π_1 are fixed: finding the mode of $p(S, P|y, \pi_0, \pi_1)$ then comes down to finding the pair (S, P) that minimizes the following weighted number of discrepancies:

$$\left(\log\frac{\pi_0}{1-\pi_1}\right) n_{10}(Y, \widehat{Y}(S, P)) \;+\; \left(\log\frac{\pi_1}{1-\pi_0}\right) n_{01}(Y, \widehat{Y}(S, P)),$$

which clearly may differ from the pair (S, P) that minimizes $D(\widehat{Y}(S, P), Y)$, particularly if π_0 and π_1 strongly differ.

3.3 Computation

In this section we describe a Metropolis-Hastings algorithm that was developed along the lines set out by Gelman et al. (2003) and that can be used to simulate the posterior distribution for either the one-error or the two-error probability model. We will immediately turn to the algorithm for obtaining a simulated posterior distribution for the parameter vector (S, P, π_0, π_1) in the two-error probability case, the algorithm for the one-error case involving some obvious simplifications only. If convenient, (S, P, π_0, π_1) is denoted by θ in this section.

We run independently m (≥ 2) parallel sequences, in each of them proceeding with the following steps:

Step 0: Initial estimates $S^{(0)}$ and $P^{(0)}$ are obtained, $s_{ik}^{(0)}$ and $p_{jk}^{(0)}$ being realizations of identically and independently distributed Bernoulli variables the probability parameters of which are chosen such that $\Pr[\hat{y}_{ij}(S^{(0)}, P^{(0)}) = 1]$ equals the proportion of 1-entries in the data Y. Subsequently, $\pi_0^{(0)}$ and $\pi_1^{(0)}$ are initialized:

$$\pi_0^{(0)} \leftarrow \frac{n_{10}(Y, \widehat{Y}(S,P)) + 1}{n_{\cdot 0}(Y, \widehat{Y}(S,P)) + 2}$$
$$\pi_1^{(0)} \leftarrow \frac{n_{01}(Y, \widehat{Y}(S,P)) + 1}{n_{\cdot 1}(Y, \widehat{Y}(S,P)) + 2},$$

where $n_{\cdot h}(Y, \widehat{Y}(S,P)) = n_{0h}(Y, \widehat{Y}(S,P)) + n_{1h}(Y, \widehat{Y}(S,P))$ $(h \in \{0, 1\})$. We add 1 in the numerator and 2 in the denominator to avoid initial values for the π's being 0 or 1 (Gelman et al., 2003). As a final step in the initialization, we set $t \leftarrow 0$.

Step 1: A candidate pair of bundle matrices (S^*, P^*) is constructed as follows: First, (S^*, P^*) is initialized with $(S^{(t)}, P^{(t)})$. Next, a strictly positive integer w is drawn from some prespecified discrete distribution (e.g., Binomial, Poisson, etc.) and, subsequently, the value in w randomly selected cells of the matrices (S^*, P^*) is switched, either from 0 to 1 or from 1 to 0. This procedure implicitly defines a jumping or proposal distribution $J(S^*, P^* | S^{(t)}, P^{(t)})$, which gives the probability of considering (S^*, P^*) as a new candidate conditionally upon the current value $S^{(t)}, P^{(t)}$ (Hastings, 1970; Gilks, Richardson, and Spiegelhalter, 1996, pp. 5–12).

Step 2: Next, we compute the importance ratio

$$r = \frac{p\left(Y \mid S^*, P^*, \pi_0^{(t)}, \pi_1^{(t)}\right)}{p\left(Y \mid \theta^{(t)}\right)} \frac{J\left(S^*, P^* | S^{(t)}, P^{(t)}\right)}{J\left(S^{(t)}, P^{(t)} | S^*, S^*\right)} \tag{9}$$

and replace $(S^{(t)}, P^{(t)})$ with (S^*, P^*) with probability $\min(1, r)$. Since the jumping distribution is symmetric, (9) reduces to:

$$r = \frac{p\left(Y \mid S^*, P^*, \pi_0^{(t)}, \pi_1^{(t)}\right)}{p\left(Y \mid \theta^{(t)}\right)}.$$

Step 3: $\pi_0^{(t)}$ and $\pi_1^{(t)}$ are updated by drawing from a Beta$\Big[n_{10}(Y, \widehat{Y}(S^{(t)}, P^{(t)})) + 1, n_{00}(Y, \widehat{Y}(S^{(t)}, P^{(t)})) + 1\Big]$ and a Beta$\Big[n_{01}(Y, \widehat{Y}(S^{(t)}, P^{(t)})) + 1, n_{11}(Y, \widehat{Y}(S^{(t)}, P^{(t)})) + 1\Big]$ distribution, respectively.

Step 4: If the pair of bundle matrices $(S^{(t)}, P^{(t)})$ is set-theoretically consistent, then we set (a) $\theta^{(t+1)} \leftarrow \theta^{(t)}$ and, subsequently, (b) $t \leftarrow t + 1$;

Steps 1 through 4 are repeated independently in each of the m sequences until for each individual parameter the sequences appear mixed. After convergence, a subset of L draws of the parameter vector is selected (from the converged part of the simulated sequences) to form the simulated posterior distribution of (S, P, π_0, π_1). To monitor convergence, we use Gelman and Rubin's (1992) $\hat{R}$-statistic, with a slight adaptation, though, to account for possible absence of within-sequence variance in the binary bundle parameters: We define $\hat{R} \equiv 1$ for a parameter that assumes the same value throughout each of the sequences and $\hat{R} \equiv +\infty$ for a parameter that in one sequence has a constant value being different from the value(s) in the other sequences.

Due to the binary nature of the parameters in the bundle matrices S and P, the convergence process of the presented algorithm shows some notable differences in comparison with other Markov Chain Monte Carlo methods. On the one hand, as changing even a single element in S^* or P^* (Step 1) generally implies a serious drop in the posterior probability, the acceptance ratio (i.e., the ratio of replacing $(S^{(t)}, P^{(t)})$ by (S^*, P^*) in Step 2) turns out to be very low (between 0.1% and 5%) and convergence may be reached after a huge number of iterations only. On the other hand, as computations involving binary operations can be combined to be executed simultaneously, the duration of each iteration usually is very short and, as a result, the whole procedure (when applied to medium-sized data matrices) often terminates in feasible time. Otherwise, even if the sequences fail to converge, the procedure described here is still an improvement over the point estimate obtained by a deterministic algorithm in that it may yield several substantively interesting pairs (S, P) with low values on (3).

As a final comment, we notice that $\hat{R}$ may indicate nonconvergence when models from different chains are identical upon a permutation of the bundles. Because these models are considered identical from a substantive point of view, we propose to permute in each simulation draw of the

Metropolis-Hastings algorithm to the solution that minimizes the difference with some prespecified reference model $(\tilde{S}, \tilde{P})$ (for example, the model obtained from the original deterministic algorithm). That is, after each iteration t a permutation of the bundles in $(S^{(t)}, P^{(t)})$ is found such that

$$\sum_{i,j} \left(\hat{y}_{ij}(S^{(t)}, P^{(t)}) - \hat{y}_{ij}(\tilde{S}, \tilde{P}) \right)^2$$

has minimal value. Whereas the choice of the reference model may affect the convergence rate, posterior inferences with respect to the association, equivalence, and hierarchy relations based on the converged simulated posterior distribution remain unaffected by the choice of $(\tilde{S}, \tilde{P})$.

3.4 Representation of the posterior distribution

In contrast with the deterministic model, in which for a particular pair of elements (or classes) the association, equivalence, and hierarchy relations either hold or do not hold, the new stochastic extension implies a (marginal) posterior probability for any individual relation. In order to represent this posterior uncertainty, one can make use both of tabular form and of an adapted version of Van Mechelen et al.'s (1995) original graphical representation.

To chart out the uncertainty in the association relation, for each row-column pair the proportion of the L simulation draws in which the row is associated with the column can be calculated and the results can be represented in an m by n table. Similarly, marginal posterior probabilities for the equivalence and hierarchy relations can be calculated and given a tabular representation, both for the row elements and the column elements.

In the adapted graphical representation, each class is represented with a box (as before), which now contains all elements for which the marginal posterior probability of belonging to the class exceeds some prespecified cut-off α. Furthermore, the uncertainty in the hierarchical relation among classes can be represented in the graph by varying the line thickness or by labeling the edges. To quantify the degree of certainty of the hierarchical relation $C_1 \prec C_2$ between any pair of row classes (C_1, C_2), we propose the following measure:

$$p(C_1 \prec_{\text{Row}} C_2) = \frac{\sum_{i=1}^{m} \sum_{i'=1}^{m} p(i \in C_1)\, p(i' \in C_2)\, p(i \prec_{\text{Row}} i')}{\sum_{i=1}^{m} \sum_{i'=1}^{m} p(i \in C_1)\, p(i' \in C_2)},$$

where $p(i \in C_k)$ is the posterior probability of row i belonging to row class C_k and $p(i \prec_{\text{Row}} i')$ is the posterior probability of row i being hierarchically below row i'. The strength of the hierarchical relations among the column classes can be calculated through a similar formula.

3.5 Model checking and drawing inferences

Within a Bayesian framework a broad range of tools for model checking is available. In this section, we will discuss the use of posterior predictive checks (PPC's) for model selection and model checking (Rubin, 1984; Gelman, Meng, and Stern, 1996). The rationale of PPC's is the comparison, via some test quantity, of the observed data with data that could have been observed if the actual experiment were replicated under the model with the same parameters that generated the observed data (Gelman, Carlin, Stern, and Rubin, 1995). The test quantity may be a statistic $T(Y)$, summarizing some aspect of the data, or, more generally, a discrepancy measure $T(Y, S, P, \pi_0, \pi_1)$ quantifying the model's deflection from the data in some respect (Meng, 1994).

The PPC procedure consists of three substeps. First, for each of the draws $\theta^l \equiv (S^l, P^l, \pi_0^l, \pi_1^l)$ $(l = 1, \ldots, L)$ from the simulated posterior distribution, a replicated data set $Y^{\text{rep}\ l}$ is simulated, each cell entry $y_{ij}^{\text{rep}\ l}$ being an independent realization of a Bernoulli variable with probability parameter $|h - \pi_h^l|$, where $h = \hat{y}_{ij}(s^l, p^l)$. Second, for each simulated draw, the value of the test quantity is calculated for both the observed data Y^{obs} (i.e., $T(Y^{\text{obs}}, \theta^l)$) and the associated replicated data $Y^{\text{rep}\ l}$ (i.e., $T(Y^{\text{rep}\ l}, \theta^l)$). As a final step, the values for the observed and the replicated data are compared and the proportion of simulation draws for which $T(Y^{\text{rep}\ l}, \theta^l) > T(Y^{\text{obs}}, \theta^l)$ is considered as an estimate of the posterior predictive p-value.

Different aspects of the model can be checked using PPC's with appropriate choices of the test quantity. In the remainder of this section, we will elaborate some PPC's for checking (1) the deterministic core of the model and (2) the error structure as implied by the stochastic part of the model. These checks will be further illustrated in the application of Section 4.

3.5.1 Checking the deterministic core of the model

Rank The PPC procedure can be used as a tool for rank selection by defining a test quantity that is sensitive to the underestimation of the rank r. That is, assuming that some true HICLAS model underlies the observed data, the test quantity should yield higher values if the rank of the fitted model is lower than the rank of the true model. To this end, we define a test statistic $T(Y)$ that is the improvement on the loss-function (3) gained by adding an extra bundle to the model in rank r. More precisely:

$$T(Y) = D(\widetilde{Y}^{r+1}, Y) - D(\widetilde{Y}^{r}, Y), \tag{10}$$

where $\widetilde{Y}^k$ $(k = r, r+1)$ is the deterministic model matrix obtained by applying the original HICLAS algorithm (De Boeck and Rosenberg, 1988; Leenen and Van Mechelen, 2001) in rank k to the data set Y. If the true rank underlying the observed data is larger than r, then the fit of a rank $r+1$ model will substantially increase compared to the fit of the rank r model, and the value of $T(Y^{\text{obs}})$ will be expected to be higher than the values $T(Y^{\text{rep }l})$ for the replicated data sets. For, the underlying true rank in the latter data sets is of rank r and the extra bundle in the $r+1$ model only fits noise.

Set-theoretical relations As a second example, we will test the conformity of the set-theoretical relations represented in the model with those in the data. Possibly the model is either too conservative or too liberal in the representation of the equivalence and hierarchy relations. To this end, we will compare, for each pair of rows i and i' [resp. columns j and j'], the presence/abscence of an implication relation $i \preceq_{\text{Row}} i'$ [resp. $j \preceq_{\text{Col}} j'$] in the model with their implicational strength in the data. As a measure for the latter, we will use conditional probabilities

$$p(\mathbf{y}_{i'\cdot}|\mathbf{y}_{i\cdot}) \equiv \frac{\sum\limits_j y_{ij}y_{i'j}}{\sum\limits_j y_{ij}} \quad \text{and} \quad p(\mathbf{y}_{\cdot j'}|\mathbf{y}_{\cdot j}) \equiv \frac{\sum\limits_i y_{ij}y_{ij'}}{\sum\limits_i y_{ij}}$$

$\big(p(\mathbf{y}_{i'\cdot}|\mathbf{y}_{i\cdot})$ $\big[$resp. $p(\mathbf{y}_{\cdot j'}|\mathbf{y}_{\cdot j})\big]$ being considered undefined whenever $\sum\limits_j y_{ij} = 0$ $\big[$resp. $\sum\limits_i y_{ij} = 0\big]\big)$. Then, the test quantity used in the PPC for the row set-theoretical relations is defined as:

$$T(Y, S) = \sum_{i=1}^{m}\sum_{i'=1}^{m}\Big|p(\mathbf{y}_{i'\cdot}|\mathbf{y}_{i\cdot}) - I_S(i \preceq_{\text{Row}} i')\Big|, \tag{11a}$$

where the first summation is across rows i with $\sum_j y_{ij} \neq 0$ and $I_S(i \preceq_{\text{Row}} i')$ takes a value of 1 if the bundle matrix S implies $i \preceq_{\text{Row}} i'$, and 0 otherwise. A similar test quantity for the column set-theoretical relations may be defined:

$$T(Y,P) = \sum_{j=1}^{n} \sum_{j'=1}^{n} \left| p(\mathbf{y}_{\cdot j'} | \mathbf{y}_{\cdot j}) - I_P(j \preceq_{\text{Col}} j') \right|, \tag{11b}$$

where the first summation is across columns j with $\sum_i y_{ij} \neq 0$.

3.5.2 Checking the error model

One-probability versus two-probability model For a check of the validity of the one-error model (4) against the alternative model (5) with two error parameters, it is straightforward to use a test quantity that compares the proportion of discrepancies in cells where $\hat{y}_{ij}(S,P) = 0$ with the proportion of discrepancies in cells where $\hat{y}_{ij}(S,P) = 1$, for example by calculating the difference:

$$T(Y,S,P) = \frac{n_{10}(Y, \widehat{Y}(S,P))}{n_{\cdot 0}(Y, \widehat{Y}(S,P))} - \frac{n_{01}(Y, \widehat{Y}(S,P))}{n_{\cdot 1}(Y, \widehat{Y}(S,P))}. \tag{12}$$

The test quantity in Eq. (12) is the difference between the maximum likelihood estimates of π_0 and π_1, respectively, for fixed S and P under the two-error model. In data sets replicated under the one-error model, the expected value of $T(Y^{\text{rep}\ l}, S^l, P^l)$ equals 0. Values of $T(Y^{\text{obs}}, S^l, P^l)$ that are found to be more extreme than $T(Y^{\text{rep}\ l}, S^l, P^l)$ may indicate a violation of the one error parameter assumption.

Homogeneity of the error process The assumption $\pi_0 = \pi_1 = \pi$ is part of a more general assumption of a homogeneous error parameter structure across all data points implied by model (4). Incidentally, the latter assumption is conceptually similar to the homoscedasticity assumption of equal error variances across observations in regression models or in analysis of variance. Within a Bayesian framework, deviations from this general assumption can be checked easily. The model, for example, implies the assumption of equal error probability parameters across rows, which may be too restrictive. This assumption can be checked with the variance of the number of discrepancies across rows,

$$T(Y,S,P) = \sum_{i=1}^{m} \left(D_{i\cdot}(\widehat{Y}(S,P),Y) \right)^2 - \frac{1}{m} \left(\sum_{i=1}^{m} D_{i\cdot}(\widehat{Y}(S,P),Y) \right)^2 \tag{13a}$$

with $D_{i\cdot}\left(\widehat{Y}(S,P),Y\right) = \sum_{j=1}^{n}\left[\hat{y}_{ij}(S,P) - y_{ij}\right]^2$, which will tend to be higher if the error probability parameter differs across rows. Similarly, the assumption of equal error probabilities across columns can be checked using the variance of the number of discrepancies across columns as a test quantity:

$$T(Y,S,P) = \sum_{j=1}^{n}\left(D_{\cdot j}(\widehat{Y}(S,P),Y)\right)^2 - \frac{1}{n}\left(\sum_{j=1}^{n} D_{\cdot j}(\widehat{Y}(S,P),Y)\right)^2 \tag{13b}$$

with $D_{\cdot j}\left(\widehat{Y}(S,P),Y\right) = \sum_{i=1}^{m}\left[\hat{y}_{ij}(S,P) - y_{ij}\right]^2$.

4 Application

In this section we present a reanalysis of person by choice object select/nonselect data (i.e., so-called pick any/n data, Coombs, 1964, p. 295), that were previously analyzed by Van Mechelen and Van Damme (1994). These data originate from each of 26 second-year psychology students being presented with 25 index cards with room descriptions from the Housing Service of the University of Leuven and being asked to select those rooms (s)he would decide to visit when looking for a student room. (Most students in Leuven rent a room during the academic year and many of them pass by the Housing Service to get an overview and descriptions of the available rooms.) This resulted in a 26×25 binary matrix Y with $y_{ij} = 1$ if student i selected room j, and $y_{ij} = 0$ otherwise.

In the original study, Van Mechelen and Van Damme analyzed these data using a deterministic conjunctive HICLAS model with an interpretation of the bundles in terms of latent choice criteria. The key idea behind their interpretation was that the rooms selected by a person are those that meet all his (her) criteria. This idea is formalized by a variant of the conjunctive HICLAS model with the bundle matrix S denoting the persons' choice criteria and the bundle matrix P the corresponding properties of the rooms and with the following association rule[2]:

$$y_{ij} = \begin{cases} 1 & \text{if } \forall k(1 \leq k \leq r) : p_{jk} \geq s_{ik} \\ 0 & \text{otherwise.} \end{cases}$$

Here we apply a Bayesian HICLAS model to the same data. First, we fitted a rank 1 model with one error probability parameter. In the Metropolis-Hastings algorithm, 5 sequences were used and in the jumping procedure the number of entries to be changed was drawn from a Poisson distribution with parameter $\lambda = 4$. Convergence was reached (for all parameters, $\hat{R} < 1.05$) after about 1.5 million of iterations (i.e., after less than ten minutes on a Pentium IV 3.06 gHz). To save computer resources, each thousandth draw in the second half of the 5 sequences (these were also the draws on which the convergence statistic was calculated), was saved, so that we are left with about 3,500 simulated posterior draws.

Figure 2(a) shows how many different pairs of bundle matrices (S, P) were found in the simulated marginal posterior distribution of the rank 1 model (with π being integrated out) at each value of $D(\widehat{Y}(S,P), Y)$. Clearly, for the data set under study, the deterministic approach would suffer from multimodality: Nine different models were found with a minimal number of discrepancies. Moreover, many more models come close to this lower bound. Yet, applying the original deterministic algorithm would return a single model, keeping the user unaware of possible alternative models with an (almost) equally low number of discrepancies.

The rank 1 model is too simple, though, to give a good account of the data: Using the PPC procedure based on the test statistic (10) for detecting underestimation of the rank, the model in rank 1 is rejected ($T = 54$, PPC-$p = .02$). Therefore, we applied the Metropolis-Hastings algorithm to obtain a simulated posterior distribution for the conjunctive rank 2 model as well. With the same settings as in the analysis for the rank 1 model (except that for the Poisson distribution, a parameter $\lambda = 6$ was used to account for the larger number of parameters in the bundle matrices), we found convergence after about 17.5 million iterations (i.e., after about one hour on a Pentium IV 3.06 gHz). Each ten thousandth draw in the second half of the 5 sequences was saved to obtain a set of about 4,400 posterior draws. A PPC procedure with statistic (10) to test a rank 2 against a rank 3 model now yielded $T = 27$ (PPC-$p = .12$); therefore, the rank 2 model was retained for further discussion. From Figure 2(b), we can read how many different pairs of bundle matrices of rank 2 were found at each value $D(\widehat{Y}(S,P), Y)$. In particular, 4 different pairs (S, P) with a minimal number of 143 discrepancies were found, and 25 other models had only one discrepancy more. Again, thanks to the new approach, multi-modality was traced.

For the obtained model in rank 2, we further illustrate the use of the three other statistics for checking some model assumptions that we presented in Section 3.5. First, with regard to the comformity of the set-theoretical relations in the model with those in the data, the PPC p-value with test quantity (11a) is found to be equal to .41 and with test quantity (11b) .40. In other words, no evidence is found for an inconsistency between the implication relations as represented in the model and the corresponding relations in the data. Second, the assumption of equal error probabilities for predicted 0's and 1's as implied by the one-error probability model is checked with test quantity (12). The mean value on the latter test quantity equals .00 for the replicated data, as expected on theoretical grounds, and $-.02$ for the observed data. Furthermore, $T(Y^{\text{obs}}, S^l, P^l)$ exceeds $T(Y^{\text{rep}\ l}, S^l, P^l)$ in 31% of the cases, from which we may conclude that the one-error probability model has not to be replaced with the more general two-error probability model. Third, the homogeneity assumption of the error across students is tested by means of the variance in the number of discrepancies across students (Eq. 13a). A higher variance for the observed data could have been likely if, for example, students differed in the level of engagement when completing the task. However, the PPC p-value of .73 provides no evidence for individually different error parameters. In the same way, a test using the variance in the number of discrepancies across rooms (Eq. 13b) is performed. Such a test could reveal, for example, differences in quality of the room discriptions. Yet, the PPC p-value of .43 does not provide any indication for extending the model with different error parameters across rooms.

Figure 3 displays a graphical representation of the simulated posterior distribution of the rank 2 model, from which we can read the three types of relations: (i) association, (ii) classification, and (iii) hierarchy. (i) The association relation can be read deterministically as in the original model: Rooms are selected by any student who is below it. For example, if we ignore the uncertainty in the association relation for a moment, it holds that the rooms in cluster C are selected by all the students, whereas those from cluster B are selected by the students in clusters b and d only. The uncertainty in the association relation is not directly represented in the graph, though. Figure 4, however, shows the marginal posterior probabilities for each room-student association using gray values, with darker values corresponding with higher selection probabilities. The dot in the center of each cell, either black or white, represent the observed data value and allows to check the degree of discrepancy between the model and the data with respect to the association relation. (ii) Concerning the classifications, row and column classes appear in the paired boxes of Figure 3, with in the solid-framed box of each pair a row (room) cluster and in the dotted-framed box a column (student) cluster. The uncertainty with respect to the clustering is represented by the marginal posterior probability for each element to belong to the particular cluster. Only elements for which the posterior probability exceeds $\alpha = .33$ are displayed in the box. So, we can read, for instance, that Room 7 is found to be in cluster C in all the simulation draws whereas Room 12 is found about evenly in the clusters A and C. (iii) The room and student hierarchies are represented by solid and dotted arrows, respectively, connecting any two clusters C_1 and C_2 for which the strength of the hierarchical relation $C_1 \prec C_2$ (as explained in Section 3.4) exceeds $\alpha = .33$. Student cluster a, for instance, is found to be hierarchically below cluster d, though weakly (.42), whereas room cluster B is strongly (.90) hierarchically below room cluster C.

Figures 3 and 4 illustrate that, although many different pairs of bundle matrices (S, P) were found to have a minimal or near-minimal value on the loss-function, the hierarchical classes structure does have a stable part: For example, in almost all simulation draws, Rooms 7, 22, and 13, were found in the same class (at the top of the hierarchy). With regard to the substantive interpretation of the model, we follow Van Mechelen and Van Damme (1994) who conceived the two bundles as latent choice criteria. From the description of the rooms on the index cards, the first bundle, on which room clusters A and C load, appears to be related to "Elementary comfort + Good quality", whereas the rooms of clusters B and C, which load on the second bundle, appear to have as a distinctive characteristic "Elementary comfort + Low price" . The rooms in cluster D, at the bottom of the hierarchy, differ from the other rooms in that these rooms lack elementary comfort (such as water and kitchen facilities). The posterior probability of belonging to cluster d being low for all students, we can conclude that all students consider elementary comfort as a necessary prerequisite for a room to be selected. The students in cluster a additionally require the rooms to be of a good quality (mainly large and quiet) and the students in cluster b have a low price as additional requisite; students in cluster c, at last, select rooms that have both good quality and low price.

5 General discussion

This paper presented a new stochastic extension of the original deterministic hierarchical classes model (De Boeck and Rosenberg, 1988; Van Mechelen et al., 1995) in response to a number of conceptual and practical problems. Conceptually, the benefit of the new extension is that the relation between the predicted values and the observed values is made explicit thanks to the inclusion of one or two error-probability parameters. This enabled us to consider the model within a Bayesian framework and to make use of a variety of tools for model estimation and model checking. As shown in the example, this approach leads to a more differentiated view on the solution space by revealing several models that fit the data (almost) equally well. This differentiation may result in a fine-tuned interpretation in which core and peripheral aspects of the resulting structure are distinguished. Unlike the probabilistic variant developed by Maris et al. (1996), this differentiation is obtained without leaving or "fuzzifying" the set-theoretical or logical framework of the original HICLAS model. As such, the new model fully retains the deterministic core of the model, including the representation of the set-theoretical relations, while by superimposing the stochastic framework, the uncertainty in those set-theoretical relations is fully accounted for. More specifically, it is possible to evaluate the strength of the implication relations (which directly follows from the hierarchies represented in the model) by making use of the simulated posterior distribution of the parameters. In this respect, both simple implications (of the type *if* A *then* B) and more complex implications (e.g., *if* A or B *then* C) can be considered.

The new model extension as presented here may be extended further in various directions. For example, the model includes only one or two probability parameters, but additional error parameters may be introduced based on some hypotheses about the error process at hand. Such hypotheses may be either a priori or may result from model checks that indicate a misfit with respect to a particular assumption (like, for example, the assumption of equal error probabilities across rows and columns). A second direction in which the model can be further extended relates to the choice of the prior distribution. We choosed a uniform distribution for the set of parameters, implying that any hierarchical classes model is considered equally likely a priori and that the posterior distribution is proportional to the likelihood. Alternative prior distributions may be considered, motivated either by former results or by restrictions one may wish to add to the HICLAS model (for example, the restriction that the hierarchies constitute total orders rather than partial orders, or that the hierarchical relationship among some elements is fixed, etc.). Such restrictions can easily be incorporated by assigning a low or zero probability in the prior distribution to models that do not satisfy them. The Metropolis-Hastings algorithm for simulating the posterior distribution can be adapted accordingly either by changing the jumping rule or by dropping models that do not satisfy the restrictions.

Finally, one may point at an interesting relation between the model extension as proposed in this paper and a general family of fixed-partition models as discussed by Bock (1996). In particular, in two-mode fixed-partition models both the set of row elements and the set of column elements is partitioned into a prespecified number of classes and it is further assumed that the entries in the data clusters that are obtained as the Cartesian products of the row and column clusters are identically and independently distributed realizations of some data cluster specific distribution (Govaert and Nadif, 2003; see also, Van Mechelen, Bock, and De Boeck, in press). The classification likelihood estimation of such models aims at retrieving the bipartition and data cluster distribution parameters that have maximal likelihood given the data. The stochastic HICLAS model as proposed in this paper can be considered a two-mode overlapping clustering counterpart of two-mode fixed-partition models with Bernoulli-distributed data-entries; the Bernoulli parameters are constrained to be functions of the row and column cluster (bundle) membership patterns and of the overall error parameter(s) of the model.

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Footnotes

[1]We limit the discussion here to bundles that are binary, although also generalizations have been proposed where the bundles are ordinal variables (Leenen, Van Mechelen, and De Boeck, 2001).

[2]Mathematically, the conjunctive HICLAS model variant used in this section comes down to the conjunctive HICLAS model as introduced in Section 2 for the transposed data.

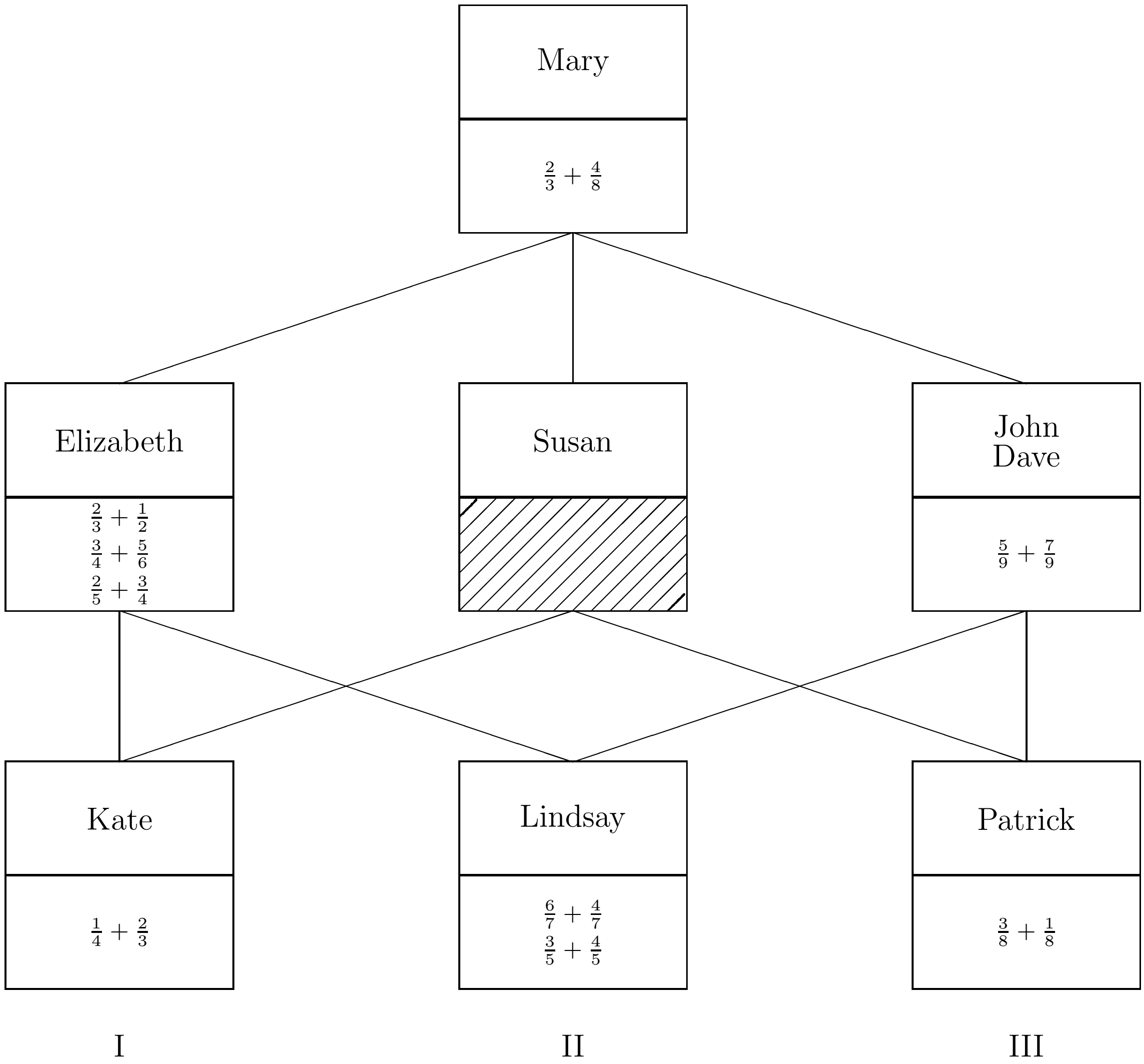
Figure captions

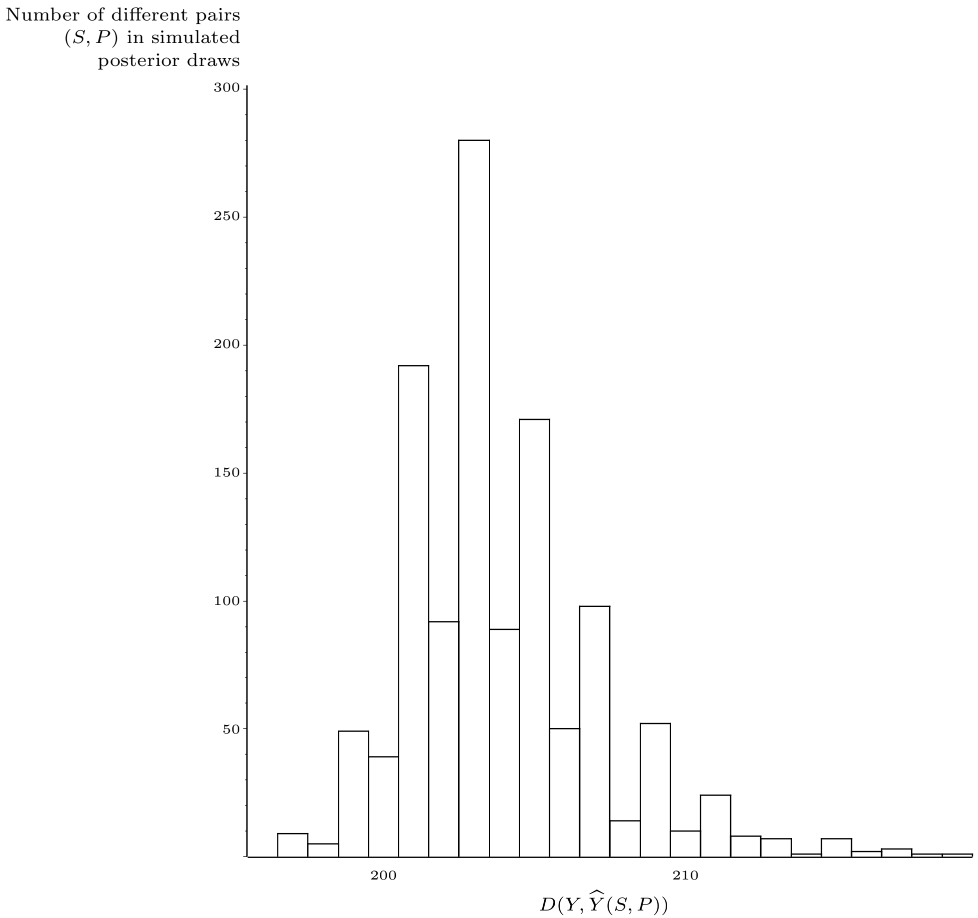
Figure 1: Graphical representation of the hierarchical classes model in Table 2.

Figure 2: Frequency of different pairs of bundle matrices (S, P) at each value of $D(\widehat{Y}(S, P), Y)$ from the simulated posterior distribution for the room data: (a) rank 1, (b) rank 2.

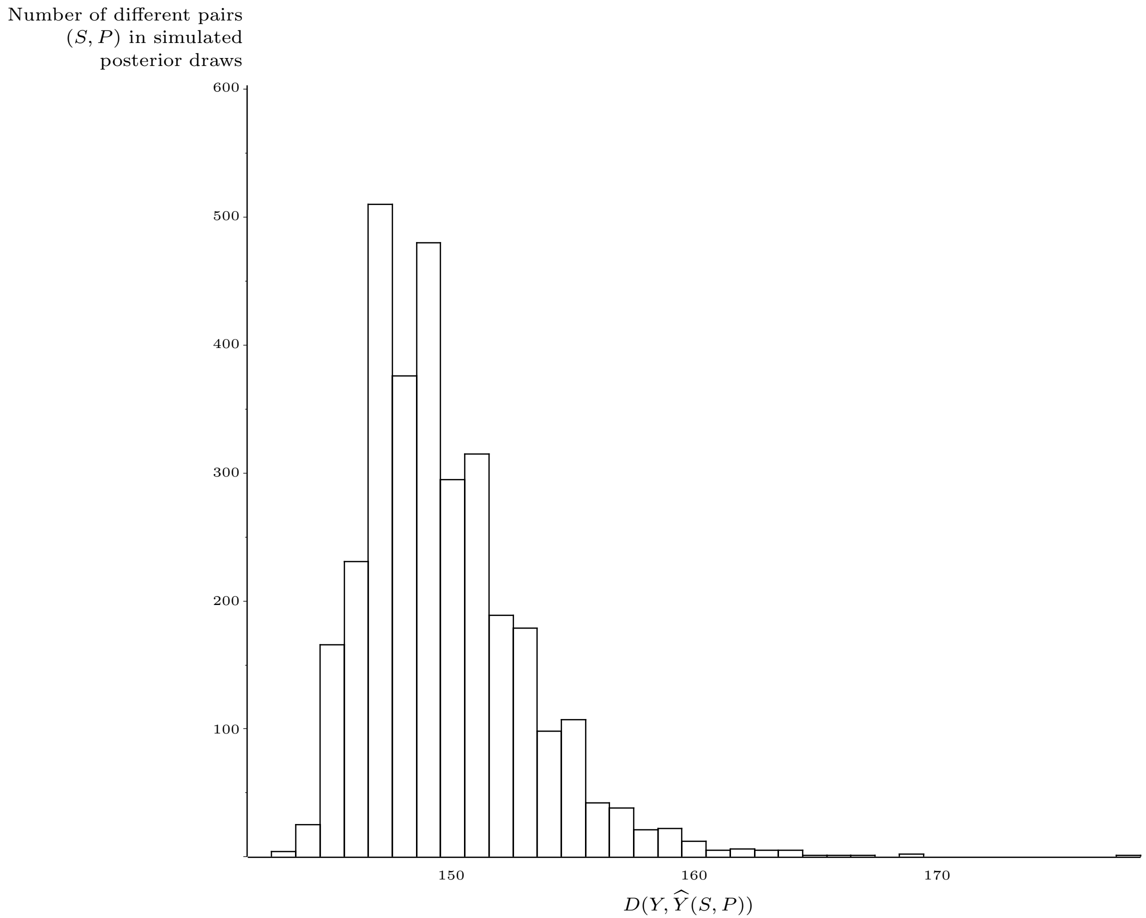
Figure 3: Graphical representation of the conjunctive Bayesian hierarchical classes model in rank 2 for the room data.

Figure 4: Posterior probabilities for the (room, student)-association relation using gray values, with darker values denoting higher probabilities. The rows [resp. columns] have been reordered in order to represent student [resp. room] classes (each element being assigned to the class for which it has the highest posterior membership probability). The circle in the middle of each cell represents the observed data value (with black and white for observed values of 1 and 0, respectively).

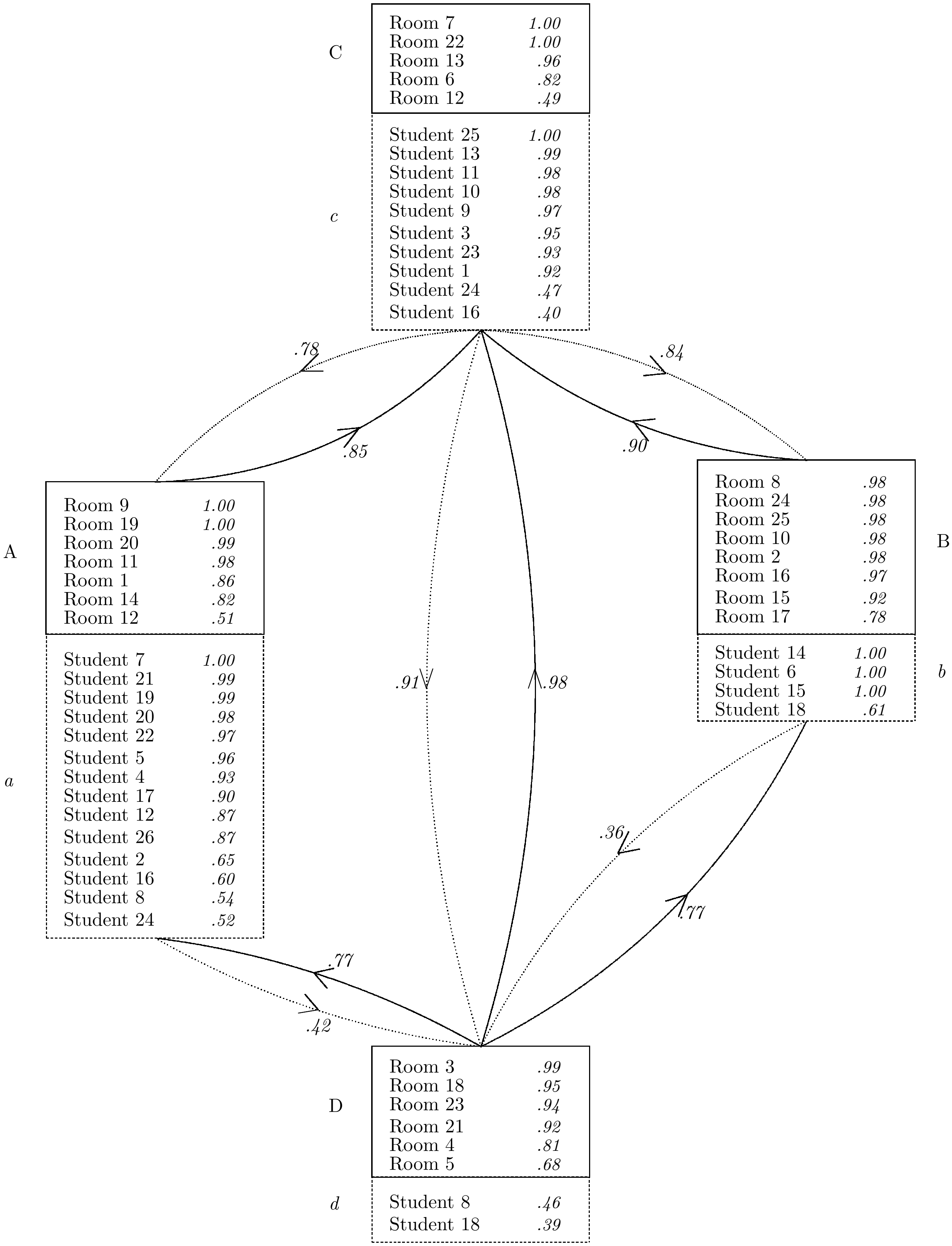

Mary
$\frac{2}{3} + \frac{4}{8}$
Elizabeth
$\frac{2}{3} + \frac{1}{2}$
$\frac{3}{4} + \frac{5}{6}$
$\frac{2}{5} + \frac{3}{4}$
Susan
John
Dave
$\frac{5}{9} + \frac{7}{9}$
Kate
$\frac{1}{4} + \frac{2}{3}$
Lindsay
$\frac{6}{7} + \frac{4}{7}$
$\frac{3}{5} + \frac{4}{5}$
Patrick
$\frac{3}{8} + \frac{1}{8}$
I
II
III



(a) Rank 1 analysis



(b) Rank 2 analysis


C
Room 7 1.00
Room 22 1.00
Room 13 .96
Room 6 .82
Room 12 .49
c
Student 25 1.00
Student 13 .99
Student 11 .98
Student 10 .98
Student 9 .97
Student 3 .95
Student 23 .93
Student 1 .92
Student 24 .47
Student 16 .40
.78
.84
.85
.90
A
Room 9 1.00
Room 19 1.00
Room 20 .99
Room 11 .98
Room 1 .86
Room 14 .82
Room 12 .51
a
Student 7 1.00
Student 21 .99
Student 19 .99
Student 20 .98
Student 22 .97
Student 5 .96
Student 4 .93
Student 17 .90
Student 12 .87
Student 26 .87
Student 2 .65
Student 16 .60
Student 8 .54
Student 24 .52
B
Room 8 .98
Room 24 .98
Room 25 .98
Room 10 .98
Room 2 .98
Room 16 .97
Room 15 .92
Room 17 .78
b
Student 14 1.00
Student 6 1.00
Student 15 1.00
Student 18 .61
.91
.98
.36
.77
.77
.42
D
Room 3 .99
Room 18 .95
Room 23 .94
Room 21 .92
Room 4 .81
Room 5 .68
d
Student 8 .46
Student 18 .39

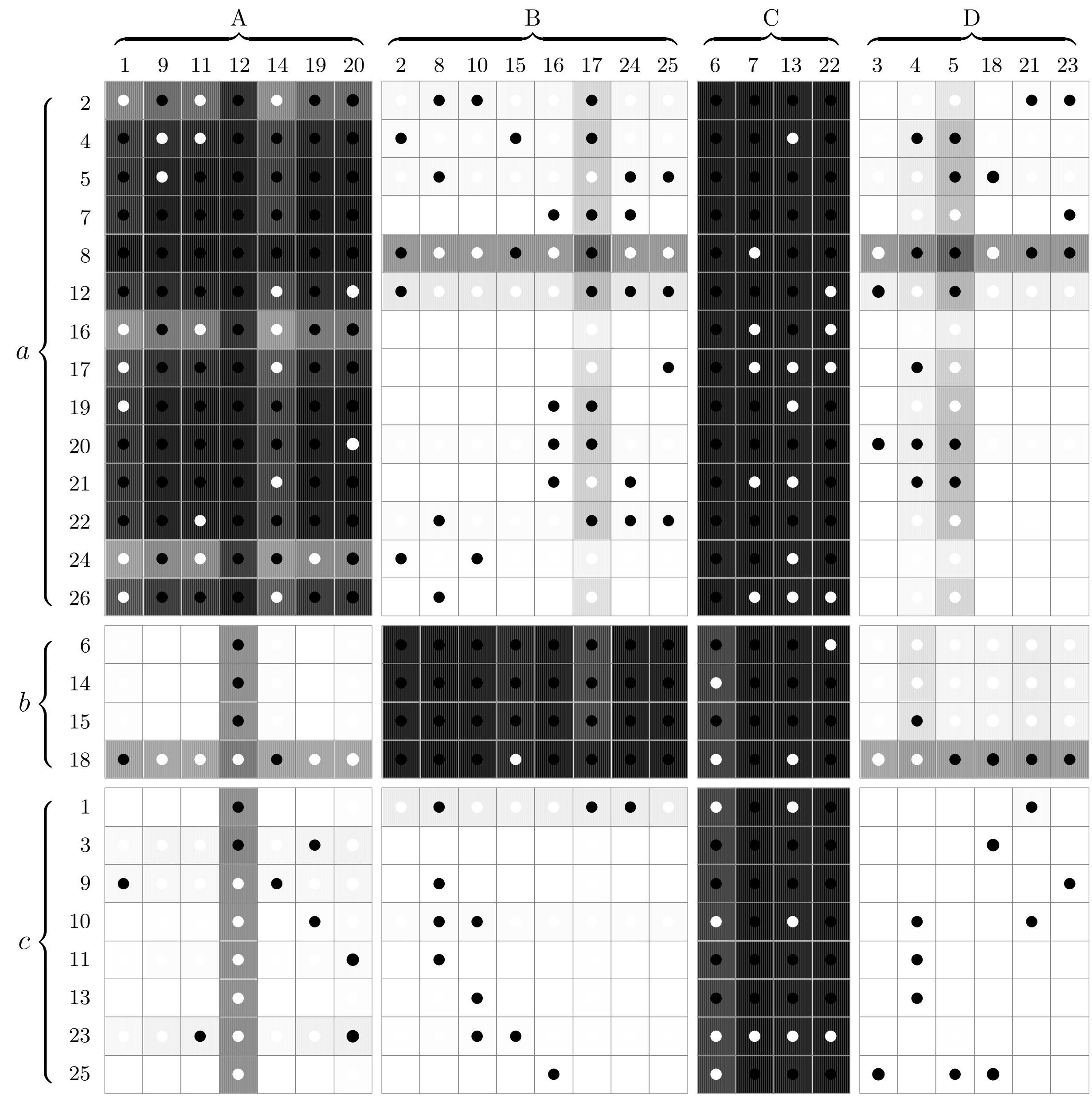

A
B
C
D
1 9 11 12 14 19 20
2 8 10 15 16 17 24 25
6 7 13 22
3 4 5 18 21 23
a
2 4 5 7 8 12 16 17 19 20 21 22 24 26
b
6 14 15 18
c
1 3 9 10 11 13 23 25

Table 1: *Hypothetical Two-Way Two-Mode Binary Matrix*

	Items								
	$\frac{5}{9}+\frac{7}{9}$	$\frac{2}{3}+\frac{1}{2}$	$\frac{3}{4}+\frac{5}{6}$	$\frac{6}{7}+\frac{4}{7}$	$\frac{3}{8}+\frac{1}{8}$	$\frac{1}{4}+\frac{2}{3}$	$\frac{2}{5}+\frac{3}{4}$	$\frac{3}{5}+\frac{4}{5}$	$\frac{2}{3}+\frac{4}{8}$
John	1	0	0	1	1	0	0	1	0
Mary	1	1	1	1	1	1	1	1	1
Lindsay	0	0	0	1	0	0	0	1	0
Elizabeth	0	1	1	1	0	1	1	1	0
Susan	0	0	0	0	1	1	0	0	0
Dave	1	0	0	1	1	0	0	1	0
Kate	0	0	0	0	0	1	0	0	0
Patrick	0	0	0	0	1	0	0	0	0

Table 2: *Hierarchical Classes Model for the Hypothethical Data in Table 1*

	Row Bundles				Column Bundles		
Rows	I	II	III	Columns	I	II	III
John	0	1	1	$\frac{5}{9}+\frac{7}{9}$	0	1	1
Mary	1	1	1	$\frac{2}{3}+\frac{1}{2}$	1	1	0
Lindsay	0	1	0	$\frac{3}{4}+\frac{5}{6}$	1	1	0
Elizabeth	1	1	0	$\frac{6}{7}+\frac{4}{7}$	0	1	0
Susan	1	0	1	$\frac{3}{8}+\frac{1}{8}$	0	0	1
Dave	0	1	1	$\frac{1}{4}+\frac{2}{3}$	1	0	0
Kate	1	0	0	$\frac{2}{5}+\frac{3}{4}$	1	1	0
Patrick	0	0	1	$\frac{3}{5}+\frac{4}{5}$	0	1	0
				$\frac{2}{3}+\frac{4}{8}$	1	1	1