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Chernoff-Savage and Hodges-Lehmann Results for Wilks' Test of Multivariate Independence

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Abstract

We show that Taskinen et al. (2003)'s normal-score rank test for multivariate independence uniformly dominates—in the Pitman sense—the classical Wilks (1935) test, which establishes the Pitman-inadmissibility of the latter. We also extend the Hodges-Lehmann (1956) result to this context, by providing, for any fixed space dimensions p, q of the marginals, the lower bound for the asymptotic relative efficiency of Taskinen et al. (2003)'s Wilcoxon type rank test with respect to Wilks' test.

Key words: Test for independence, Pitman non-admissibility, Rank-based inference, Chernoff-Savage results, Hodges-Lehmann results, Multivariate signs and ranks.
1991 MSC: 62G20.

1 Introduction.

Rank-based inference long has been considered as a somewhat heteroclite collection of "quick-and-easy" methods applicable under a broad range of assumptions, but showing poor performances when compared to their parametric competitors. This opinion was partly dispelled by two famous papers—Hodges and Lehmann (1956) and Chernoff and Savage (1958)—establishing that, contrary to widespread opinion, rank-based methods, with adequate score functions, not only compete very well, but even outperform their parametric counterparts.

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In their celebrated ".864 result", Hodges and Lehmann (1956) proved that, in the two-sample location model (but this extends to most location problems, such as one-sample, c-sample, ANOVA, regression problems, etc.), the Pitman asymptotic relative efficiency (ARE) of Wilcoxon (i.e., linear-score) rank tests with respect to their normal-theory competitors (namely, standard two-sample t-tests) is never less than .864. In other words, Wilcoxon tests, asymptotically never—that is, irrespective of the distribution of the underlying noise—need more than 14.6% observations more than t-tests to achieve the same power (see, e.g., Pratt and Gibbons (1981) for a more formal definition of ARE). No less celebrated is the Chernoff and Savage (1958) result proving the amazing fact that, in the same type of models, the ARE of van der Waerden (i.e., normal-score) rank tests, still with respect to the corresponding standard Gaussian tests, is always larger than 1, and that this minimal value is reached at Gaussian distributions only. It should be stressed, however, that these results deal with the "worst cases": both for the Wilcoxon and the van der Waerden tests, it is possible to show that there is no "best case", that is, it is possible to construct a sequence of underlying distributions along which AREs (still with respect to standard Gaussian tests) go to infinity.

One should not believe, though, that these Chernoff-Savage and Hodges-Lehmann results are just some isolated miracles, a happy accident specific to *location* problems involving *univariate* observations. Indeed, Hallin (1994) showed that the van der Waerden version of the *serial* rank tests proposed by Hallin and Puri (1994) also uniformly beats (in the Pitman sense) the corresponding everyday practice parametric Gaussian test. These serial rank tests allow for testing for randomness against serial dependence, for testing the adequacy of an ARMA model, or for testing linear restrictions on the parameter of an ARMA model. As for the extension of the Hodges-Lehmann (1956) result to this time series setup, the lower bound of the linear-score version of those tests, still with respect to the parametric Gaussian tests, was shown to be .856 by Hallin and Tribel (2000).

Extensions to (possibly serial) problems involving *multivariate* observations were recently obtained by Hallin and Paindaveine (2002a, b, and 2003), who showed that their various multivariate van der Waerden rank tests uniformly dominate the corresponding parametric Gaussian procedures in a broad class of problems (culminating in the problem of testing linear restrictions on the parameter of the multivariate general linear model with vector ARMA errors); the Pitman non-admissibility of the associated everyday practice Gaussian tests (one-sample and two-sample Hotelling tests, multivariate F-tests, multivariate Portmanteau and Durbin-Watson tests, etc.) follows. Hallin and Paindaveine (2002a) (resp., Hallin and Paindaveine (2002b)) also extended Hodges-Lehmann's result to the multivariate location (resp., serial) setup, providing, for any fixed dimension of the observations, the lower bound for the AREs of the proposed multivariate linear-score rank tests with respect to the parametric Gaussian tests.

In this paper, we consider the problem of testing for multivariate independence between two (elliptically symmetric) random vectors, and focus on the asymptotically distribution-free rank score tests recently proposed by Taskinen et al. (2003). We prove two results confirming the excellent asymptotic efficiency behavior of the van der Waerden and Wilcoxon versions of their tests. The first one is a Chernoff-Savage result, showing that the parametric Gaussian test—namely Wilks (1935) test—is uniformly dominated by their van der Waerden test, which establishes the Pitman-inadmissibility of Wilks' test (hence, in the univariate case, of the classical correlation test). The second one is an extension of the Hodges-Lehmann (1956) ".864 result", providing, for any fixed space dimensions p, q of the marginals, the lower bound for the asymptotic relative efficiency of the Taskinen et al. (2003) Wilcoxon test with respect to Wilks' test.

The paper is organized as follows. In Section 2, we define the notation to be used in the sequel, describe the problem of testing for independence between elliptically symmetric marginals, and briefly recall the rank score tests developed by Taskinen et al. (2003). In Section 3, we establish the Pitman non-admissibility of Wilks' test for multivariate independence. The analog of Hodges-Lehmann (1956)'s result for the problem under study is derived in Section 4.

2 Rank score test for multivariate independence.

2.1 Elliptical symmetry.

Recall that the distribution of a random k-vector $\mathbf{X}$ is said to be elliptically symmetric with parameters $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, and f, if and only if its pdf is given by

$$\underline{f}_{\boldsymbol{\mu},\boldsymbol{\Sigma};f}(\mathbf{x}) := c_{k,f}\,(\det\boldsymbol{\Sigma})^{-1/2} f\left(\left((\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)^{1/2}\right), \quad \mathbf{x}\in\mathbb{R}^k, \qquad (1)$$

for some k-vector $\boldsymbol{\mu}$ (the centre of the distribution), some symmetric positive definite real $k \times k$ matrix $\boldsymbol{\Sigma} = (\Sigma_{ij})$ with $\Sigma_{11} = 1$, and some function $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}^+$ such that $f > 0$ a.e. and $\mu_{k-1;f} := \int_0^\infty r^{k-1} f(r)\, dr < \infty$ ($c_{k,f}$ is a normalization factor depending on the dimension k and f). We will denote this distribution by $\mathcal{E}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}, f)$.

The *shape parameter* $\boldsymbol{\Sigma}$ determines the orientation and shape of the equidensity contours associated with $\underline{f}_{\boldsymbol{\mu},\boldsymbol{\Sigma};f}$, which are hyper-ellipsoids centered at $\boldsymbol{\mu}$. The problem of testing for multivariate independence is invariant under (block-

)affine transformations, and so are all tests considered in this paper. Therefore we can restrict—without loss of generality—to the class of centered spherical distributions, for which $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ do coincide with the origin $\mathbf{0}$ in $\mathbb{R}^k$ and the k-dimensional identity matrix $\mathbf{I}_k$, respectively.

Under $\mathcal{E}_k(\mathbf{0}, \mathbf{I}_k, f)$, the *radial function* f determines the distribution of $\|\mathbf{X}\|$. More precisely, the probability density function of $\|\mathbf{X}\|$ is $\tilde{f}_k(r) := (\mu_{k-1;f})^{-1} r^{k-1} f(r)\, I_{[r>0]}$ (I_A stands for the indicator function of the set A); denote by $\tilde{F}_k$ the corresponding distribution function.

To guarantee that (1) is a density, we need to assume that $\mu_{k-1;f} < \infty$. The classical Gaussian procedure for testing multivariate independence—Wilks (1935)'s test—requires the underlying distribution to have a finite variance; consequently, when considering AREs with respect to Wilks' test, we will restrict to radial functions satisfying the stronger condition $\mu_{k+1;f} := \int_0^\infty r^{k+1} f(r)\, dr < \infty$, under which the distribution $\mathcal{E}_k(\mathbf{0}, \mathbf{I}_k, f)$ has finite second-order moments. One can associate with each radial function f the *radial function type of* f defined as the class $\{f_a,\, a > 0\}$, where $f_a(r) := f(ar)$, for all $r > 0$. By affine-invariance, one could restrict to parameters of the form $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, f) = (\mathbf{0}, \mathbf{I}_k, f_{a_0})$ for which the variance of the associated elliptical distributions is equal to $\mathbf{I}_k$. However, it will be convenient in the sequel to consider all possible radial functions, so that we will only fix $(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (\mathbf{0}, \mathbf{I}_k)$ but let f range over its radial function type. Some extremely mild smoothness conditions on f—that we will throughout assume to be fulfilled—are required to derive AREs. We refer to Taskinen et al. (2003) for details.

The radial function f is said to be Gaussian if and only if $f = \phi_a$ for some $a > 0$, where $\phi(r) := \exp(-r^2/2)$. Under $\mathcal{E}_k(\mathbf{0}, \mathbf{I}_k, \phi)$, the pdf of $\|\mathbf{X}\|$ is $\tilde{\phi}_k(r) := (2^{(k-2)/2}\Gamma(k/2))^{-1} r^{k-1}\phi(r) I_{[r>0]}$ (where $\Gamma(.)$ stands for the Euler gamma function), and we denote by $\tilde{\Phi}_k$ the associated cdf. Under $\mathcal{E}_k(\mathbf{0}, \mathbf{I}_k, \phi)$, the distribution of $\|\mathbf{X}\|^2 = (\tilde{\Phi}_k^{-1}(U))^2$ (throughout, U stands for a random variable that is uniformly distributed over $(0, 1)$) is χ^2_k, so that the cdf of $\|\mathbf{X}\|$ is simply $\tilde{\Phi}_k(r) = \Psi_k(r^2)$, where Ψ_k denotes the distribution function of a chi-square variable with k degrees of freedom.

2.2 Testing for multivariate independence.

Consider an i.i.d. sample $(\boldsymbol{x}_{11}^T, \boldsymbol{x}_{21}^T)^T, (\boldsymbol{x}_{12}^T, \boldsymbol{x}_{22}^T)^T, \ldots, (\boldsymbol{x}_{1n}^T, \boldsymbol{x}_{2n}^T)^T$ of $(p+q)$-random vectors with the same distribution as $(\boldsymbol{x}_1^T, \boldsymbol{x}_2^T)^T$, where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are elliptically symmetric random vectors, with distribution $\mathcal{E}_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1, f)$ and $\mathcal{E}_q(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2, g)$, respectively. The problem we consider is that of testing, on the basis of this sample, the null hypothesis of independence between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. As already metioned, there is no loss—since the testing problem under study is invariant under block-diagonal affine transformations—in restricting to centered spherical marginal distributions, that is, assuming $(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) = (\mathbf{0}, \mathbf{I}_p)$ and $(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2) = (\mathbf{0}, \mathbf{I}_q)$.

The standard parametric Gaussian procedure (the likelihood ratio test) is Wilks (1935)'s test $\phi_{\mathcal{N}}$, which rejects the null (at asymptotic level α) as soon as

$$-n \log \frac{|\mathbf{S}|}{|\mathbf{S}_{11}||\mathbf{S}_{22}|} > \chi^2_{pq,1-\alpha},$$

where we write

$$\mathbf{S} := \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{pmatrix}$$

for the partitioned sample covariance matrix, and where $\chi^2_{pq,1-\alpha}$ denotes the α-upper quantile of the chi-square distribution with pq degrees of freedom.

Taskinen et al. (2003) recently proposed the following rank score competitors of Wilks' test. Define the standardized subvectors $\hat{\boldsymbol{z}}_{11}, \ldots, \hat{\boldsymbol{z}}_{1n}$ associated with original subvectors $\boldsymbol{x}_{11}, \ldots, \boldsymbol{x}_{1n}$ as

$$\hat{\boldsymbol{z}}_{1i} := \hat{\boldsymbol{\Sigma}}_1^{-1/2} \left(\boldsymbol{x}_{1i} - \hat{\boldsymbol{\mu}}_1 \right), \quad i = 1, \ldots, n,$$

where $\hat{\boldsymbol{\mu}}_1$ and $\hat{\boldsymbol{\Sigma}}_1^{-1/2}$ are chosen in such a way that the so-called *standardized spatial signs* $\hat{\boldsymbol{u}}_{1i} = \hat{\boldsymbol{z}}_{1i}/\|\hat{\boldsymbol{z}}_{1i}\|$ satisfy

$$\text{ave}_i \ \hat{\boldsymbol{u}}_{1i} = \mathbf{0} \quad \text{and} \quad \text{ave}_i \left(\hat{\boldsymbol{u}}_{1i} \hat{\boldsymbol{u}}_{1i}^T \right) = \frac{1}{p} \mathbf{I}_p;$$

denote further by $\hat{R}_{1i}$ the rank of $\|\hat{\boldsymbol{z}}_{1i}\|$ among $\|\hat{\boldsymbol{z}}_{11}\|, \ldots, \|\hat{\boldsymbol{z}}_{1n}\|$. The statistics $\hat{\boldsymbol{u}}_{2i}$ and $\hat{R}_{2i}$ are defined in the same way within the sample $\boldsymbol{x}_{21}, \ldots, \boldsymbol{x}_{2n}$. Letting $K_1, K_2 : (0,1) \to \mathbb{R}$ be two square-integrable *score functions*, the (K_1, K_2)-score version of the rank test statistics for multivariate independence proposed in Taskinen et al. (2003) is

$$T_{K_1,K_2} := \frac{npq}{\sigma^2_{K_1} \sigma^2_{K_2}} \left\| \text{ave}_i \left\{ K_1 \left(\frac{\hat{R}_{1i}}{n+1} \right) K_2 \left(\frac{\hat{R}_{2i}}{n+1} \right) \hat{\boldsymbol{u}}_{1i} \, \hat{\boldsymbol{u}}_{2i}^T \right\} \right\|^2,$$

where $\sigma^2_K := \mathrm{E}[K^2(U)]$ (U uniformly distributed over $(0,1)$) and where $\|\mathbf{A}\|^2 = \mathrm{tr}\left(\mathbf{A}\mathbf{A}^T\right)$ is the squared Frobenius norm of $\mathbf{A}$. Under the null distribution of independence (with elliptical marginals), this rank score statistic is asymptotically chi-square with pq degrees of freedom, and the associated test ϕ_{K_1,K_2} rejects the null as soon as $T_{K_1,K_2} > \chi^2_{pq,1-\alpha}$ (at asymptotic level α).

As shown in the sequel, two particular cases (corresponding to two specific types of score functions) of the above rank score tests exhibit a remarkably good uniform efficiency behaviour.

3 Pitman non-admissibility of Wilks' test.

The van der Waerden (normal-score) version of the above rank score test statistics is obtained with the score functions $K_1 = \tilde{\Phi}_p^{-1} = (\Psi_p^{-1})^{1/2}$, $K_2 = \tilde{\Phi}_q^{-1} = (\Psi_q^{-1})^{1/2}$:

$$T_{vdW} := n \left\| \text{ave}_i \left\{ \tilde{\Phi}_p^{-1} \left(\frac{\hat{R}_{1i}}{n+1} \right) \tilde{\Phi}_q^{-1} \left(\frac{\hat{R}_{2i}}{n+1} \right) \hat{\boldsymbol{u}}_{1i} \, \hat{\boldsymbol{u}}_{2i}^T \right\} \right\|^2 .$$

To compute asymptotic relative efficiencies of the van der Waerden test ϕ_{vdW} with respect to Wilks' test $\phi_{\mathcal{N}}$, a model of dependence (and related local alternatives) must be adopted. As in Gieser and Randles (1997) and Taskinen et al. (2003), we consider the local alternatives of the model of dependence generated by the transformation

$$\begin{pmatrix} \boldsymbol{y}_{1i} \\ \boldsymbol{y}_{2i} \end{pmatrix} = \begin{pmatrix} (1 - n^{-1/2}\delta)\,\mathbf{I}_p & n^{-1/2}\delta\,\mathbf{M}_1 \\ n^{-1/2}\delta\,\mathbf{M}_2 & (1 - n^{-1/2}\delta)\,\mathbf{I}_q \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{1i} \\ \boldsymbol{x}_{2i} \end{pmatrix}, \quad i = 1, \ldots, n, \qquad (2)$$

of mutually independent random vectors $\boldsymbol{x}_{1i}$, $\boldsymbol{x}_{2i}$, with respective elliptic distributions $\mathcal{E}_p(\mathbf{0}, \mathbf{I}_p, f)$ and $\mathcal{E}_q(\mathbf{0}, \mathbf{I}_q, g)$, say. Above, $\mathbf{M}_1, \mathbf{M}_2$ are fixed non-random arrays with appropriate dimensions. If one restricts to the special case for which $\mathbf{M}_1 = \mathbf{M}_2^T = \mathbf{M}$, the asymptotic relative efficiency of the rank score test ϕ_{K_1,K_2} with respect to Wilks' test does not depend on $\mathbf{M}$. In particular, as shown in Taskinen et al. (2003), the asymptotic relative efficiency of the van der Waerden test ϕ_{vdW} based on T_{vdW} with respect to Wilks' test, under the sequence of local alternatives in (2), is

$$\text{ARE}_{q,g}^{p,f}(\phi_{vdW}/\phi_{\mathcal{N}}) = \frac{1}{4p^2q^2} \Big(D_p(\phi, f) C_q(\phi, g) + D_q(\phi, g) C_p(\phi, f) \Big)^2,$$

where, denoting by $\varphi_f(r) := -f'(r)/f(r)$ the optimal location score function associated with some radial function f, we let

$$C_k(\phi, f) := \text{E}\Big[\tilde{\Phi}_k^{-1}(U)\, \varphi_f(\tilde{F}_k^{-1}(U))\Big] \quad \text{and} \quad D_k(\phi, f) := \text{E}\Big[\tilde{\Phi}_k^{-1}(U)\, \tilde{F}_k^{-1}(U)\Big].$$

Some numerical values of these AREs, under multivariate t- and normal distributions, are provided in Table 1. All these values are larger or equal than 1, and seem to be equal to 1 only if both marginals are Gaussian.

Taskinen et al. (2003) pointed out that it is remarkable that, in the multinormal case, the limiting Pitman efficiency of the van der Waerden score test ϕ_{vdW} equals that of Wilks' test. But, it is even more remarkable that, as we shall see, the multinormal case is actually the least favourable one to the above van der Waerden procedure. Theorem 1 below indeed states that, as soon as one of the marginal is not Gaussian, the van der Waerden test strictly beats Wilks' test (Table 1 provides an empirical corroboration of this result). The Pitman inadmissibility of Wilks' test follows.

Theorem 1 *For all integers $p, q \geq 1$ and all radial functions f, g satisfying $\mu_{p+1;f} < \infty$ and $\mu_{q+1;g} < \infty$, we have*

$$\mathrm{ARE}^{p,f}_{q,g}(\phi_{vdW}/\phi_{\mathcal{N}}) \geq 1,$$

where equality holds iff f and g do coincide and are Gaussian.

To prove this theorem, we need the following intermediate result; see Paindaveine (2003) for an elementary proof.

Lemma 1 *For all integer $k \geq 1$ and all radial function f satisfying $\mu_{k+1;f} < \infty$, we have $D_k(\phi, f)C_k(\phi, f) \geq k^2$, where equality holds iff f is Gaussian.*

Proof of Theorem 1. The proof is based on the decomposition

$$\Big(D_p(\phi, f)C_q(\phi, g) + D_q(\phi, g)C_p(\phi, f)\Big)^2 = A^{p,f}_{q,g} + B^{p,f}_{q,g},$$

where we let

$$A^{p,f}_{q,g} := 4\, D_p(\phi, f)C_p(\phi, f)D_q(\phi, g)C_q(\phi, g), \ \text{ and}$$

$$B^{p,f}_{q,g} := \Big(D_p(\phi, f)C_q(\phi, g) - D_q(\phi, g)C_p(\phi, f)\Big)^2.$$

It directly follows from Lemma 1 that

$$\mathrm{ARE}^{p,f}_{q,g}(\phi_{vdW}/\phi_{\mathcal{N}}) \geq \frac{1}{4p^2q^2}\, A^{p,f}_{q,g} \geq 1. \tag{3}$$

Let us now show that equality holds iff f and g do coincide and are Gaussian, i.e., are Gaussian with the same scale. For the equality to hold, we need to have $A^{p,f}_{q,g} = 4p^2q^2$ and $B^{p,f}_{q,g} = 0$. From Lemma 1, $A^{p,f}_{q,g} = 4p^2q^2$ implies that both f and g are Gaussian ($f = \phi_a$ and $g = \phi_b$, say). Now, since $D_k(\phi, \phi_a) = a^{-1}D_k(\phi) = a^{-1}k$ and $C_k(\phi, \phi_a) = aC_k(\phi) = aD_k(\phi) = ak$ for all k, we have $B^{p,\phi_a}_{q,\phi_b} = p^2q^2((b/a) - (a/b))^2$, which is equal to zero iff $a = b$. Consequently, equality holds iff $f = g = \phi_a$, for some $a > 0$. □

4 A Hodges-Lehmann result for multivariate independence.

Consider now the Wilcoxon test statistic

$$T_W := \frac{9npq}{(n+1)^4} \left\| \mathrm{ave}_i \left\{ \hat{R}_{1i} \, \hat{R}_{2i} \, \hat{\boldsymbol{u}}_{1i} \, \hat{\boldsymbol{u}}_{2i}^T \right\} \right\|^2 ,$$

which is associated with the score functions $K_1(u) = K_2(u) = u$ for all $u \in (0, 1)$. The asymptotic relative efficiency of the corresponding Wilcoxon test ϕ_W with respect to Wilks' test $\phi_{\mathcal{N}}$, under the sequence of local alternatives in (2) (still with $\mathbf{M}_1 = \mathbf{M}_2^T$), is given by

$$\mathrm{ARE}_{q,g}^{p,f}(\phi_W/\phi_{\mathcal{N}}) = \frac{9}{4pq} \Big(D_p(I,f) C_q(I,g) + D_q(I,q) C_p(I,f) \Big)^2 ,$$

where we let

$$C_k(I,f) := \mathrm{E}\Big[U \, \varphi_f(\tilde{F}_k^{-1}(U)) \Big] \quad \text{and} \quad D_k(I,f) := \mathrm{E}\Big[U \, \tilde{F}_k^{-1}(U) \Big].$$

Some numerical values of these AREs, under multivariate t- and normal distributions, are provided in Table 2. The uniformly good asymptotic efficiency behavior of the Wilcoxon test in Table 2 holds more generally, as shown by the following result which provides the lower bound of these AREs for any fixed values of the dimensions p, q of the marginals (some numerical values of this lower bound are presented in Table 3).

Theorem 2 *Let $p, q \geq 1$ be two integers. Then, letting*

$$c_k := \inf \left\{ x > 0 \;\middle|\; \left(\sqrt{x} \, J_{\sqrt{2k-1}/2}(x) \right)' = 0 \right\}, \qquad k \in \mathbb{N}_0,$$

where J_r denotes the first-kind Bessel function of order r, the lower bound for the asymptotic relative efficiency of ϕ_W with respect to $\phi_{\mathcal{N}}$, for fixed subvector dimensions p, q, is

$$\inf_{f,g} \mathrm{ARE}_{q,g}^{p,f}(\phi_W/\phi_{\mathcal{N}}) = \frac{9}{2^{10} pq c_p^2 c_q^2} \left(2\, c_p^2 + p - 1 \right)^2 \left(2\, c_q^2 + q - 1 \right)^2 , \qquad (4)$$

where the infimum is taken over the collection of radial functions f, g for which $\mu_{p+1;f} < \infty$ and $\mu_{q+1;g} < \infty$. The infimum is reached at the couples of radial functions

$$(f,g) \in \left\{ \Big(h_{p,\sigma}(r), h_{q,\sigma}(r) \Big) := \Big(h_{p,1}(\sigma r), h_{q,1}(\sigma r) \Big), \sigma > 0 \right\},$$

where $h_{k,1}$ denotes "the" radial function associated with the radial cumulative *distribution function*

$$H_{k,1}(r) := \frac{\sqrt{r}\, J_{\sqrt{2k-1}/2}(r)}{\sqrt{c_k}\, J_{\sqrt{2k-1}/2}(c_k)}\, \mathcal{I}_{[0<r\le c_k]} + \mathcal{I}_{[r>c_k]}$$

($\mathcal{I}_A$ denotes the indicator function of the set A).

To prove this theorem, we need the following result, which is established in the proof of Proposition 7 in Hallin and Paindaveine (2002b).

Lemma 2 *Let $k \geq 1$ be a fixed integer. Then,*

$$\inf_f \Big\{ D_k(I,f) C_k(I,f) \Big\} = \frac{1}{2^5 c_k^2} \left(2\, c_k^2 + k - 1\right)^2,$$

where the infimum is taken over the collection of radial functions f for which $\mu_{k+1;f} < \infty$, and the infimum is reached at the radial functions $f \in \{h_{k,\sigma}(r), \sigma > 0\}$. Moreover, letting $\omega_k := (2\, c_k^2 + k - 1)/(8c_k)$, we have $D_k(I, h_{k,\omega_k}) = 1$.

Since we have $D_k(I, f_a) = a^{-1} D_k(I,f)$ and $C_k(I,f_a) = aC_k(I,f)$ for all k, the quantity $D_k(I,f_a)C_k(I,f_a)$ does not depend on a. This allows for identifying a particular member h_{k,σ_k} of the radial function type $\{h_{k,\sigma}(r), \sigma > 0\}$ such that $D_k(I, h_{k,\sigma_k}) = 1$. According to Lemma 2, $\sigma_k = \omega_k$.

Proof of Theorem 2. Proceeding as in the proof of Theorem 1, we consider the decomposition

$$\mathrm{ARE}^{p,f}_{q,g}(\phi_W/\phi_{\mathcal{N}}) = A^{p,f}_{q,g} + B^{p,f}_{q,g},$$

where

$$A^{p,f}_{q,g} := \frac{9}{pq} D_p(I,f)C_p(I,f)D_q(I,g)C_q(I,g),$$
$$B^{p,f}_{q,g} := \frac{9}{4pq}\Big(D_p(I,f)C_q(I,g) - D_q(I,g)C_p(I,f)\Big)^2.$$

Lemma 2 directly yields that, for all couple (f,g) of radial functions,

$$\mathrm{ARE}^{p,f}_{q,g}(\phi_W/\phi_{\mathcal{N}}) \geq A^{p,f}_{q,g} \geq \frac{9}{2^{10} pq c_p^2 c_q^2}\left(2\, c_p^2 + p - 1\right)^2 \left(2\, c_q^2 + q - 1\right)^2. \quad (5)$$

We now show that the right hand side in (5) does actually coincide with the infimum, by determining the (non-empty) collection of couples (f,g) achieving the bound in (4). For the couple (f,g) to achieve the bound, we only need equalities in (5) to hold, i.e., we need to have

$$A^{p,f}_{q,g} = \frac{9}{2^{10} pq c_p^2 c_q^2}\left(2\, c_p^2 + p - 1\right)^2 \left(2\, c_q^2 + q - 1\right)^2, \quad \text{and} \quad (6)$$
$$B^{p,f}_{q,g} = 0. \quad (7)$$

Lemma 2 shows that (6) holds if and only if

$$(f, g) \in \Big\{ \Big(h_{p,a}(r), h_{q,b}(r) \Big) := \Big(h_{p,1}(ar), h_{q,1}(br) \Big), a, b > 0 \Big\}.$$

Now, using the fact that $D_k(I, h_{k,1}) = \omega_k$ for all k, we have

$$\begin{aligned} B^{p,h_{p,a}}_{q,h_{q,b}} &= \frac{9}{4pq} \Big(\frac{b}{a} D_p(I, h_{p,1}) C_q(I, h_{q,1}) - \frac{a}{b} D_q(I, h_{q,1}) C_p(I, h_{p,1}) \Big)^2 \\ &= \frac{9}{4pq} \Big(\frac{\omega_p b}{a} C_q(I, h_{q,1}) - \frac{\omega_q a}{b} C_p(I, h_{p,1}) \Big)^2 \\ &= \frac{9}{4pq} \Big(\frac{\omega_p b}{\omega_q a} D_q(I, h_{q,1}) C_q(I, h_{q,1}) - \frac{\omega_q a}{\omega_p b} D_p(I, h_{p,1}) C_p(I, h_{p,1}) \Big)^2 \\ &= 0 \end{aligned}$$

if and only if

$$\Big(\frac{\omega_q a}{\omega_p b} \Big)^2 = \frac{D_q(I, h_{q,1}) C_q(I, h_{q,1})}{D_p(I, h_{p,1}) C_p(I, h_{p,1})} = \frac{c_p^2 (2\, c_q^2 + q - 1)^2}{c_q^2 (2\, c_p^2 + p - 1)^2} = \Big(\frac{\omega_q}{\omega_p} \Big)^2,$$

i.e., if and only if $a = b$. □

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		ν_p							ν_p				
q	ν_q	3	4	6	12	∞	q	ν_q	3	4	6	12	∞
1	3	1.378	1.295	1.266	1.281	1.339	4	3	1.430	1.332	1.292	1.298	1.348
	4	1.293	1.190	1.141	1.135	1.167		4	1.336	1.223	1.167	1.156	1.183
	6	1.267	1.144	1.078	1.054	1.067		6	1.294	1.165	1.096	1.069	1.080
	12	1.285	1.141	1.058	1.019	1.016		12	1.295	1.149	1.064	1.024	1.020
	∞	1.343	1.174	1.072	1.017	1.000		∞	1.343	1.174	1.072	1.017	1.000
2	3	1.400	1.311	1.277	1.289	1.343	6	3	1.448	1.345	1.301	1.304	1.351
	4	1.311	1.204	1.152	1.144	1.174		4	1.353	1.236	1.177	1.163	1.189
	6	1.277	1.152	1.085	1.060	1.072		6	1.306	1.175	1.103	1.075	1.085
	12	1.289	1.144	1.060	1.021	1.017		12	1.300	1.153	1.068	1.027	1.023
	∞	1.343	1.174	1.072	1.017	1.000		∞	1.343	1.174	1.072	1.017	1.000
3	3	1.417	1.323	1.286	1.294	1.346	10	3	1.471	1.361	1.312	1.311	1.353
	4	1.325	1.214	1.161	1.150	1.179		4	1.375	1.252	1.190	1.173	1.196
	6	1.286	1.159	1.091	1.065	1.076		6	1.323	1.188	1.114	1.084	1.092
	12	1.292	1.146	1.062	1.023	1.019		12	1.308	1.159	1.073	1.032	1.027
	∞	1.343	1.174	1.072	1.017	1.000		∞	1.343	1.174	1.072	1.017	1.000

Table 1
AREs of the van der Waerden rank score test ϕ_{vdW} for multivariate independence with respect to Wilks' test $\phi_{\mathcal{N}}$, under standard multivariate Student (with 3, 4, 6, and 12 degrees of freedom) and standard Gaussian densities, for subvector dimensions $p = 2$ and $q = 1, 2, 3, 4, 6,$ and 10, respectively.

		ν_p							ν_p				
q	ν_q	3	4	6	12	∞	q	ν_q	3	4	6	12	∞
1	3	1.305	1.227	1.193	1.193	1.222	4	3	1.248	1.192	1.175	1.191	1.235
	4	1.239	1.147	1.099	1.085	1.098		4	1.174	1.101	1.068	1.066	1.090
	6	1.208	1.104	1.044	1.018	1.018		6	1.149	1.059	1.011	0.993	1.002
	12	1.204	1.086	1.015	0.978	0.969		12	1.165	1.056	0.992	0.961	0.955
	∞	1.219	1.087	1.006	0.959	0.940		∞	1.228	1.095	1.013	0.966	0.947
2	3	1.305	1.237	1.211	1.219	1.257	6	3	1.211	1.161	1.150	1.168	1.215
	4	1.237	1.152	1.111	1.102	1.121		4	1.134	1.067	1.038	1.039	1.066
	6	1.211	1.111	1.055	1.033	1.037		6	1.105	1.022	0.978	0.963	0.974
	10	1.219	1.102	1.033	0.997	0.989		12	1.122	1.019	0.959	0.930	0.927
	∞	1.257	1.121	1.037	0.989	0.970		∞	1.198	1.068	0.988	0.943	0.924
3	3	1.274	1.213	1.193	1.206	1.248	10	3	1.173	1.129	1.121	1.144	1.193
	4	1.203	1.125	1.089	1.084	1.106		4	1.090	1.029	1.005	1.009	1.038
	6	1.179	1.084	1.032	1.013	1.020		6	1.056	0.979	0.940	0.929	0.941
	12	1.192	1.079	1.013	0.980	0.973		12	1.069	0.973	0.918	0.892	0.891
	∞	1.245	1.110	1.027	0.979	0.960		∞	1.158	1.033	0.955	0.911	0.893

Table 2
AREs of the Wilcoxon the rank score test ϕ_W for multivariate independence with respect to Wilks' test $\phi_{\mathcal{N}}$, under multivariate Student (with 3, 4, 6, and 12 degrees of freedom) and normal densities, for subvector dimensions $p = 2$ and $q = 1, 2, 3, 4, 6,$ and 10, respectively

p/q	1	2	3	4	6	10	∞
1	0.856	0.884	0.867	0.850	0.826	0.797	0.694
2		0.913	0.895	0.878	0.853	0.823	0.717
3			0.878	0.861	0.836	0.807	0.703
4				0.845	0.820	0.792	0.689
6					0.797	0.769	0.669
10						0.742	0.646
∞							0.563

Table 3
Some numerical values, for various values of the dimensions p, q of the subvectors, of the lower bound for the asymptotic relative efficiency of the Wilcoxon rank score test ϕ_W for multivariate independence with respect to Wilks' test $\phi_{\mathcal{N}}$.