T E C H N I C A L
R E P O R T

0433

ON AGGREGATION OF SCALE ELASTICITIES ACROSS FIRMS

FARE, R. and V. ZELENYUK

I A P S T A T I S T I C S
N E T W O R K

INTERUNIVERSITY ATTRACTION POLE

http://www.stat.ucl.ac.be/IAP

ON AGGREGATION OF SCALE ELASTICITIES ACROSS FIRMS

Rolf Färe[*] and Valentin Zelenyuk[**]

Abstract

In this paper we propose new aggregate or 'group' primal and dual *scale elasticity* measures of an economic system (e.g., industry). The new aggregate measures are the *weighted* averages of individual scale elasticities. Remarkably, the aggregation function and the aggregation weights are *not* ad hoc but derived from economic theory—duality theory of Shephard (1953) and the aggregation theory of Koopmans (1957), Färe, Grosskopf and Zelenyuk (2002), and Färe and Zelenyuk (2003).

Key Words: Scale Elasticity, Aggregation, Duality

JEL Code: D24

April 15, 2004

[*] — Department of Economics and Department of Agricultural and Resource Economics, Oregon State University (Corvallis, OR 97331, USA.), USA. e-mail: rolf.fare@orst.edu

****** — Institut de Statistique, Université Catholique de Louvain, 20 voie du Roman Pays, B 1348, Louvain-la-Neuve, Belgium; and UPEG at EROC and EERC Kiev, Ukraine. e-mail: zelenyuk@stat.ucl.ac.be

Acknowledgement: We would like to thank Tome Coupé, Roy Gardner and Volodymyr Vakhitov for their comments. Research support from the "Interuniversity Attraction Pole", Phase V (No.P5/24) from the Belgian Government (Belgian Science Policy) is acknowledged. All views expressed remains responsibility of the authors.

Introduction

Applied economists are often interested in measuring economies of scale present in an industry that they analyze, and are often resorting to estimating such measure as scale elasticity. To infer on the economies of scale for the industry (or its representative sample) as a whole, researchers often estimate the elasticity *at* the mean of the data or, alternatively, take the (equally-weighted) mean of the *individual* (for each observation) estimates of scale elasticities. Both measures in general are not equal but both have some theoretical appeal. In this study we propose another theoretical measure of the scale elasticity of a *group* that is based on the aggregation result derived from economic principles.

1. Technology Characterization

Consider an example of an industry where each firm k (k = *1, 2, … ,* K) uses vector of N inputs $x^k = (x_1^k,...,x_N^k)^T \in \Re_+^N$ to produce vector of M *outputs*, which we denote by $y^k = (y_1^k,..., y_M^k)^T \in \Re_+^M$. Following Shephard (1953) and Färe and Primont (1995), suppose technology of firm k can be characterized by the input requirement set,

$$L^k(y^k) \equiv \{x: \ x \ can \ produce \ y^k\}, \qquad y^k \in \Re_+^M \tag{2.1}$$

which we assume satisfies the standard regularity axioms of production theory (see Färe and Primont (1995) for details), so that this technology can be characterized (dually) via the *cost* function,

$$C^k(y^k, \ w) \equiv \min_x \{wx \quad : \quad x \in L^k(y^k)\}, \tag{2.2}$$

where $w \equiv (w_1,...,w_N) \in \Re_{++}^N$ denotes the vector of input prices. In a single-output case (M=*1*) technology of firm k can also be characterized by the *production function*. When M>*1*, the *input orientated* Shephard's (1953) distance function $D_i^k : \Re_+^M \times \Re_+^N \to \Re_+^1 \cup \{\infty\}$, defined as

$$D_i^k(y^k, x^k) \equiv \sup_\theta \{\theta > 0: \ (x^k / \theta) \in L^k(y^k)\}, \tag{2.3}$$

gives an alternative (primal) complete characterization of the technology of firm k, in the sense that

$$D_i^k(y^k, x^k) \geq 1 \quad \Leftrightarrow \quad x^k \in L^k(y^k). \tag{2.4}$$

Both the *dual*, given in (2.2), and the *primal*, given in (2.3), characterizations are often used in measuring properties of a technology. The focus of our paper is on the *scale economies*, which can be inferred from the notion of *scale elasticity*, defined for the *dual* and *primal* frameworks as (Färe and Primont, 1995):

$$e_c(y^k, w) \equiv \left.\frac{\partial \ln C^k(y^k\theta, w)}{\partial \ln \theta}\right|_{\theta=1} = \frac{\nabla^T_{y^k} C^k(y^k, w)\, y^k}{C^k(y^k, w)} \quad , \tag{2.5}$$

and

$$e_i(y^k, x^k) \equiv \left.\frac{\partial \ln \lambda}{\partial \ln \theta}\right|_{\theta=1,\lambda=1} = -\nabla^T_{y^k} D^k_i(y^k, x^k)\, y^k \ \text{ such that } D^k_i(y^k\theta, x^k\lambda) = 1 \tag{2.6}$$

respectively. Letting x^{*k} be a solution to (2.2), we get the dual and primal measures being equal, i.e.,

$$e_c(y^k, w) = e_i(y^k, x^{*k}), \qquad x^{*k} \equiv \arg\min_x \{wx \quad : \quad x \in L^k(y^k)\}. \tag{2.7}$$

This result states that the same information about the scale elasticity of an *individual* firm k can be obtained from primal and dual approaches. In the next section we show analogous results for the measures of scale elasticity of a *group* and its relationship to these individual measures.

3. Aggregation of Scale Elasticities

Define the *group* (e.g., industry, sub-industry, etc) input requirement set to be

$$\overline{L}(y^1,..., \ y^K) = \sum_{k=1}^{K} L^k(y^k). \tag{3.1}$$

The *group* cost function, which is the *group* analog of the (2.2), is then can be defined as

$$\overline{C}(y^1,..., y^K, \ w) \equiv \min_x \{wx \quad : \quad x \in \overline{L}(y^1,..., y^K)\}, \tag{3.2}$$

and the *group* input oriented distance function—the *group* analog of (2.3)—can be defined as

$$\overline{D}_i(y^1,..., y^K, \sum_{k=1}^{K} x^k) \equiv \sup_\theta \{\theta > 0 : \ (\sum_{k=1}^{K} x^k / \theta) \in \overline{L}(y^1,..., y^K)\}. \tag{3.3}$$

The *scale economies* for the *group* can be inferred from measures of scale elasticities defined for the *aggregate* technology in the same manner as for the individual technologies, e.g., for the dual framework, we have

$$\overline{E}_c(y^1,..., y^K, w) \equiv \left.\frac{\partial \ln \overline{C}^k(y^1\theta,..., y^K\theta, w)}{\partial \ln \theta}\right|_{\theta=1} = \frac{\nabla^T_Y \overline{C}(y^1,..., y^K, w) Y}{\overline{C}(y^1,..., y^K, w)}, \tag{3.4}$$

where $\nabla^T_Y \overline{C}(y^1,..., y^K, w) \equiv (\partial \overline{C}(y^1,..., y^K, w)/\partial y^1, \dots, \partial \overline{C}(y^1,..., y^K, w)/\partial y^K)$, and $Y \equiv (y^1,..., y^K)^T$.

On the other hand, for the primal framework, given $\overline{D}_i(y^1\theta,..., y^K\theta, \sum_{k=1}^{K} x^k\lambda) = 1$, we have

$$\overline{E}_i(y^1,\ldots,y^K,\sum_{k=1}^{K} x^k) \equiv \left.\frac{\partial \ln \lambda}{\partial \ln \theta}\right|_{\theta=1,\lambda=1} = -\nabla_Y^T \overline{D}_i(y^1,\ldots,y^K,\sum_{k=1}^{K} x^k)Y\ . \tag{3.5}$$

Let x^* be a solution to (3.2), then the dual and primal measures of group scale elasticities are equal, i.e.,

$$\overline{E}_c(y^1,\ldots,y^K,w) = \overline{E}_i(y^1,\ldots,y^K,x^*), \tag{3.6}$$

where

$$x^* \equiv \arg\min_x \{wx \quad : \quad x \in \overline{L}\ (y^1,\ldots,y^K)\}\ .$$

This is an analog to the relationship (2.7) that was derived for the individual level by Färe and Primont (1995) and for the group level it is established in a similar manner.

We now want to establish the relationship between the aggregate (group) and the individual scale elasticities and the key instrument for this would be the following theorem.

Theorem. Given definitions (2.1), (2.2), (3.1) and (3.2), we have

$$\overline{C}(y^1,\ldots,\ y^K,\ w) = \sum_{k=1}^{K} C^k(y^k,w)\ . \tag{3.7}$$

In words, the industry cost function is the sum of individual cost functions of all firms in this industry. This theorem is from Färe, Grosskopf and Zelenyuk (2002) and is the cost analog of the Koopmans (1957) theorem for profit functions. (Färe and Zelenyuk (2003) provide revenue analog.)

While treating expression (3.7) as an identity, we can differentiate both sides of it along the ray from the origin through the point $Y \equiv (y^1,\ldots,y^K)^T$ (i.e., looking at the change in costs due to infinitesimal *equi-proportional* change of *all* outputs) and obtain the desired aggregation result. Specifically, such differentiation of the l.h.s. of (3.7) along the ray gives

$$\left.\frac{\partial \overline{C}(y^1\theta,\ldots,\ y^K\theta,\ w)}{\partial \theta}\right|_{\theta=1} = \nabla_Y^T \overline{C}(y^1,\ldots,\ y^K,\ w)Y\ . \tag{3.8}$$

And, differentiating the r.h.s. along the same ray gives

$$\left.\partial\left(\sum_{k=1}^{K} C^k(y^k\theta,w)\right)\Big/\partial\theta\right|_{\theta=1} = \sum_{k=1}^{K} \nabla_{y^k}^T C^k(y^k,w)\,y^k \quad . \tag{3.9}$$

Putting the two sides together yields the desired result:

$$\overline{E}_c(y^1,\ldots, y^K, w) = \sum_{k=1}^{K} e_c(y^k, w)\cdot S^k, \qquad S^k \equiv C^k(y^k, w)\Big/\sum_{k=1}^{K} C^k(y^k, w) \qquad (3.10)$$

This mathematical fact has quite intuitive economic appeal: The dual scale elasticity of the group can be obtained as the *weighted* sum of the *individual* dual scale elasticities of all firms in this group, where the weights are the cost shares. We would like to emphasize that the weights here are not ad hoc, but derived from economic criterion—agents' optimization behavior.

We can also obtain the aggregation result for the *primal* measures of scale elasticities (based on the group and individual input distance functions). By recalling (2.7) and (3.6), we have

$$\overline{E}_i(y^1,\ldots, y^K, x^*) = \sum_{k=1}^{K} e_i(y^k, x^{*k})\cdot S^k, \qquad S^k \equiv C^k(y^k, w)\Big/\sum_{k=1}^{K} C^k(y^k, w). \qquad (3.11)$$

where $x^* \equiv \arg\min_x \{wx \ : \ x \in \overline{L}(y^1,\ldots, y^K)\}$, and $x^{*k} \equiv \arg\min_x \{wx \ : \ x \in L^k(y^k)\}$.

The last result gives us a practical way of obtaining the (primal) *group* scale elasticity measure immediately from the (primal) *individual* scale elasticity measures. Specifically, it states that the primal group (aggregate) scale efficiency measure can be obtained by taking the *weighted* arithmetic average of individual scale elasticities of all firms in this group, where the weights are the cost shares—the same as in the dual framework, and again, are not ad hoc but derived from economic principles.

When the primal information is used, a researcher may have no price information available to obtain the cost functions used to compute the weights. A feasible option then might be to use the shadow prices, estimated from the primal information. Alternatively, the unavailability of price information can be circumvented by deriving the price independent weights in the manner similar to Färe and Zelenyuk (2003). This would require imposing additional standardization,

$$w_i \sum\nolimits_{k=1}^{K} x_i^{*k} \Big/ \left(\sum\nolimits_{i=1}^{N} w_i \sum\nolimits_{k=1}^{K} x_i^{*k}\right) = a_i, \qquad i = 1, \ldots, N, \qquad (3.12)$$

where a_i is a known (or estimated) constant between zero and unity. Intuitively, (3.12) says that the share of the *industry* expenditures on input *i* in the industry total cost equals a_i. Now, letting $\varpi_i^k = x_i^k / \sum\nolimits_{k=1}^{K} x_i^k$ be the share of *k's* firm in the group in terms of i^{th}-input, and using (3.11) with the standardization (3.12) we obtain the price-independent weights, with an intuitive economic meaning,

$$S^k = \sum\nolimits_{i=1}^{N} \varpi_i^k a_i, \qquad k = 1, \ldots, K. \qquad (3.13)$$

Intuitively, (3.13) says that firm's weight is the weighted average over all input-shares of this firm in the group, where the weights are the shares of the industry expenditures on input i in the industry total cost.

Conclusion

In this paper we introduced the *aggregate* primal and dual measures of scale elasticities and showed that they can be obtained by weighted aggregation of *individual* scale elasticities. Specifically, when economic agents follow cost minimizing behavior, the weights would be the cost shares. We also suggested price independent weights—when dual (price or cost) information is not available. Similar analysis can be applied to the context of the revenue and the profit maximization, in which case the output distance function and the directional distance functions, respectively, might be used to characterize technology (These results can be derived from the aggregation theorems of Färe and Zelenyuk (2003) and Färe, Grosskopf and Zelenyuk (2002)). These results can also be extended to the case of aggregation *within* distinct *sub-groups* (e.g., public and private, regulated and non-regulated, etc) and then aggregation *between* these sub-groups into a larger group (For the related theorem, see Simar and Zelenyuk (2003)). Finally, similar analysis can be applied for other 'derivatives' of the cost, revenue and profit functions—to obtain aggregation of demand and supply functions (using Shephard's/Hotelling lemmas), which we have not focused on in this paper. Overall, the aggregate measures of scale elasticity that account for importance of each economic agent via weights derived from economic criterion, which we proposed, shall serve as useful measures for empirical researchers while analyzing and presenting their results.

References

Färe, R., S. Grosskopf and V. Zelenyuk (2002), "Aggregation of Cost Efficiency Indicators and Indexes Across Firms", *Scientific Report 2002:*1, R.R. Institute of Applied Economics, Sweden, also *forthcoming* in *Academia Economic Papers* (Academia Sinica, Taiwan).

Färe, R. and D. Primont (1995), *Multi-Output Production and Duality: Theory and Applications*, Kluwer Academic Publishers, Boston.

Färe, R. and V. Zelenyuk (2003), "On Aggregate Farrell Efficiency Scores," *European Journal of Operations Research* 146:3, 615-620.

Koopmans, T.C. (1957), *Three Essays on the State of Economic Analysis*, McGraw-Hill, New York.

Shephard (1970), *Theory of Cost and Production Function*s, Princeton: Princeton University Press.

Simar L. and V. Zelenyuk (2003), "Statistical Inference For Aggregates Of Farrell-Type Efficiencies," Discussion Paper (0324) of Institute of Statistics, University Catholique de Louvain.