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A PICKANDS TYPE ESTIMATOR OF THE EXTREME VALUE INDEX

L. GARDES and S. GIRARD

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Abstract

One of the main goals of extreme value analysis is to estimate the probability of rare events given a sample from an unknown distribution. The upper tail behavior of this distribution is described by the extreme value index. We present a new estimator of the extreme value index adapted to any domain of attraction. Its construction is similar to the one of Pickands' estimator. Its weak consistency and its asymptotic distribution are established and a bias reduction method is proposed. Our estimator is compared with classical extreme value index estimators through a simulation study.

Contact information

Laurent Gardes and Stéphane Girard: SMS/LMC, Université Grenoble I, BP 53,
38041 Grenoble cedex 9, France. `Laurent.Gardes@imag.fr`, `Stephane.Girard@imag.fr`

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1 Introduction

Suppose one is given a sequence $X_1, \dots, X_n$ of independent and identically distributed (i.i.d.) observations from some distribution function F. Suppose there exist sequences $a_n > 0$ and b_n and some $\xi \in \mathbf{R}$ such that:

$$\lim_{n\to\infty} \mathrm{P}\left[\frac{\max(X_1,\dots,X_n) - b_n}{a_n} \le x\right] = G_\xi(x), \tag{1}$$

with $G_\xi(x) = \exp[-(1+\xi x)_+^{-1/\xi}]$ if $\xi \neq 0$ and $G_0(x) = \exp[-\mathrm{e}^{-x}]$, where $y_+ = \max(0, y)$. Necessary and sufficient conditions on F for the convergence (1) to the extreme value distribution G_ξ can be found in [15]. The aim of this paper is the definition of a new estimator of the extreme value index $\xi \in \mathbf{R}$. This parameter drives the decay of the tail distribution: as a power function if $\xi > 0$ (Pareto, Burr, Student's, Log-gamma distributions, etc ...), exponentially if $\xi = 0$ (Exponential, Normal, Log-normal, Gamma distributions, etc ...) and with finite right endpoint if $\xi < 0$ (Uniform, Beta, Reversed Pareto, Reversed Burr distributions, etc ...). The knowledge of ξ is for example of high interest for extreme quantile estimation which arises in a lot of applications [12] such as finance, insurance, hydrology, etc ... There is a substantial number of publications dedicated to the estimation of this extreme value index, especially on the heavy tailed distribution context ($\xi > 0$) (see Beirlant et *al.* [3], Feueverger and Hall [13] and, for a recent overview of this literature, see Csörgo and Viharos [6]). The most popular estimator in this case is the Hill estimator [19] defined by:

$$\hat\xi^H_{k,n} = \frac{1}{k}\sum_{i=1}^{k} \ln(X_{n-i+1,n}) - \ln(X_{n-k,n}), \text{ for } k = 1,\dots,n-1,$$

where $X_{1,n} \le \dots \le X_{n,n}$ correspond to the random variables $X_1, \dots, X_n$ rearranged in ascending order. The consistency and the asymptotic normality of this estimator are proved for example by Davis and Resnick [8], Csörgo and Mason [5], etc ...

The general case $\xi \in \mathbf{R}$ has been less extensively studied. Dekkers, Einmahl and de Haan [11] have adapted the estimator proposed by Hill to this situation. Another estimator was proposed by Pickands [20]:

$$\hat\xi^P_{k,n} = \frac{1}{\ln(2)} \ln\left(\frac{X_{n-k+1,n} - X_{n-2k+1,n}}{X_{n-2k+1,n} - X_{n-4k+1,n}}\right), \text{ for } k = 1,\dots,\lfloor n/4\rfloor,$$

where $\lfloor x\rfloor$ denotes the integer part of x. Weak and strong consistency as well as asymptotic normality of $\hat\xi^P_{k,n}$ were established by Dekkers and de Haan [10]. Proofs are based on the following well known result: Let U be the tail quantile function of the distribution function F defined by

$$U(x) = \left(\frac{1}{1-F(x)}\right)^{\leftarrow}$$

(the arrow means inverse function) and let

$$\varphi_t(x) = \int_1^x u^{t-1} du, \ x > 0, \ t \in \mathbf{R}.$$

Relation (1) holds if and only if (see de Haan [9]), there exist a positive measurable function a such that uniformly locally on $x > 0$,

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{a(t)} = \varphi_\xi(x). \tag{2}$$

Clearly, relation (2) implies that uniformly locally on $x, y > 0$, $y \neq 1$,

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{U(ty) - U(t)} = \frac{\varphi_\xi(x)}{\varphi_\xi(y)}. \tag{3}$$

Thus, by substituting in (3) U by $\hat{U}_n = [1/(1-\hat{F}_n)]^{\leftarrow}$ ($\hat{F}_n$ denoting the empirical distribution function), t by $n/(2k)$, x by $1/2$ and y by 2, and remarking that $\hat{U}_n(n/k) = X_{n-k+1,n}$, we have asymptotically

$$2^{-\xi} \frac{X_{n-k+1,n} - X_{n-2k+1,n}}{X_{n-2k+1,n} - X_{n-4k+1,n}} = 1. \tag{4}$$

Pickands' estimator $\hat{\xi}^P_{k,n}$ is the solution of the equation (4). One can notice that this estimator does not take into account of the extreme observations $X_{n-k+2,n}, \dots, X_{n,n}$.

In the next section, we define a new estimator of the extreme value index ξ when $\xi \in \mathbf{R}$. This estimator is similar to the one of Pickands but exploiting the information given by the spacing between $X_{n-k+1,n}$ and $X_{n,n}$. Weak consistency and asymptotic distribution are established in section 3 and a bias corrected estimator is introduced. Section 4 is devoted to the proofs of the main results and a simulation study is presented in section 5.

2 Estimation of the extreme value index

We propose to estimate the extreme value index $\xi \in \mathbf{R}$ by $\hat{\xi}_{k,n}$ defined as the root of the equation in θ:

$$\left\{\frac{\varphi_\theta(1/k')}{\varphi_\theta(1/k)}\right\} \frac{X_{n-k+1,n} - X_{n,n}}{X_{n-k'+1,n} - X_{n,n}} = 1, \text{ for } 1 < k' < k < n. \tag{5}$$

We can show (see Gardes [14], Appendix B) that (5) admits an unique solution. This estimator applies to all real ξ and, as Pickands' estimator, remains unaffected when the scale or location of the data are changed. Furthermore, as we will see on a simulation study, the behavior of $\hat{\xi}_{k,n}$ is less influenced by the parameter k than Pickands' estimator. One can justify the definition of $\hat{\xi}_{k,n}$ by the two following lemmas:

Lemma 1 Suppose that relation (1) holds. Then, if $\xi < 0$,

$$\lim_{t\to\infty,\ x\to\infty} \frac{U(tx) - U(t)}{a(t)} = -\frac{1}{\xi}$$

and if $\xi \geq 0$,

$$\lim_{t\to\infty,\ x\to\infty} \frac{U(tx) - U(t)}{a(t)} = +\infty.$$

As a consequence of Lemma 1, we have:

Lemma 2 Suppose that relation (1) holds. Then,

$$\left\{\frac{\varphi_\xi(y)}{\varphi_\xi(x)}\right\} \frac{U(tx) - U(t)}{U(ty) - U(t)} \to 1, \tag{6}$$

as $t \to \infty$, $x \to 0$ with $ty \to \infty$ and $x/y \to d > 0$.

Lemma 1 and Lemma 2 can be seen as an extension of respectively (2) and (3) when x and y are going to zero or infinity. The proofs of Lemma 1 and Lemma 2 are postponed to the Appendix. By substituting in (6) U by $\hat{U}_n$, t by n, x by $1/k'(n) = 1/k'$ and y by $1/k(n) = 1/k$ with $k/k' \to c > 1$, $k \to \infty$ and $k/n \to 0$ as $n \to \infty$, we have asymptotically:

$$\left\{ \frac{\varphi_\xi(1/k')}{\varphi_\xi(1/k)} \right\} \frac{X_{n-k+1,n} - X_{n,n}}{X_{n-k'+1,n} - X_{n,n}} = 1,$$

which is an intuitive justification for the definition of $\hat{\xi}_{k,n}$. The next section is dedicated to the study of $\hat{\xi}_{k,n}$ asymptotical properties.

3 Main results

3.1 Asymptotic properties

We first state the weak consistency of $\hat{\xi}_{k,n}$ under some conditions on k and k'.

Theorem 1 Suppose that relation (1) holds. If $k/k' \to c > 1$, $k \to \infty$ and $k/n \to 0$ as $n \to \infty$, then $\hat{\xi}_{k,n} \xrightarrow{\mathrm{P}} \xi$.

Remark 1 Similar conditions on k are used by Dekkers and de Haan [10] to prove that Pickands' estimator $\hat{\xi}^P_{k,n}$ is weakly consistent.

To establish the asymptotic distribution of the estimator $\hat{\xi}_{k,n}$, additionnary conditions are introduced. The first of them is a cornerstone in all proofs of asymptotic normality for extreme value estimators.

$(H1)$ – U has a positive derivative and there exist a slowly varying function ℓ such that $U'(x) = x^{\xi-1}\ell(x)$.

We refer to [4] for more details on slow variation theory. The next condition controls the uniform rate of convergence of $\ell(tx)/\ell(x)$ to 1 as $x \to \infty$. Let $\delta = \min(-\xi, 1/2)$ and introduce the random variables $K_{k,n} = \bar{F}(X_{n-k+1,n})/\bar{F}(X_{n,n})$ and $N_n = 1/\bar{F}(X_{n,n})$ where $\bar{F}$ is the survival function ($\bar{F} = 1 - F$).

$(H2)$ –

$$\varphi_\delta(k') \sup_{t \in [1, K_{k',n}]} \left| \frac{\ell(tN_n/K_{k',n})}{\ell(N_n/K_{k',n})} - 1 \right| \xrightarrow{\mathrm{P}} 0.$$

Our second main result is the following:

Theorem 2 Let $V_k(\xi) = \varphi_\delta(k)[(\ln(k)-1)\mathbf{I}\{\xi \geq 0\}+1]$. Under the conditions of Theorem 1 (with $k = ck'$) and if $(H1)$ and $(H2)$ are satisfied, we have for all $t \in \mathbf{R}$:

$$\lim_{n\to\infty} \mathrm{P}[V_k(\xi)(\hat{\xi}_{k,n} - \xi) \leq t] = \begin{cases} \exp(-\mathrm{e}^{-t}) & \text{if } 0 < \xi, \\ \exp(-\mathrm{e}^{-t/2}) & \text{if } \xi = 0, \\ \exp\left[-\left[1 + t\ln(c)/\varphi_\xi(1/c)\right]^{-1/\xi}\right] & \text{if } -1/2 < \xi < 0, \\ \Phi\left[-tc^{-\xi}\ln(c)/(2\xi\sigma)\right] & \text{if } \xi < -1/2, \end{cases} \quad (7)$$

where $\sigma = c^{-\xi}(c-1)^{1/2}$ and Φ is the cumulative distribution function of the standard normal distribution.

Remark 2

i) Theorem 2 states that the asymptotic distribution of $\hat{\xi}_{k,n}$ is Gaussian if $\xi < -1/2$ and an extreme value distribution if $\xi > -1/2$. If $\xi = -1/2$, we prove that $V_k(\xi)(\hat{\xi}_{k,n} - \xi)$ converges to a non-degenerate distribution with non explicit cumulative distribution function. In fact, as it will appear in the next section, the limit distribution of $\hat{\xi}_{k,n}$ is driven by $k'/K_{k',n}$ if $\xi < -1/2$, by Y_n if $\xi > -1/2$ with

$$Y_n = \sqrt{k}/\sigma[\varphi_\xi(K_{k',n}/K_{k,n}) - \varphi_\xi(1/c)]$$

and by both of them if $\xi = -1/2$.

ii) $(H1)$ and $(H2)$ are second order conditions on the tail quantile function U. Similar conditions are used by Dekkers and de Haan [10] to establish the asymptotic distribution of Pickands' estimator.

iii) Let $\mu(\xi) = \gamma \mathbf{I}\{\xi > 0\} - [1 - \Gamma(1-\xi)]\varphi_\xi(1/c)/\ln(c)\mathbf{I}\{-1/2 < \xi < 0\}$, where γ is the Euler constant and Γ is the gamma function. Theorem 2 entails that $V_k(\xi)(\hat{\xi}_{k,n} - \xi)$ converges to a distribution of mean $\mu(\xi)$ if $\xi \neq 0$ and $\xi \neq -1/2$. This suggests to define the bias corrected estimator:

$$\hat{\xi}^*_{k,n} = \hat{\xi}_{k,n} - \frac{\mu(\hat{\xi}_{k,n})}{V_k(\hat{\xi}_{k,n})}.$$

As we will see on a simulation study (see section 5.1), this bias correction improves the behavior of our estimator in most finite sample situations.

3.2 Examples

Let α, $\beta > 0$, $\theta \in \mathbf{R}\backslash\{0\}$ and define $\ln_2(x) = \ln(\ln(x))$, $x > 1$. The two following models of slowly varying functions ℓ are considered:

$$\ell(x) = \alpha + \theta x^{-\beta} + o(x^{-\beta}), \quad (\textbf{Model A}),$$

$$\ell(x) = \theta[\ln(x)]^{-\beta}\left\{1 + O\left(\frac{\ln_2(x)}{\ln(x)}\right)\right\}, \quad (\textbf{Model B}).$$

Model A has been first introduced by Hall [18]. In both models, the parameter β tunes the decay of the slowly varying function ℓ. The conditions that should be satisfied by models A and B to insure convergence (7) are given in Corollary 1. In both cases, the best rate of convergence of $\hat{\xi}_{k,n}$ is also established. Some examples of distributions satisfying the assumptions of Corollary 1 are presented in Table 1. In the sequel, the following notation is adopted. Let (u_n) and (v_n) be two non negative deterministic sequences. The notation $u_n \asymp v_n$ means that

$$0 < \liminf \frac{u_n}{v_n} \leq \limsup \frac{u_n}{v_n} < \infty.$$

Corollary 1 Suppose that $k = ck'$, $k \to \infty$ and that F satisfies assumption $(H1)$ with a slowly varying function asymptotically monotone.

i) If ℓ belongs to Model A and if $\varphi_\delta(k')\,(n/k')^{-\beta} \to 0$ then convergence (7) holds. In this case, the best rate of convergence of $\hat{\xi}_{k,n}$ is given by:

$$V_k(\xi) \asymp \begin{cases} \ln(n) & \text{if } 0 < \xi, \\ \ln^2(n) & \text{if } \xi = 0, \\ n^{\delta/[1+\delta/\beta]-\varepsilon} & \text{if } \xi < 0, \end{cases}$$

where $\varepsilon \in]0, \delta/(1+\delta/\beta)[$ is arbitrarily small.

ii) If ℓ belongs to Model B and if $\varphi_\delta(k')\ln(k')/\ln(n) \to 0$ and $\varphi_\delta(k')\ln_2(n)/\ln(n) \to 0$ then, convergence (7) holds. Furthermore, the best rate of convergence of $\hat{\xi}_{k,n}$ is given by:

$$V_k(\xi) \asymp \begin{cases} \ln_2(n) & \text{if } 0 < \xi, \\ \ln_2^2(n) & \text{if } \xi = 0, \\ \ln^{1-\varepsilon}(n) & \text{if } \xi < 0, \end{cases}$$

where $\varepsilon \in]0,1[$ is arbitrarily small.

Remark 3 This corollary points out the fact that the case $\xi < 0$ is more favorable to our estimator, i.e. its convergence is faster than in the case $\xi \geq 0$. This is illustrated by the simulation study (see section 5).

4 Proofs of the main results

This section is devoted to the proof of Theorem 1 and Theorem 2. Proofs of lemmas are postponed to the appendix.

4.1 Preliminary results

The following function will play an important role. Let

$$H_n(x) = \left\{\frac{\varphi_x(1/k')}{\varphi_x(1/k)}\right\} \frac{X_{n-k+1,n} - X_{n,n}}{X_{n-k'+1,n} - X_{n,n}} = \left\{\frac{\varphi_x(1/k')}{\varphi_x(1/k)}\right\}(1 + Z_n),$$

with $Z_n = (X_{n-k+1,n} - X_{n-k'+1,n})/(X_{n-k'+1,n} - X_{n,n})$.

Lemma 3 Under the conditions of Theorem 1,

i) $Z_n \stackrel{d}{=} [U(N_n/K_{k,n}) - U(N_n/K_{k',n})]/[U(N_n/K_{k',n}) - U(N_n)]$.

ii) $N_n \xrightarrow{\text{a.s}} +\infty$, $K_{k,n} \xrightarrow{\text{a.s}} +\infty$, $K_{k,n}/K_{k',n} \xrightarrow{\text{a.s}} c$, $K_{k,n}/N_n \xrightarrow{\text{a.s}} 0$ and $Z_n \xrightarrow{\text{a.s}} \max(0, c^{-\xi} - 1)$ as $n \to \infty$.

iii) $k'/K_{k',n} \xrightarrow{d} \text{Exp}(1)$. If moreover $k = ck'$, then $Y_n \xrightarrow{d} \mathcal{N}(0,1)$.

4.2 Proof of Theorem 1

We shall need the following result:

Lemma 4 Suppose that relation (1) holds. If $\xi > 0$, for all $\eta_1 \in]0,\xi[$, $\eta_2 > 0$, there exist t_0, $\beta_1, \beta_2, \tilde{\beta}_1, \tilde{\beta}_2 > 0$ such that, for all $t \geq t_0$ and $x > 0$,

$$\beta_1 x^{\xi-\eta_1} - \tilde{\beta}_1 \leq \frac{U(tx) - U(t)}{a(t)} \leq \beta_2 x^{\xi+\eta_2} - \tilde{\beta}_2.$$

If $\xi = 0$, for all $\eta > 0$, there exist t_0, $\beta > 0$ such that, for all $t \geq t_0$ and $x > 0$,

$$\frac{U(tx) - U(t)}{a(t)} \leq \beta x^{\eta}.$$

Proof of Theorem 1 – We have to show that, for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathrm{P}[|\hat{\xi}_{k,n} - \xi| > \varepsilon] = 0. \tag{8}$$

Remark that if $\xi \neq 0$, proving (8) for all $\varepsilon > 0$ reduces to demonstrate (8) for all $0 < \varepsilon < |\xi|$. Since H_n is a non-decreasing function (see [14], Appendix B) and since $H_n(\hat{\xi}_{k,n}) = 1$, we have:

$$\mathrm{P}[|\hat{\xi}_{k,n} - \xi| > \varepsilon] = \mathrm{P}[H_n(\xi+\varepsilon) < 1] + \mathrm{P}[H_n(\xi-\varepsilon) > 1].$$

To prove Theorem 1 it is sufficient to establish that $\mathrm{P}[H_n(\xi+\varepsilon) \geq 1] \to 1$ and $\mathrm{P}[H_n(\xi-\varepsilon) \leq 1] \to 1$ as $n \to \infty$. The two following expansions hold:

$$\frac{\varphi_t(1/k')}{\varphi_t(1/k)} = 1 + k^{-t} - k'^{-t} + o(k^{-t}) \text{ if } t > 0 \text{ and } \frac{\varphi_t(1/k')}{\varphi_t(1/k)} \to c^t \text{ as } n \to \infty \text{ if } t < 0. \tag{9}$$

The two following cases are considered separately:
If $\xi \geq 0$, (9) and Lemma 3 ii) imply that $H_n(\xi+\varepsilon) = 1 + Z_n + k^{-(\xi+\varepsilon)} - k'^{-(\xi+\varepsilon)} + o_{\mathrm{P}}[k^{-(\xi+\varepsilon)}]$ since $\xi + \varepsilon > 0$. Furthermore, from Lemma 3 i),

$$Z_n \stackrel{d}{=} \underbrace{\frac{U(N_n/K_{k,n}) - U(N_n/K_{k',n})}{a(N_n/K_{k',n})}}_{Z_{1,n}} \times \underbrace{\frac{a(N_n/K_{k',n})}{U(N_n/K_{k',n}) - U(N_n)}}_{Z_{2,n}}, \tag{10}$$

with

$$Z_{1,n} \xrightarrow{\text{a.s.}} \varphi_\xi(c^{-1}), \tag{11}$$

from (2) and Lemma 3 ii) and

$$\beta_1 K_{k',n}^{\xi-\eta_1} - \tilde{\beta}_1 \leq -\frac{1}{Z_{2,n}} \leq \beta_2 K_{k',n}^{\xi+\eta_2} - \tilde{\beta}_2, \tag{12}$$

from Lemma 4. Since, from Lemma 3 iii), $k'/K_{k',n}$ converges to a standard exponential distribution, we deduce from (10)-(12) that $k'^{-(\xi+\varepsilon)}/Z_n \xrightarrow{\mathrm{P}} 0$ which entails that $H_n(\xi+\varepsilon) - 1 \stackrel{\mathrm{P}}{\sim} Z_n > 0$ i.e. that $\mathrm{P}[H_n(\xi+\varepsilon) \geq 1] \to 1$. Similarly, we prove that $\mathrm{P}[H_n(\xi-\varepsilon) \leq 1] \to 1$.
If $\xi < 0$, expansions (9) and Lemma 3 ii) imply that $H_n(\xi+\varepsilon) \to c^{\varepsilon} > 1$ since $\xi + \varepsilon > 0$. Thus, $\mathrm{P}[H_n(\xi+\varepsilon) \geq 1] \to 1$. In the same way, we prove that $\mathrm{P}[H_n(\xi-\varepsilon) \leq 1] \to 1$. ♠

4.3 Proof of Theorem 2

Let us define the function:

$$\varphi_t^*(x) = \begin{cases} 1 + tx & \text{if } t \neq 0, \\ \mathrm{e}^x & \text{if } t = 0. \end{cases}$$

To prove Theorem 2, two auxiliary results are necessary. Lemma 5 is dedicated to the study of the function φ_t^*.

Lemma 5

I) For $x \in (0, \infty)$, $\varphi_t^*[\varphi_t(x)] = x^{t+\mathbf{1}\{t=0\}}$.

Let (u_n) and (v_n) be two sequences such that $u_n \sim v_n$ (i.e. $u_n/v_n \to 1$).

II) Let $t = 0$. If $u_n \to \infty$ and $u_n - v_n \to \alpha$ then $\varphi_t^*(v_n) \sim \varphi_t^*(u_n)\mathrm{e}^{-\alpha}$.

III) Let $t \neq 0$. If $u_n \to \infty$ then $\varphi_t^*(v_n) \sim \varphi_t^*(u_n)$.

IV) Let $t \neq 0$. If $u_n \to -1/t$ with $v_n = u_n(1 + \varepsilon_n)$, then:

i) If moreover $\varepsilon_n/\varphi_t^*(u_n) \sim \alpha_n$ where α_n does not converge to ∞ or to 1, then $\varphi_t^*(v_n) \sim \varphi_t^*(u_n)(1-\alpha_n)$.

ii) If moreover $\varepsilon_n/\varphi_t^*(u_n) \to \infty$ then $\varphi_t^*(v_n) \sim -\varepsilon_n$.

The proof of this basic result is not detailed here. Clearly, the distribution of $H_n(x)$ is determined by Z_n. The following lemma provides the asymptotic distribution of Z_n.

Lemma 6 Under the conditions of Theorem 2,

$$\lim_{n\to\infty} \mathrm{P}\left[k'^{\delta-\mathbb{1}\{\xi=0\}}\varphi_\xi^*\left(-\frac{\varphi_\xi(1/c)}{Z_n}\right)\le t\right] = \begin{cases} \exp(-t^{-1/\xi}) & \text{if } 0<\xi, \\ \exp(-t^{-1}) & \text{if } \xi=0, \\ 1-\exp(-t^{-1/\xi}) & \text{if } -1/2<\xi<0, \\ \mathrm{P}[T<t\sqrt{c}] & \text{if } \xi=-1/2, \\ \Phi\left[-t\varphi_\xi(1/c)\sqrt{c}/\sigma\right] & \text{if } \xi<-1/2, \end{cases}$$

where the random variable T is defined as the limit in distribution of

$$T_n = \sqrt{\frac{k}{K_{k',n}}} + \frac{\sigma}{\varphi_\xi(1/c)}Y_n,$$

which is non-degenerate from Lemma 3 iii).

Proof of Theorem 2 – Let $F_n(t) = \mathrm{P}[V_k(\xi)(\hat{\xi}_{k,n}-\xi)\le t]$. We have,

$$F_n(t) = \mathrm{P}\left[\hat{\xi}_{k,n}\le \xi + t/V_k(\xi)\right] = \mathrm{P}[H_n(\xi+t/V_k(\xi))\ge 1] = \mathrm{P}\left[(1+Z_n)\frac{\varphi_{-\xi-t/V_k(\xi)}(k/c)}{\varphi_{-\xi-t/V_k(\xi)}(k)}\ge 1\right],$$

since H_n is a non-decreasing function and since $H_n(\hat{\xi}_{k,n})=1$. Routine calculations yield:

$$F_n(t) = \mathrm{P}\left[-\frac{\varphi_\xi(1/c)}{Z_n}\le t_n\right],$$

with

$$t_n = -\varphi_\xi(1/c)\frac{\varphi_{-\xi-t/V_k(\xi)}(k/c)}{\varphi_{-\xi-t/V_k(\xi)}(k)-\varphi_{-\xi-t/V_k(\xi)}(k/c)}.$$

Remarking that φ_ξ^* is an increasing function for $\xi\ge 0$ and decreasing for $\xi<0$, we have,

$$F_n(t) = \begin{cases} \mathrm{P}\left[(k')^{\delta-\mathbb{1}\{\xi=0\}}\varphi_\xi^*\left(-\frac{\varphi_\xi(1/c)}{Z_n}\right)\le (k')^{\delta-\mathbb{1}\{\xi=0\}}\varphi_\xi^*(t_n)\right] & \text{if } \xi\ge 0, \\ \mathrm{P}\left[(k')^{\delta}\varphi_\xi^*\left(-\frac{\varphi_\xi(1/c)}{Z_n}\right)\ge (k')^{\delta}\varphi_\xi^*(t_n)\right] & \text{if } \xi<0. \end{cases} \tag{13}$$

The asymptotic behavior of the left hand side random term

$$(k')^{\delta-\mathbb{1}\{\xi=0\}}\varphi_\xi^*\left(-\frac{\varphi_\xi(1/c)}{Z_n}\right)$$

is given by Lemma 6. Let us now focus on the right hand side deterministic term $(k')^{\delta-\mathbb{1}\{\xi=0\}}\varphi_\xi^*(t_n)$. Different cases have to be considered:

<u>If $\xi>0$</u>, the following sequence of asymptotic equivalences holds

$$t_n \sim k^{\xi+t/V_k(\xi)}\frac{\varphi_\xi(1/c)}{1-c^{\xi+t/V_k(\xi)}} \sim k^{\xi+t/V_k(\xi)}c^{-\xi}/\xi.$$

Since $V_k(\xi) \sim \ln(k)/\xi$, we have that $k^{t/V_k(\xi)} \to \mathrm{e}^{t\xi}$ as $n \to \infty$. Thus, $t_n \sim (k')^\xi/\xi \mathrm{e}^{t\xi} \to \infty$ as $n \to \infty$. Lemma 5 III) implies that $\varphi_\xi^*(t_n) \sim 1 + (k')^\xi \mathrm{e}^{t\xi} \sim (k')^\xi \mathrm{e}^{t\xi}$. Thus,

$$\lim_{n\to\infty} (k')^{\delta - \mathbb{1}\{\xi=0\}} \varphi_\xi^*(t_n) = \mathrm{e}^{t\xi}. \tag{14}$$

<u>If $\xi = 0$</u>, we have

$$t_n = \ln(c) \frac{c^{t/\ln^2(k)} - k^{t/\ln^2(k)}}{1 - c^{t/\ln^2(k)}}.$$

Using the expansions,

$$c^{t/\ln^2(k)} = 1 + t\frac{\ln(c)}{\ln^2(k)} + O\left(\frac{1}{\ln^4(k)}\right) \text{ and } k^{t/\ln^2(k)} = 1 + t\frac{1}{\ln(k)} + t^2 \frac{1}{2\ln^2(k)} + o\left(\frac{1}{\ln^2(k)}\right),$$

we find that

$$t_n = \ln(k)\left[1 + \frac{t/2 - \ln(c)}{\ln(k)} + o\left(\frac{1}{\ln(k)}\right)\right].$$

Since $t_n \to \infty$ and $\ln(k) - t_n \to -t/2 + \ln(c)$ as $n \to \infty$, Lemma 5 II) implies that

$$\varphi_\xi^*(t_n) \sim \varphi_\xi^*(\ln(k)) \exp[t/2 - \ln(c)] = k' \exp(t/2).$$

Thus,

$$\lim_{n\to\infty} (k')^{\delta - \mathbb{1}\{\xi=0\}} \varphi_\xi^*(t_n) = \mathrm{e}^{t/2}. \tag{15}$$

When <u>$\xi < 0$</u>, we have:

$$t_n = -\frac{1}{\xi} \frac{c^{-\xi} - 1}{c^{-\xi} - c^{t/V_k(\xi)}} c^{t/V_k(\xi)} [1 - (k')^{\xi + t/V_k(\xi)}].$$

Remarking that

$$c^{t/V_k(\xi)} = 1 + t\frac{\ln(c)}{V_k(\xi)} + O\left(\frac{1}{V_k(\xi)}\right),$$

and that $(k')^{t/V_k(\xi)} = 1 + o(1)$ lead to the following expansion:

$$\begin{aligned} t_n &= -\frac{1}{\xi}\left[1 + t\frac{\ln(c)}{V_k(\xi)}\right]\left[1 + o\left(\frac{1}{V_k(\xi)}\right)\right]\left[1 + t\frac{\ln(c)}{(c^{-\xi}-1)V_k(\xi)} + o\left(\frac{1}{V_k(\xi)}\right)\right][1 - (k')^\xi + o(k^\xi)] \\ &= u_n(1 + \varepsilon_n), \end{aligned}$$

with $u_n = -1/\xi[1 + t\ln(c)/V_k(\xi)]$ and

$$\varepsilon_n = t\frac{\delta \ln(c)}{c^{-\xi} - 1} k^{-\delta} - (k')^\xi + o(k^{-\delta}) + o(k^\xi).$$

Two situations have to be considered:
<u>If $-1/2 \le \xi < 0$</u>, it follows that

$$\lim_{n\to\infty} \frac{\varepsilon_n}{\varphi_\xi^*(u_n)} = -\frac{1}{t\xi\ln(c)}\left[\frac{t\xi\ln(c)}{c^{-\xi} - 1} + c^{-\xi}\right],$$

and Lemma 5 IV) i) implies

$$\varphi_\xi^*(t_n) \sim \varphi_\xi^*(u_n)\left[1 + \frac{1}{t\xi\ln(c)}\left(\frac{t\xi\ln(c)}{c^{-\xi} - 1} + c^{-\xi}\right)\right]$$

and thus

$$\lim_{n\to\infty}(k')^{\delta-\mathbb{1}\{\xi=0\}}\varphi_\xi^*(t_n) = 1 + t\frac{\xi\ln(c)}{c^{-\xi}-1}. \tag{16}$$

If $\xi < -1/2$, $\varepsilon_n/\varphi_\xi^*(u_n) \to (1-c^{-\xi})^{-1}$ as $n\to\infty$. Thus, Lemma 5 IV) i) yields

$$\lim_{n\to\infty}(k')^{\delta-\mathbb{1}\{\xi=0\}}\varphi_\xi^*(t_n) = -\frac{t}{2}\ln(c)\frac{c^{-\xi-1/2}}{1-c^{-\xi}}. \tag{17}$$

Collecting (13)-(17) with Lemma 6 concludes the proof. ♠

5 Simulation study

In this section, the improvement brought by the bias correction is illustrated through a simulation study. Next, a comparison with classical extreme value estimators is proposed. For each of the distributions considered in this section, $N = 100$ random samples of size $n = 500$ are generated.

5.1 Bias corrected estimator behavior

We first study the behavior of the bias corrected estimator $\hat{\xi}^*_{k,n}$ versus the estimator $\hat{\xi}_{k,n}$. In this aim, the following distributions are considered: (see Table 1 for their parameterizations)

- Case $\xi > 0$, Fréchet distribution with $\xi = 3$.
- Case $\xi = 0$, Weibull distribution with $(\lambda,\tau) = (1, 1/2)$, $(\lambda,\tau) = (1,1)$ and $(\lambda,\tau) = (1, 3/2)$.
- Case $-1/2 < \xi < 0$, Weibull_M distribution with $\xi = -1/3$.
- Case $\xi < -1/2$, Weibull_M with $\xi = -1$.

In Figure 1, the empirical mean over the N samples of $\hat{\xi}^*_{k,n}$ and $\hat{\xi}_{k,n}$ is represented as a function of the number k of upper order statistics ("Hill plot"). The true value of ξ is represented by a straight line. To compute these estimators, we choose $c = k/k' = 4$. If $\xi > 0$ (Fréchet distribution, Figure 1 (a)), the behavior of $\hat{\xi}_{k,n}$ is improved by the bias correction. If $\xi = 0$ (Weibull distribution), the estimation is highly influenced by the parameter τ which controls the rate of convergence of the slowly varying function ℓ (see table 1). If $\tau \le 1$ (Figure 1 (b), (c)), $\hat{\xi}^*_{k,n}$ is less biased than $\hat{\xi}_{k,n}$ and if $\tau > 1$ (Figure 1 (d)), the bias correction does not improve the behavior of $\hat{\xi}_{k,n}$. If $-1/2 < \xi < 0$ (Figure 1 (e)), $\hat{\xi}^*_{k,n}$ is slightly more biased than the estimator $\hat{\xi}_{k,n}$. Finally, if $\xi < -1/2$ (Weibull_M distribution, Figure 1 (f)), there is no correction ($\hat{\xi}_{k,n} = \hat{\xi}^*_{k,n}$). As a conclusion, it seems that the bias correction improves (or at least does not really degrade) the behavior of our estimator. Thus, in the sequel, we focus on the behavior of $\hat{\xi}^*_{k,n}$.

5.2 Comparison with other estimators

The estimator $\hat{\xi}^*_{k,n}$ is now compared with the following well known estimators: **Pickands' estimator** $\hat{\xi}^P_{k,n}$, the **moment estimator** proposed by Dekkers, Einmahl and de Haan [11] and defined by:

$$\hat{\xi}^M_{k,n} = \hat{\xi}^H_{k,n} + 1 - \frac{1}{2}\left[1 - \frac{(\hat{\xi}^H_{k,n})^2}{S_{k,n}}\right]^{-1},$$

where $S_{k,n} = 1/k\sum_{i=1}^k[\ln(X_{n-i+1,n}) - \ln(X_{n-k,n})]^2$ and the **generalized Zipf estimator** [2] defined by:

$$\hat{\xi}^Z_{k,n} = \frac{\sum_{j=1}^k \ln[(k+1)/j]\ln UH_{j,n} - \frac{1}{k}\sum_{j=1}^k\ln[(k+1)/j]\sum_{j=1}^k\ln UH_{j,n}}{\sum_{j=1}^k\ln^2[(k+1)/j] - \frac{1}{k}\left(\sum_{j=1}^k\ln[(k+1)/j]\right)^2},$$

with

$$UH_{j,n} = X_{n-j,n}\left(\frac{1}{j}\sum_{i=1}^{j}\ln X_{n-i+1,n} - \ln X_{n-j,n}\right).$$

The following distributions are considered: (see table 1 for their parameterization)

- Case $\xi > 0$, Burr distribution for which $\xi = 1/(\tau\lambda)$ with $(\beta, \tau, \lambda) = (1, 1, 1)$.
- Case $\xi = 0$, standard normal distribution.
- Case $-1/2 < \xi < 0$, Weibull$_{\mathrm{M}}$ distribution with $\xi = -1/4$.
- Case $\xi < -1/2$, Weibull$_{\mathrm{M}}$ with $\xi = -2$.

In Figures 2-4, the empirical mean and the empirical Mean Squared Error (MSE) of each estimator are represented as functions of k and we also choose $c = 4$. If $\xi > 0$ (Burr distribution), $\hat{\xi}^*_{k,n}$ is less biased than the other estimators (Figure 2 (a)) but it suffers from a high variance (Figure 2 (b)). If $\xi = 0$ (Gaussian distribution, Figure 2 (c), (d)), all estimates yield very poor results. If $-1/2 < \xi < 0$ (Weibull$_{\mathrm{M}}$ distribution, Figure 2 (e), (f)), $\hat{\xi}^*_{k,n}$ provides the best estimation and if $\xi < -1/2$ (Figure 3), generalized Zipf estimator and $\hat{\xi}^*_{k,n}$ are equivalent from the MSE point of view.

Finally, let us focus on the influence of the rate of convergence of the slowly varying function on the estimation of ξ. In this aim, we consider the reversed Burr distribution for which $\xi = -1/(\lambda\tau)$ (see Table 1 for its parameterization). Here, the parameter λ controls the rate of convergence of the slowly varying function (see Section 3.2). The larger is λ, the slower ℓ converges to a constant. This is illustrated in Figure 4 for $x_F = 10$, $w = 1$, $\tau = 1/\lambda$ with $\lambda = 1$, $\lambda = 2$ and $\lambda = 3$. In all cases, $\hat{\xi}^*_{k,n}$ performed better than Pickands' and moment estimators and the best estimation is provided by the generalized Zipf estimator.

As a conclusion, $\hat{\xi}^*_{k,n}$ performes well in the Weibull domain of attraction ($\xi < 0$) and it is competitive with Pickands' and moment estimator if $\xi \geq 0$.

Remark 4 The high volatility of the previous Hill plots points out the importance of the choice of the number k of upper order statistics. A lot of methods to select this parameter have been proposed (see Danielsson et *al.* [7], Gomes and Oliveira [16], Guillou and Hall [17], ...). A part of our future work will consist in the adaptation of these methods to our estimator. However, note that in all cases, our estimator is less influenced by this choice than other estimators.

Distribution	Cumulative distribution function	Model	β	Best rate of convergence
$\text{Weibull}_{\text{M}}$ ($\xi \in \mathbf{R}$)	$\exp[-(1+\xi x)^{-1/\xi}]$, for x such that $1+\xi x > 0$.	A	1	$\ln(n)$ if $0 < \xi$ $\ln^2(n)$ if $\xi = 0$ $n^{\delta/(1+\delta/\beta)-\varepsilon}$ if $\xi < 0$
Burr ($\xi > 0$)	$1-[w/(w+x^\tau)]^\lambda$, for $x > 0$, with w, λ, $\tau > 0$, $\xi = 1/(\lambda\tau)$,	A	$1/\lambda$	$\ln(n)$
Fréchet ($\xi > 0$)	$\exp(-x^{-1/\xi})$ for $x > 0$,	A	1	$\ln(n)$
Weibull ($\xi = 0$)	$1-\exp(-\lambda x^\tau)$, for $x > 0$, with λ, $\tau > 0$.	B	$1-1/\tau$	$\ln_2^2(n)$
Normal ($\xi = 0$)	$\int_{-\infty}^{x} 1/\sqrt{2\pi}\mathrm{e}^{-t^2/2}dt$	B	$1/2$	$\ln_2^2(n)$.
Reversed Burr ($\xi < 0$)	$1-[w/(w+(x_F-x)^{-\tau})]^\lambda$, for $x < x_F$, with $\xi = 1/(\lambda\tau)$. w, λ, $\tau > 0$, $x_F \in \mathbf{R}$.	A	$1/\lambda$	$n^{\delta/(1+\delta/\beta)-\varepsilon}$

Table 1: Examples of distributions satisfying the assumptions of Corollary 1

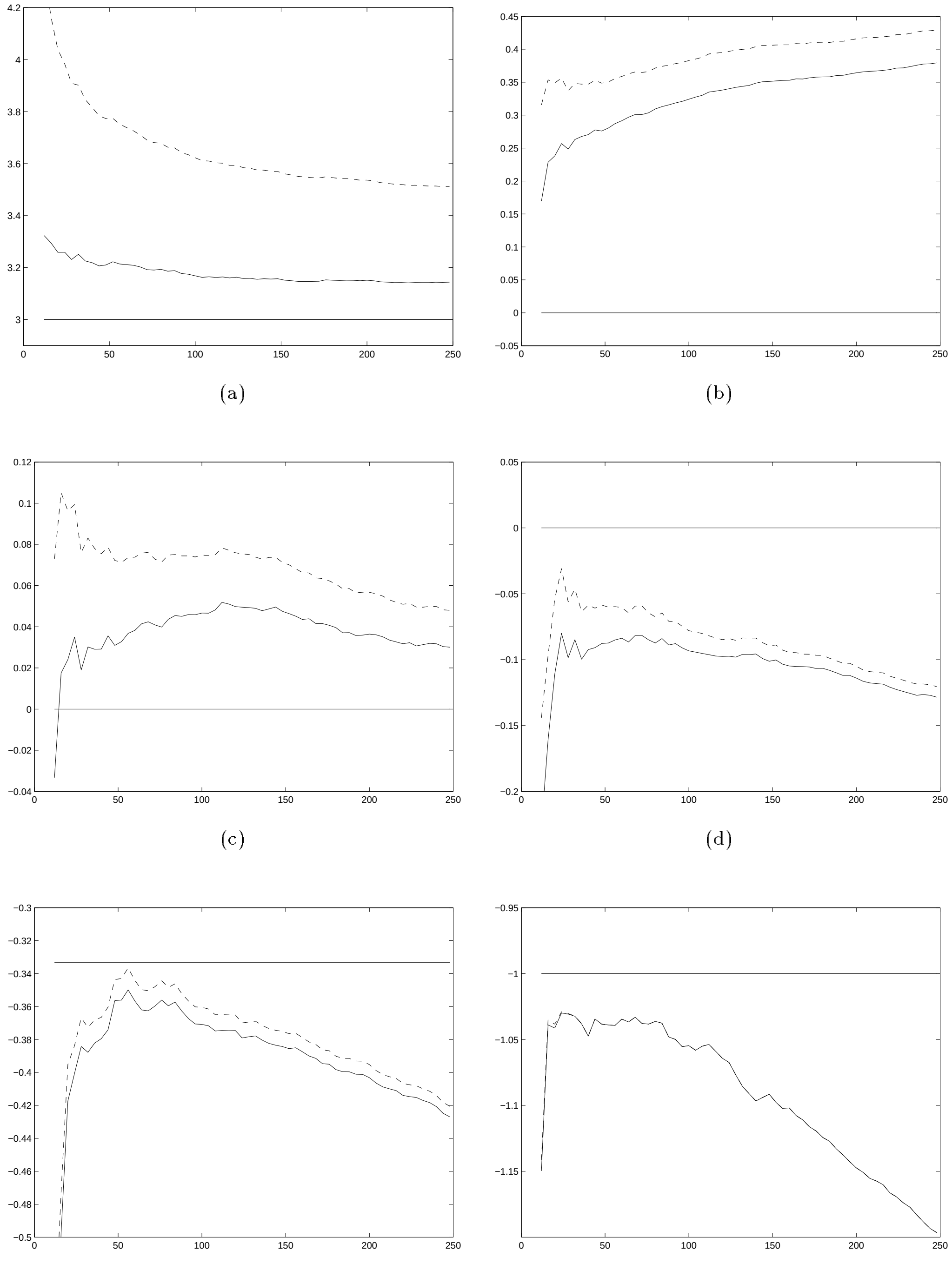
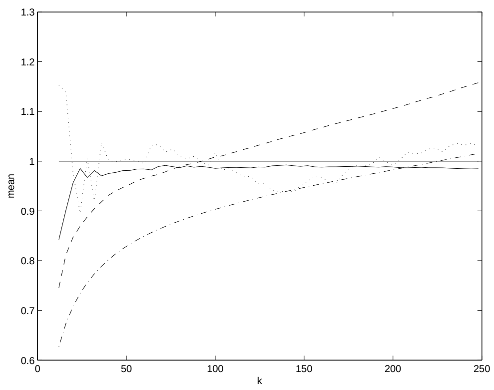
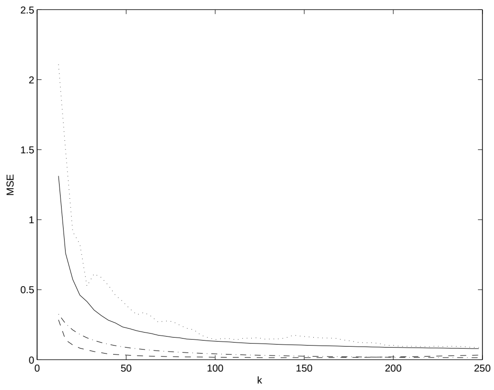


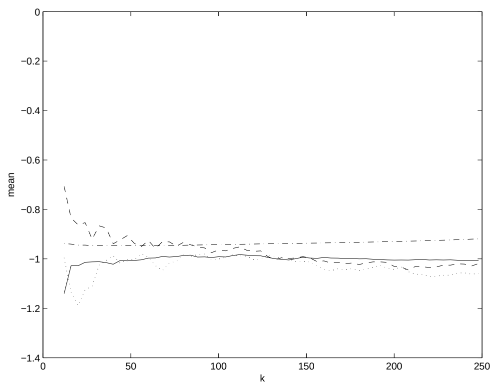
Figure 1: Comparison of the bias corrected estimator $\hat{\xi}^*_{k,n}$ (solid line) and $\hat{\xi}_{k,n}$ (dashed line) for (a) the Fréchet distribution, (b) the Weibull distribution with $(\lambda, \tau) = (1, 1/2)$, (c) the Weibull distribution with $(\lambda, \tau) = (1, 1)$, (d) the Weibull distribution with $(\lambda, \tau) = (1, 3/2)$, (e) the $\mathrm{Weibull_M}$ distribution with $\xi = -1/3$ and (f) the $\mathrm{Weibull_M}$ distribution with $\xi = -1$.



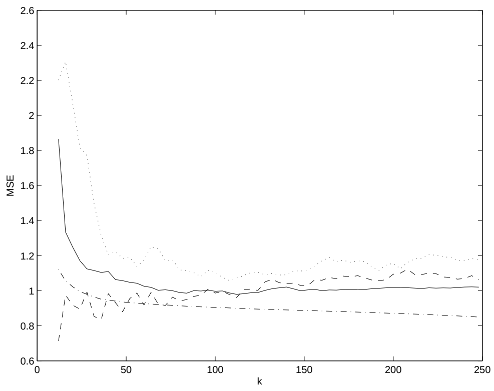
(a) mean



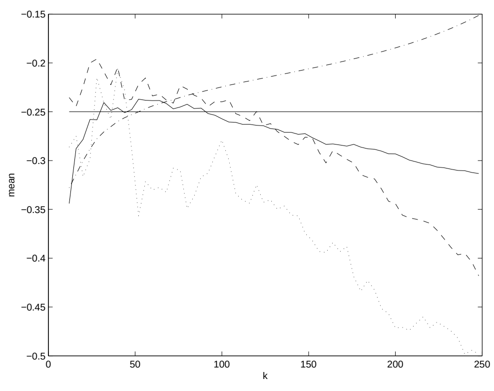
(b) MSE



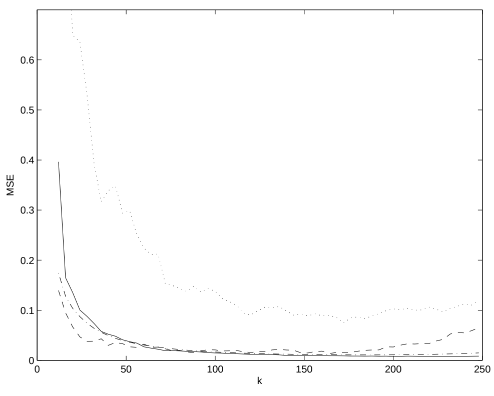
(c) mean



(d) MSE

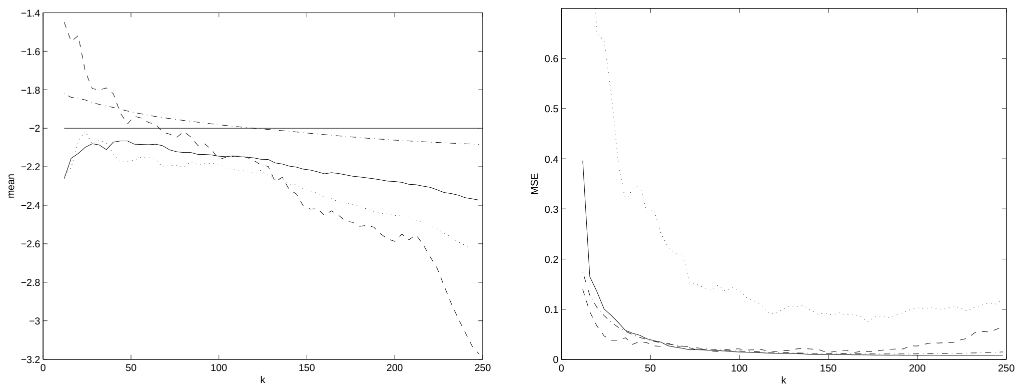


(e) mean



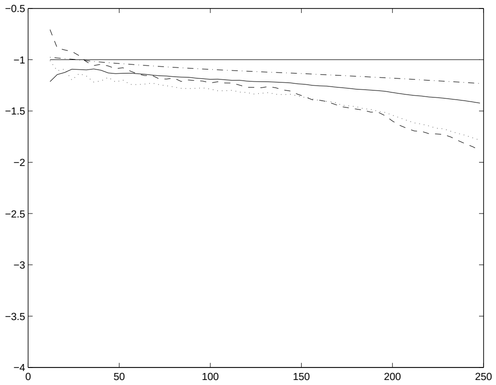
(f) MSE

Figure 2: Comparison of the estimator $\hat{\xi}^*_{k,n}$ (solid line), moment estimator (dashed line), Pickands' estimator (dotted line) and the generalized Zipf estimator (dash-dot line) for (a), (b) the Burr distribution with $(\beta, \tau, \lambda) = (1, 1, 1)$, (c), (d) the standard normal distribution and (e), (f) the $\text{Weibull}_{\text{M}}$ distribution with $\xi = -1/4$.

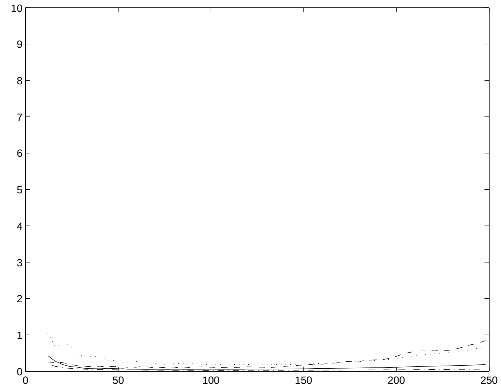


(a) Mean (b) MSE

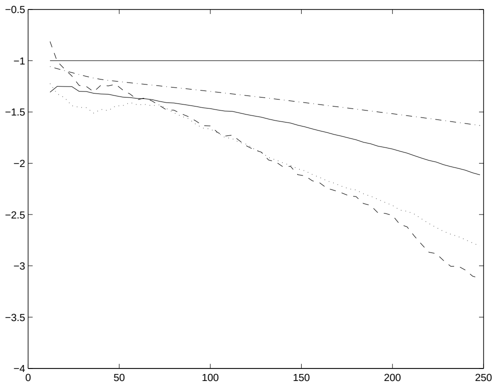
Figure 3: Comparison of the estimator $\hat{\xi}^*_{k,n}$ (solid line), moment estimator (dashed line), Pickands' estimator (dotted line) and the generalized Zipf estimator (dash-dot line) for the $\mathrm{Weibull}_{\mathrm{M}}$ distribution with $\xi = -2$.



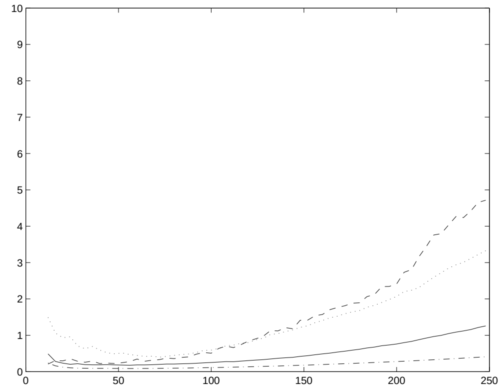
(a) mean



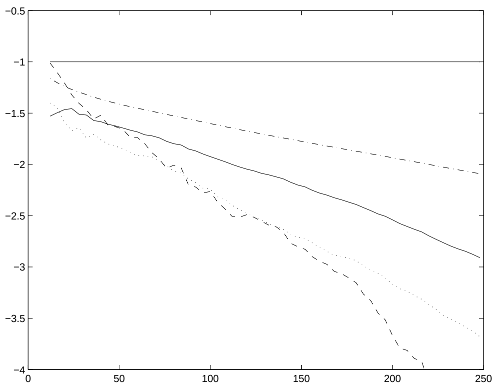
(b) MSE



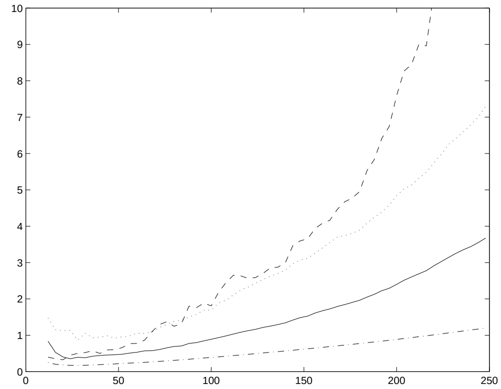
(c) mean



(d) MSE



(e) mean



(f) MSE

Figure 4: Comparison of the estimator $\hat{\xi}^*_{k,n}$ (solid line), moment estimator (dashed line), Pickands' estimator (dotted line) and the generalized Zipf estimator (dash-dot line) for the reversed Burr distribution with (a), (b) $\lambda = 1$, (c), (d) $\lambda = 2$ and (e), (f) $\lambda = 3$.

APPENDIX

Proof of Lemma 1 – First, we focus on the case $\xi < 0$. Recall that one can take in relation (2) $a(t) = \xi[U(t) - x_F]$ where x_F is the right endpoint of the distribution function F. Thus,

$$\frac{U(tx) - U(t)}{a(t)} = -\frac{1}{\xi} - \frac{x_F - U(tx)}{\xi[U(t) - x_F]} < -\frac{1}{\xi}, \tag{18}$$

since $x_F > U(t)$ for all t. Furthermore, since U is a non-decreasing function, we have that for all $A > 0$ and $x \geq A$,

$$\frac{U(tx) - U(t)}{a(t)} \geq \frac{U(tA) - U(t)}{a(t)}.$$

Using (2), we have that for all $A, \varepsilon > 0$ there exist T such that for all $t \geq T$ and $x \geq A$,

$$\frac{U(tx) - U(t)}{a(t)} \geq -\frac{1}{\xi}(1+\varepsilon)(1 - A^{\xi}). \tag{19}$$

Using (18) and (19), we conclude the proof for $\xi < 0$. Second, suppose that $\xi = 0$ (the proof for $\xi > 0$ is quite similar). Using relation (2) we have that for all $A, \varepsilon > 0$ there exist T such that for all $t \geq T$ and $x \geq A$,

$$\frac{U(tx) - U(t)}{a(t)} \geq (1+\varepsilon)\ln(A),$$

which concludes the proof. ♠

Proof of Lemma 2 – Remark that

$$\frac{U(tx) - U(t)}{U(ty) - U(t)} = 1 + z(t, x, y),$$

with

$$z(t, x, y) = \frac{U(tx) - U(ty)}{U(ty) - U(t)} = \underbrace{\frac{U(tyx/y) - U(ty)}{a(ty)}}_{z_1(t,x,y)} \times \underbrace{\frac{a(ty)}{U(ty) - U(ty/y)}}_{z_2(t,x,y)}.$$

Since $y \to \infty$ and $x/y \to d$, relation (2) implies that $z_1(t, x, y)$ converges to $\varphi_\xi(d)$. Since $y \to \infty$ and $ty \to \infty$, Lemma 1 implies that $z_2(t, x, y)$ converges to $\min(0, \xi)$. Thus, $1 + z(t, x, y) \to 1$ if $\xi \geq 0$ and $1 + z(t, x, y) \to c^{\xi}$ if $\xi < 0$. Remarking that $\varphi_\xi(y)/\varphi_\xi(x) \to 1$ if $\xi \geq 0$ and $\varphi_\xi(y)/\varphi_\xi(x) \to c^{-\xi}$ if $\xi < 0$ concludes the proof. ♠

Proof of Lemma 3 – Let $W_1, \ldots, W_n$ be independent standard uniform random variables and $W_{1,n} \leq \ldots \leq W_{n,n}$ the corresponding order statistics.

i) Remarking that

$$(1 - K_{i,n}/N_n)_{1 \leq i \leq n} = (F(X_{n-i+1,n}))_{1 \leq i \leq n} \stackrel{d}{=} (W_{n-i+1,n})_{1 \leq i \leq n},$$

it follows that

$$\begin{aligned}(U(N_n/K_{i,n}))_{1 \leq i \leq n} = (F^{\leftarrow}(1 - K_{i,n}/N_n))_{1 \leq i \leq n} &\stackrel{d}{=} (F^{\leftarrow}(W_{n-i+1,n}))_{1 \leq i \leq n} \\ &\stackrel{d}{=} (X_{n-i+1,n})_{1 \leq i \leq n},\end{aligned}$$

which conclude the demonstration.

ii) We have $K_{k,n} \stackrel{d}{=} W_{k,n}/W_{1,n}$. Using Rényi representation [1] p.72, $K_{k,n} \stackrel{d}{=} T_k/T_1$ where T_k is the sum of k independent standard exponential random variables. Another use of Rényi representation leads to $K_{k,n} \stackrel{d}{=} 1/W_{1,k-1}$. Since $W_{1,k-1} \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$, we prove that $K_{k,n} \xrightarrow{\text{a.s.}} \infty$ as $n \to \infty$. Similarly, $N_n \xrightarrow{\text{a.s.}} \infty$ as $n \to \infty$.
Now, remark that $K_{k,n}/K_{k',n} \stackrel{d}{=} W_{k,n}/W_{k',n}$. Rényi representation yields $K_{k,n}/K_{k',n} \stackrel{d}{=} T_k/T_{k'} = (k/k')[T_k/k]/[T_{k'}/k'] \xrightarrow{\text{a.s.}} c$ as $n \to \infty$. The proof of $K_{k,n}/N_n \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$ is similar.
Finally, Lemma 2 and Lemma 3 i) ii) imply that:

$$(1+Z_n)\frac{\varphi_\xi(1/k')}{\varphi_\xi(1/k)} \xrightarrow{\text{P}} 1,$$

which concludes the proof using $k/k' \to c$.

iii) Since $1/K_{k,n} \stackrel{d}{=} W_{1,k-1} \stackrel{d}{=} 1 - W_{k-1,k-1}$, and remarking that

$$k(1 - W_{k-1,k-1}) \xrightarrow{d} \text{Exp}(1),$$

it follows that $k/K_{k,n}$ has asymptotically a standard exponential distribution.
Now, let $E_1, \dots, E_n$ be independent standard exponential random variables and $E_{1,n} \le \dots \le E_{n,n}$ the corresponding order statistics. Dekkers and de Haan ([10], Lemma 2.1) shows that

$$\sqrt{\frac{k}{c-1}}(E_{n-k/c+1,n} - E_{n-k+1,n} - \ln(c)) \xrightarrow{d} \mathcal{N}(0,1).$$

Since $K_{k,n} \stackrel{d}{=} \exp(E_{n,n} - E_{n-k+1,n})$ and $K_{k',n} \stackrel{d}{=} \exp(E_{n,n} - E_{n-k/c+1,n})$, the δ-method applied to the function $\varphi_\xi(\mathrm{e}^{-x})$ concludes the proof. ♠

Proof of Lemma 4 – This proof is inspired by the one of [21], Lemma 0.13. We only give the proof for $\xi > 0$ (the case $\xi = 0$ is quite similar). For $j \in \mathbb{N}$, let

$$\beta_j = \exp\left[\frac{j}{\xi}\ln(1+\xi)\right].$$

Using relation (2), we have that for any $\varepsilon \in]0, 1-(1+\xi)^{-1}[$ there exist t_0 such that for all $t \ge t_0$ and for all $j \in \mathbb{N}\backslash\{0\}$,

$$1-\varepsilon \le \frac{U(t\beta_j) - U(t\beta_{j-1})}{a(t\beta_{j-1})} \le 1+\varepsilon. \tag{20}$$

Furthermore, since $a(.)$ is regularly varying at infinity with index ξ (see [21], Proposition 0.8 v) and Proposition 0.12), we have that for any $\varepsilon \in]0, 1-(1+\xi)^{-1}[$ there exist t_0 such that for all $t \ge t_0$ and for all $j \in \mathbb{N}\backslash\{0\}$,

$$\beta_1^\xi(1-\varepsilon) = (1+\xi)(1-\varepsilon) \le \frac{a(t\beta_j)}{a(t\beta_{j-1})} \le (1+\xi)(1+\varepsilon). \tag{21}$$

Let $N \in \mathbb{N}$. We have:

$$\frac{U(t\beta_N) - U(t)}{a(t)} = \sum_{j=1}^{N} \frac{U(t\beta_j) - U(t\beta_{j-1})}{a(t\beta_{j-1})}\frac{a(t\beta_{j-1})}{a(t)}.$$

Using (20), we find that for $t \ge t_0$:

$$(1-\varepsilon)\sum_{j=1}^{N}\frac{a(t\beta_{j-1})}{a(t)} \le \frac{U(t\beta_N) - U(t)}{a(t)} \le (1+\varepsilon)\sum_{j=1}^{N}\frac{a(t\beta_{j-1})}{a(t)}. \tag{22}$$

Remarking that

$$\frac{a(t\beta_{j-1})}{a(t)} = \prod_{i=1}^{j-1} \frac{a(t\beta_i)}{a(t\beta_{i-1})},$$

(21) and (22) imply that there exist $\tilde{\beta}_1, \tilde{\beta}_2 > 0$ such that for $t \geq t_0$:

$$\tilde{\beta}_1[(1+\xi)(1-\varepsilon)]^N - \tilde{\beta}_1 \leq \frac{U(t\beta_N) - U(t)}{a(t)} \leq \tilde{\beta}_2[(1+\xi)(1+\varepsilon)]^N - \tilde{\beta}_2. \tag{23}$$

Let $N_x = [\xi/\ln(1+\xi)]\ln(x)$ (i.e. $x = \exp[N_x/\xi \ln(1+\xi)]$). Remarking that $\lfloor N_x \rfloor \leq N_x \leq \lfloor N_x \rfloor + 1$ implies that $\beta_{\lfloor N_x \rfloor} \leq x \leq \beta_{\lfloor N_x \rfloor + 1}$. Since U is a non-decreasing function, we find that

$$\frac{U(t\beta_{\lfloor N_x \rfloor}) - U(t)}{a(t)} \leq \frac{U(tx) - U(t)}{a(t)} \leq \frac{U(t\beta_{\lfloor N_x \rfloor + 1}) - U(t)}{a(t)},$$

and, using (23),

$$\tilde{\beta}_1[(1+\xi)(1-\varepsilon)]^{\lfloor N_x \rfloor} - \tilde{\beta}_1 \leq \frac{U(tx) - U(t)}{a(t)} \leq \tilde{\beta}_2[(1+\xi)(1+\varepsilon)]^{\lfloor N_x \rfloor + 1} - \tilde{\beta}_2.$$

Thus, there exist β_1 and β_2 such that:

$$\beta_1[(1+\xi)(1-\varepsilon)]^{N_x} - \tilde{\beta}_1 \leq \frac{U(tx) - U(t)}{a(t)} \leq \beta_2[(1+\xi)(1+\varepsilon)]^{N_x} - \tilde{\beta}_2. \tag{24}$$

Remarking that

$$[(1+\xi)(1-\varepsilon)]^{N_x} = x^{\xi - \eta_1},$$

with $\eta_1 = -\xi \ln(1-\varepsilon)/\ln(1+\xi) \in]0, \xi[$ and

$$[(1+\xi)(1+\varepsilon)]^{N_x} = x^{\xi + \eta_2},$$

with $\eta_2 = \xi \ln(1+\varepsilon)/\ln(1+\xi) > 0$ concludes the proof. ♠

Proof of Lemma 6 – The first step of the proof consists in establishing the following expansion:

$$-\frac{\varphi_\xi(1/c)}{Z_n} = \varphi_\xi(K_{k',n}) \left[1 - \frac{\sigma}{\sqrt{k}\varphi_\xi(K_{k',n}/K_{k,n})} Y_n\right] \left[1 + o_{\mathrm{P}}\left(\frac{1}{\varphi_\delta(k)}\right)\right]. \tag{25}$$

Let $V(t) = U(\mathrm{e}^t)$. From Lemma 3 i), we have:

$$Z_n \stackrel{d}{=} \underbrace{\frac{V'[\ln(N_n/K_{k',n})]}{U(N_n/K_{k',n}) - U(N_n)}}_{Z_{1,n}} \times \underbrace{\frac{U(N_n/K_{k,n}) - U(N_n/K_{k',n})}{V'[\ln(N_n/K_{k',n})]}}_{Z_{2,n}}.$$

Clearly,

$$-\frac{1}{Z_{1,n}} = \int_0^{\ln(K_{k',n})} \frac{V'[\ln(N_n/K_{k',n}) + s]}{V'[\ln(N_n/K_{k',n})]} ds,$$

and conditions $(H1)$ and $(H2)$ imply that uniformly on $[0, \ln(K_{k',n})]$,

$$\frac{V'[\ln(N_n/K_{k',n}) + s]}{V'[\ln(N_n/K_{k',n})]} = \mathrm{e}^{\xi s}\left[1 + o_{\mathrm{P}}\left(\frac{1}{\varphi_\delta(k)}\right)\right].$$

Thus,

$$-\frac{1}{Z_{1,n}} = \left[1 + o_{\mathrm{P}}\left(\frac{1}{\varphi_\delta(k)}\right)\right] \int_0^{\ln(K_{k',n})} \mathrm{e}^{\xi s} ds = \varphi_\xi(K_{k',n})\left[1 + o_{\mathrm{P}}\left(\frac{1}{\varphi_\delta(k)}\right)\right]. \tag{26}$$

The proof of

$$Z_{2,n} = \varphi_\xi(K_{k',n}/K_{k,n}) \left[1 + o_{\mathrm{P}}\left(\frac{1}{\varphi_\delta(k)}\right)\right], \tag{27}$$

follows the same lines. Collecting (26) and (27) yields

$$-\frac{\varphi_\xi(K_{k',n}/K_{k,n})}{Z_n} = \varphi_\xi(K_{k',n}) \left[1 + o_{\mathrm{P}}\left(\frac{1}{\varphi_\delta(k)}\right)\right],$$

which proves (25) by remarking that

$$\frac{\varphi_\xi(1/c)}{\varphi_\xi(K_{k',n}/K_{k,n})} = 1 - \frac{\sigma}{\sqrt{k}\varphi_\xi(K_{k',n}/K_{k,n})} Y_n.$$

Now, remark that (25) can be rewritten $v_n = u_n(1+\varepsilon_n)$ with $u_n = \varphi_\xi(K_{k',n})$, $v_n = -\varphi_\xi(1/c)/Z_n$ and

$$\varepsilon_n = \frac{-\sigma}{\sqrt{k}\varphi_\xi(K_{k',n}/K_{k,n})} Y_n + o_{\mathrm{P}}\left(\frac{1}{\varphi_\delta(k)}\right) = O_{\mathrm{P}}\left(k^{-1/2}\right) + o_{\mathrm{P}}\left(\frac{1}{\varphi_\delta(k)}\right) \xrightarrow{\mathrm{P}} 0,$$

from Lemma 3 iii). The second step of the proof is dedicated to the study of $\varphi_\xi^*(-\varphi_\xi(1/c)/Z_n)$. Five cases have to be considered:

<u>If $\xi > 0$</u>, since $u_n \xrightarrow{\mathrm{P}} +\infty$ as $n \to \infty$ (Lemma 3 iii)), Lemma 5 III) entails that $\varphi_\xi^*(u_n) \overset{\mathrm{P}}{\sim} \varphi_\xi^*(v_n)$, i.e.,

$$\varphi_\xi^*\left(-\frac{\varphi_\xi(1/c)}{Z_n}\right) \overset{\mathrm{P}}{\sim} \varphi_\xi^*[\varphi_\xi(K_{k',n})] = (K_{k',n})^{\xi + \mathbb{1}\{\xi=0\}}, \tag{28}$$

by Lemma 5 I).

<u>If $\xi = 0$</u>, we have $u_n \xrightarrow{\mathrm{P}} +\infty$ as $n \to \infty$ and $u_n - v_n = u_n\varepsilon_n = o_{\mathrm{P}}[\ln(K_{k',n})\ln(k)] = o_{\mathrm{P}}(1)$. Thus, Lemma 5 II) implies (28).

<u>If $-1/2 < \xi < 0$</u>, we have from Lemma 3 ii) that $u_n \xrightarrow{\mathrm{P}} -1/\xi$ as $n \to \infty$. Remarking that $\varepsilon_n = o_{\mathrm{P}}[1/\varphi_\delta(k)] = o_{\mathrm{P}}(k^\xi)$ implies

$$\frac{\varepsilon_n}{\varphi_\xi^*(u_n)} = o_{\mathrm{P}}[(k/K_{k',n})^\xi] = o_{\mathrm{P}}(1),$$

which entails (28) by Lemma 5 IV) i).

<u>If $\xi = -1/2$</u>, we have $u_n \xrightarrow{\mathrm{P}} -1/\xi$ as $n \to \infty$. Remarking that $\varepsilon_n \overset{\mathrm{P}}{\sim} -\sigma/[\sqrt{k}\varphi_\xi(1/c)]Y_n$ yields:

$$\frac{\varepsilon_n}{\varphi_\xi^*(u_n)} \overset{\mathrm{P}}{\sim} -\frac{\sigma c^{-1/2}}{\varphi_\xi(1/c)} Y_n \sqrt{\frac{K_{k',n}}{k'}} = \alpha_n,$$

where α_n does not converges in probability to ∞ or 1 as $n \to \infty$ (see Lemma 3 iii)). Thus, from Lemma 5 IV) i), we have $\varphi_\xi^*(u_n) \overset{\mathrm{P}}{\sim} \varphi_\xi^*(v_n)(1-\alpha_n)$ i.e.

$$\varphi_\xi^*\left(-\frac{\varphi_\xi(1/c)}{Z_n}\right) \overset{\mathrm{P}}{\sim} (K_{k',n})^{-1/2}\left(1 + \frac{\sigma c^{-1/2}}{\varphi_\xi(1/c)} Y_n \sqrt{\frac{K_{k',n}}{k'}}\right) = k^{-1/2} T_n. \tag{29}$$

<u>If $\xi < -1/2$</u>, we have $u_n \xrightarrow{\mathrm{P}} -1/\xi$ as $n \to \infty$, $\varepsilon_n \overset{\mathrm{P}}{\sim} -\sigma/[\sqrt{k}\varphi_\xi(1/c)]Y_n$ and $\varepsilon_n/\varphi_\xi^*(u_n) \xrightarrow{\mathrm{P}} \infty$ as $n \to \infty$. Thus, Lemma 5 IV) ii) implies that $\varphi_\xi^*(v_n) \overset{\mathrm{P}}{\sim} -\varepsilon_n$ i.e.

$$\varphi_\xi^*\left(-\frac{\varphi_\xi(1/c)}{Z_n}\right) \overset{\mathrm{P}}{\sim} \frac{\sigma k^{-1/2}}{\varphi_\xi(1/c)} Y_n. \tag{30}$$

Lemma 3 iii) and (28)-(30) conclude the proof. ♠

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