T E C H N I C A L
R E P O R T

0305

SERIAL AND NONSERIAL SIGN-AND-RANK STATISTICS ASYMPTOTIC REPRESENTATION AND ASYMPTOTIC NORMALITY

HALLIN, M., VERMANDELE, C. and B. WERKER

Σ T A T

I A P S T A T I S T I C S
N E T W O R K

INTERUNIVERSITY ATTRACTION POLE

http://www.stat.ucl.ac.be/IAP

Serial and Nonserial Sign-and-Rank Statistics
Asymptotic Representation and Asymptotic Normality

Marc Hallin*
I.S.R.O., E.C.A.R.E.S., and Département de Mathématique
Université Libre de Bruxelles, Brussels, Belgium

Catherine Vermandele
I.S.R.O. and Faculté des Sciences Psychologiques et de l'Education
Université Libre de Bruxelles, Brussels, Belgium

and

Bas Werker
Econometrics and Finance Group, CentER
Tilburg University, Tilburg, the Netherlands

Abstract

The classical theory of rank-based inference is entirely based either on *ordinary* ranks, which do not allow for considering location nor intercept parameters, or on signed ranks, which require an assumption of symmetry. If the median, in the absence of a symmetry assumption, is considered as a location parameter, the maximal invariance property of ordinary ranks is lost to the ranks and the signs. As shown in Hallin and Werker (2003), conditioning on a maximal invariant in such situations is essential if semiparametric efficiency is to be reached. This new maximal invariant thus suggests a new class of statistics, based on ordinary ranks *and* signs. An asymptotic representation theory *à la* Hájek is developed here for such statistics, both in the nonserial and in the serial case. The corresponding asymptotic normality results clearly show how the signs are adding a separate contribution to the asymptotic variance, hence, potentially, to asymptotic efficiency. Applications to semiparametric inference in regression and time series models with median restrictions are treated in detail in a companion paper (Hallin, Werker, and Vermandele 2003).

AMS 1980 subject classification : 62G10, 62M10

Key words and phrases : Ranks, signs, Hájek representation, median regression, median restrictions, maximal invariant.

1 Introduction

The classical theory of rank-based inference is entirely based either on *ordinary ranks*, or on *signed ranks*. Ranks indeed are maximal invariant with respect to the group of continuous order-preserving transformations, a group that generates the null hypothesis of absolutely continuous independent white noise (no location restriction), whereas signed ranks (that is, the signs along

*Research supported by a P.A.I. contract of the Belgian Federal Government and an Action de Recherche Concertée of the Communauté française de Belgique.

with the ranks of absolute values) are maximal invariant under the subgroup generating the subhypothesis of symmetric (with respect to the origin) independent white noise.

Now, a location parameter is usually specified to be zero for the error term in most statistical models : regression and analysis of variance models, autoregressive-moving average models, etc. Symmetric white noise allows for such an identification, at the expense, however, of a symmetry assumption which in practice is often quite unrealistic. And, the trouble with independent white noise without further restrictions is that it does not allow for identifying any location parameter.

This location parameter in general is the mean—an heritage of Gaussian models—but could be the median as well. Zero-median noise is certainly as natural as zero-mean noise. In a semiparametric context, it is even more satisfactory, as it does not require any moment assumption on the densities under consideration. Median-regression and autoregression models therefore recently have attracted much attention : see, for instance, Jung (1996), Koenker (2000), Zhao (2001), McKeague, Subramian, and Sun (2001), Horowitz and Spokoiny (2002), to quote only a few.

Moreover, from the point of view of statistical inference, the assumption of zero-median noise is also more convenient, since it induces more structure. The hypothesis of zero-mean white noise indeed is not invariant under any nontrivial group of transformations, so that group invariance arguments cannot be invoked in models involving zero-mean noise. The situation is quite different for the hypothesis of zero-median noise, which is generated by the group of all continuous order-preserving transformations g such that $g(0) = 0$. A maximal invariant for that group is the vector of ordinary ranks, along with the vector of signs. Hallin and Werker (2003) have shown that, in such a situation, semiparametric efficiency is achieved by conditioning with respect to a *maximal* invariant. Maximality of the invariant here is essential : conditioning, e.g., on the ranks when the signs-and-ranks, not the ranks alone, are maximal invariant, induces an avoidable loss of efficiency.

Invariance and semiparametric efficiency arguments in such models thus lead to a new concept of rank-based statistics, involving both the signs and the ranks. This new concept is more natural than the traditional ranks in all models involving a location parameter, or an intercept, but also in models such as ARMA models, where the noise is inherently centered. The objective of this paper is to present a detailed study of the class of linear sign-and-rank statistics, for which we provide Hájek-type asymptotic representation and asymptotic normality results. These results readily allow for building new rank-based tests for a variety of problems in one-, two-, and k-sample location, regression, $ARMA$, and related models, without making any symmetry assumptions on the underlying error densities. They also form a basis for the construction of semiparametrically efficient procedures in median constrained models (see Hallin, Vermandele, and Werker 2003).

The paper is organized as follow. Section 2 briefly introduces several concepts of white noise: *independent*, *independent with zero mean*, *independent with zero median*, and *independent symmetric* white noises, showing how the invariance principle in each case yields a different concept of ranks. Sections 3 and 4 propose a systematic investigation of the linear nonserial and serial sign-and-rank statistics. These new statistics, measurable with respect to the vectors of ranks and signs or, equivalently, with respect to the vector of ranks and the number of negative or positive residuals, are studied along the same lines as the classical linear rank statistics (see, for example, Hájek and Šidák (1967) for the nonserial context, Hallin, Ingenbleek, and Puri (1985), Hallin and Puri (1991) for the serial context) and the linear signed-rank statistics (see Hájek and Šidák (1967), Hušková (1970) for the nonserial context, Hallin and Puri (1991) for the serial context). However, the non-independence between the ranks and the signs requires a more delicate treatment.

2 White noise and group invariance.

2.1 White noise and semiparametric statistical models

Whatever the concept of ranks, rank-based inference applies in the context of semiparametric models under which the distribution of some observed n-tuple $\mathbf{Y}^{(n)} := (Y_1^{(n)}, \ldots, Y_n^{(n)})'$ belongs to a family of distributions of the form

$$\left\{ \mathrm{P}^{(n)}_{f;\boldsymbol{\theta}},\, \boldsymbol{\theta} \in \boldsymbol{\Theta} \subseteq \mathbb{R}^K,\, f \in \mathcal{F} \right\},$$

where $\boldsymbol{\theta}$ denotes some finite-dimensional parameter of interest, and f some unspecified density (densities throughout are tacitly taken with respect to the Lebesgue measure over the real line), playing the role of a nonparametric nuisance. This distribution $\mathrm{P}^{(n)}_{f;\boldsymbol{\theta}}$ in general is described by means of

(i) a *residual function*, namely, a family of invertible functions $\mathfrak{Z}^{(n)}_{\boldsymbol{\theta}}$ indexed by n and $\boldsymbol{\theta}$, mapping the observation $\mathbf{Y}^{(n)}$ onto a n-tuple of *residuals*

$$\mathfrak{Z}^{(n)}_{\boldsymbol{\theta}} \left(\mathbf{Y}^{(n)} \right) = \mathbf{Z}^{(n)}(\boldsymbol{\theta}) := (Z_1^{(n)}(\boldsymbol{\theta}), \ldots, Z_n^{(n)}(\boldsymbol{\theta}))',$$

and

(ii) a concept of *white noise* with marginal density f

such that $\mathbf{Y}^{(n)}$ has distribution $\mathrm{P}^{(n)}_{f;\boldsymbol{\theta}}$ iff $\mathbf{Z}^{(n)}(\boldsymbol{\theta})$ is white noise with density f (in the sense of (ii)). The concept of white noise thus plays a fundamental role in most semiparametric models.

As an example, consider the first-order autoregressive model with unspecified innovation density under which $\mathbf{Y}^{(n)}$ is a realization of length n of some solution of

$$Y_t = \theta Y_{t-1} + \varepsilon_t, \quad t = 1, \ldots n,$$

with $\theta \in \Theta := (-1, 1)$. Assuming, for simplicity, that $Y_0 = 0$, the residual function here is $\mathfrak{Z}^{(n)}_{\theta} \left(\mathbf{Y}^{(n)} \right) := (Z_1^{(n)}(\theta), \ldots, Z_n^{(n)}(\theta))'$, with $Z_t^{(n)}(\theta) := Y_t - \theta Y_{t-1}$.

2.2 White noise, group invariance, ranks, signed-ranks, and signs-and-ranks

For simplicity, let us concentrate on four particular forms of white noise. Defining $\mathcal{F} := \{ f : f(x) > 0, x \in \mathbb{R} \}$ as the set of all nonvanishing probability densities over the real line, let $\mathcal{F}_* := \left\{ f \in \mathcal{F} : \mu_f := \int_{-\infty}^{\infty} z f(z) dz = 0 \right\}$ be the subset of all densities in $\mathcal{F}$ having mean zero, $\mathcal{F}_0 := \left\{ f \in \mathcal{F} : \int_{-\infty}^{0} f(z) dz = \int_{0}^{\infty} f(z) dz = 1/2 \right\}$ the set of densities in $\mathcal{F}$ having zero median, and $\mathcal{F}_+ := \{ f \in \mathcal{F} : f(-z) = f(z), z \in \mathbb{R} \}$ the set of densities in $\mathcal{F}$ that are symmetric with respect to the origin. Denote by

(a) (independent white noise) $\mathcal{H}^{(n)}_f$ the hypothesis under which the random vector $\mathbf{Z}^{(n)} = (Z_1^{(n)}, \ldots, Z_n^{(n)})'$ is a realization of length n of an *independent* white noise, i.e., $Z_i^{(n)}$, $i = 1, \ldots, n$, are independent and identically distributed (i.i.d.) with density $f \in \mathcal{F}$,

(b) (zero mean independent white noise) $\mathcal{H}^{(n)}_{*;f}$ the hypothesis under which $\mathbf{Z}^{(n)}$ is a realization of length n of an *independent with zero mean* white noise, i.e., $Z^{(n)}_i$, $i = 1, \ldots, n$, are i.i.d. with density $f \in \mathcal{F}_*$,

(c) (zero median independent white noise) $\mathcal{H}^{(n)}_{0;f}$ the hypothesis under which $\mathbf{Z}^{(n)}$ is a realization of length n of an *independent with zero median* white noise, i.e., $Z^{(n)}_i$, $i = 1, \ldots, n$, are i.i.d. with density $f \in \mathcal{F}_0$, and by

(d) (symmetric independent white noise) $\mathcal{H}^{(n)}_{+;f}$ the hypothesis under which $\mathbf{Z}^{(n)}$ is a realization of length n of an *independent symmetrical* white noise, i.e., $Z^{(n)}_i$, $i = 1, \ldots, n$, are i.i.d. with density $f \in \mathcal{F}_+$.

The notation $\mathcal{H}^{(n)}$, $\mathcal{H}^{(n)}_*$, $\mathcal{H}^{(n)}_0$, and $\mathcal{H}^{(n)}_+$ is used whenever the underlying density function f remains unspecified within $\mathcal{F}$, $\mathcal{F}_*$, $\mathcal{F}_0$, and $\mathcal{F}_+$, respectively. In practice, of course, the role of the random variables $Z^{(n)}_i$ is actually played by the residuals $Z^{(n)}_i(\boldsymbol{\theta})$ ($i = 1, \ldots, n$) associated with a specific value $\boldsymbol{\theta}$ of the parameter in the statistical model under consideration.

The independent white noise hypothesis $\mathcal{H}^{(n)}$ is of course most general, but does not allow for identifying location parameters. A classical attitude, when location is to be identified, consists in assuming that the underlying white noise density has zero mean, i.e., adopting $\mathcal{H}^{(n)}_*$ as a concept of white noise. As already explained, an equally natural solution requires the median (instead of the mean) of the white noise density to be zero, leading to $\mathcal{H}^{(n)}_0$. The additional assumption of symmetry yields $\mathcal{H}^{(n)}_+$.

Let $\mathcal{E}^{(n)} := \left(\mathbb{R}^n, \mathcal{B}^n, \mathcal{P}^{(n)} := \{\mathrm{P}^{(n)}_{\boldsymbol{\theta};f}, \boldsymbol{\theta} \in \boldsymbol{\Theta}, f \in \mathcal{F}\}\right)$ be characterized (in the sense of Section 2.1) by the residual function $\mathfrak{Z}^{(n)}_{\boldsymbol{\theta}}$ and the white noise concept $\mathcal{H}^{(n)}_f$. Denoting by $\mathfrak{G}$ the set of all continuous, strictly monotone increasing functions $g : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to\pm\infty} g(x) = \pm\infty$, define

$$\mathcal{G}^{(n)}_g : \mathbf{z} = (z_1, \ldots, z_n)' \in \mathbb{R}^n \mapsto \mathcal{G}^{(n)}_g(\mathbf{z}) := (g(z_1), \ldots, g(z_n))' \in \mathbb{R}^n,$$

and consider the group (acting on $\mathbb{R}^n$)

$$\mathcal{G}^{(n)}_{\boldsymbol{\theta}}, \circ := \left\{ \left(\mathfrak{Z}^{(n)}_{\boldsymbol{\theta}}\right)^{-1} \circ \mathcal{G}^{(n)}_g \circ \mathfrak{Z}^{(n)}_{\boldsymbol{\theta}} \ , \ g \in \mathfrak{G} \right\}, \circ.$$

This group (called the group of order-preserving transformations of residuals) clearly is a generating group for the fixed-$\boldsymbol{\theta}$ submodel $\mathcal{E}^{(n)}(\boldsymbol{\theta}) := \left(\mathbb{R}^n, \mathcal{B}^n, \mathcal{P}^{(n)}(\boldsymbol{\theta}) := \{\mathrm{P}^{(n)}_{\boldsymbol{\theta};f}, f \in \mathcal{F}\}\right)$ of $\mathcal{E}^{(n)}$, with maximal invariant the vector $\mathbf{R}^{(n)}(\boldsymbol{\theta}) := (R^{(n)}_1(\boldsymbol{\theta}), \ldots, R^{(n)}_n(\boldsymbol{\theta}))'$, where $R^{(n)}_i(\boldsymbol{\theta})$ denotes the rank of the residual $Z^{(n)}_i(\boldsymbol{\theta})$ among $Z^{(n)}_1(\boldsymbol{\theta}), \ldots, Z^{(n)}_n(\boldsymbol{\theta})$.

Similarly, let $\mathfrak{G}_+ := \{g \in \mathfrak{G} : g(-z) = -g(z)\}$, and denote by $\mathcal{G}^{(n)}_{\boldsymbol{\theta};+}$ the corresponding subgroup of $\mathcal{G}^{(n)}_{\boldsymbol{\theta}}$. This group (the group of *symmetric* order-preserving transformations of residuals) is a generating group for $\mathcal{E}^{(n)}_+(\boldsymbol{\theta}) := \left(\mathbb{R}^n, \mathcal{B}^n, \mathcal{P}^{(n)}_+(\boldsymbol{\theta}) := \{\mathrm{P}^{(n)}_{\boldsymbol{\theta};f}, f \in \mathcal{F}_+\}\right)$, the submodel of $\mathcal{E}^{(n)}(\boldsymbol{\theta})$ resulting from restricting to symmetric densities $f \in \mathcal{F}_+$. A maximal invariant here is the vector $\mathbf{R}^{(n)}_+(\boldsymbol{\theta}) := (s_1(\boldsymbol{\theta})R^{(n)}_{+;1}(\boldsymbol{\theta}), \ldots, s_n(\boldsymbol{\theta})R^{(n)}_{+;n}(\boldsymbol{\theta}))'$, where $R^{(n)}_{+;i}(\boldsymbol{\theta})$ denotes the rank of the absolute value $|Z^{(n)}_i(\boldsymbol{\theta})|$ among $|Z^{(n)}_1(\boldsymbol{\theta})|, \ldots, |Z^{(n)}_n(\boldsymbol{\theta})|$, and $s_i(\boldsymbol{\theta})$ is the sign of $Z^{(n)}_i(\boldsymbol{\theta})$.

Turning to the model $\mathcal{E}^{(n)}_0 := \left(\mathbb{R}^n, \mathcal{B}^n, \mathcal{P}^{(n)}_0 := \{\mathrm{P}^{(n)}_{\boldsymbol{\theta};f}, \boldsymbol{\theta} \in \boldsymbol{\Theta}, f \in \mathcal{F}_0\}\right)$ characterized by the residual function $\mathfrak{Z}^{(n)}_{\boldsymbol{\theta}}$ and the zero median white noise concept $\mathcal{H}^{(n)}_{0;f}$, it is easy to see that a generating group for (with obvious notation) $\mathcal{E}_0^{(n)}(\boldsymbol{\theta})$ is obtained by considering the subgroup of $\mathcal{G}_{\boldsymbol{\theta}}^{(n)}$ corresponding to $\mathfrak{G}_0 := \{g \in \mathfrak{G} : g(0) = 0\}$, with maximal invariant the vectors $\mathbf{s}^{(n)}(\boldsymbol{\theta}) := (s_1(\boldsymbol{\theta}), \ldots, s_n(\boldsymbol{\theta}))'$ of residual signs and $\mathbf{R}^{(n)}(\boldsymbol{\theta})$ of residual ranks.

Except for the condition that residuals should have finite first-order moments, the model $\mathcal{E}_*^{(n)} := \left(\mathbb{R}^n, \mathcal{B}^n, \mathcal{P}_*^{(n)} := \{\mathrm{P}_{\boldsymbol{\theta};f}^{(n)}, \boldsymbol{\theta} \in \boldsymbol{\Theta}, f \in \mathcal{F}_*\}\right)$ characterized by the same residual function $\mathfrak{Z}_{\boldsymbol{\theta}}^{(n)}$ as $\mathcal{E}_0^{(n)}$, but zero mean rather than zero median white noise, globally coincides (as a nonparametric statistical model) with $\mathcal{E}_0^{(n)}$, in the sense that both models involve the same family of distributions $\mathcal{P}^{(n)}$ over $(\mathbb{R}^n, \mathcal{B}^n)$. Actually, they only differ by the way the parameter of interest and the nuisance are separated from each other. However, the invariance structure underlying $\mathcal{E}_0^{(n)}$ allows for a rank-based approach of testing problems, an approach that cannot be considered for $\mathcal{E}_*^{(n)}$. The median, in this respect, seems more appropriate than the mean as a location parameter.

The theory of tests and estimation based on ranks or signed ranks offers a pretty complete toolkit of methods in the analysis of linear models with independent observations (see Hájek, Šidák and Sen (1999), or Puri and Sen (1985) for a systematic account and state-of-the-art in this context), as well as in the analysis of linear time series models (see Dufour *et al.* (1982), Hallin *et al.* (1985) and Hallin and Puri (1988, 1991, 1994)).

The importance of considering maximal invariants—signs and ranks, thus, in models with median zero white noise—has been substantiated in Hallin and Werker (2003), who show that, in a very broad class of models, semiparametrically efficient inference procedures can be obtained by conditioning with respect to the maximal invariant σ-algebra. It is somewhat surprising, therefore, that sign-and-rank statistics never have been considered so far in the vast literature devoted to rank-based inference. The purpose of this paper is to fill this gap.

2.3 Sign-and-rank statistics : definitions and notation

A *sign-and-rank statistic* is a $\left(\mathbf{s}^{(n)}, \mathbf{R}^{(n)}\right)$-measurable statistic, where $\mathbf{s}^{(n)} = (s_1, \ldots, s_n)'$ and $\mathbf{R}^{(n)} = \left(R_1^{(n)}, \ldots, R_n^{(n)}\right)'$ are the vector of signs and the vector of ranks, respectively, associated with some n-dimensional random vector $\mathbf{Z}^{(n)}$.

Denote by $N_-^{(n)} := \sum_{i=1}^n I[Z_i^{(n)} < 0] = \sum_{i=1}^n I[s_i = -1]$ and by $N_+^{(n)} := \sum_{i=1}^n I[Z_i^{(n)} > 0] = \sum_{i=1}^n I[s_i = 1]$ the numbers of negative and positive components in $\mathbf{Z}^{(n)}$ (in $\mathbf{s}^{(n)}$), respectively. Under $\mathcal{H}_0^{(n)}$, $N_+^{(n)}$ is binomial $Bin(n, 1/2)$. Letting $\mathbf{N}^{(n)} := (N_-^{(n)}, N_+^{(n)})$, note that $\sigma\left(\mathbf{N}^{(n)}\right) = \sigma\left(N_-^{(n)}\right) = \sigma\left(N_+^{(n)}\right)$, as $N_+^{(n)} = n - N_-^{(n)}$ with probability one. Since $s_i = I[Z_i^{(n)} > 0] - I[Z_i^{(n)} < 0] = I[R_i^{(n)} > n - N_+^{(n)}] - I[R_i^{(n)} \leq N_-^{(n)}]$, for all $i = 1, \ldots, n$, the couple $\left(\mathbf{N}^{(n)}, \mathbf{R}^{(n)}\right)$ is thus another maximal invariant for $\mathcal{H}_0^{(n)}$.

Defining the sets $\mathcal{N}_-^{(n)} := \{i \in \{1, \ldots, n\} : s_i = -1\} = \{i_1^- < \cdots < i_{N_-^{(n)}}^-\}$ and $\mathcal{N}_+^{(n)} := \{i \in \{1, \ldots, n\} : s_i = 1\} = \{i_1^+ < \cdots < i_{N_+^{(n)}}^+\}$, the distribution of $(\mathbf{s}^{(n)}, \mathbf{R}^{(n)})$ under $\mathcal{H}_0^{(n)}$ is conveniently characterized as follows: the marginal distribution of $\mathbf{s}^{(n)}$ is uniform over the 2^n elements of $\{-1, 1\}^n$, and the conditional distribution of $\mathbf{R}^{(n)}$ given $\mathbf{s}^{(n)}$ is such that $(R_{i_1^-}^{(n)}, R_{i_2^-}^{(n)}, \ldots, R_{i_{N_-^{(n)}}^-}^{(n)}; R_{i_1^+}^{(n)}, R_{i_2^+}^{(n)}, \ldots, R_{i_{N_+^{(n)}}^+}^{(n)})$ is (conditionally) uniformly distributed over the $(N_-^{(n)}!)(N_+^{(n)}!)$ possible combinations of a permutation of $\{1, \ldots, N_-^{(n)}\}$ with a permutation of

$\{(n - N_+^{(n)}) + 1, \dots, n\}$.

Let us finally denote by $\mathbf{Z}_{(\cdot)-}^{(N_-^{(n)})}$ and $\mathbf{Z}_{(\cdot)+}^{(N_+^{(n)})}$ the vectors of order statistics associated with the negative and positive elements of $\mathbf{Z}^{(n)}$, respectively. These two vectors—the first one of length $N_-^{(n)}$ and the second one of length $N_+^{(n)}$—constitute a natural (random) decomposition of the vector of order statistics $\mathbf{Z}_{(\cdot)}^{(n)}$ associated with $\mathbf{Z}^{(n)}$.

3 Nonserial linear sign-and-rank statistics

3.1 Definition and conditional asymptotic representation

A *linear nonserial sign-and-rank statistic* is a statistic of the form

$$S_{\mathbf{c}}^{(n)} := \frac{1}{n} \sum_{i=1}^{n} c_i^{(n)} a^{(n)} \left(\mathbf{N}^{(n)}; R_i^{(n)} \right), \tag{3.1}$$

where $a^{(n)}(\cdot;\cdot)$ is a real-valued *score function* defined over $\{((\nu,\eta);i) : \nu,\eta \in \{0,1,\dots,n\},$ $\eta \le n-\nu,\ i \in \{1,\dots,n\}\}$; note that each summand in (3.1) is allowed to depend on the sign s_i of $Z_i^{(n)}$, but also, via $\mathbf{N}^{(n)}$, on the other signs. As usual, the $c_i^{(n)}$'s ($i = 1,\dots,n$) denote nonrandom *regression constants.*

The exact mean $\mathrm{E}\left[S_{\mathbf{c}}^{(n)}\right]$ and the exact variance $\mathrm{Var}\left[S_{\mathbf{c}}^{(n)}\right]$ of $S_{\mathbf{c}}^{(n)}$ under $\mathcal{H}_0^{(n)}$ are easily obtained from elementary combinatorial arguments : letting $\bar{c}^{(n)} := n^{-1}\sum_{i=1}^{n} c_i^{(n)}$, we obtain

$$\mathrm{E}\left[S_{\mathbf{c}}^{(n)}\right] = (n2^n)^{-1}\bar{c}^{(n)} \sum_{j=1}^{n}\sum_{\nu=0}^{n} \binom{n}{\nu} a^{(n)}((\nu, n-\nu); j)$$

and

$$\mathrm{Var}\left[S_{\mathbf{c}}^{(n)}\right] = \frac{1}{n(n-1)2^n} \sum_{i=1}^{n} (c_i^{(n)} - \bar{c}^{(n)})^2 \times \sum_{\nu=0}^{n} \binom{n}{\nu} \left\{ \sum_{i=1}^{n} \left[a^{(n)}((\nu,n-\nu);i)\right]^2 - \frac{1}{n}\left[\sum_{i=1}^{n} a^{(n)}((\nu,n-\nu);i)\right]^2 \right\},$$

respectively.

If asymptotic results are to be obtained, some stability of the scores $a^{(n)}$ is required as n increases. We therefore will assume the existence of a *score-generating function* : a function $\varphi : (0,1) \to \mathbb{R}$ is called a *score-generating function* for the score function $a^{(n)}$ if

$$\mathrm{E}\left[\left\{ a^{(n)}\left(\mathbf{N}^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right) \right\}^2 \middle| \mathbf{Z}_{(\cdot)}^{(n)} \right] = o_{\mathrm{P}}(1), \tag{3.2}$$

under $\mathcal{H}_{0;f}^{(n)}$, as $n \to \infty$. Note that (3.2) is automatically satisfied if, under $\mathcal{H}_{0;f}^{(n)}$,

$$\mathrm{E}\left[\left\{ a^{(n)}\left(\mathbf{N}^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right) \right\}^2 \middle| \mathbf{N}^{(n)} \right] = o_{\mathrm{P}}(1), \tag{3.3}$$

as $n \to \infty$. Indeed, we almost surely have

$$\begin{aligned}
&\mathrm{E}\left[\left\{a^{(n)}\left(\mathbf{N}^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| \mathbf{N}^{(n)}\right] \\
&\quad = \mathrm{E}\left[\mathrm{E}\left[\left\{a^{(n)}\left(\mathbf{N}^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| \mathbf{N}^{(n)}, \mathbf{Z}_{(\cdot)}^{(n)}\right] \middle| \mathbf{N}^{(n)}\right] \\
&\quad = \mathrm{E}\left[\mathrm{E}\left[\left\{a^{(n)}\left(\mathbf{N}^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| \mathbf{Z}_{(\cdot)}^{(n)}\right] \middle| \mathbf{N}^{(n)}\right], \qquad (3.4)
\end{aligned}$$

where the latter (almost sure) equality follows from the fact that $\mathbf{N}^{(n)} = \left(N_-^{(n)}, N_+^{(n)}\right)$ is measurable with respect to $\mathbf{Z}_{(\cdot)}^{(n)}$. Hence, (3.3) implies the convergence in probability to zero of the conditional expectation (3.4) and, consequently, the convergence (3.2).

No asymptotic results for $S_{\mathbf{c}}^{(n)}$ can be obtained without some assumptions on the asymptotic behavior of regression constants $c_i^{(n)}, i = 1, \ldots, n$. We will assume that the classical *Noether condition* holds :

(N) The constants $c_i^{(n)}$, $i = 1, \ldots, n$ are not all equal, and

$$\lim_{n\to\infty} \frac{\max_{1\le i\le n}\left(c_i^{(n)} - \bar{c}^{(n)}\right)^2}{\sum_{j=1}^n \left(c_j^{(n)} - \bar{c}^{(n)}\right)^2} = 0.$$

Finally, the following central limit theorem for independent random variables will be useful.

Theorem 3.1 *Let $W_i^{(n)}$, $i = 1, \ldots, n$, be i.i.d., with mean $\mathrm{E}(W_i^{(n)}) := \mu_W$ and variance $0 < \mathrm{Var}(W_i^{(n)}) := \sigma_W^2 < \infty$. Define $T^{(n)} := \sum_{i=1}^n d_i^{(n)} W_i^{(n)}$, where the constants $d_i^{(n)}$ satisfy the Noether condition (N). Then,*

$$\left(T^{(n)} - \mu_T^{(n)}\right)/\sigma_T^{(n)} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1),$$

as $n \to \infty$, with $\mu_T^{(n)} := \mathrm{E}(T^{(n)}) = \mu_W \sum_{i=1}^n d_i^{(n)}$ and $0 < \left(\sigma_T^{(n)}\right)^2 := \mathrm{Var}(T^{(n)}) = \sigma_W^2 \sum_{i=1}^n \left(d_i^{(n)}\right)^2 < \infty$.

Proof. The proof simply consists in checking for Lindeberg's classical condition. □

We may now state a first asymptotic representation and asymptotic normality result. This result however is a *conditional* one, in the sense that the centering in (3.5) and (3.6) below, is a conditional centering, and will serve as an intermediate step in the derivation of the main result (of an unconditional nature) in Section 3.3. Contrary to the unconditional one, which requires *exact* or *approximate* scores, the conditional result however holds for any scores satisfying (3.2).

Lemma 3.1 *Let $\varphi : (0,1) \to \mathbb{R}$ be a non-constant square-integrable score-generating function for $a^{(n)}$, and let the regression constants $c_i^{(n)}$ ($i = 1, \ldots, n$) satisfy the Noether condition (N). Assume moreover that $\sum_{i=1}^n (c_i^{(n)} - \bar{c}^{(n)})^2 = O(n)$, as $n \to \infty$. Then,*
(i) (asymptotic representation) under $\mathcal{H}_{0;f}^{(n)}$, as $n \to \infty$,

$$S_{\mathbf{c}}^{(n)} - \mathrm{E}\left[S_{\mathbf{c}}^{(n)} \middle| \mathbf{N}^{(n)}\right] = T_{\varphi;f}^{(n)} - \mathrm{E}\left[T_{\varphi;f}^{(n)} \middle| \mathbf{Z}_{(\cdot)}^{(n)}\right] + o_{\mathrm{P}}(1/\sqrt{n}), \qquad (3.5)$$

where $T^{(n)}_{\varphi;f} := \frac{1}{n}\sum_{i=1}^n c_i^{(n)}\varphi\left(F(Z_i^{(n)})\right)$ *(F stands for the distribution function associated with f);*
(ii) (asymptotic normality) under $\mathcal{H}_0^{(n)}$*, as* $n\to\infty$*,*

$$\sqrt{n}\left(\frac{S_{\mathbf{c}}^{(n)} - \mathrm{E}\left[S_{\mathbf{c}}^{(n)}\middle|\mathbf{N}^{(n)}\right]}{\sqrt{\frac{1}{n}\sum_{i=1}^n(c_i^{(n)}-\bar{c}^{(n)})^2}}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,\sigma_\varphi^2), \tag{3.6}$$

where $0<\sigma_\varphi^2 := \int_0^1\varphi^2(u)du - \left(\int_0^1\varphi(u)du\right)^2 < \infty$.

Before turning to the proof of this proposition, observe that, almost surely under $\mathcal{H}_0^{(n)}$,

$$\begin{aligned}
\mathrm{E}\left[S_{\mathbf{c}}^{(n)}\middle|\mathbf{N}^{(n)}\right] &= \frac{1}{n}\sum_{i=1}^n c_i^{(n)}\mathrm{E}\left[\mathrm{E}\left[a^{(n)}\left(\mathbf{N}^{(n)};R_i^{(n)}\right)\middle|\mathbf{s}^{(n)}\right]\middle|\mathbf{N}^{(n)}\right]\\
&= \frac{1}{n}\sum_{i=1}^n c_i^{(n)}\ \mathrm{E}\Big[I\left[s_i=-1\right]\tfrac{1}{N_-^{(n)}}\textstyle\sum_{j=1}^{N_-^{(n)}} a^{(n)}\left(\mathbf{N}^{(n)};j\right)\\
&\qquad\qquad + I\left[s_i=1\right]\tfrac{1}{N_+^{(n)}}\textstyle\sum_{j=(n-N_+^{(n)})+1}^{n} a^{(n)}\left(\mathbf{N}^{(n)};j\right)\Big|\mathbf{N}^{(n)}\Big]\\
&= \frac{1}{n}\sum_{i=1}^n c_i^{(n)}\ \Big\{\mathrm{P}\left[s_i=-1|\,\mathbf{N}^{(n)}\right]\tfrac{1}{N_-^{(n)}}\textstyle\sum_{j=1}^{N_-^{(n)}} a^{(n)}\left(\mathbf{N}^{(n)};j\right)\\
&\qquad\qquad + \mathrm{P}\left[s_i=1|\,\mathbf{N}^{(n)}\right]\tfrac{1}{N_+^{(n)}}\textstyle\sum_{j=(n-N_+^{(n)})+1}^{n} a^{(n)}\left(\mathbf{N}^{(n)};j\right)\Big\}\\
&= \bar{c}^{(n)}\left(\frac{1}{n}\sum_{j=1}^n a^{(n)}\left(\mathbf{N}^{(n)};j\right)\right) = \bar{c}^{(n)}\left(\frac{1}{n}\sum_{i=1}^n a^{(n)}\left(\mathbf{N}^{(n)};R_i^{(n)}\right)\right),
\end{aligned}$$

and

$$\mathrm{E}\left[T^{(n)}_{\varphi;f}\middle|\mathbf{Z}^{(n)}_{(\cdot)}\right] = \frac{1}{n}\sum_{i=1}^n c_i^{(n)}\mathrm{E}\left[\varphi\left(F(Z_i^{(n)})\right)\middle|\mathbf{Z}^{(n)}_{(\cdot)}\right] = \bar{c}^{(n)}\left(\frac{1}{n}\sum_{i=1}^n\varphi\left(F(Z_i^{(n)})\right)\right). \tag{3.7}$$

Hence, part (i) of Lemma 3.1 actually states that

$$\begin{aligned}
S_{\mathbf{c}}^{(n)} - \mathrm{E}\left[S_{\mathbf{c}}^{(n)}\middle|\mathbf{N}^{(n)}\right] &= \frac{1}{n}\sum_{i=1}^n\left(c_i^{(n)}-\bar{c}^{(n)}\right)a^{(n)}\left(\mathbf{N}^{(n)};R_i^{(n)}\right)\\
&= \frac{1}{n}\sum_{i=1}^n\left(c_i^{(n)}-\bar{c}^{(n)}\right)\varphi\left(F(Z_i^{(n)})\right) + o_{\mathrm{P}}(1/\sqrt{n}),
\end{aligned} \tag{3.8}$$

under $\mathcal{H}_{0;f}^{(n)}$, as $n\to\infty$. Note that the expression in the right-hand side of (3.8) coincides with the asymptotic representation of the purely rank-based statistic $\frac{1}{n}\sum_{i=1}^n\left(c_i^{(n)}-\bar{c}^{(n)}\right)a_\varphi^{(n)}\left(R_i^{(n)}\right)$, where $a_\varphi^{(n)}(R_i^{(n)})$ are, for instance, the traditional exact scores $\mathrm{E}\left[\varphi(F(Z_i^{(n)}))|R_i^{(n)}\right]$ associated with the score-generating function φ. The sign-and-rank statistic $S_{\mathbf{c}}^{(n)}$ thus asymptotically decomposes into two parts; one of them (namely, $S_{\mathbf{c}}^{(n)} - \mathrm{E}\left[S_{\mathbf{c}}^{(n)}\middle|\mathbf{N}^{(n)}\right]$) asymptotically does not depend on $\mathbf{N}^{(n)}$, and represents the contribution of the ranks, while the second one ($\mathrm{E}\left[S_{\mathbf{c}}^{(n)}\middle|\mathbf{N}^{(n)}\right] - \mathrm{E}\left[S_{\mathbf{c}}^{(n)}\right]$) constitutes the contribution of the signs. Moreover, the ranks and $\mathbf{N}^{(n)}$ being mutually independent, these two quantities are orthogonal to each other, and contribute additively to the unconditional asymptotic variance (see the proof of Proposition 3.2 below).

Proof of Lemma 3.1. Part (i) of the lemma follows if we show that, under $\mathcal{H}^{(n)}_{0;f}$,

$$\mathrm{E}\left[\left\{D^{(n)}_{\mathbf{c}}\right\}^2 \Big| \mathbf{Z}^{(n)}_{(\cdot)}\right] = o_{\mathrm{P}}(1), \tag{3.9}$$

as $n \to \infty$, where $D^{(n)}_{\mathbf{c}} := \sqrt{n}\left(S^{(n)}_{\mathbf{c}} - \mathrm{E}\left[S^{(n)}_{\mathbf{c}} \Big| \mathbf{N}^{(n)}\right]\right) - \sqrt{n}\left(T^{(n)}_{\varphi;f} - \mathrm{E}\left[T^{(n)}_{\varphi;f} \Big| \mathbf{Z}^{(n)}_{(\cdot)}\right]\right)$. Obviously,

$$\begin{aligned} S^{(n)}_{\mathbf{c}} - T^{(n)}_{\varphi;f} &= \frac{1}{n}\sum_{i=1}^{n} c^{(n)}_i \left[a^{(n)}\left(\mathbf{N}^{(n)}; R^{(n)}_i\right) - \varphi\left(F(Z^{(n)}_i)\right)\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} c^{(n)}_i \left[a^{(n)}\left(\mathbf{N}^{(n)}; R^{(n)}_i\right) - \varphi\left(F(Z^{(n)}_{\left(R^{(n)}_i\right)})\right)\right] \end{aligned}$$

is, conditionally on $\mathbf{Z}^{(n)}_{(\cdot)}$ (hence also on $\mathbf{N}^{(n)}$), a linear nonserial *rank* statistic that may be written as

$$S^{(n)}_{\mathbf{c}} - T^{(n)}_{\varphi;f} = \frac{1}{n}\sum_{i=1}^{n} c^{(n)}_i \alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)}}\left(R^{(n)}_i\right),$$

with $\alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)}}(i) := a^{(n)}\left(\mathbf{N}^{(n)}; i\right) - \varphi\left(F(Z^{(n)}_{(i)})\right)$, $i \in \{1, \ldots, n\}$. Define $\overline{\alpha}^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)}} := \frac{1}{n}\sum_{i=1}^{n} \alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)}}(i)$.

Now, the maximal invariant $\left(\mathbf{N}^{(n)}, \mathbf{R}^{(n)}\right)$ depends on $\mathbf{Z}^{(n)}_{(\cdot)}$ only through $\mathbf{N}^{(n)}$, and hence, $\mathrm{E}\left[S^{(n)}_{\mathbf{c}} \Big| \mathbf{N}^{(n)}\right] = \mathrm{E}\left[S^{(n)}_{\mathbf{c}} \Big| \mathbf{Z}^{(n)}_{(\cdot)}\right]$. So, we actually have

$$\begin{aligned} \mathrm{E}\left[\left\{D^{(n)}_{\mathbf{c}}\right\}^2 \Big| \mathbf{Z}^{(n)}_{(\cdot)}\right] &= n\mathrm{E}\left[\left\{\left(S^{(n)}_{\mathbf{c}} - T^{(n)}_{\varphi;f}\right) - \mathrm{E}\left[S^{(n)}_{\mathbf{c}} - T^{(n)}_{\varphi;f} \Big| \mathbf{Z}^{(n)}_{(\cdot)}\right]\right\}^2 \Big| \mathbf{Z}^{(n)}_{(\cdot)}\right] \\ &= n\mathrm{Var}\left[S^{(n)}_{\mathbf{c}} - T^{(n)}_{\varphi;f} \Big| \mathbf{Z}^{(n)}_{(\cdot)}\right] = \mathrm{Var}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} c^{(n)}_i \alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)}}\left(R^{(n)}_i\right) \Big| \mathbf{Z}^{(n)}_{(\cdot)}\right]. \end{aligned}$$

Consequently, by Theorem II.3.1.c of Hájek and Šidák (1967, p.61),

$$\begin{aligned} \mathrm{E}\left[\left\{D^{(n)}_{\mathbf{c}}\right\}^2 \Big| \mathbf{Z}^{(n)}_{(\cdot)}\right] &= \frac{1}{n}\sum_{i=1}^{n}\left(c^{(n)}_i - \overline{c}^{(n)}\right)^2 \cdot \frac{1}{n-1}\sum_{j=1}^{n}\left(\alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)}}(j) - \overline{\alpha}^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)}}\right)^2 \\ &\leq \frac{1}{n}\sum_{i=1}^{n}\left(c^{(n)}_i - \overline{c}^{(n)}\right)^2 \cdot \frac{n}{n-1} \cdot \frac{1}{n}\sum_{j=1}^{n}\left(\alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)}}(j)\right)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(c^{(n)}_i - \overline{c}^{(n)}\right)^2 \cdot \frac{n}{n-1} \cdot \mathrm{E}\left[\left\{\alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)}}\left(R^{(n)}_1\right)\right\}^2 \Big| \mathbf{Z}^{(n)}_{(\cdot)}\right] \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(c^{(n)}_i - \overline{c}^{(n)}\right)^2 \cdot \frac{n}{n-1} \cdot \mathrm{E}\left[\left\{a^{(n)}\left(\mathbf{N}^{(n)}; R^{(n)}_1\right) - \varphi\left(F(Z^{(n)}_1)\right)\right\}^2 \Big| \mathbf{Z}^{(n)}_{(\cdot)}\right]. \end{aligned}$$

The assumptions made on the constants $c^{(n)}_i$ ($i = 1, \ldots, n$) and the function φ ensure that the latter expression is $o_{\mathrm{P}}(1)$, under $\mathcal{H}^{(n)}_{0;f}$, as $n \to \infty$. This completes the proof of (3.9).

Part (ii) of the lemma is a direct corollary of (3.5) and Theorem 3.1. □

3.2 Exact and approximate scores.

Let $U^{(n)}_1, \ldots, U^{(n)}_n$ be a n-tuple of i.i.d. random variables uniformly distributed over $(0, 1)$. Define $s_{U^{(n)}_i} = I\left[U^{(n)}_i > 1/2\right] - I\left[U^{(n)}_i < 1/2\right]$, $N^{(n)}_{\mathbf{U};-} = \sum_{i=1}^{n} I\left[U^{(n)}_i < 1/2\right]$, and $N^{(n)}_{\mathbf{U};+} =$

$\sum_{i=1}^n I\left[U_i^{(n)} > 1/2\right]$. Denote by $R_{U_i}^{(n)}$ the rank of $U_i^{(n)}$ among $U_1^{(n)}, \ldots, U_n^{(n)}$, by $U_{(i)-}^{(\nu)}$ $(i = 1, \ldots, \nu)$ the ith order statistic associated with a sample of ν i.i.d. random variables uniformly distributed over $(0, 1/2)$, and by $U_{(i)+}^{(\nu)}$ $(i = 1, \ldots, \nu)$ the ith order statistic associated with a sample of ν i.i.d. random variables uniformly distributed over $(1/2, 1)$. Note that the conditional distribution of $U_i^{(n)}$ given the event $s_{U_i^{(n)}} = -1$ (resp. $s_{U_i^{(n)}} = 1$) is uniform over $(0, 1/2)$ (resp. $(1/2, 1)$). The linear nonserial sign-and-rank statistics constructed from the *exact* and *approximate* scores associated with φ are defined by

$$\begin{aligned} S_{\mathbf{c};\varphi;\text{ex/appr}}^{(n)} &= \frac{1}{n}\sum_{i=1}^n c_i^{(n)} a_{\varphi;\text{ex/appr}}^{(n)}\left(\mathbf{N}^{(n)}; R_i^{(n)}\right) \\ &= \frac{1}{n}\sum_{i=1}^n c_i^{(n)} \left\{ I\left[s_i = -1\right] a_{\varphi;-;\text{ex/appr}}^{(n)}\left(N_-^{(n)}; R_i^{(n)}\right) \right. \\ &\qquad \left. + I\left[s_i = 1\right] a_{\varphi;+;\text{ex/appr}}^{(n)}\left(N_+^{(n)}; R_i^{(n)} - (n - N_+^{(n)})\right)\right\}, \end{aligned} \tag{3.10}$$

where the score functions $a_{\varphi;-;\text{ex}}^{(n)}$, $a_{\varphi;-;\text{appr}}^{(n)}$, $a_{\varphi;+;\text{ex}}^{(n)}$, and $a_{\varphi;+;\text{appr}}^{(n)}$, all defined on the set $\{(\nu; i); \nu, i \in \{1, \ldots, n\} \text{ with } i \leq \nu\}$, are given by

$$a_{\varphi;-;\text{ex}}^{(n)}(\nu; i) = \mathrm{E}\left[\varphi\left(U_1^{(n)}\right) \middle| N_{\mathbf{U};-}^{(n)} = \nu, R_{U_1}^{(n)} = i\right] = \mathrm{E}\left[\varphi\left(U_{(i)-}^{(\nu)}\right)\right], \tag{3.11}$$

$$a_{\varphi;-;\text{appr}}^{(n)}(\nu; i) = \varphi\left(\mathrm{E}\left[U_{(i)-}^{(\nu)}\right]\right) = \varphi\left(\frac{i}{2(\nu+1)}\right), \tag{3.12}$$

$$a_{\varphi;+;\text{ex}}^{(n)}(\nu; i) = \mathrm{E}\left[\varphi\left(U_1^{(n)}\right) \middle| N_{\mathbf{U};+}^{(n)} = \nu, R_{U_1}^{(n)} = (n-\nu) + i\right] = \mathrm{E}\left[\varphi\left(U_{(i)+}^{(\nu)}\right)\right], \tag{3.13}$$

and

$$a_{\varphi;+;\text{appr}}^{(n)}(\nu; i) = \varphi\left(\mathrm{E}\left[U_{(i)+}^{(\nu)}\right]\right) = \varphi\left(\frac{1}{2} + \frac{i}{2(\nu+1)}\right). \tag{3.14}$$

Observe that, under $\mathcal{H}_{0;f}^{(n)}$, $S_{\mathbf{c};\varphi;\text{ex}}^{(n)} = \mathrm{E}\left[T_{\varphi;f}^{(n)} \middle| \mathbf{N}^{(n)}, \mathbf{R}^{(n)}\right] = \mathrm{E}\left[T_{\varphi;f}^{(n)} \middle| \mathbf{s}^{(n)}, \mathbf{R}^{(n)}\right]$.

We then have the following proposition.

Proposition 3.1 *Let $\varphi : (0,1) \to \mathbb{R}$ be a non-constant square-integrable function. Then, φ is a score-generating function for $a_{\varphi;\text{ex}}^{(n)}$. If moreover φ is the difference of two nondecreasing square-integrable functions, then φ is also a score-generating function for $a_{\varphi;\text{appr}}^{(n)}$.*

Proof. (a) Let us first consider the *exact* scores defined by relations (3.10), (3.11), and (3.13), and let us show that, under $\mathcal{H}_{0;f}^{(n)}$,

$$\mathrm{E}\left[\left\{a_{\varphi;\text{ex}}^{(n)}\left(\mathbf{N}^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| \mathbf{N}^{(n)}\right] = o_{\mathrm{P}}(1), \tag{3.15}$$

as $n \to \infty$. By the definition of $a_{\varphi;\text{ex}}^{(n)}$, we have

$$\begin{aligned} &\mathrm{E}\left[\left\{a_{\varphi;\text{ex}}^{(n)}\left(\mathbf{N}^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| \mathbf{N}^{(n)}\right] \\ &\quad = \mathrm{E}\left[\left\{I\left[s_1 = -1\right]\left(a_{\varphi;-;\text{ex}}^{(n)}\left(N_-^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right)\right)\right.\right. \\ &\qquad \left.\left. + I\left[s_1 = 1\right]\left(a_{\varphi;+;\text{ex}}^{(n)}\left(N_+^{(n)}; R_1^{(n)} - (n - N_+^{(n)})\right) - \varphi\left(F(Z_1^{(n)})\right)\right)\right\}^2 \middle| \mathbf{N}^{(n)}\right] \\ &\quad = \mathrm{E}\left[I\left[s_1 = -1\right]\left\{a_{\varphi;-;\text{ex}}^{(n)}\left(N_-^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| N_-^{(n)}\right] \\ &\qquad + \mathrm{E}\left[I\left[s_1 = 1\right]\left\{a_{\varphi;+;\text{ex}}^{(n)}\left(N_+^{(n)}; R_1^{(n)} - (n - N_+^{(n)})\right) - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| N_+^{(n)}\right]. \end{aligned}$$

Hence, condition (3.15) holds if, under $\mathcal{H}_{0;f}^{(n)}$,

$$\mathrm{E}\left[I\left[s_1=-1\right] \left\{ a_{\varphi;-;\mathrm{ex}}^{(n)}\left(N_-^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| N_-^{(n)}\right]$$

and

$$\mathrm{E}\left[I\left[s_1=1\right] \left\{ a_{\varphi;+;\mathrm{ex}}^{(n)}\left(N_+^{(n)}; R_1^{(n)} - (n-N_+^{(n)})\right) - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| N_+^{(n)}\right]$$

are $o_{\mathrm{P}}(1)$, as $n \to \infty$. Since both terms may be treated similarly, we only consider the first one. Observe that

$$\begin{aligned}
\mathrm{E}&\left[I\left[s_1=-1\right] \left\{ a_{\varphi;-;\mathrm{ex}}^{(n)}\left(N_-^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| N_-^{(n)}\right] \\
&= \mathrm{E}\left[\mathrm{E}\left[I\left[s_1=-1\right] \left\{ a_{\varphi;-;\mathrm{ex}}^{(n)}\left(N_-^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| N_-^{(n)}, s_1\right] \middle| N_-^{(n)}\right] \\
&= \mathrm{E}\left[I\left[s_1=-1\right] \sum_{\nu=1}^{n} I\left[N_-^{(n)}=\nu\right] \right.\\
&\qquad \left. \times \mathrm{E}\left[\left\{ a_{\varphi;-;\mathrm{ex}}^{(n)}\left(N_-^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| N_-^{(n)}=\nu, s_1=-1\right] \middle| N_-^{(n)}\right] \\
&= \sum_{\nu=1}^{n} I\left[N_-^{(n)}=\nu\right] \mathrm{E}\left[\left\{ a_{\varphi;-;\mathrm{ex}}^{(n)}\left(N_-^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| N_-^{(n)}=\nu, s_1=-1\right] \\
&\qquad \times \mathrm{E}\left[I\left[s_1=-1\right] \middle| N_-^{(n)}\right].
\end{aligned}$$

Recall that $\mathrm{E}\left[I\left[s_1=-1\right] \middle| N_-^{(n)}\right] = \frac{N_-^{(n)}}{n}$ which, by the strong law of large numbers, converges almost surely to $\frac{1}{2}$ under $\mathcal{H}_{0;f}^{(n)}$, as $n \to \infty$. Hence, the required convergence condition holds if

$$\mathrm{E}\left[\left\{ a_{\varphi;-;\mathrm{ex}}^{(n)}\left(N_-^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| N_-^{(n)}, s_1=-1\right] = o_{\mathrm{P}}(1), \qquad (3.16)$$

under $\mathcal{H}_{0;f}^{(n)}$, as $n \to \infty$. By the definition of $a_{\varphi;-;\mathrm{ex}}^{(n)}$, we actually need to show that

$$\mathrm{E}\left[\left\{ \mathrm{E}\left[\varphi\left(F(Z_1^{(n)})\right) \middle| s_1=-1, N_-^{(n)}, R_1^{(n)}\right] - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| N_-^{(n)}, s_1=-1\right] = o_{\mathrm{P}}(1), \qquad (3.17)$$

under $\mathcal{H}_{0;f}^{(n)}$, as $n \to \infty$. Now, since $F(Z_1^{(n)})$ is, under $\mathcal{H}_{0;f}^{(n)}$ and conditionally on $s_1=-1$, uniform over the interval $(0, 1/2)$, (3.17) follows directly from a slight generalization of Theorem V.1.4.a of Hájek and Šidák (1967, p.157). More precisely, let $U_{(a,b)1}, U_{(a,b)2}, \ldots$ be independent and uniformly distributed over (a, b) $(0 < a < b < 1)$. Let $R_{(a,b)i}^{(N)}$ denote the rank of $U_{(a,b)i}$ $(1 \le i \le N)$ in N-tuple $U_{(a,b)1}, \ldots, U_{(a,b)N}$. Then, if $\varphi : (0, 1) \to \mathbb{R}$ is square-integrable,

$$\lim_{N\to\infty} \mathrm{E}\left[\left\{ \mathrm{E}\left[\varphi\left(U_{(a,b)1}\right) \middle| R_{(a,b)1}^{(N)}\right] - \varphi\left(U_{(a,b)1}\right)\right\}^2 \right] = 0.$$

(b) Let us now consider the *approximate* scores defined by (3.10), (3.12), and (3.14). Using the same arguments as in part (a) of this proof, we see that (3.3) holds for $a_{\varphi;\mathrm{appr}}^{(n)}$ if, under $\mathcal{H}_{0;f}^{(n)}$,

$$\mathrm{E}\left[\left\{ a_{\varphi;-;\mathrm{appr}}^{(n)}\left(N_-^{(n)}; R_1^{(n)}\right) - \varphi\left(F(Z_1^{(n)})\right)\right\}^2 \middle| N_-^{(n)}, s_1=-1\right]$$

and

$$\mathrm{E}\left[\left\{a^{(n)}_{\varphi;+;\mathrm{appr}}\left(N^{(n)}_{+};R^{(n)}_{1}-(n-N^{(n)}_{+})\right)-\varphi\left(F(Z^{(n)}_{1})\right)\right\}^{2}\middle|\, N^{(n)}_{+},s_1=1\right]$$

are $o_{\mathrm{P}}(1)$, as $n\to\infty$. Again, we only consider the first term. We have

$$\begin{aligned}
&\mathrm{E}\left[\left\{a^{(n)}_{\varphi;-;\mathrm{appr}}\left(N^{(n)}_{-};R^{(n)}_{1}\right)-\varphi\left(F(Z^{(n)}_{1})\right)\right\}^{2}\middle|\, N^{(n)}_{-},s_1=-1\right]\\
&\quad=\mathrm{E}\Big[\Big\{\left(a^{(n)}_{\varphi;-;\mathrm{appr}}\left(N^{(n)}_{-};R^{(n)}_{1}\right)-a^{(n)}_{\varphi;-;\mathrm{ex}}\left(N^{(n)}_{-};R^{(n)}_{1}\right)\right)\\
&\qquad\quad+\left(a^{(n)}_{\varphi;-;\mathrm{ex}}\left(N^{(n)}_{-};R^{(n)}_{1}\right)-\varphi\left(F(Z^{(n)}_{1})\right)\right)\Big\}^{2}\Big|\, N^{(n)}_{-},s_1=-1\Big]\\
&\quad\leq 2\mathrm{E}\left[\left\{a^{(n)}_{\varphi;-;\mathrm{appr}}\left(N^{(n)}_{-};R^{(n)}_{1}\right)-a^{(n)}_{\varphi;-;\mathrm{ex}}\left(N^{(n)}_{-};R^{(n)}_{1}\right)\right\}^{2}\middle|\, N^{(n)}_{-},s_1=-1\right]\\
&\qquad\quad+2\mathrm{E}\left[\left\{a^{(n)}_{\varphi;-;\mathrm{ex}}\left(N^{(n)}_{-};R^{(n)}_{1}\right)-\varphi\left(F(Z^{(n)}_{1})\right)\right\}^{2}\middle|\, N^{(n)}_{-},s_1=-1\right].
\end{aligned}$$

The second term in this sum has been treated before. Hence, it just remains to show that

$$\mathrm{E}\left[\left\{a^{(n)}_{\varphi;-;\mathrm{appr}}\left(N^{(n)}_{-};R^{(n)}_{1}\right)-a^{(n)}_{\varphi;-;\mathrm{ex}}\left(N^{(n)}_{-};R^{(n)}_{1}\right)\right\}^{2}\middle|\, N^{(n)}_{-},s_1=-1\right]=o_{\mathrm{P}}(1), \tag{3.18}$$

under $\mathcal{H}^{(n)}_{0;f}$, as $n\to\infty$. Denoting by $\lfloor x\rfloor$ the integer part of x ($x\in\mathbb{R}^{+}$), we may write

$$\begin{aligned}
&\mathrm{E}\left[\left\{a^{(n)}_{\varphi;-;\mathrm{appr}}\left(N^{(n)}_{-};R^{(n)}_{1}\right)-a^{(n)}_{\varphi;-;\mathrm{ex}}\left(N^{(n)}_{-};R^{(n)}_{1}\right)\right\}^{2}\middle|\, N^{(n)}_{-},s_1=-1\right]\\
&\quad=\frac{1}{N^{(n)}_{-}}\sum_{i=1}^{N^{(n)}_{-}}\left\{a^{(n)}_{\varphi;-;\mathrm{appr}}\left(N^{(n)}_{-};i\right)-a^{(n)}_{\varphi;-;\mathrm{ex}}\left(N^{(n)}_{-};i\right)\right\}^{2}\\
&\quad=\int_0^1\left\{a^{(n)}_{\varphi;-;\mathrm{appr}}\left(N^{(n)}_{-};1+\left\lfloor N^{(n)}_{-}u\right\rfloor\right)-a^{(n)}_{\varphi;-;\mathrm{ex}}\left(N^{(n)}_{-};1+\left\lfloor N^{(n)}_{-}u\right\rfloor\right)\right\}^{2}du\\
&\quad=\int_0^1\Big\{\left(a^{(n)}_{\varphi;-;\mathrm{appr}}\left(N^{(n)}_{-};1+\left\lfloor N^{(n)}_{-}u\right\rfloor\right)-\varphi\left(u/2\right)\right)\\
&\qquad\quad+\left(\varphi\left(u/2\right)-a^{(n)}_{\varphi;-;\mathrm{ex}}\left(N^{(n)}_{-};1+\left\lfloor N^{(n)}_{-}u\right\rfloor\right)\right)\Big\}^{2}du\\
&\quad\leq 2\int_0^1\left\{a^{(n)}_{\varphi;-;\mathrm{appr}}\left(N^{(n)}_{-};1+\left\lfloor N^{(n)}_{-}u\right\rfloor\right)-\varphi\left(u/2\right)\right\}^{2}du\\
&\qquad\quad+2\int_0^1\left\{a^{(n)}_{\varphi;-;\mathrm{ex}}\left(N^{(n)}_{-};1+\left\lfloor N^{(n)}_{-}u\right\rfloor\right)-\varphi\left(u/2\right)\right\}^{2}du.
\end{aligned}$$

The required convergence (3.18) then follows from an obvious adaptation of Lemma V.1.6.a (p.164) and Theorem V.1.4.b (p.158) in Hájek and Šidák (1967). □

3.3 Asymptotic representation and asymptotic normality

We now can state, for the nonserial case, the main result of this paper.

Proposition 3.2 *Let $\varphi:(0,1)\to\mathbb{R}$ be a non-constant square-integrable score-generating function for $S^{(n)}_{\mathbf{c};\varphi;\mathrm{ex/appr}}$, and let the regression constants $c^{(n)}_i$ ($i=1,\dots,n$) satisfy the Noether*

condition (N). Whenever approximate scores are considered, assume that φ is the difference of two non-decreasing square-integrable functions. Assume moreover that $\bar{c}^{(n)} = O(1)$ and $\sum_{i=1}^n (c_i^{(n)} - \bar{c}^{(n)})^2 = O(n)$, as $n \to \infty$. Let $\mu_\varphi^- := \int_0^{1/2} \varphi(u)du$, $\mu_\varphi^+ := \int_{1/2}^1 \varphi(u)du$, and $\mu_\varphi := \int_0^1 \varphi(u)du$. Then, writing $S_{\mathbf{c}}^{(n)}$ for either $S_{\mathbf{c};\varphi;\mathrm{ex}}^{(n)}$ or $S_{\mathbf{c};\varphi;\mathrm{appr}}^{(n)}$,
(i) (asymptotic representation) under $\mathcal{H}_{0;f}^{(n)}$, as $n \to \infty$,

$$S_{\mathbf{c}}^{(n)} - \mathrm{E}\left[S_{\mathbf{c}}^{(n)}\right] = \frac{1}{n}\sum_{i=1}^n (c_i^{(n)} - \bar{c}^{(n)})\varphi(F(Z_i^{(n)})) + \bar{c}^{(n)}\left\{2\frac{N_-^{(n)}}{n}\mu_\varphi^- + 2\frac{N_+^{(n)}}{n}\mu_\varphi^+ - \mu_\varphi\right\} + o_{\mathrm{P}}(1/\sqrt{n}), \tag{3.19}$$

and
(ii) (asymptotic normality) under $\mathcal{H}_0^{(n)}$, as $n \to \infty$,

$$\sqrt{n}\left(\frac{S_{\mathbf{c}}^{(n)} - \mathrm{E}\left[S_{\mathbf{c}}^{(n)}\right]}{\sqrt{\frac{\sigma_\varphi^2}{n}\sum_{i=1}^n (c_i^{(n)} - \bar{c}^{(n)})^2 + \left[\bar{c}^{(n)}\left(\mu_\varphi^- - \mu_\varphi^+\right)\right]^2}}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1). \tag{3.20}$$

Proof. (i) We first establish (3.19) for exact scores. From (3.5) and (3.7), we have

$$S_{\mathbf{c};\varphi;\mathrm{ex}}^{(n)} - \mathrm{E}\left[S_{\mathbf{c};\varphi;\mathrm{ex}}^{(n)}\right] = \frac{1}{n}\sum_{i=1}^n (c_i^{(n)} - \bar{c}^{(n)})\varphi(F(Z_i^{(n)})) + \mathrm{E}\left[S_{\mathbf{c};\varphi;\mathrm{ex}}^{(n)}\middle|\mathbf{N}^{(n)}\right] - \mathrm{E}\left[S_{\mathbf{c};\varphi;\mathrm{ex}}^{(n)}\right] + o_{\mathrm{P}}(1/\sqrt{n}). \tag{3.21}$$

Since

$$\begin{aligned}
\mathrm{E}\left[S_{\mathbf{c};\varphi;\mathrm{ex}}^{(n)}\middle|\mathbf{N}^{(n)}\right] &= \mathrm{E}\left[\mathrm{E}\left[T_{\varphi;f}^{(n)}\middle|\mathbf{N}^{(n)}, \mathbf{R}^{(n)}\right]\middle|\mathbf{N}^{(n)}\right] = \mathrm{E}\left[T_{\varphi;f}^{(n)}\middle|\mathbf{N}^{(n)}\right] \\
&= \mathrm{E}\left[\mathrm{E}\left[T_{\varphi;f}^{(n)}\middle|\mathbf{s}^{(n)}\right]\middle|\mathbf{N}^{(n)}\right] = \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^n c_i^{(n)}\mathrm{E}\left[\varphi(F(Z_i^{(n)}))\middle| s_i\right]\middle|\mathbf{N}^{(n)}\right],
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{E}\left[\varphi(F(Z_i^{(n)}))\middle| s_i\right] &= I[s_i = -1]\mathrm{E}\left[\varphi(F(Z_i^{(n)}))\middle| s_i = -1\right] \\
&\quad + I[s_i = 1]\mathrm{E}\left[\varphi(F(Z_i^{(n)}))\middle| s_i = 1\right] \\
&= I[s_i = -1]\int_0^{1/2}\varphi(u)2du \\
&\quad + I[s_i = 1]\int_{1/2}^1 \varphi(u)2du \\
&= 2I[s_i = -1]\mu_\varphi^- + 2I[s_i = 1]\mu_\varphi^+,
\end{aligned}$$

it follows that

$$\mathrm{E}\left[S_{\mathbf{c};\varphi;\mathrm{ex}}^{(n)}\middle|\mathbf{N}^{(n)}\right] = \frac{2}{n}\sum_{i=1}^n c_i^{(n)}\mathrm{E}\left[I[s_i = -1]\mu_\varphi^- + I[s_i = 1]\mu_\varphi^+\middle|\mathbf{N}^{(n)}\right] = 2\bar{c}^{(n)}\left(\frac{N_-^{(n)}}{n}\mu_\varphi^- + \frac{N_+^{(n)}}{n}\mu_\varphi^+\right)$$

and

$$\begin{aligned}
\mathrm{E}\left[S_{\mathbf{c};\varphi;\mathrm{ex}}^{(n)}\middle|\mathbf{N}^{(n)}\right] - \mathrm{E}\left[S_{\mathbf{c};\varphi;\mathrm{ex}}^{(n)}\right] &= 2\bar{c}^{(n)}\left(\frac{N_-^{(n)}}{n}\mu_\varphi^- + \frac{N_+^{(n)}}{n}\mu_\varphi^+\right) - 2\bar{c}^{(n)}\mathrm{E}\left(\frac{N_-^{(n)}}{n}\mu_\varphi^- + \frac{N_+^{(n)}}{n}\mu_\varphi^+\right) \\
&= \bar{c}^{(n)}\left(2\frac{N_-^{(n)}}{n}\mu_\varphi^- + 2\frac{N_+^{(n)}}{n}\mu_\varphi^+ - \mu_\varphi\right) \qquad (3.22)
\end{aligned}$$

which, along with (3.21), establishes (3.19) for exact scores.

Turning to approximate scores, we can assume without loss of generality that φ is non-decreasing. Since (3.21) also holds if approximate scores are substituted for the exact ones, it is sufficient, in order for (3.19) to hold for approximate scores, to show that the difference

$$E^{(n)} := \left\{ \mathrm{E}\left[S^{(n)}_{\mathbf{c};\varphi;\mathrm{appr}} \Big| \mathbf{N}^{(n)} \right] - \mathrm{E}\left[S^{(n)}_{\mathbf{c};\varphi;\mathrm{appr}} \right] \right\} - \left\{ \mathrm{E}\left[S^{(n)}_{\mathbf{c};\varphi;\mathrm{ex}} \Big| \mathbf{N}^{(n)} \right] - \mathrm{E}\left[S^{(n)}_{\mathbf{c};\varphi;\mathrm{ex}} \right] \right\} \tag{3.23}$$

is $o_{\mathrm{P}}(1/\sqrt{n})$. Note that

$$\begin{aligned} \mathrm{E}\left[S^{(n)}_{\mathbf{c};\varphi;\mathrm{appr}} \Big| \mathbf{N}^{(n)} \right] &= \bar{c}^{(n)} \frac{1}{n} \left\{ \sum_{j=1}^{N_-^{(n)}} \varphi\left(\frac{j}{2(N_-^{(n)}+1)} \right) + \sum_{j=1}^{N_+^{(n)}} \varphi\left(\frac{1}{2} + \frac{j}{2(N_+^{(n)}+1)} \right) \right\} \\ &= \bar{c}^{(n)} \left\{ 2\frac{N_-^{(n)}}{n} D^-_{N_-^{(n)}} + 2\frac{N_+^{(n)}}{n} D^+_{N_+^{(n)}} \right\}, \end{aligned} \tag{3.24}$$

where $D^-_m := \frac{1}{2m}\sum_{j=1}^m \varphi\left(\frac{j}{2(m+1)}\right)$ and $D^+_m := \frac{1}{2m}\sum_{j=1}^m \varphi\left(\frac{1}{2} + \frac{j}{2(m+1)}\right)$ are Riemann sums for the integrals $\mu^-_\varphi := \int_0^{1/2} \varphi(u)du$ and $\mu^+_\varphi := \int_{1/2}^1 \varphi(u)du$, respectively. Since φ is square-integrable, any term in the Riemann sum $\frac{1}{2m}\sum_{j=1}^m \varphi^2\left(\frac{1}{2} + \frac{j}{2(m+1)}\right)$ associated with $\int_{1/2}^1 \varphi^2(u)du$ is $o(1)$ as $m \to \infty$. This implies that $\frac{1}{2m}\varphi\left(\frac{1}{2} + \frac{m}{2(m+1)}\right)$ is $o(1/\sqrt{m})$, hence, in view of the fact that $N_+^{(n)} = O_{\mathrm{P}}(n)$, that $\frac{1}{2N_+^{(n)}}\varphi\left(\frac{1}{2} + \frac{N_+^{(n)}}{2(N_+^{(n)}+1)}\right) = o_{\mathrm{P}}(1/\sqrt{n})$ as $n \to \infty$. The same reasoning shows that any finite sum of Riemann terms in $D^-_{N_-^{(n)}}$ or $D^+_{N_+^{(n)}}$ actually is $o_{\mathrm{P}}(1/\sqrt{n})$ as $n \to \infty$.

Now, any Riemann sum D^+_m for μ^+_φ satisfies, since φ is non-decreasing, the double inequality $\underline{D}^+_m \leq D^+_m \leq \bar{D}^+_m$, where $\underline{D}^+_m := \frac{1}{2m}\sum_{j=0}^{m-1} \varphi(\frac{1}{2} + \frac{j}{2(m+1)})$ and $\bar{D}^+_m := \frac{1}{2m}\sum_{j=1}^m \varphi(\frac{1}{2} + \frac{j}{2(m+1)})$ are the upper and lower Darboux sums associated with $\int_{1/2}^1 \varphi(u)du$. The difference $\bar{D}^+_m - \underline{D}^+_m$ clearly is $\frac{1}{2m}(\varphi(\frac{1}{2} + \frac{m}{2(m+1)}) - \varphi(\frac{1}{2}))$, which is $o(1/\sqrt{m})$ as $m \to \infty$. Hence, for any Riemann sum, $D^+_m - \mu^+_\varphi$ is also $o(1/\sqrt{m})$, so that $D^+_{N_+^{(n)}} - \mu^+_\varphi = o_{\mathrm{P}}(1/\sqrt{n})$ as $n \to \infty$.

Further, since the sequence $D^+_m - \mu^+_\varphi$ converges to zero, it is bounded, so that $D^+_{N_+^{(n)}} - \mu^+_\varphi$ is uniformly integrable , and

$$\mathrm{E}\left[\frac{N_+^{(n)}}{n} D^+_{N_+^{(n)}} - \frac{1}{2}\mu^+_\varphi \right] = o(1/\sqrt{n}) \quad \text{as} \quad n \to \infty.$$

A similar reasoning of course holds for $D^-_{N_-^{(n)}}$ and μ^-_φ. Going back to (3.24) and recalling that $\bar{c}^{(n)} = O(1)$, we thus obtain the desired result that $E^{(n)}$ is $o_{\mathrm{P}}(1/\sqrt{n})$. This completes the proof of part (i) of the proposition.

(ii) As for asymptotic normality, elementary calculations yield

$$\sqrt{n}\, \bar{c}^{(n)} \left(2\frac{N_-^{(n)}}{n}\mu^-_\varphi + 2\frac{N_+^{(n)}}{n}\mu^+_\varphi - \mu_\varphi \right) = \bar{c}^{(n)} \left(2\left(\mu^-_\varphi - \mu^+_\varphi\right) \left(\frac{N_-^{(n)}}{n} - \frac{1}{2} \right) / \sqrt{1/4n} \right) \sqrt{1/4},$$

which, since $\left(\frac{N_-^{(n)}}{n} - \frac{1}{2}\right)/\sqrt{1/4n}$ is asymptotically standard normal, is also asymptotically normal, with mean zero and asymptotic variance $\left[\bar{c}^{(n)}(\mu^-_\varphi - \mu^+_\varphi)\right]^2$. The remark (right after

Lemma 3.1) on the orthogonality between the two parts of the asymptotic representation of $S_{\mathbf{c}}^{(n)}$ completes the proof. □

Test statistics related to "regression coefficients" naturally involve "regression constants" $c_i^{(n)}$ that are not all equal. Quite on the contrary, test statistics related to location and intercepts do not involve any constants—more precisely, they are still of the form $S_{\mathbf{c}}^{(n)}$, but with constants $c_i^{(n)}$ all equal to 1. Proposition 3.2 thus does not apply. For the sake of completeness, this case is treated now.

Proposition 3.3 *Let φ be a non-constant square-integrable score-generating function for $S_{\varphi;\mathrm{ex/appr}}^{(n)} := \frac{1}{n}\sum_{i=1}^n a_{\varphi;\mathrm{ex/appr}}^{(n)}(\mathbf{N}^{(n)}; R_i^{(n)})$. Whenever approximate scores are considered, assume that φ is the difference of two non-decreasing square-integrable functions. Let μ_φ^-, μ_φ^+, and μ_φ be defined as in Proposition 3.2. Then,*

(i) (asymptotic representation) under $\mathcal{H}_{0;f}^{(n)}$, as $n \to \infty$,

$$\begin{aligned} S_{\varphi;\mathrm{ex}}^{(n)} - \mathrm{E}\left[S_{\varphi;\mathrm{ex}}^{(n)}\right] &= 2\frac{N_-^{(n)}}{n}\mu_\varphi^- + 2\frac{N_+^{(n)}}{n}\mu_\varphi^+ - \mu_\varphi \\ &= S_{\varphi;\mathrm{appr}}^{(n)} - \mathrm{E}\left[S_{\varphi;\mathrm{appr}}^{(n)}\right] + o_{\mathrm{P}}(1/\sqrt{n}) \end{aligned} \tag{3.25}$$

and

(ii) (asymptotic normality) under $\mathcal{H}_0^{(n)}$, as $n \to \infty$,

$$\sqrt{n}\left(S_{\varphi;\mathrm{ex/appr}}^{(n)} - \mathrm{E}\left[S_{\varphi;\mathrm{ex/appr}}^{(n)}\right]\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \left(\mu_\varphi^- - \mu_\varphi^+\right)^2\right).$$

Proof. Clearly,

$$S_{\varphi;\mathrm{ex}}^{(n)} = \frac{1}{n}\mathrm{E}\left[\sum_{i=1}^n \varphi(F(Z_i^{(n)}))\middle| \mathbf{N}^{(n)}, \mathbf{R}^{(n)}\right] = \frac{1}{n}\mathrm{E}\left[\sum_{i=1}^n \varphi(F(Z_i^{(n)}))\middle| \mathbf{N}^{(n)}\right].$$

Thus, since $\mathbf{N}^{(n)}$ is $\mathbf{s}^{(n)}$-measurable,

$$\begin{aligned} S_{\varphi;\mathrm{ex}}^{(n)} &= \frac{1}{n}\sum_{i=1}^n \mathrm{E}\left[\mathrm{E}\left[\varphi(F(Z_i^{(n)}))\middle|\mathbf{s}^{(n)}\right]\middle|\mathbf{N}^{(n)}\right] \\ &= \frac{1}{n}\sum_{i=1}^n \mathrm{E}\left[I[s_i=-1]\int_0^{1/2}\varphi(u)2du + I[s_i=1]\int_{1/2}^1 \varphi(u)2du \middle| \mathbf{N}^{(n)}\right] \\ &= 2\frac{N_-^{(n)}}{n}\mu_\varphi^- + 2\frac{N_+^{(n)}}{n}\mu_\varphi^+ \end{aligned} \tag{3.26}$$

and $\mathrm{E}\left[S_{\varphi;\mathrm{ex}}^{(n)}\right] = \mu_\varphi$. This takes care of the exact-score part of (i)—which provides an exact representation of $\sqrt{n}S_{\varphi;\mathrm{ex}}^{(n)}$, not just an asymptotic one. Let us now consider the approximate scores. We have

$$\begin{aligned} S_{\varphi;\mathrm{appr}}^{(n)} &= \frac{1}{n}\sum_{i=1}^n \left\{ I[s_i=-1]\varphi\left(\frac{R_i^{(n)}}{2(N_-^{(n)}+1)}\right) + I[s_i=1]\varphi\left(\frac{1}{2} + \frac{R_i^{(n)} - (n - N_+^{(n)})}{2(N_+^{(n)}+1)}\right)\right\} \\ &= \frac{1}{n}\sum_{j=1}^{N_-^{(n)}} \varphi\left(\frac{j}{2(N_-^{(n)}+1)}\right) + \frac{1}{n}\sum_{j=1}^{N_+^{(n)}} \varphi\left(\frac{1}{2} + \frac{j}{2(N_+^{(n)}+1)}\right). \end{aligned}$$

Hence, recalling (3.26) and the definitions of $D^-_{N^{(n)}_-}$ and $D^+_{N^{(n)}_+}$ given in the proof of Proposition 3.2,

$$\begin{aligned} S^{(n)}_{\varphi;\text{appr}} - S^{(n)}_{\varphi;\text{ex}} &= \left\{ \frac{1}{n} \sum_{j=1}^{N^{(n)}_-} \varphi \left(\frac{j}{2(N^{(n)}_- + 1)} \right) + \frac{1}{n} \sum_{j=1}^{N^{(n)}_+} \varphi \left(\frac{1}{2} + \frac{j}{2(N^{(n)}_+ + 1)} \right) \right\} \\ &- \left\{ 2\frac{N^{(n)}_-}{n} \mu^-_\varphi + 2\frac{N^{(n)}_+}{n} \mu^+_\varphi \right\} \\ &= 2\frac{N^{(n)}_-}{n} \left(D^-_{N^{(n)}_-} - \mu^-_\varphi \right) + 2\frac{N^{(n)}_+}{n} \left(D^+_{N^{(n)}_+} - \mu^+_\varphi \right). \end{aligned}$$

The arguments developed in the proof of Proposition 3.2 establish that $S^{(n)}_{\varphi;\text{appr}} - S^{(n)}_{\varphi;\text{ex}} = o_{\text{P}}(1/\sqrt{n})$ and $\text{E}\left[S^{(n)}_{\varphi;\text{appr}}\right] - \text{E}\left[S^{(n)}_{\varphi;\text{ex}}\right] = o(1/\sqrt{n})$, as $n \to \infty$. This completes the proof of part (i) of Proposition 3.3.

As for part (ii), asymptotic normality readily follows from the de Moivre-Laplace version of the Central Limit Theorem, with an obvious computation of the mean and variance of $S^{(n)}$. □

4 Serial linear sign-and-rank statistics

4.1 Definition and conditional asymptotic representation

Nonserial sign-and-rank statistics, just as their traditional rank-based counterparts, are inefficient in the context of dependent observations : only serial statistics can capture the effects of serial dependence. Define a linear serial sign-and-rank statistic of order k ($k \in \{1, \ldots, n-1\}$) as a statistic of the form

$$S^{(n)}_k := \frac{1}{n-k} \sum_{t=k+1}^{n} a^{(n)}_k \left(\mathbf{N}^{(n)}; R^{(n)}_t, \ldots, R^{(n)}_{t-k} \right),$$

where $a^{(n)}_k(\cdot; \cdot, \ldots, \cdot)$ is defined over the product of the set $\{(\nu, \eta); \nu, \eta \in \{0, 1, \ldots, n\}, \eta \leq n - \nu\}$ with the set of all $(k+1)$-tuples of distinct integers in $\{1, \ldots, n\}$. The asymptotic mean and variance of $S^{(n)}_k$ are given in the subsequent Proposition 4.1.

Here also, an asymptotic representation result is proved, establishing the asymptotic equivalence between $S^{(n)}_k$ and a "parametric" serial statistic $T^{(n)}_k$. The asymptotic normality of $T^{(n)}_k$ then entails that of $S^{(n)}_k$. A function $\varphi_k : (0,1)^{k+1} \to \mathbb{R}$ is a score-generating function for the serial score function $a^{(n)}_k$ if

$$\text{E}\left[\left\{ a^{(n)}_k \left(\mathbf{N}^{(n)}; R^{(n)}_{k+1}, \ldots, R^{(n)}_1 \right) - \varphi_k \left(F(Z^{(n)}_{k+1}), \ldots, F(Z^{(n)}_1) \right) \right\}^2 \middle| \mathbf{Z}^{(n)}_{(\cdot)} \right] = o_{\text{P}}(1), \tag{4.1}$$

under $\mathcal{H}^{(n)}_{0;f}$, as $n \to \infty$. Obviously, (4.1) automatically holds if, under $\mathcal{H}^{(n)}_{0;f}$,

$$\text{E}\left[\left\{ a^{(n)}_k \left(\mathbf{N}^{(n)}; R^{(n)}_{k+1}, \ldots, R^{(n)}_1 \right) - \varphi_k \left(F(Z^{(n)}_{k+1}), \ldots, F(Z^{(n)}_1) \right) \right\}^2 \middle| \mathbf{N}^{(n)} \right] = o_{\text{P}}(1), \tag{4.2}$$

as $n \to \infty$. We then have the following conditional asymptotic representation and asymptotic normality results, which is the serial counterpart of Lemma 3.1.

Lemma 4.1 *Let $\varphi_k : (0,1)^{k+1} \to \mathbb{R}$ be a score-generating function for $a_k^{(n)}$. Then,*
(i) (asymptotic representation) under $\mathcal{H}_{0;f}^{(n)}$, as $n \to \infty$,

$$S_k^{(n)} - \mathrm{E}\left[S_k^{(n)} \middle| \mathbf{N}^{(n)} \right] = T_{\varphi_k;f;k}^{(n)} - \mathrm{E}\left[T_{\varphi_k;f;k}^{(n)} \middle| \mathbf{Z}_{(\cdot)}^{(n)} \right] + o_{\mathrm{P}}(1/\sqrt{n}), \tag{4.3}$$

where

$$T_{\varphi_k;f;k}^{(n)} := \frac{1}{n-k} \sum_{t=k+1}^{n} \varphi_k \left(F(Z_t^{(n)}), \ldots, F(Z_{t-k}^{(n)}) \right)$$

and

$$\mathrm{E}\left[T_{\varphi_k;f;k}^{(n)} \middle| \mathbf{Z}_{(\cdot)}^{(n)} \right] = [n(n-1)\ldots(n-k)]^{-1} \sum_{1 \le t_1 \neq \ldots \neq t_{k+1} \le n} \ldots \sum \varphi_k \left(F(Z_{t_1}^{(n)}), \ldots, F(Z_{t_{k+1}}^{(n)}) \right);$$

(ii) (asymptotic normality) if moreover $0 < \int_{(0,1)^{k+1}} |\varphi_k (u_{k+1}, \ldots, u_1)|^{2+\delta} du_1 \ldots du_{k+1} < \infty$ for some $\delta > 0$, then, under $\mathcal{H}_0^{(n)}$, as $n \to \infty$,

$$\sqrt{n-k} \left(S_k^{(n)} - \mathrm{E}\left[S_k^{(n)} \middle| \mathbf{N}^{(n)} \right] \right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, V^2\right),$$

where, denoting by $U_1, U_2, \ldots$ an i.i.d. sequence uniformly distributed over $(0,1)$,

$$V^2 := \mathrm{E}\left[\left\{ \varphi_k^* (U_{k+1}, \ldots, U_1) \right\}^2 \right] + 2 \sum_{j=1}^{k} \mathrm{E}\left[\varphi_k^* (U_{k+1}, \ldots, U_1) \, \varphi_k^* (U_{k+1+j}, \ldots, U_{1+j}) \right] \tag{4.4}$$

with, for $u_1, \ldots, u_{k+1} \in (0,1)$,

$$\varphi_k^* (u_{k+1}, \ldots, u_1) := \varphi_k (u_{k+1}, \ldots, u_1) - \sum_{l=1}^{k+1} \mathrm{E}\left[\varphi_k (U_{k+1}, \ldots, U_1) \middle| U_l = u_1 \right] + k\mathrm{E}\left[\varphi_k (U_{k+1}, \ldots, U_1) \right].$$

Proof. In order to prove part (i) of the proposition, we first show that, under $\mathcal{H}_{0;f}^{(n)}$, as $n \to \infty$, $\mathrm{E}\left[\left\{ D_k^{(n)} \right\}^2 \middle| \mathbf{Z}_{(\cdot)}^{(n)} \right] = o_{\mathrm{P}}(1)$, where

$$D_k^{(n)} := \sqrt{n-k} \left\{ \left(S_k^{(n)} - \mathrm{E}\left[S_k^{(n)} \middle| \mathbf{N}^{(n)} \right] \right) - \left(T_{\varphi_k;f;k}^{(n)} - \mathrm{E}\left[T_{\varphi_k;f;k}^{(n)} \middle| \mathbf{Z}_{(\cdot)}^{(n)} \right] \right) \right\}.$$

Since the maximal invariant $\left(\mathbf{N}^{(n)}, \mathbf{R}^{(n)} \right)$ depends on $\mathbf{Z}_{(\cdot)}^{(n)}$ only through $\mathbf{N}^{(n)}$, and hence, $\mathrm{E}\left[S_k^{(n)} \middle| \mathbf{N}^{(n)} \right] = \mathrm{E}\left[S_k^{(n)} \middle| \mathbf{Z}_{(\cdot)}^{(n)} \right]$, we actually have

$$\begin{aligned} \mathrm{E}\left[\left\{ D_k^{(n)} \right\}^2 \middle| \mathbf{Z}_{(\cdot)}^{(n)} \right] &= (n-k)\mathrm{E}\left[\left\{ \left(S_k^{(n)} - T_{\varphi_k;f;k}^{(n)} \right) - \mathrm{E}\left[S_k^{(n)} - T_{\varphi_k;f;k}^{(n)} \middle| \mathbf{Z}_{(\cdot)}^{(n)} \right] \right\}^2 \middle| \mathbf{Z}_{(\cdot)}^{(n)} \right] \\ &= (n-k)\mathrm{Var}\left[S_k^{(n)} - T_{\varphi_k;f;k}^{(n)} \middle| \mathbf{Z}_{(\cdot)}^{(n)} \right]. \end{aligned}$$

Obviously,

$$\begin{aligned} S_k^{(n)} - T_{\varphi_k;f;k}^{(n)} &= \frac{1}{n-k} \sum_{t=k+1}^{n} \left[a_k^{(n)} \left(\mathbf{N}^{(n)}; R_t^{(n)}, \ldots, R_{t-k}^{(n)} \right) - \varphi_k \left(F(Z_t^{(n)}), \ldots, F(Z_{t-k}^{(n)}) \right) \right] \\ &= \frac{1}{n-k} \sum_{t=k+1}^{n} \left[a_k^{(n)} \left(\mathbf{N}^{(n)}; R_t^{(n)}, \ldots, R_{t-k}^{(n)} \right) - \varphi_k \left(F(Z_{\left(R_t^{(n)}\right)}^{(n)}), \ldots, F(Z_{\left(R_{t-k}^{(n)}\right)}^{(n)}) \right) \right] \end{aligned}$$

is, conditionally on $\mathbf{Z}^{(n)}_{(\cdot)}$ (and hence on $\mathbf{N}^{(n)}$), a linear serial *rank* statistic in the sense of Hallin et al. (1985). Defining, for distinct integers $i_1, \ldots, i_{k+1} \in \{1, \ldots, n\}$,

$$\alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)};k}(i_1, \ldots, i_{k+1}) := a^{(n)}_k \left(\mathbf{N}^{(n)}; i_1, \ldots, i_{k+1}\right) - \varphi_k\left(F(Z^{(n)}_{(i_1)}), \ldots, F(Z^{(n)}_{(i_{k+1})})\right),$$

we obtain $S^{(n)}_k - T^{(n)}_{\varphi_k;f;k} = \frac{1}{n-k}\sum_{t=k+1}^n \alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)};k}\left(R^{(n)}_t, \ldots, R^{(n)}_{t-k}\right)$.

We may now complete the proof using arguments similar to those developed in Section 4.1 of Hallin et al. (1985). Corollary 2 of Lemma 2, and Lemma 4 (Appendix 3) of that paper imply that there exists a constant K (not depending on n) such that

$$\begin{aligned}
\mathrm{E}\left[\left\{D^{(n)}_k\right\}^2 \middle| \mathbf{Z}^{(n)}_{(\cdot)}\right] &= \mathrm{Var}\left[\frac{1}{\sqrt{n-k}}\sum_{t=k+1}^n \alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)};k}\left(R^{(n)}_t, \ldots, R^{(n)}_{t-k}\right) \middle| \mathbf{Z}^{(n)}_{(\cdot)}\right] \\
&= \frac{1}{n-k}\mathrm{Var}\left[\sum_{t=k+1}^n \alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)};k}\left(R^{(n)}_t, \ldots, R^{(n)}_{t-k}\right) \middle| \mathbf{Z}^{(n)}_{(\cdot)}\right] \\
&\leq (2k+1)\mathrm{Var}\left[\alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)};k}\left(R^{(n)}_{k+1}, \ldots, R^{(n)}_1\right) \middle| \mathbf{Z}^{(n)}_{(\cdot)}\right] \\
&\quad + \frac{n}{n-k}\left|\mathrm{Cov}\left[\alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)};k}\left(R^{(n)}_{2k+2}, \ldots, R^{(n)}_{k+2}\right), \alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)};k}\left(R^{(n)}_{k+1}, \ldots, R^{(n)}_1\right) \middle| \mathbf{Z}^{(n)}_{(\cdot)}\right]\right| \\
&\leq (2k+1)\mathrm{Var}\left[\alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)};k}\left(R^{(n)}_{k+1}, \ldots, R^{(n)}_1\right) \middle| \mathbf{Z}^{(n)}_{(\cdot)}\right] \\
&\qquad + \frac{K}{n-k}\mathrm{E}\left[\left\{\alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)};k}\left(R^{(n)}_{k+1}, \ldots, R^{(n)}_1\right)\right\}^2 \middle| \mathbf{Z}^{(n)}_{(\cdot)}\right] \\
&\leq \left(2k+1+\frac{K}{n-k}\right)\mathrm{E}\left[\left\{\alpha^{(n)}_{\mathbf{Z}^{(n)}_{(\cdot)};k}\left(R^{(n)}_{k+1}, \ldots, R^{(n)}_1\right)\right\}^2 \middle| \mathbf{Z}^{(n)}_{(\cdot)}\right] \\
&= \left(2k+1+\frac{K}{n-k}\right) \\
&\quad \mathrm{E}\left[\left\{a^{(n)}_k\left(\mathbf{N}^{(n)}; R^{(n)}_{k+1}, \ldots, R^{(n)}_1\right) - \varphi_k\left(F(Z^{(n)}_{k+1}), \ldots, F(Z^{(n)}_1)\right)\right\}^2 \middle| \mathbf{Z}^{(n)}_{(\cdot)}\right].
\end{aligned}$$

By (4.1), the last term converges to zero in probability, under $\mathcal{H}^{(n)}_{0;f}$, as $n \to \infty$. This completes the proof of (4.3).

The asymptotic normality of $\sqrt{n-k}\left(T^{(n)}_{\varphi_k;f;k} - \mathrm{E}\left[T^{(n)}_{\varphi_k;f;k}\middle| \mathbf{Z}^{(n)}_{(\cdot)}\right]\right)$ (part (ii) of Lemma 4.1), hence also that of $\sqrt{n-k}\left(S^{(n)}_k - \mathrm{E}\left[S^{(n)}_k \middle| \mathbf{N}^{(n)}\right]\right)$, is established in Hallin et al. (1985), and follows from Yoshihara (1976)'s central limit theorem for U-statistics constructed from absolutely regular processes. This central limit theorem requires the $(2+\delta)$-integrability of the score-generating function φ_k. □

Note that the right hand side in (4.3) is exactly the same as in the asymptotic representation of the purely rank-based serial statistic

$$(n-k)^{-1}\sum_{t=k+1}^n \varphi_k\left(\frac{R^{(n)}_t}{n+1}, \ldots, \frac{R^{(n)}_{t-k}}{n+1}\right) - \mathrm{E}\left[(n-k)^{-1}\sum_{t=k+1}^n \varphi_k\left(\frac{R^{(n)}_t}{n+1}, \ldots, \frac{R^{(n)}_{t-k}}{n+1}\right)\right].$$

This remark, which is analogous to the remark made, in the nonserial case, just before the proof of Lemma 3.1, will play a crucial role in the proof of the asymptotic normality part of Proposition 4.1 (ii).

4.2 Exact and approximate scores.

As in the nonserial case, two types of scores, the exact and the approximate ones, are naturally associated with a given score-generating function. Define (referring to Section 3.2 for notation)

$$S^{(n)}_{\varphi_k;\mathrm{ex/appr}} := \frac{1}{n-k}\sum_{t=k+1}^{n} a^{(n)}_{\varphi_k;\mathrm{ex/appr}}(\mathbf{N}^{(n)}; R^{(n)}_t, R^{(n)}_{t-1}, \ldots, R^{(n)}_{t-k})$$

where, for $(\eta,\nu) \in \{0,1,\ldots,n\}^2$, $\nu \leq n-\eta$, and $1 \leq i_1 \neq i_2 \neq \ldots \neq i_{k+1} \leq n$,

$$\begin{aligned} &a^{(n)}_{\varphi_k;\mathrm{ex}}((\eta,\nu); i_1,\ldots,i_{k+1}) \\ &\qquad := \mathrm{E}\left[\varphi_k(U^{(n)}_1,\ldots,U^{(n)}_{k+1}) \middle| N^{(n)}_{\mathbf{U};-} = \eta, N^{(n)}_{\mathbf{U};+} = \nu, R^{(n)}_{U_1} = i_1, \ldots, R^{(n)}_{U_{k+1}} = i_{k+1}\right] \end{aligned}$$

and

$$\begin{aligned} &a^{(n)}_{\varphi_k;\mathrm{appr}}((\eta,\nu); i_1,\ldots,i_{k+1}) \\ &\quad := \varphi_k\Big(\mathrm{E}\left[U^{(n)}_1 \middle| N^{(n)}_{\mathbf{U};-} = \eta, N^{(n)}_{\mathbf{U};+} = \nu, R^{(n)}_{U_1} = i_1\right], \\ &\qquad\qquad \ldots, \mathrm{E}\left[U^{(n)}_{k+1} \middle| N^{(n)}_{\mathbf{U};-} = \eta, N^{(n)}_{\mathbf{U};+} = \nu, R^{(n)}_{U_{k+1}} = i_{k+1}\right]\Big) \\ &\quad = \varphi_k\Big(I[i_1 \leq \eta]\left(\frac{i_1}{2(\eta+1)}\right) + I[i_1 > n-\nu]\left(\frac{1}{2} + \frac{i_1-(n-\nu)}{2(\nu+1)}\right), \\ &\qquad\qquad \ldots, I[i_{k+1} \leq \eta]\left(\frac{i_{k+1}}{2(\eta+1)}\right) + I[i_{k+1} > n-\nu]\left(\frac{1}{2} + \frac{i_{k+1}-(n-\nu)}{2(\nu+1)}\right)\Big). \end{aligned}$$

The following lemma provides sufficient conditions for φ_k being a score-generating function for $a^{(n)}_{\varphi_k;\mathrm{ex}}$ and $a^{(n)}_{\varphi_k;\mathrm{appr}}$, respectively.

Lemma 4.2 *Let $\varphi_k : (0,1)^{k+1} \longrightarrow \mathbb{R}$ be non-constant and square-integrable. Then φ_k is a score-generating function for $a^{(n)}_{\varphi_k;\mathrm{ex}}$. If moreover φ_k is a linear combination of a finite number of square-integrable functions which are monotone in all their arguments, then φ_k is also a score-generating function for $a^{(n)}_{\varphi_k;\mathrm{appr}}$.*

Proof. The proof easily follows along the same lines as in the nonserial case, and is left to the reader. □

4.3 Unconditional asymptotic representation

Lemma 4.1 was only an intermediate, conditional result; the following proposition provides the corresponding unconditional asymptotic representation and asymptotic normality.

Proposition 4.1 *Let φ_k be a non-constant square-integrable score-generating function for $S^{(n)}_{\varphi_k;\mathrm{ex/appr}}$. Whenever approximate scores are considered, assume that φ_k is a linear combination of square-integrable functions which are monotone in all their arguments. Then, writing $S^{(n)}_k$ for either $S^{(n)}_{\varphi_k;\mathrm{ex}}$ or $S^{(n)}_{\varphi_k;\mathrm{appr}}$,* *(i) (asymptotic representation) under $\mathcal{H}^{(n)}_{0;f}$, as $n \to \infty$,*

$$
\begin{aligned}
S^{(n)}_k - \mathrm{E}\left[S^{(n)}_k\right] = T^{(n)}_{\varphi_k;f;k} - \mathrm{E}\left[T^{(n)}_{\varphi_k;f;k}\Big|\mathbf{Z}^{(n)}_{(\cdot)}\right] \\
&+2^{k+1}[n(n-1)\ldots(n-k)]^{-1}\Big\{I[N^{(n)}_- \geq k+1]N^{(n)}_-(N^{(n)}_- - 1)\ldots(N^{(n)}_- - k)\mu^{(0)}_{\varphi_k} \\
&+\sum_{\nu=1}^{k} I[k+1-\nu \leq N^{(n)}_- \leq n-\nu]N^{(n)}_-(N^{(n)}_- - 1)\ldots(N^{(n)}_- - k+\nu) \\
&\qquad\qquad \times N^{(n)}_+(N^{(n)}_+ - 1)\ldots(N^{(n)}_+ - \nu+1)\mu^{(\nu)}_{\varphi_k} \\
&+I[N^{(n)}_+ \geq k+1]N^{(n)}_+(N^{(n)}_+ - 1)\ldots(N^{(n)}_+ - k)\mu^{(k+1)}_{\varphi_k}\Big\} - \mu_{\varphi_k} + o_{\mathrm{P}}(1/\sqrt{n}), \quad (4.5)
\end{aligned}
$$

where

$$
\mu_{\varphi_k} := \int_{[0,1]^{k+1}} \varphi_k(u_0, \ldots, u_k)du_0\ldots du_k
$$

and, for $\nu = 0, 1, \ldots, k+1$,

$$
\mu^{(\nu)}_{\varphi_k} := \sum_{0\leq i_1<\ldots<i_\nu\leq k}\ldots\sum \int_{(u_{i_1},\ldots,u_{i_\nu})\in[1/2,1]^\nu}\int_{(u_j,0\leq j\leq k, j\neq i_1,\ldots,\, i_\nu)\in[0,1/2]^{k+1-\nu}} \varphi_k(u_0,\ldots,u_k)du_0\ldots du_k.
$$

(ii) (asymptotic normality) if moreover φ_k is $(2+\delta)-$integrable for some $\delta > 0$, then, under $\mathcal{H}^{(n)}_0$, as $n \to \infty$,

$$
\sqrt{n-k}\left(\frac{S^{(n)}_k - \mathrm{E}\left[S^{(n)}_k\right]}{\sqrt{V^2 + (k+1)^2\left[\mu_{\varphi_k} - 2\sum_{\nu=1}^{k+1}\nu\mu^{(\nu)}_{\varphi_k}/(k+1)\right]^2}}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1), \quad (4.6)
$$

with V^2 given in (4.4).

Proof. As in the nonserial case, we first prove the asymptotic representation result for exact scores. From the definition of exact scores, we obtain, for $S^{(n)}_k = S^{(n)}_{\varphi_k;\mathrm{ex}}$, writing $T^{(n)}_k$ for $T^{(n)}_{\varphi_k;f;k} := \frac{1}{n-k}\sum_{t=k+1}^n \varphi_k\left(F(Z^{(n)}_t), \ldots, F(Z^{(n)}_{t-k})\right)$,

$$
\begin{aligned}
\mathrm{E}\left[S^{(n)}_k\Big|\mathbf{N}^{(n)}\right] &= \mathrm{E}\left[\mathrm{E}\left[T^{(n)}_k\Big|\mathbf{R}^{(n)}, \mathbf{N}^{(n)}\right]\Big|\mathbf{N}^{(n)}\right] = \mathrm{E}\left[T^{(n)}_k\Big|\mathbf{N}^{(n)}\right] \\
&= \mathrm{E}\left[\frac{1}{n-k}\sum_{t=k+1}^n \mathrm{E}\left[\varphi_k\left(F(Z^{(n)}_t),\ldots,F(Z^{(n)}_{t-k})\right)\Big|\mathbf{N}^{(n)}, s_t,\ldots,s_{t-k}\right]\Bigg|\mathbf{N}^{(n)}\right]
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathrm{E}\left[\varphi_k\left(F(Z^{(n)}_t),\ldots,F(Z^{(n)}_{t-k})\right)\Big|\mathbf{N}^{(n)}, s_t,\ldots,s_{t-k}\right] \quad (4.7) \\
&\quad = 2^{k+1}\int_{[0,1]^{k+1}} \varphi_k(u_0,\ldots,u_k)I\left[\mathrm{sign}(u_0 - \frac{1}{2}) = s_t, \ldots, \mathrm{sign}(u_k - \frac{1}{2}) = s_{t-k}\right]du_0\ldots du_k.
\end{aligned}
$$

The asymptotic representation (4.5) (for exact scores) follows from combining (4.7) and part (i) of Lemma 4.1. Turning to approximate scores, it is sufficient, for (4.5) to hold, that

$$
E^{(n)} := \left\{\mathrm{E}\left[S^{(n)}_{\varphi_k;\mathrm{appr}}\Big|\mathbf{N}^{(n)}\right] - \mathrm{E}\left[S^{(n)}_{\varphi_k;\mathrm{appr}}\right]\right\} - \left\{\mathrm{E}\left[S^{(n)}_{\varphi_k;\mathrm{ex}}\Big|\mathbf{N}^{(n)}\right] - \mathrm{E}\left[S^{(n)}_{\varphi_k;\mathrm{ex}}\right]\right\} \quad (4.8)
$$

be $o_{\mathrm{P}}(1/\sqrt{n})$. Note that

$$\mathrm{E}\left[S^{(n)}_{\varphi_k;\mathrm{appr}}\Big|\mathbf{N}^{(n)}\right] = [n(n-1)\ldots(n-k)]^{-1}$$

$$\times \sum\ldots\sum_{1\le i_1\neq\ldots\neq i_{k+1}\le n}\varphi_k\Bigg(I[i_1\le N^{(n)}_-]\Big(\frac{i_1}{2(N^{(n)}_-+1)}\Big)+I[i_1>N^{(n)}_-]\Big(\frac{1}{2}+\frac{i_1-N^{(n)}_-}{2(N^{(n)}_++1)}\Big),$$

$$\ldots, I[i_{k+1}\le N^{(n)}_-]\Big(\frac{i_{k+1}}{2(N^{(n)}_-+1)}\Big)+I[i_{k+1}>N^{(n)}_-]\Big(\frac{1}{2}+\frac{i_{k+1}-N^{(n)}_-}{2(N^{(n)}_++1)}\Big)\Bigg).$$

For notational simplicity, let us consider the case $k=1$; the general case follows along the same ideas. For $k=1$, we have

$$\mathrm{E}\left[S^{(n)}_{\varphi_1;\mathrm{appr}}\Big|\mathbf{N}^{(n)}\right]-\mathrm{E}\left[S^{(n)}_{\varphi_1;\mathrm{ex}}\Big|\mathbf{N}^{(n)}\right]=\frac{1}{n(n-1)}$$

$$\times\Bigg\{\sum\sum_{1\le i\neq j\le N^{(n)}_-}\varphi_1\Big(\frac{i}{2(N^{(n)}_-+1)},\frac{j}{2(N^{(n)}_-+1)}\Big)+\sum_{i=1}^{N^{(n)}_-}\sum_{j=N^{(n)}_-+1}^{n}\varphi_1\Big(\frac{i}{2(N^{(n)}_-+1)},\frac{1}{2}+\frac{j-N^{(n)}_-}{2(N^{(n)}_++1)}\Big)$$

$$+\sum_{i=N^{(n)}_-+1}^{n}\sum_{j=1}^{N^{(n)}_-}\varphi_1\Big(\frac{1}{2}+\frac{i-N^{(n)}_-}{2(N^{(n)}_++1)},\frac{j}{2(N^{(n)}_-+1)}\Big)+\sum\sum_{N^{(n)}_-+1\le i\neq j\le n}\varphi_1\Big(\frac{1}{2}+\frac{i-N^{(n)}_-}{2(N^{(n)}_++1)},\frac{1}{2}+\frac{j-N^{(n)}_-}{2(N^{(n)}_++1)}\Big)\Bigg\}$$

$$-\frac{4}{n(n-1)}\Big\{I[N^{(n)}_-\ge 2]N^{(n)}_-(N^{(n)}_--1)\mu^{(0)}_{\varphi_1}+I[1\le N^{(n)}_-\le n-1]N^{(n)}_-N^{(n)}_+\mu^{(1)}_{\varphi_1}$$

$$+I[N^{(n)}_+\ge 2]N^{(n)}_+(N^{(n)}_+-1)\mu^{(2)}_{\varphi_1}\Big\}$$

$$=\frac{4N^{(n)}_-(N^{(n)}_--1)^+}{n(n-1)}\Bigg\{\frac{\left(N^{(n)}_-\right)^2}{N^{(n)}_-(N^{(n)}_--1)}D^{--}_{N^{(n)}_-,N^{(n)}_-}-\mu^{--}_{\varphi_1}$$

$$-\frac{\left(N^{(n)}_-\right)^2}{N^{(n)}_-(N^{(n)}_--1)}\frac{1}{4\left(N^{(n)}_-\right)^2}\sum_{i=1}^{N^{(n)}_-}\varphi_1\Big(\frac{i}{2(N^{(n)}_-+1)},\frac{i}{2(N^{(n)}_-+1)}\Big)\Bigg\}$$

$$+\frac{4N^{(n)}_-N^{(n)}_+}{n(n-1)}\Big\{D^{-+}_{N^{(n)}_-,N^{(n)}_+}-\mu^{-+}_{\varphi_1}\Big\}+\frac{4N^{(n)}_+N^{(n)}_-}{n(n-1)}\Big\{D^{+-}_{N^{(n)}_+,N^{(n)}_-}-\mu^{+-}_{\varphi_1}\Big\}$$

$$+\frac{4N^{(n)}_+(N^{(n)}_+-1)^+}{n(n-1)}\Bigg\{\frac{\left(N^{(n)}_+\right)^2}{N^{(n)}_+(N^{(n)}_+-1)}D^{++}_{N^{(n)}_+,N^{(n)}_+}-\mu^{++}_{\varphi_1}\tag{4.9}$$

$$-\frac{\left(N^{(n)}_+\right)^2}{N^{(n)}_+(N^{(n)}_+-1)}\frac{1}{4\left(N^{(n)}_+\right)^2}\sum_{i=1}^{N^{(n)}_+}\varphi_1\Big(\frac{1}{2}+\frac{i}{2(N^{(n)}_++1)},\frac{1}{2}+\frac{i}{2(N^{(n)}_++1)}\Big)\Bigg\}$$

where $x^+ := \max(0, x)$,

$$D^{--}_{\ell,m} := \frac{1}{4\ell m}\sum_{i=1}^{\ell}\sum_{j=1}^{m}\varphi_1\Big(\frac{i}{2(\ell+1)}, \frac{j}{2(m+1)}\Big), \qquad D^{-+}_{\ell,m} := \frac{1}{4\ell m}\sum_{i=1}^{\ell}\sum_{j=1}^{m}\varphi_1\Big(\frac{i}{2(\ell+1)}, \frac{1}{2}+\frac{j}{2(m+1)}\Big),$$

$$D^{+-}_{\ell,m} := \frac{1}{4\ell m}\sum_{i=1}^{\ell}\sum_{j=1}^{m}\varphi_1\Big(\frac{1}{2}+\frac{i}{2(\ell+1)}, \frac{j}{2(m+1)}\Big), \text{and } D^{++}_{\ell,m} := \frac{1}{4\ell m}\sum_{i=1}^{\ell}\sum_{j=1}^{m}\varphi_1\Big(\frac{1}{2}+\frac{i}{2(\ell+1)}, \frac{1}{2}+\frac{j}{2(m+1)}\Big)$$

are Riemann sums for the integrals

$$\mu^{--}_{\varphi_1} := \int_0^{1/2}\int_0^{1/2}\varphi_1(u_0,u_1)\,du_0du_1, \qquad \mu^{-+}_{\varphi_1} := \int_0^{1/2}\int_{1/2}^{1}\varphi_1(u_0,u_1)\,du_0du_1,$$
$$\mu^{+-}_{\varphi_1} := \int_{1/2}^{1}\int_0^{1/2}\varphi_1(u_0,u_1)\,du_0du_1, \text{ and } \mu^{++}_{\varphi_1} := \int_{1/2}^{1}\int_{1/2}^{1}\varphi_1(u_0,u_1)\,du_0du_1,$$

respectively. Here again, due to the fact that φ_1 is square-integrable, the function $(u,v) \mapsto \varphi_1^*(u,v) := \varphi_1(u,v)I[u=v]$, $(u,v) \in [1/2,1]^2$ which vanishes except over the diagonal of the unit square is integrable, and has integral zero. Hence, $(1/4m^2)\sum_{i=1}^m \varphi_1^2\Big(\frac{1}{2}+\frac{i}{2(m+1)}, \frac{1}{2}+\frac{i}{2(m+1)}\Big)$, as a Riemann sum for the integral of φ_1^* over $[1/2,1]^2$, is $o(1)$. Since

$$\left[\sum_{i=1}^{m}\varphi_1\Big(\frac{1}{2}+\frac{i}{2(m+1)}, \frac{1}{2}+\frac{i}{2(m+1)}\Big)\right]^2 \le m\sum_{i=1}^{m}\varphi_1^2\Big(\frac{1}{2}+\frac{i}{2(m+1)}, \frac{1}{2}+\frac{i}{2(m+1)}\Big),$$

it follows that $(1/4m^2)\sum_{i=1}^m \varphi_1\Big(\frac{1}{2}+\frac{i}{2(m+1)}, \frac{1}{2}+\frac{i}{2(m+1)}\Big)$ is $o(1/\sqrt{m})$, as $m \to \infty$. A similar result holds for $(1/4m^2)\sum_{i=1}^m \varphi_1\Big(\frac{i}{2(m+1)}, \frac{i}{2(m+1)}\Big)$, as well of course for any individual terms, such as $(1/4m^2)\varphi_1\Big(\frac{1}{2}+\frac{m}{2(m+1)}, \frac{1}{2}+\frac{m}{2(m+1)}\Big)$. Thus, (4.9) as $n \to \infty$ takes the form

$$\begin{aligned}
&\mathrm{E}\left[S^{(n)}_{\varphi_1;\mathrm{appr}}\middle|\mathbf{N}^{(n)}\right] - \mathrm{E}\left[S^{(n)}_{\varphi_1;\mathrm{ex}}\middle|\mathbf{N}^{(n)}\right]\\
&= \frac{4N_-^{(n)}(N_-^{(n)}-1)^+}{n(n-1)}\left[D^{--}_{N_-^{(n)},N_-^{(n)}} - \mu^{--}_{\varphi_1}\right] + \frac{4N_-^{(n)}N_+^{(n)}}{n(n-1)}\left[D^{-+}_{N_-^{(n)},N_+^{(n)}} - \mu^{-+}_{\varphi_1}\right]\\
&\quad + \frac{4N_+^{(n)}N_-^{(n)}}{n(n-1)}\left[D^{+-}_{N_+^{(n)},N_-^{(n)}} - \mu^{+-}_{\varphi_1}\right] + \frac{4N_+^{(n)}(N_+^{(n)}-1)^+}{n(n-1)}\left[D^{++}_{N_+^{(n)},N_+^{(n)}} - \mu^{++}_{\varphi_1}\right] + o_{\mathrm{P}}(1/\sqrt{n}).
\end{aligned}$$

Considering the difference $D^{++}_{m,m} - \mu^{++}_{\varphi_1}$, we have

$$\begin{aligned}
&D^{++}_{m,m} - \mu^{++}_{\varphi_1}\\
&= \frac{1}{4m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}\varphi_1\Big(\frac{1}{2}+\frac{i}{2(m+1)}, \frac{1}{2}+\frac{j}{2(m+1)}\Big) - \int\int_{[\frac{1}{2},1]^2}\varphi_1(u_0,u_1)du_0du_1\\
&= \frac{1}{4m^2}\sum_{i=1}^{m}\varphi_1\Big(\frac{1}{2}+\frac{i}{2(m+1)}, \frac{1}{2}+\frac{m}{2(m+1)}\Big) + \frac{1}{4m^2}\sum_{j=1}^{m-1}\varphi_1\Big(\frac{1}{2}+\frac{m}{2(m+1)}, \frac{1}{2}+\frac{j}{2(m+1)}\Big)\\
&\quad + \frac{1}{4m^2}\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\varphi_1\Big(\frac{1}{2}+\frac{i}{2(m+1)}, \frac{1}{2}+\frac{j}{2(m+1)}\Big) - \int\int_{[\frac{1}{2},1]^2}\varphi_1(u_0,u_1)du_0du_1 \qquad (4.10)\\
&= \frac{1}{4m^2}\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\varphi_1\Big(\frac{1}{2}+\frac{i}{2(m+1)}, \frac{1}{2}+\frac{j}{2(m+1)}\Big) - \int\int_{[\frac{1}{2},1]^2}\varphi_1(u_0,u_1)du_0du_1 + o(1/\sqrt{m})
\end{aligned}$$

since, in view of the same argument as above, the two first sums in (4.10) are $o(1/\sqrt{m})$. As in the proof of Proposition 3.1, due to the fact that φ_1 can be assumed to be non-decreasing in its two arguments, the sum appearing in this latter expression is comprised between the two Darboux sums

$$\underline{D}^{++}_{m,m} := \frac{1}{4m^2}\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\varphi_1\Big(\frac{1}{2}+\frac{i-1}{2m},\frac{1}{2}+\frac{j-1}{2m}\Big) \text{ and } \bar{D}^{++}_{m,m} := \frac{1}{4m^2}\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\varphi_1\Big(\frac{1}{2}+\frac{i}{2m},\frac{1}{2}+\frac{j}{2m}\Big).$$

These Darboux sums also converge to the integral $\int\int_{[\frac{1}{2},1]^2}\varphi_1(u_0,u_1)du_0du_1$, and their difference is

$$\bar{D}^{++}_{m,m} - \underline{D}^{++}_{m,m} = \frac{1}{4m^2}\left[\varphi_1\Big(\frac{1}{2}+\frac{m-1}{2m},\frac{1}{2}+\frac{m-1}{2m}\Big) - \varphi_1\Big(\frac{1}{2},\frac{1}{2}\Big)\right];$$

the same argument still implies that this difference, hence also $D^{++}_{m,m} - \mu^{++}_{\varphi_1}$, is $o(1/\sqrt{m})$. The other three quantities of the same type can be treated similarly. Uniform integrability and the fact that $N^{(n)}_{\pm}$ are $O_{\rm P}(n)$, as in the proof of Proposition 3.1, complete the proof that (4.8) is indeed $o_{\rm P}(1/\sqrt{n})$.

To conclude, we now prove the asymptotic normality result. Denote by Π_{k+1} the set of permutations π of $\{1,\ldots,k+1\}$. Then,

$$\begin{aligned}\mathrm{E}\left[S^{(n)}_{\varphi_k;\mathrm{ex}}\middle|\mathbf{N}^{(n)}\right] &= \binom{n}{k+1}^{-1}\sum\cdots\sum_{1\leq t_1<\ldots<t_{k+1}\leq n}\Bigg\{\sum_{\nu=0}^{k+1}\frac{1}{(k+1)!}2^{k+1}\mu^{(\nu)}_{\varphi_k}\\ &\quad\times\sum_{\pi\in\Pi_{k+1}} I\left[s_{t_{\pi(1)}}=1,\ldots,s_{t_{\pi(\nu)}}=1,s_{t_{\pi(\nu+1)}}=-1,\ldots,s_{t_{\pi(k+1)}}=-1\right]\Bigg\};\end{aligned}$$

hence, $\mathrm{E}\left[S^{(n)}_{\varphi_k;\mathrm{ex}}\middle|\mathbf{N}^{(n)}\right]$ is a U-statistic computed from the n-tuple $Z^{(n)}_1,\ldots,Z^{(n)}_n$, with kernel

$$h_k(z_1,\ldots,z_{k+1}) = \sum_{\nu=0}^{k+1}\frac{2^{k+1}\mu^{(\nu)}_{\varphi_k}}{(k+1)!}\sum_{\pi\in\Pi_{k+1}} I\left[z_{\pi(1)}>0,\ldots,z_{\pi(\nu)}>0,z_{\pi(\nu+1)}\leq 0,\ldots,z_{\pi(k+1)}\leq 0\right].$$

Routine calculation yields, under $\mathcal{H}^{(n)}_{0;f}$,

$$\mathrm{E}\left[h_k(Z^{(n)}_1,\ldots,Z^{(n)}_{k+1})\middle| Z^{(n)}_1\right] = 2I[Z^{(n)}_1>0]\sum_{\nu=0}^{k+1}\frac{\nu}{k+1}\mu^{(\nu)}_{\varphi_k} + 2I[Z^{(n)}_1\leq 0]\sum_{\nu=0}^{k+1}\frac{k+1-\nu}{k+1}\mu^{(\nu)}_{\varphi_k}$$

and

$$\mathrm{Var}\left(\mathrm{E}\left[h_k(Z^{(n)}_1,\ldots,Z^{(n)}_{k+1})\middle| Z^{(n)}_1\right]\right) = \left\{\mu_{\varphi_k} - 2\sum_{\nu=1}^{k+1}\frac{\nu}{k+1}\mu^{(\nu)}_{\varphi_k}\right\}^2,$$

which is strictly positive. Classical results on U-statistics (see, e.g., Serfling 1980) then imply that, under $\mathcal{H}^{(n)}_{0;f}$, as $n\to\infty$,

$$(n-k)^{1/2}\left(\mathrm{E}\left[S^{(n)}_{\varphi_k;\mathrm{ex}}\middle|\mathbf{N}^{(n)}\right] - \mathrm{E}\left[S^{(n)}_{\varphi_k;\mathrm{ex}}\right]\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0,(k+1)^2\left\{\mu_{\varphi_k} - 2\sum_{\nu=1}^{k+1}\frac{\nu}{k+1}\mu^{(\nu)}_{\varphi_k}\right\}^2\right).$$

The same argument as in the nonserial case can be invoked in order to establish the asymptotic independence of the right hand side in the conditional asymptotic representation (4.3) and $(n-k)^{1/2}\left(\mathrm{E}\left[S^{(n)}_{\varphi_k;\mathrm{ex}}\middle|\mathbf{N}^{(n)}\right] - \mathrm{E}\left[S^{(n)}_{\varphi_k;\mathrm{ex}}\right]\right)$. The result follows. □

5 Conclusion.

Semiparametric efficiency can be reached, in models involving white noise with unspecified density, by conditioning upon maximal invariants. In median regression and median time series models, a maximal invariant is the vector of residual ranks along with the vector of residual signs. Conditioning with respect to this maximal invariant yields sign-and-rank statistics, which so far have not been considered in the literature. Asymptotic representation and asymptotic normality results are obtained for this new class of statistics, both in the nonserial and in the serial case. Optimal tests based on these statistics (for median regression and median time series models) are the subject of a companion paper (Hallin, Vermandele, and Werker 2003). The variance in the asymptotic normal distributions obtained here breaks into two distinct parts, associated with the ranks and the signs, respectively; this decomposition of asymptotic variances provides a quantitative evaluation of the advantage of sign-and-rank statistics over the more classical rank or signed-rank ones.

References

[1] Chakraborty, B. (1999). On multivariate median regression, *Bernoulli* **5**, 683-703.

[2] Dufour, J.-M. (1981). Rank tests for serial dependence, *Journal of Time Series Analysis* **2**, 117-128.

[3] Dufour, J.-M., Y. Lepage, and H. Zeidan (1982). Nonparametric testing for time series: a bibliography, *Canadian Journal of Statistic* **10**, 1-38.

[4] Hájek, J. and Z. Šidák (1967). *Theory of Rank Tests*, Academic Press, New York.

[5] Hájek, J., Z. Šidák, and P. K. Sen (1999). *Theory of Rank Tests*, Academic Press, New York.

[6] Hallin, M., J.-Fr. Ingenbleek, and M. L. Puri (1985). Linear serial rank tests for randomness against *ARMA* alternatives, *Annals of Statistic* **13**, 1156-1181.

[7] Hallin, M. and G. Mélard (1988). Rank-based tests for randomness against first-order serial dependence, *Journal of the American Statistical Association* **83**, 1117-1128.

[8] Hallin, M. and M. L. Puri (1988). Optimal rank-based procedures for time-series analysis : testing an *ARMA* model against other *ARMA* models, *Annals of Statistics* **16**, 402-432.

[9] Hallin, M. and M. L. Puri (1991). Time-series analysis via rank-order theory: signed-rank tests for *ARMA* models. *Journal of Multivariate Analysis* **39**, 1-29.

[10] Hallin, M. and M. L. Puri (1992). Rank tests for time series analysis: a survey, in *New Directions in Time Series Analysis* (Vol. 45), D. Brillinger *et al.* eds., Springer Verlag, New York, 111-153.

[11] Hallin, M. and M. L. Puri (1994). Aligned rank tests for linear models with autocorrelated error terms, *Journal of Multivariate Analysis* **50**, 175-237.

[12] Hallin, M. and B.J.M. Werker (1999). Optimal testing for semi-parametric AR models: from Gaussian Lagrange multipliers to autoregression rank scores and adaptive tests, in *Asymptotics, Nonparametrics and Time Series* (Ghosh, S. ed.), M. Dekker, New York, 295-350.

[13] Hallin, M. and B.J.M. Werker (2003). Semiparametric efficiency, distribution-freeness, and invariance, *Bernoulli* **9**, to appear.

[14] Hallin, M., C. Vermandele, and B.J.M. Werker (2003). Rank-and-sign statistics for median (auto)regression, unpublished manuscript.

[15] Horowitz, J. L. and V.G. Spokoiny (2002). An adaptive rate-optimal test of linearity for median regression models, *Journal of the American Statistical Association* **97**, 822-835.

[16] Hušková, M. (1970). Asymptotic distribution of simple linear rank statistics for testing symmetry, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **14**, 308-322.

[17] Jung, S. H. (1996). Quasi-likelihood for median regression models, *Journal of the American Statistical Association* **91**, 251-257.

[18] Koenker, R. (2000). Galton, Edgeworth, Frisch, and prospects for quantile regression in econometrics, *Journal of Econometrics* **95**, 347-374.

[19] McKeague I. W., S. B. Subramanian, and Y. Q. Sun (2001). Median regression and the missing information principle, *Journal of Nonparametric Statistics* **13**, 709-727.

[20] Puri, M. L. and P. K. Sen (1971). *Nonparametric Methods in Multivariate Analysis*, John Wiley, New York.

[21] Puri, M. L. and P. K. Sen (1985). *Nonparametric Methods in General Linear Models*, John Wiley, New York.

[22] Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics*, John Wiley, New York.

[23] Yoshihara, K. I. (1976). Limiting behaviour of U-statistics for stationary absolutely regular processes, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **35**, 237-252.

[24] Zhao, Q. S. (2001). Asymptotically efficient median regression in the presence of heteroskedasticity of unknown form, *Econometric Theory* **17**, 765-784.