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HIERARCHICAL CLASSES MODELING OF RATING DATA

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Running head: HICLAS-R

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Abstract

Hierarchical classes (HICLAS) models constitute a distinct family of structural models for *N*-way *N*-mode data. All members of the family include *N* simultaneous and linked classifications of the elements of the *N* modes implied by the data; those classifications are organized in terms of hierarchical, if-then type relations. Moreover, the models go with comprehensive, insightful graphical representations. Up to now the hierarchical classes family has been limited to dichotomous or dichotomized data. In the present paper we propose a novel extension of it to two-way two-mode rating data (HICLAS-R). The HICLAS-R model preserves the representation of simultaneous and linked classifications as well as of generalized if-then type relations, and keeps going with a comprehensive graphical representation. It is shown to bear interesting relationships with classical real-valued two-way component analysis and with methods of optimal scaling.

The family of hierarchical classes models (acronym: HICLAS) as introduced by De Boeck and Rosenberg (1988), and further extended by Van Mechelen, De Boeck, and Rosenberg (1995), Leenen, Van Mechelen, De Boeck, and Rosenberg (1999), and Ceulemans, Van Mechelen, and Leenen (2002), constitutes a distinct family of structural models for binary *N*-way *N*-mode data D. Hierarchical classes analysis of a binary $I_1 \times I_2 \ldots \times I_N$ data array D approximates D with a same-sized binary reconstructed data array M that is represented by a hierarchical classes model. A hierarchical classes model implies *N* binary $I_n \times P_n$ matrices A_n ($n = 1, \ldots, N$) and possibly one binary $P_1 \times P_2 \times \ldots \times P_N$ array G. The matrices A_n are called bundle matrices, the array G is called the core and (P_1, P_2, ... , P_N) is called the rank of the hierarchical classes model. Three types of relations implied by M are represented by the bundle matrices and the core: equivalence, hierarchy and association.

Equivalence relations are defined on each of the modes of M as follows: Each element of the *n*-th mode corresponds with an $I_1 \times \ldots \times I_{n-1} \times I_{n+1} \times \ldots \times I_N$ subarray of M; two elements of the *n*-th mode are equivalent iff they correspond with identical such subarrays. By the representation of these equivalence relations and the resulting partitions into equivalence classes, a hierarchical classes model includes *N* simultaneous classifications of the elements of each of the modes of M.

Similarly, *hierarchical relations* are defined on each of the modes of M. The latter relations are of the if-then type and can be defined as quasi-orders on the elements of each mode, or as partial orders on the corresponding equivalence classes. One element/class of the *n*-th mode is hierarchically below a second element/class of that mode if the subarray corresponding to the first is less than or equal to the subarray corresponding to the second (in terms of the natural order defined on arrays: $\underline{A} \leq \underline{A}'$ iff $\forall i_1,\ldots,i_v : a_{i_1\ldots i_v} \leq a'_{i_1\ldots i_v}$). By the representation of these hierarchical relations, the *N* simultaneous classifications included in a hierarchical classes model turn into *N* hierarchically organized partitions. One may note that, although asymmetric, implicational relations are often of key substantive interest in psychological research, the formal models and methods of analysis that can properly deal with them are only few in number. The representation of the *N* if-then type hierarchies may therefore be considered an important contribution of the hierarchical classes approach.

The *association relation* is the *N*-ary relation among the *N* modes of $\underline{M}$ as defined by the 1-entries in $\underline{M}$. Alternatively, this relation may be defined in terms of classes as an *N*-ary relation between the *N* partitions implied by $\underline{M}$; the association relation may therefore be considered a linkage system between the *N* hierarchically organized partitions. By the representation of the association relation, $\underline{M}$ may be fully reconstructed from a hierarchical classes model.

Various hierarchical classes models have been proposed, which differ in the way they represent the three types of relations as outlined above. Each of the models goes with an insightful graphic representation that gives a comprehensive account of the three types of relations.

Up to now, the hierarchical classes family has been limited to binary (0/1) data. To deal with polytomous data, and in particular with rating data with integer values ranging from 0 to *V*, up to now a dichotomization of the raw data was done prior to the actual hierarchical classes analysis. In order to avoid the resulting loss of information, one may consider to use simultaneous multiple dichotomizations that, for example, in the case of two-way two-mode object by attribute data, comes down to a replacement of each single attribute by a series of dummy variables; this strategy, however, as a side-effect results into an expansion of one of the modes of the data, which may typically hamper the transparency and interpretability of the corresponding hierarchical classes models.

In the present paper, we propose a novel extension of the hierarchical classes approach to two-way two-mode rating data (acronym: HICLAS-R). The extended hierarchical classes model includes the representation of natural generalizations of the relations of equivalence, hierarchy, and association; moreover, it goes with a comprehensive graphical representation. The extended model further can be shown to bear interesting links with the two-way real-valued principal component analysis model as well as with methods of optimal scaling.

The remainder of this paper is organized as follows: Section 1 briefly recapitulates two major existing instances of hierarchical classes models for binary data as well as the associated data analysis. Section 2 introduces the novel hierarchical classes model and associated data analysis for two-way two-mode rating data. The model is illustrated with an empirical application in Section 3. Section 4 discusses the relation of the novel hierarchical classes model with other models both inside and outside the hierarchical classes family, as well as possible model extensions.

1. Hierarchical Classes Models and Associated Data Analysis for Binary Data

1.1 Models

We recapitulate the essentials of the hierarchical classes modeling of binary data through two major representatives of the HICLAS family: the disjunctive HICLAS model for two-way two-mode data, as originally introduced by De Boeck and Rosenberg (1988), and the disjunctive Tucker3-HICLAS model for three-way three-mode binary data, as proposed by Ceulemans, Van Mechelen, and Leenen (2002). As will further appear, especially the latter model for binary three-way data will turn out to play a key role in the modeling of two-way rating data.

1.1.1 The disjunctive two-way HICLAS model. A rank P disjunctive two-way HICLAS model for an $I \times J$ reconstructed data matrix M implies a binary $I \times P$ bundle matrix A and a binary $J \times P$ bundle matrix B. The relations of equivalence, hierarchy, and association are represented by these bundle matrices as follows: (a) Equivalent elements of Mode 1 (2) have identical rows in A (B). (b) Two elements/classes of Mode 1 (2) are hierarchically related iff for their row vectors in A (B) it holds that the vector of the first is less than or equal to the vector of the second. (c) For the association of the i-th element of Mode 1 with the j-th element of Mode 2, it holds that:

$$m_{ij} = \bigoplus_{p=1}^{P} a_{ip}\, b_{jp} \tag{1}$$

where ⊕ denotes a Boolean sum. Rule (1) means that the disjunctive two-way HICLAS model may be considered to imply a one-to-one relationship between the Mode 1 and Mode 2 bundles, with elements of Mode 1 and Mode 2 being associated iff they belong to at least one pair of corresponding bundles. Note further that from (1) one may immediately derive the association relation between the classes of the two modes.

To illustrate, we will make use of the leftmost matrix in Table 1 as a hypothetical reconstructed data matrix M. Table 2 contains a rank 2 disjunctive HICLAS model for M. A graphic representation of this model is presented in Figure 1. The upper half of the figure represents the partition of the first mode, and the lower half that of the second mode. The hierarchical relations are represented by the links between the boxes that denote the equivalence classes; note that the hierarchy of the second mode is represented upside down. The association relation can be read from the figure as follows: An element/class of the first mode is associated with an element/class of the second iff it is linked to it via a downward path of links and a dashed line.

1.1.2 The disjunctive Tucker3-HICLAS model. A rank (P,Q,R) disjunctive Tucker3-HICLAS model for an $I \times J \times K$ reconstructed data array M̲ implies binary $I \times P$, $J \times Q$, and $K \times R$ bundle matrices A, B and C and a binary $P \times Q \times R$ core array G̲. The relations of equivalence, hierarchy, and association are represented by the bundle matrices and the core as follows: (a) Equivalent elements of Mode 1 (2, 3) have identical rows in A (B, C). (b) Two elements/classes of Mode 1 (2, 3) are hierarchically related iff for their row vectors in A (B, C) it holds that the vector of the first is less than or equal to the vector of the second. (c) For the association of the *i*-th element of Mode 1, the *j*-th element of Mode 2 and the *k*-th element of Mode 3, it holds that:

$$m_{ijk} = \bigoplus_{p=1}^{P} \bigoplus_{q=1}^{Q} \bigoplus_{r=1}^{R} a_{ip}\, b_{jq}\, c_{kr}\, g_{pqr} \qquad (2)$$

The latter means that, according to the Tucker3-HICLAS model, elements of the three modes are associated iff there is at least one triplet of bundles to which the three elements belong and that are associated in the core G.

To illustrate, we will make use of the two matrices in Table 1 as a hypothetical reconstructed data array M, with Mode 3 being equal to $\{\alpha,\beta,\gamma\}$. Table 3 contains a disjunctive rank (3,2,2) Tucker3-HICLAS model for M. A graphic representation of this model is presented in Figure 2. The upper half of the figure represents the partition of the first mode and the lower half that of the second mode, whereas the partition of the third mode is represented separately at the right side of the figure. The hierarchical relations are represented by the links between the boxes that denote the equivalence classes; note that the hierarchy of the second mode is represented upside down. Regarding the association relation, the bottom classes of each pair of a Mode 1 bundle *p* and a Mode 2 bundle *q* that pertain to at least one nonzero core element g_{pqr} are linked with a dashed line; the link is further labeled with a diamond that contains all Mode 3 elements belonging to some Mode 3 bundle *r* with $g_{pqr} = 1$. The association relation then can be read from the figure as follows: An element/class of Mode 1 is associated with an element/class of Mode 2 according to some element/class of Mode 3, iff the first is linked to the second via a downward path of links and a diamond that contains the third.

1.2 Data Analysis

The aim of a disjunctive two-way hierarchical classes analysis in rank *P* of a data matrix D is to approximate D as closely as possible with a reconstructed data matrix M that can be represented by a rank *P* disjunctive hierarchical classes model. In particular, closeness is formalized in terms of the (least absolute deviation or, equivalently, least squares) loss function L,

$$L = \sum_{i,j} \left| m_{ij} - d_{ij} \right| . \tag{3}$$

The two-way HICLAS algorithm comprises two routines. In the first routine, the two bundle matrices are alternatingly re-estimated, conditional on the other bundle matrix, and starting from an initial estimate for one of the bundle matrices; the estimation is done such that in each step the reconstructed data matrix M implied by the two matrices according to (1), yields a minimal value for (3). In the second routine, the final bundle matrices resulting from the first routine are transformed such as to correctly represent the relations of equivalence and hierarchy in M; the latter can be done without affecting the value of the loss function L.

Similarly, the aim of a disjunctive Tucker3-HICLAS analysis in rank (*P*,*Q*,*R*) of a data array D is to approximate D as closely as possible by a reconstructed data array M, in terms of the loss function L,

$$L = \sum_{i,j,k} \left| m_{ijk} - d_{ijk} \right|, \tag{4}$$

and such that M can be represented by a rank (*P*,*Q*,*R*) Tucker3 hierarchical classes model. Again, the Tucker3-HICLAS algorithm comprises two routines. In the first routine, each of the bundle matrices and the core is re-estimated, conditional upon all the others, and starting from an initial estimate for all bundle matrices; the estimation is done such that in each step the value of L in (4) is minimal for the array M obtained through (2) from the bundle matrices and the core array. In the second routine, the bundle matrices resulting from the first routine are once again transformed such as to correctly represent the relations of equivalence and hierarchy in M; the latter again can be done without affecting the value of the loss function L.

2. A Hierarchical Classes Model and Associated Data Analysis for Two-Way Rating Data

2.1 Model

We assume a two-way two-mode $I \times J$ data matrix D with integer values ranging from 0 to *V*. Hierarchical classes analysis will approximate D by an $I \times J$ reconstructed data matrix M with integer values ranging from 0 to *V* that can be represented by a HICLAS-R model. As a guiding example for the remainder of this section, we consider the hypothetical reconstructed rating data matrix that is presented in Table 4.

We want the HICLAS-R model to represent three aspects of M that constitute straightforward generalizations of the three relations represented by HICLAS models of binary data: (a) equivalence, (b) hierarchy, and (c) association. In particular: (a) Two Mode 1 (2) elements are *equivalent* iff they correspond with identical rows (columns) in M. (b) One element/class of Mode 1 (2) is *hierarchically below* another element/class of that mode iff the row (column) in M defined by the first is less than or equal to the row (column) defined by the second; the latter means that, *if* the first element/class is associated with an element/class *e* of the other mode at some level of association strength, *then* the second element/class is associated with *e* at a level of association strength that is at least as high. (c) *Association* refers to the mapping defined by the entries in M, which maps each pair of row and column elements/classes onto the corresponding value in the value set {0, 1, ..., *V*}. For example, from Table 4, it appears that b and c are equivalent, that c is hierarchically below d, and that the pair (c,β) is mapped onto 2.

In order to introduce the HICLAS-R model, we first define the following recoding function t that maps the set of all $I \times J$ matrices with entries in {0, 1, ..., *V*} into the set of all binary $I \times J \times V$ arrays:

$$\begin{aligned} \mathbb{R}^{I\times J} &\rightarrow \{0,1\}^{I\times J\times V} \\ \mathrm{M} &\mapsto \mathrm{t}(\mathrm{M}),\ \text{with}\ \ \mathrm{t}(\mathrm{M})_{ijv} = \begin{cases} 1 & \text{if } m_{ij} \geq v \\ 0 & \text{otherwise} \end{cases},\ v \in \{1,\ldots,V\}. \end{aligned}$$

The transformation t can be considered a standard dummy recoding according to an ordinal coding scheme, with, however, the arrangement of the recoded data into a three-way array as a special feature. As an example, Table 5 contains the image t(M) of the rating data matrix of Table 4.

Note that the recoded matrices satisfy the following property:

$$\forall v' \leq v:\ \mathrm{t(M)}_{ijv} \leq \mathrm{t(M)}_{ijv'}. \tag{5}$$

Note further that the transformation t preserves the relations of equivalence in the two data modes, in that two elements of Mode 1 (2) are equivalent in M iff the corresponding elements of Mode 1 (2) in t(M) are equivalent; similarly, t also preserves the hierarchical relations in that one element of Mode 1 (2) is hierarchically below a second element of that mode iff the corresponding elements in t(M) are hierarchically related.

A rank (P,Q,R) HICLAS-R model for an $I \times J$ reconstructed rating data matrix M now can be defined as a rank (P,Q,R) disjunctive Tucker3-HICLAS model of the recoded binary $I \times J \times V$ array t(M). The latter implies binary $I \times P$, $J \times Q$ and $V \times R$ bundle matrices A, B and C and a binary $P \times Q \times R$ core array $\underline{\mathrm{G}}$. The relations of equivalence and hierarchy in t(M) are represented by the bundle matrices A, B, and C. Since t preserves equivalence and hierarchy, it immediately follows that the relations of equivalence and hierarchy in M are also represented by A and B. Regarding association, it holds that:

$$\mathrm{t(M)}_{ijv} = \bigoplus_{p=1}^{P}\bigoplus_{q=1}^{Q}\bigoplus_{r=1}^{R} a_{ip}\, b_{jq}\, c_{vr}\, g_{pqr}\,, \tag{6}$$

and therefore:

$$m_{ij} = \sum_{v=1}^{V}\bigoplus_{p=1}^{P}\bigoplus_{q=1}^{Q}\bigoplus_{r=1}^{R} a_{ip}\, b_{jq}\, c_{vr}\, g_{pqr}\,. \tag{7}$$

The latter can be rewritten as:

$$m_{ij} = \sum_{v=1}^{V} \underset{p=1,\ldots,P}{\mathrm{Max}}\ \underset{q=1,\ldots,Q}{\mathrm{Max}}\ \underset{r=1,\ldots,R}{\mathrm{Max}}\ a_{ip}\, b_{jq}\, c_{vr}\, g_{pqr}\,. \tag{8}$$

Given that all entries at the right-hand side of (8) are 0/1, and given that C represents the hierarchy of the value mode (which, in view of (5), implies that $\forall v' \leq v:\ c_{vr} \leq c_{v'r}$), it follows that (8) can be further rewritten as:

$$\begin{aligned} m_{ij} &= \underset{p=1,\ldots,P}{\mathrm{Max}}\ \underset{q=1,\ldots,Q}{\mathrm{Max}}\ \underset{r=1,\ldots,R}{\mathrm{Max}}\ a_{ip}\, b_{jq}\, g_{pqr} \sum_{v=1}^{V} c_{vr} \\ &= \underset{p=1,\ldots,P}{\mathrm{Max}}\ \underset{q=1,\ldots,Q}{\mathrm{Max}}\ \underset{r=1,\ldots,R}{\mathrm{Max}}\ a_{ip}\, b_{jq}\, g_{pqr}\, \tilde{c}_r \\ &= \underset{p=1,\ldots,P}{\mathrm{Max}}\ \underset{q=1,\ldots,Q}{\mathrm{Max}}\ a_{ip}\, b_{jq}\, \underset{r=1,\ldots,R}{\mathrm{Max}}\, g_{pqr}\, \tilde{c}_r\,, \end{aligned}$$

with $\tilde{c}_r = \sum_{v=1}^{V} c_{vr}$. Or,

$$m_{ij} = \underset{p=1,\ldots,P}{\text{Max}} \; \underset{q=1,\ldots,Q}{\text{Max}} \; a_{ip}\, b_{jq}\, \tilde{g}_{pq}\,, \tag{9}$$

with $\tilde{g}_{pq} = \underset{r=1,\ldots,R}{\text{Max}}\, g_{pqr}\, \tilde{c}_r$. Note that, in (9), A and B are binary matrices whereas $\tilde{G}$ is a matrix with rating values. Note also that from the above derivation follows that $\tilde{G}$, in addition to the value of 0, takes exactly R different values. As a matter of fact, it is easy to show that, provided the latter constraint, equations (8) and (9) are equivalent. As a key equation of the HICLAS-R model, (9) implies that a Mode 1 element is associated with a Mode 2 element at the maximum level of association indicated in the core matrix for a pair of bundles to which the two elements belong.

Table 6 contains a disjunctive rank (3,3,3) HICLAS-R model for the reconstructed rating data matrix of Table 4. More in particular, the table contains the bundle matrices, the binary core array $\underline{G}$ and the rating-valued core matrix $\tilde{G}$.

The HICLAS-R model can be given a comprehensive graphic representation. The latter comes down to the graphic representation of the corresponding Tucker3-HICLAS model, with the value mode being represented in the diamonds between the Mode 1 and Mode 2 hierarchies; note that the representation of the value mode may be simplified by entering in each diamond the maximum value of the corresponding value bundles only. As an example, Figure 3 contains a graphic representation of the HICLAS-R model of Table 6. Note that a HICLAS-R graphic representation can be derived in a straightforward way from the bundle matrices A and B and the rating-valued core matrix $\tilde{G}$ of the model: Hasse diagrams can be derived from the quasi-orders implied by A (B), with the addition of possibly empty bottom classes of both hierarchies; next, a dashed link is to be drawn between each pair of bundles/bottom classes of the Mode 1 and Mode 2 hierarchies corresponding with a nonzero value in $\tilde{G}$; the latter value is further entered as a label in the diamond on the link. From the graphic representation, one may immediately read, in the usual way, the relations of equivalence and hierarchy in the two modes of the reconstructed rating data matrix. Regarding association, it holds that two elements are associated at the level of the maximum of the values in the diamonds on all downward paths linking them with one another.

2.2 Data Analysis

The aim of a disjunctive HICLAS-R analysis in rank (*P*,*Q*,*R*) of a rating-valued data matrix D is to approximate D as closely as possible with a reconstructed rating-valued data matrix M that can be represented by a rank (*P*,*Q*,*R*) HICLAS-R model. Again, closeness is formalized in terms of the loss function L,

$$L = \sum_{i,j} \left| m_{ij} - d_{ij} \right| . \tag{10}$$

L, however, now is a least absolute deviation but not a least squares loss function.

As a naive algorithmic strategy to achieve this goal, one may consider to apply the Tucker3-HICLAS algorithm to the recoded data t(D). Application of this strategy in 5 different ranks to each of 180 simulated data sets, however, resulted in a small amount of the cases (.55%) in reconstructed binary three-way arrays that did not satisfy property (5); the latter implies that the reconstructed arrays do not belong to the range of the transformation t, and, hence, that they cannot be backtransformed to reconstructed rating data matrices M. One may wonder whether these failures are due to a weakness of the Tucker3-HICLAS algorithm (which might, e.g., sometimes end up in local minima, whereas, for the global minimum, property (5) might be conjectured to hold). However, the counterintuitive proposition holds that there exist rating-valued data matrices D and ranks (*P*,*Q*,*R*), such that the optimal rank (*P*,*Q*,*R*) Tucker3-HICLAS model of t(D) does not satisfy property (5), whereas all Tucker3-HICLAS models of t(D) in the same rank that do satisfy (5) take a higher value on the loss function (10). To illustrate, the appendix contains an example of such a rating-valued data matrix D as well as the associated optimal Tucker3-HICLAS model of t(D) that does not satisfy (5).

As a way out for this problem, a modification of the Tucker3-HICLAS algorithm was developed. In the first routine of the modified algorithm, each conditional re-estimation of the value bundle matrix C was constrained to solutions of the Guttman scale type that inversely reflect the order of the values (i.e., if $v' \leq v$ then $\forall r = 1, \ldots, R : c_{v'r} \geq c_{vr}$); the latter further implies that (5) is satisfied for the reconstructed binary data array. The performance of this algorithm was evaluated in an extensive simulation study (Ceulemans & Van Mechelen, 2002a). From the latter, it appeared that the modified Tucker3-HICLAS algorithm has a good performance in terms of goodness of fit (i.e., in terms of the algorithm's primary objective, the minimization of the loss function (10)), and a satisfactory performance in terms of goodness of recovery (i.e., in terms of retrieval of the truth underlying the error-perturbed data); regarding the latter, it should be noted, however, that goodness of recovery appears to be somewhat less good for the combination of small-sized data sets with high error levels (20% or more).

3. Illustrative Application

We fitted the HICLAS-R model to data from a study on helping behavior. A group of 102 students was presented an experimental list with 16 descriptions of everyday emergency situations with a victim that could possibly be helped by the subject. The list was constructed on the basis of a facet-theoretic design, the facets being: extent of the victim's distress (low vs. high) and the subject's expectation to get something in return for possible help (no vs. yes). The students were asked to rate each situation with respect to the extent they would be willing to help the victim in it. For this purpose they had to use a rating scale from 0 (definitely not) through 6 (definitely yes).

The resulting 102 by 16 rating data matrix was subjected to HICLAS-R analyses in ranks (1,1,1) through (6,6,6). A generalized scree test on the resulting proportions of discrepancies (Ceulemans, Van Mechelen & Leenen, 2002) suggested that either a (2,2,2) or a (2,3,3) solution was to be preferred. On the basis of interpretational considerations, we finally retained the (2,3,3) solution. The latter had 13.9 % discrepancies and a Jaccard goodness-of-fit value of .83.

Figure 4 contains a graphic representation of the (2,3,3) model. Regarding the situation hierarchy, it immediately appears from the figure that the situations constitute a three-level Guttman scale, which means that they imply a quantitative dimension (Gati & Tversky, 1982). In order to derive a substantive psychological interpretation for this dimension, the position on it (quantified as 1, 2, 3) was correlated with external ratings of the situations as obtained from expert judges. The two highest correlations were obtained for ratings of the extent to which the situation was frustrating ($r = -.74$) and of the extent to which it was emotionally threatening for the potential helper ($r = -.73$). These correlations are remarkably high, especially given the fact that the situation Guttman scale comprised three different levels only. Apparently, overall extent of willingness to help in an emergency situation, unlike what one might expect, does not primarily depend on straightforward situation characteristics such as extent of the victim's distress (for which $r = .29$ only). Rather, willingness to help appears to be especially low in emergency situations that are frustrating or unpleasant for the potential helper.

Regarding the value hierarchy, in line with the rank of the value mode, the HICLAS-R model contains, in addition to zero, three values only from the original seven point rating scale (0-6). These values are: 3 (the scale midpoint), 4 (the value just above the scale midpoint), and 6 (the maximum value). We may conclude that our analysis sheds light on how the rating scale was used by our subjects, the major distinctions being: refusal to help (0), doubt (3), weakly positive answer (4) and clear willingness to help (6).

Regarding the person hierarchy, we observe that three person types can be distinguished. Unlike the situation classes, they do not constitute a Guttman scale. The characteristic reponse profiles of each of the person types may be read from the graphic representation in Figure 4. The profiles can also be given an alternative graphic representation, as shown in Figure 5; the construction and interpretation of this alternative representation is facilitated by the quantitative dimension underlying the situation hierarchy. One may first note that all response profiles in Figure 5 are nondecreasing; the latter necessarily follows from the Guttman scale structure of the situation classes. Furthermore, the response profile of Person Type I reflects a rather clear-cut, categorical nature; the persons of this type are willing to admit clearly that they do not intend to help in highly unpleasant situations, whereas, at the same time, they are also willing to express a definite intention to help in lowly unpleasant situations; moreover, they do not leave room for doubt (3: midpoint score) in their response profile. Persons of Person Type II display a large amount of doubt and avoid extreme responses of any kind; as a result, they do not differentiate considerably between situations at distinct levels of unpleasantness. Finally, persons of Person Type III do not want to give clearly negative answers, whereas they do express a definite intention to help in lowly unpleasant situations.

4. Discussion

In this section we first discuss the novel HICLAS-R model in relation with other models, both within and outside the hierarchical classes family (4.1). Next we discuss possible model extensions (4.2).

4.1 Relation with Other Models

4.1.1 Relation with other HICLAS models. In line with the other HICLAS models, the novel HICLAS-R model preserves the representation of simultaneous and linked classifications as well as of generalized if-then type relations. Moreover, the HICLAS-R model keeps going with a comprehensive graphical representation.

It should further be clear that the HICLAS-R model naturally extends the disjunctive two-way HICLAS model for binary data as developed by De Boeck and Rosenberg (1988). In this respect, one may note that, in case of binary reconstructed data M, the core matrix $\tilde{G}$ in (9) is binary; furthermore, in the same case, the Max-operators in (9) come down to Boolean sums. Hence, (9) can then be rewritten as follows:

$$
\begin{aligned}
m_{ij} &= \operatorname*{Max}_{p=1,\ldots,P} \ \operatorname*{Max}_{q=1,\ldots,Q} \ a_{ip}\, b_{jq}\, \tilde{g}_{pq} \\
&= \bigoplus_{p=1}^{P} \bigoplus_{q=1}^{Q} a_{ip}\, b_{jq}\, \tilde{g}_{pq} \\
&= \bigoplus_{p=1}^{P} a_{ip} \bigoplus_{q=1}^{Q} b_{jq}\, \tilde{g}_{pq} \\
&= \bigoplus_{p=1}^{P} a_{ip}\, \tilde{b}_{jp},
\end{aligned}
$$

with $\tilde{b}_{jp} = \bigoplus_{q=1}^{Q} b_{jq}\, \tilde{g}_{pq}$. The latter makes clear that (9) is equivalent to the association rule (1) of the disjunctive two-way HICLAS model.

In spite of the striking similarities between the HICLAS and HICLAS-R models, one should take care not to erroneously transfer features of the original HICLAS model to its HICLAS-R counterpart. For example, if in a regular HICLAS model the hierarchy of one mode constitutes a Guttman scale, then the same necessarily also holds for the other mode; this, however, is not the case for the HICLAS-R model, as illustrated by the model of the helping data of Section 3. Moreover, whereas in the regular two-way HICLAS model the hierarchies of both modes necessarily have the same complexity (i.e., the same number of bottom classes or underlying bundles), and whereas there is a one-to-one relationship between the bundles/bottom classes of both modes, this does not have to be the case in the HICLAS-R model. As a matter of fact, the HICLAS-R model may be considered to inherit both features of the regular two-way HICLAS model (viz., the inclusion of two bundle matrices) and of the Tucker3-HICLAS model (viz., the possibility of different numbers of bundles for different modes, and the inclusion of a core).

One may note that the HICLAS-R model can be constrained such as to re-install a one-to-one relationship between the Mode 1 and Mode 2 bundles. The latter can be achieved by constraining the core matrix $\tilde{G}$ in (9) to a diagonal matrix. A sufficient (though not necessary) condition for this is to assume that the three-way core array $\underline{G}$ in (6) is a so-called "superidentity" array, which means that t(M) is represented by an INDCLAS model (i.e., the CANDECOMP/PARAFAC counterpart of the Tucker3-HICLAS model: see Leenen, Van Mechelen, De Boeck, & Rosenberg, 1999).

4.1.2 Relation with classical real-valued principal component analysis. The regular disjunctive two-way HICLAS model bears a natural relationship to the model of real-valued principal component analysis. Indeed, apart from the distinction between Boolean and non-Boolean sums, and apart from the fact that the bundle matrices of the HICLAS model are constrained to be binary, association rule (1) is identical to the model of principal component analysis. For the HICLAS-R model, at first sight, the link with principal component analysis may seem less obvious, especially given the presence of a core matrix in association rule (9). Interestingly, however, the model of real-valued two-way component analysis may be reformulated such as to include a core matrix as well (Levin, 1966). The model then reads as follows:

$$m_{ij} = \sum_{p=1}^{P} \sum_{q=1}^{P} a_{ip}\, b_{jq}\, \tilde{g}_{pq} \ . \qquad (11)$$

The latter comes very close to association rule (9), especially if one takes into account that the Max-operators in (9) can be considered generalized (Boolean) sum-operators. Apart from the constraint, implied by the HICLAS-R model, for the matrices A en B in (9) to be binary, the only remaining key difference between association rule (9) and the principal component model, is that, in the principal component model, the number of Mode 1 components necessarily equals the number of Mode 2 components (which further implies that the core is a square matrix). Indeed, if the two numbers of components in (11) would be unequal, then it is easy to show that the model equation can be rewritten with identical numbers of components; this, however, is not the case for association rule (9).

4.1.3 Relation with methods of optimal scaling. A reconstructed rating data matrix of a rank (*P*,*Q*,*R*) HICLAS-R model contains, in addition to zero, exactly *R* different rating values. --Note that from the general Tucker3-HICLAS theory it follows that $R \leq P \times Q$ (Ceulemans, Van Mechelen, & Leenen, 2002).-- In practice, the number *R* will often be smaller than *V* (i.e., the number of nonzero values in the value set). Hence, the HICLAS-R model may be considered to imply a reduction of the value set to a more coarse subset of values. The latter reduction may highlight the most important distinctions on the rating scale that is being used. As such, this may reveal useful information on, for example, how the rating scale has been dealt with psychologically by the subjects under study. This is nicely illustrated by the analysis of the helping data.

One may observe a striking parallel between the latter type of substantive inferences implied by the HICLAS-R modeling and similar inferences derived from methods of optimal scaling (GIFI, 1990; Van de Geer, 1993). Interestingly, however, the HICLAS-R and optimal scaling strategies seem to arrive at those similar inferences, starting from almost opposite strategies: In optimal scaling methods, inferences on the most important scale distinctions and on scale use are typically based on real-valued (and, hence, more refined) quantifications of the original categorical variables, whereas in HICLAS-R the same type of inferences are based on a *reduction* of the rating value set; one may note that the latter reduction strategy links up in a more straightforward way with the type of final conclusions on psychologically important scale distinctions that are derived.

4.2 Possible Extensions of the HICLAS-R Model

Various possible extensions of the HICLAS-R model could be considered, both from the point of view of the data and from that of the model.

Regarding the *data*, in the present paper, HICLAS-R has been advanced as a model for two-way rating-valued data with integer values ranging from zero to some maximum value V. The proposed approach, however, can be extended to *real-valued* data, ranging from zero to some maximum integer value V. Such real-valued data may be approximated by reconstructed data that take values from an equally-spaced grid of $W+1$ values ranging from 0 through V, $\left\{\frac{wV}{W} \,\middle|\, w = 0, 1, \ldots, W\right\}$; as to the latter, the value of W, which indexes the resolution of the grid, is to be specified by the user. For the extended HICLAS-R model, association rule (9) can again be used (with $\tilde{G}$ now taking values from the grid). For the associated data analysis, the loss function (10) again applies. One may note that, unless the data do not contain but values from the grid, too, now only approximate HICLAS-R models can be obtained --unlike for the two-valued rating data discussed in Section 2, for which there always exists a perfect HICLAS-R model in some rank (P,Q,R)--. As a second possible data-related model extension, one may wish to represent integer-valued data with a minimum data value υ different from zero. The most straightforward HICLAS-R model extension one may consider to capture this type of data is to extend model equation (9) with an offset term υ:

$$m_{ij} = \upsilon + \underset{p=1,\ldots,P}{\text{Max}} \; \underset{q=1,\ldots,Q}{\text{Max}} \; a_{ip} \, b_{jq} \, \tilde{g}_{pq} \, .$$

From a *modeling* viewpoint, similar to the case of the regular two-way HICLAS model for binary data (Van Mechelen, De Boeck, & Rosenberg, 1995), one may formulate a conjunctive variant of the disjunctive HICLAS-R model as described in the present paper. Yet, for such a conjunctive HICLAS-R model (like for the conjunctive Tucker3-HICLAS model), a comprehensive graphic representation as in Figure 4 is not yet available (although alternative graphic representations as the one in Figure 5 still apply). Finally, one might wish to consider various types of constrained HICLAS-R models, for instance, in a confirmatory approach to test a priori hypotheses stemming from substantive theories or from previous empirical research (for an extensive discussion of this topic, see Ceulemans & Van Mechelen, 2002b). As an example, rather than deriving “by accident” a Guttman scale structure for one of the data modes (as was the case in the analysis of the helping data in Section 3), one may wish to impose such a structure in an a priori way. Otherwise, as a side effect, a Guttman scale constraint may also facilitate the construction of alternative graphic representations of the resulting HICLAS-R models as shown in Figure 5.

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TABLE 1

Hypothetical Reconstructed Binary Data Array

α					β,γ				
	Mode 2					Mode 2			
Mode 1	a	b	c	d	Mode 1	a	b	c	d
1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	2	1	1	1	1
3	1	0	0	1	3	1	0	0	1
4	0	1	1	1	4	0	1	1	1
5	0	0	0	0	5	0	1	1	1
6	0	0	0	0	6	0	1	1	1

TABLE 2

Disjunctive Rank 2 Hierarchical Classes Model for Leftmost Reconstructed Binary Data Matrix in Table 1

A			B		
1	1	1	a	1	0
2	1	1	b	0	1
3	1	0	c	0	1
4	0	1	d	1	1
5	0	0			
6	0	0			

TABLE 3

Disjunctive Rank (3,2,2) Tucker3 Hierarchical Classes Model for Reconstructed Data Array in Table 1

A			
1	1	1	0
2	1	1	0
3	1	0	0
4	0	1	0
5	0	0	1
6	0	0	1

B		
a	1	0
b	0	1
c	0	1
d	1	1

C		
α	1	0
β	1	1
γ	1	1

$G_{..1}$	
1	0
0	1
0	0

$G_{..2}$	
1	0
0	1
0	1

TABLE 4

Hypothetical
Reconstructed Rating
Data Matrix

	Mode 2		
Mode 1	α	β	γ
a	1	0	1
b	0	2	1
c	0	2	1
d	0	3	1

TABLE 5

Recoding t(M) of Rating Data Matrix M of Table 4

$t(M)_{..1}$			$t(M)_{..2}$			$t(M)_{..3}$		
1	0	1	0	0	0	0	0	0
0	1	1	0	1	0	0	0	0
0	1	1	0	1	0	0	0	0
0	1	1	0	1	0	0	1	0

TABLE 6

Disjunctive Rank (3,3,3) HICLAS-R Model for Reconstructed Rating Data Matrix of Table 4

A			
a	1	0	0
b	0	1	0
c	0	1	0
d	0	1	1

B			
α	1	0	0
β	0	0	1
γ	1	1	0

C			
1	1	1	1
2	0	1	1
3	0	0	1

$G_{..1}$		
1	0	0
0	1	0
0	0	0

$G_{..2}$		
0	0	0
0	0	1
0	0	0

$G_{..3}$		
0	0	0
0	0	0
0	0	1

$\tilde{G}$		
1	0	0
0	1	2
0	0	3

TABLE 7

Rating Data Matrix D

	Mode 2		
Mode 1	α	β	γ
a	3	3	3
b	0	0	2
c	3	3	4
d	3	3	4

TABLE 8

Disjunctive Rank (2,2,2) Tucker3-HICLAS Model of Binary Three-Way Recoding t(D) of Rating Data Matrix of Table 7

	A	
a	1	0
b	0	1
c	1	1
d	1	1

	B	
α	1	0
β	1	0
γ	1	1

	C	
1	1	1
2	1	1
3	1	0
4	0	1

$G_{..1}$	
1	0
0	0

$G_{..2}$	
0	0
0	1

Figure Captions

Figure 1. Graphic representation of disjunctive two-way HICLAS model of Table 2.

Figure 2. Graphic representation of disjunctive Tucker3-HICLAS model of Table 3.

Figure 3. Graphic representation of HICLAS-R model of Table 6.

Figure 4. Graphic representation of rank (2,3,3) HICLAS-R model of helping data.

Figure 5. Alternative graphic representation of helping behavior profiles of three person types from rank (2,3,3) HICLAS-R model of helping data.

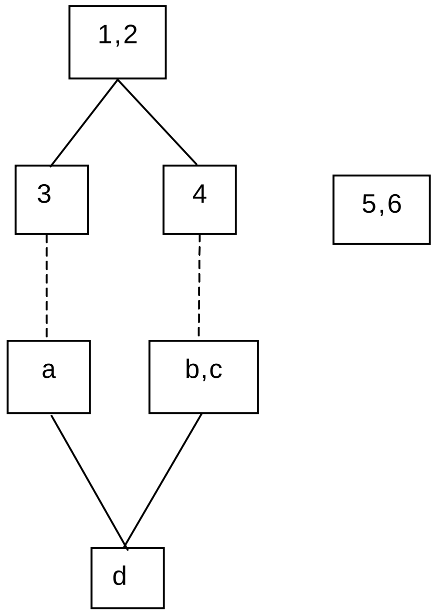

Figure 1

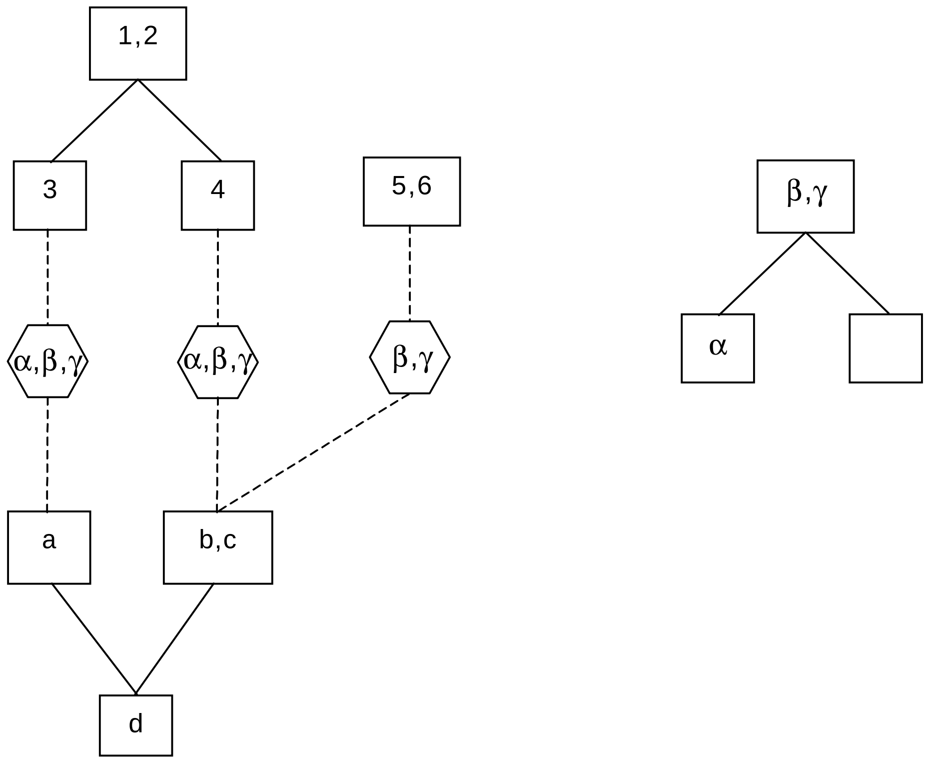

Figure 2

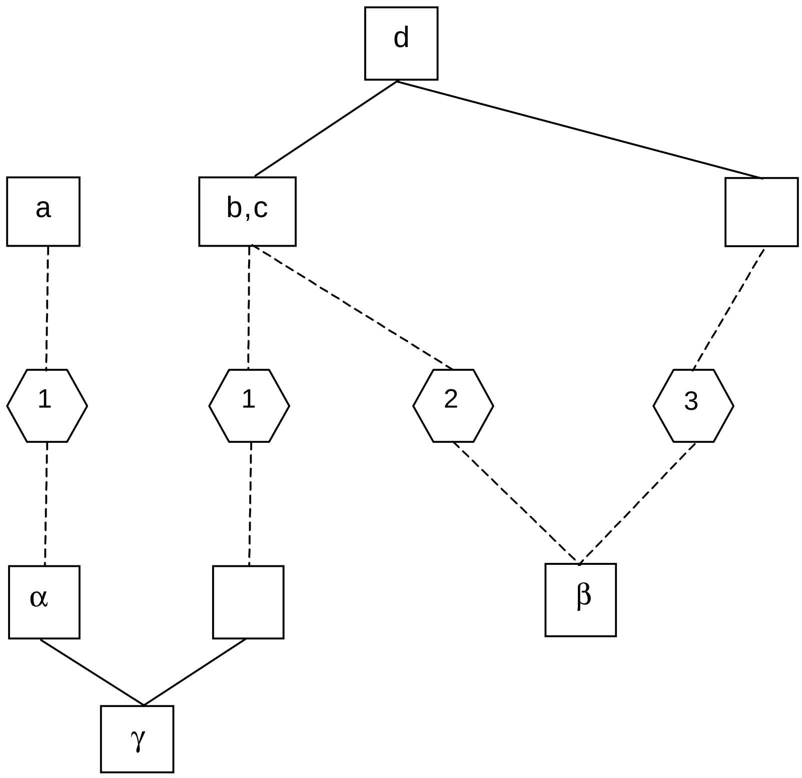

Figure 3

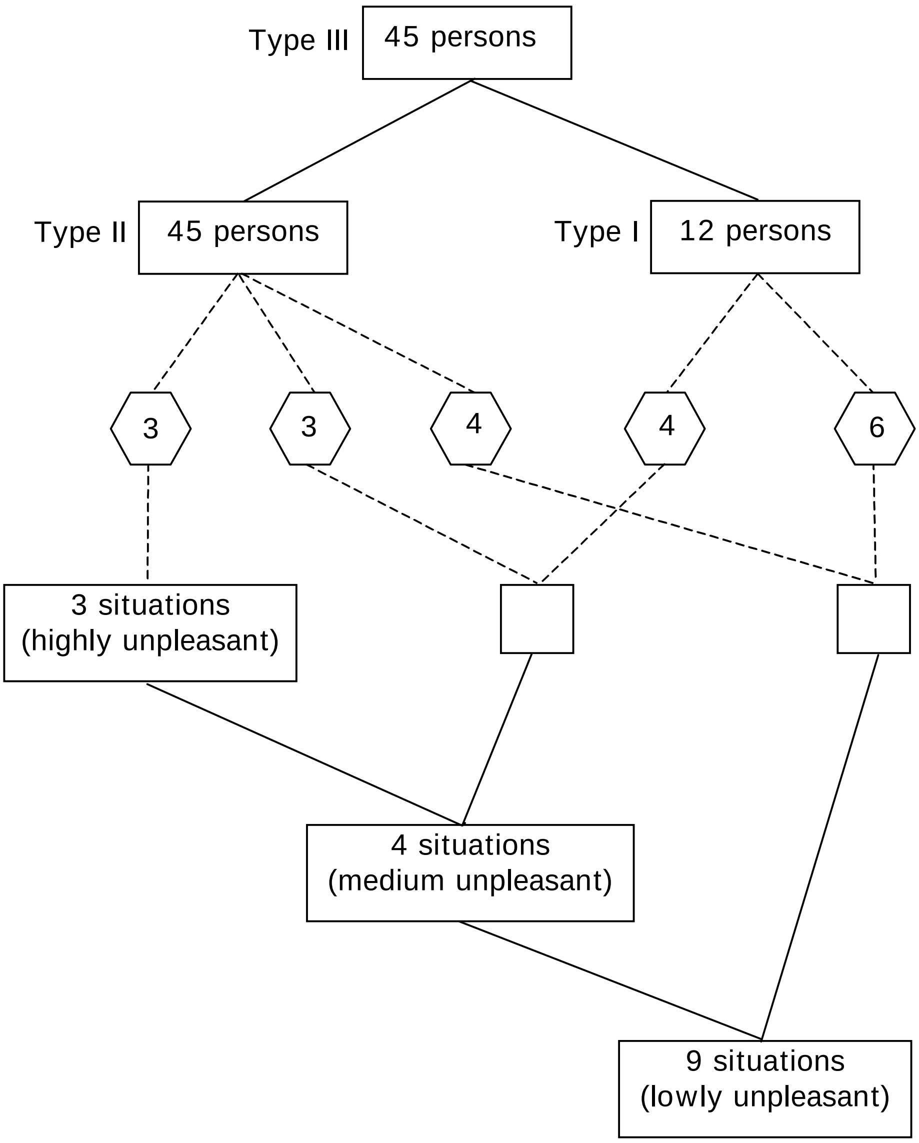
Type III
45 persons
Type II
45 persons
Type I
12 persons
3
3
4
4
6
3 situations
(highly unpleasant)
4 situations
(medium unpleasant)
9 situations
(lowly unpleasant)

Figure 4

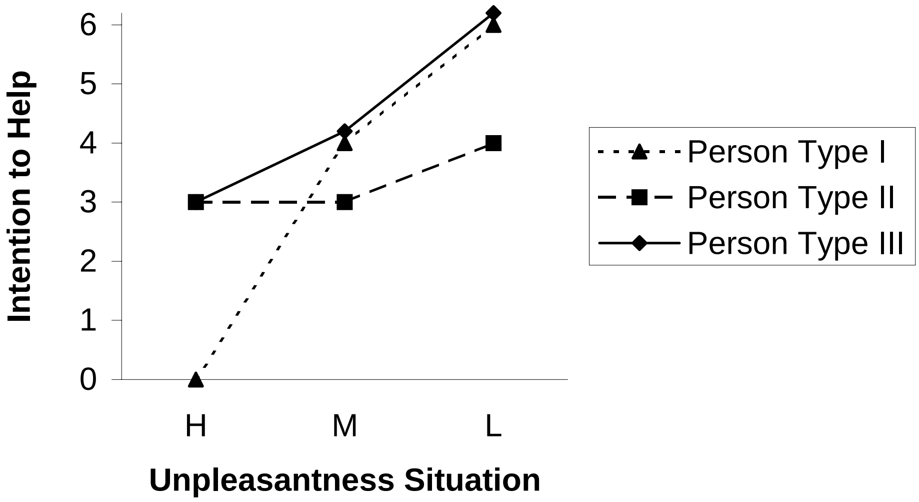
6
5
4
3
2
1
0
Intention to Help
H
M
L
Unpleasantness Situation
Person Type I
Person Type II
Person Type III

Figure 5

Appendix

Consider the following 4 x 3 rating valued data matrix in Table 7. Table 8 contains an optimal rank (2,2,2) Tucker3-HICLAS model of t(D). The model has 1 discrepancy with respect to the recoded data t(D). It does not satisfy property (5), as also can be derived from bundle matrix C. By enumerating all rank (2,2,2) Tucker3-HICLAS models of size 4 x 3 x 4 it can easily be checked that there is no model of t(D) with one discrepancy or less that satisfies (5).