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Affine-invariant Aligned Rank Tests for the Multivariate General Linear Model with ARMA Errors

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Abstract

We develop optimal rank-based procedures for testing affine-invariant linear hypotheses on the parameters of a multivariate general linear model with elliptical VARMA errors. We propose a class of optimal procedures that are based either on residual (pseudo-)Mahalanobis signs and ranks, or on *absolute interdirections* and *lift-interdirection ranks*, i.e., on hyperplane-based signs and ranks. The Mahalanobis versions of these procedures are strictly affine-invariant, while the hyperplane-based ones are asymptotically affine-invariant. Both versions generalize the univariate signed rank procedures proposed by Hallin and Puri (1994), and are locally asymptotically most stringent under correctly specified radial densities. Their AREs with respect to Gaussian procedures are shown to be convex linear combinations of the AREs obtained in Hallin and Paindaveine (2002a, 2002b) for the pure location and purely serial models, respectively. The resulting test statistics are provided under closed form for several important particular cases, including generalized Durbin-Watson tests, VARMA order identification tests, etc. The key technical result is a multivariate asymptotic linearity result proved in Hallin and Paindaveine (2002f).

1 Introduction.

In this paper, we consider the multivariate general linear model with VARMA error terms

$$\mathbf{Y}^{(n)} = \mathbf{X}^{(n)}\boldsymbol{\beta} + \mathbf{U}^{(n)}, \qquad (1)$$

where

$$\mathbf{X}^{(n)} := \begin{pmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,m} \\ \vdots & \vdots & & \vdots \\ x_{n,1} & x_{n,2} & \dots & x_{n,m} \end{pmatrix} := \begin{pmatrix} \mathbf{x}'_1 \\ \vdots \\ \mathbf{x}'_n \end{pmatrix} \quad \text{and} \quad \boldsymbol{\beta} := \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \dots & \beta_{1,k} \\ \vdots & \vdots & & \vdots \\ \beta_{m,1} & \beta_{m,2} & \dots & \beta_{m,k} \end{pmatrix} := \begin{pmatrix} \boldsymbol{\beta}'_1 \\ \vdots \\ \boldsymbol{\beta}'_m \end{pmatrix}$$

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denote an $n \times m$ matrix of constants (the design matrix), and the $m \times k$ regression parameter, respectively. Instead of the traditional assumption that the error term

$$\mathbf{U}^{(n)} := \begin{pmatrix} U_{1,1} & U_{1,2} & \dots & U_{1,k} \\ \vdots & \vdots & & \vdots \\ U_{n,1} & U_{n,2} & \dots & U_{n,k} \end{pmatrix} := \begin{pmatrix} \mathbf{U}_1' \\ \vdots \\ \mathbf{U}_n' \end{pmatrix}$$

is white noise, we rather assume $(\mathbf{U}_t,\ t = 1, \dots, n)$ to be a finite realization (of length n) of a solution of the multivariate linear stochastic difference equation (a VARMA(p_1, q_1) model)

$$\mathbf{A}(L)\,\mathbf{U}_t = \mathbf{B}(L)\,\boldsymbol{\varepsilon}_t, \qquad t \in \mathbb{Z}, \tag{2}$$

where, writing $\mathbf{I}_k$ for the k-dimensional identity matrix, $\mathbf{A}(L) := \mathbf{I}_k - \sum_{i=1}^{p_1} \mathbf{A}_i L^i$ and $\mathbf{B}(L) := \mathbf{I}_k + \sum_{i=1}^{q_1} \mathbf{B}_i L^i$ for some $(p_1 + q_1)$-tuple of $k \times k$ real matrices $(\mathbf{A}_1, \dots, \mathbf{A}_{p_1}, \mathbf{B}_1, \dots, \mathbf{B}_{q_1})$, L stands for the lag operator, and $\{\boldsymbol{\varepsilon}_t \,|\, t \in \mathbb{Z}\}$ is a k-dimensional white-noise process. Under this model, the observation

$$\mathbf{Y}^{(n)} := \begin{pmatrix} Y_{1,1} & Y_{1,2} & \dots & Y_{1,k} \\ \vdots & \vdots & & \vdots \\ Y_{n,1} & Y_{n,2} & \dots & Y_{n,k} \end{pmatrix} := \begin{pmatrix} \mathbf{Y}_1' \\ \vdots \\ \mathbf{Y}_n' \end{pmatrix}$$

is the realization of a k-variate VARMA process $\{\mathbf{Y}_t,\ t \in \mathbb{Z}\}$ with trend $\mathrm{E}[\mathbf{Y}_t] = \boldsymbol{\beta}'\mathbf{x}_t$.

Denote by

$$\boldsymbol{\theta} := \Big((\mathrm{vec}\,\boldsymbol{\beta}')', (\mathrm{vec}\,\mathbf{A}_1)', \dots, (\mathrm{vec}\,\mathbf{A}_{p_1})', (\mathrm{vec}\,\mathbf{B}_1)', \dots, (\mathrm{vec}\,\mathbf{B}_{q_1})'\Big)' \in \mathbb{R}^K := \mathbb{R}^{km + k^2(p_1+q_1)},$$

the parameter of this model. We consider the problem of testing linear hypotheses about $\boldsymbol{\theta}$. Writing $\mathcal{M}(\boldsymbol{\Upsilon})$ for the vector space spanned by the columns of some (full-rank) matrix $\boldsymbol{\Upsilon}$, such null hypotheses can be written as $\mathcal{H}_0 : \boldsymbol{\theta} - \boldsymbol{\theta}_0 \in \mathcal{M}(\boldsymbol{\Upsilon})$, for some specified K-vector $\boldsymbol{\theta}_0$ and full-rank $(K \times r)$ matrix $\boldsymbol{\Upsilon}$. Linear constraints that imply VARMA orders less than p_1 and/or q_1 however require a special treatment. Therefore, we denote by $p_0 \le p_1$ and $q_0 \le q_1$ respectively the orders, under $\mathcal{H}_0$, of the autoregressive and moving average operators (meaning that under $\mathcal{H}_0$ all entries in rows $km + k^2 p_0 + 1, \dots, km + k^2 p_1$ and rows $km + k^2(p_1 + q_0) + 1, \dots, K$ of $\boldsymbol{\Upsilon}$ are zeros); $p_0 = p_1$ and $q_0 = q_1$ thus simply means that the orders of the model are not an issue.

In the sequel, we restrict to the class of linear hypotheses $\mathcal{H}_0$ that are invariant under affine transformations in the following sense. For any $k \times k$ full-rank matrix $\mathbf{M}$, the affine transformation $\boldsymbol{\varepsilon}_t \mapsto \mathbf{M}\boldsymbol{\varepsilon}_t$ of the noise induces the transformation

$$(\boldsymbol{\beta}, \mathbf{A}_1, \dots, \mathbf{A}_{p_1}, \mathbf{B}_1, \dots, \mathbf{B}_{q_1}) \mapsto (\boldsymbol{\beta}\,\mathbf{M}', \mathbf{M}\mathbf{A}_1\mathbf{M}^{-1}, \dots, \mathbf{M}\mathbf{A}_{p_1}\mathbf{M}^{-1}, \mathbf{M}\mathbf{B}_1\mathbf{M}^{-1}, \dots, \mathbf{M}\mathbf{B}_{q_1}\mathbf{M}^{-1})$$

of the parameter. In terms of $\boldsymbol{\theta}$, this induced transformation is $\boldsymbol{\theta} \mapsto \mathbf{g}_{\mathbf{M}}^{(m, p_1+q_1)}\boldsymbol{\theta}$, where

$$\mathbf{g}_{\mathbf{M}}^{(r_1, r_2)} := \begin{pmatrix} \mathbf{I}_{r_1} \otimes \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{r_2} \otimes (\mathbf{M}'^{-1} \otimes \mathbf{M}) \end{pmatrix}.$$

Letting $\mathcal{G}_{r_2}^{r_1}(k) := \{\mathbf{g}_{\mathbf{M}}^{(r_1, r_2)}, \mathbf{M} \text{ of full-rank}\}$, we say that the null hypothesis $\mathcal{H}_0 : \boldsymbol{\theta} - \boldsymbol{\theta}_0 \in \mathcal{M}(\boldsymbol{\Upsilon})$ is invariant under affine transformations iff $\mathbf{g}_{\mathbf{M}}^{(m, p_1+q_1)}(\boldsymbol{\theta}_0 + \mathcal{M}(\boldsymbol{\Upsilon})) = \boldsymbol{\theta}_0 + \mathcal{M}(\boldsymbol{\Upsilon})$, for all $\mathbf{g}_{\mathbf{M}}^{(m, p_1+q_1)} \in \mathcal{G}_{p_1+q_1}^{m}(k)$.

Let $\mathbf{l}_k := \text{vec}\,\mathbf{I}_k$, $\mathbf{L}_k := \mathbf{l}_k\mathbf{l}_k'$, and denote by $\mathbf{P}_k$ the $k^2 \times (k^2-1)$ array obtained by deleting the last column in $\mathbf{I}_{k^2} - \mathbf{L}_k$. Then Hallin and Paindaveine (2002c) showed that the null hypothesis $\mathcal{H}_0 : \boldsymbol{\theta} - \boldsymbol{\theta}_0 \in \mathcal{M}(\boldsymbol{\Upsilon})$ is invariant under affine transformations iff

$$\boldsymbol{\Upsilon} = \begin{pmatrix} \boldsymbol{\Upsilon}_I & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Upsilon}_{II} \end{pmatrix} = \begin{pmatrix} \mathbf{Z} \otimes \mathbf{I}_k & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V} \otimes \mathbf{P}_k & \mathbf{W} \otimes \mathbf{l}_k \end{pmatrix} \mathbf{G}$$

and

$$\boldsymbol{\theta}_0 = \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \otimes \mathbf{l}_k \end{pmatrix} + \boldsymbol{\Upsilon}\boldsymbol{\omega},$$

where $\mathbf{w}$ and $\boldsymbol{\omega}$ denote arbitrary vectors with dimensions p_1+q_1 and r, respectively, $\mathbf{Z}$, $\mathbf{V}$, and $\mathbf{W}$ are (possibly void) full-rank matrices with dimensions $m \times r_Z$, $(p_1+q_1) \times r_V$, and $(p_1+q_1) \times r_W$, respectively, and (letting $r = r_I + r_{II}$, where $r_I := r_Z k$ and $r_{II} := r_V(k^2-1) + r_W$) $\mathbf{G}$ is an invertible $r \times r$ matrix. Since $\mathcal{M}(\boldsymbol{\Upsilon}) = \mathcal{M}(\boldsymbol{\Upsilon}\mathbf{G})$ for any such $\mathbf{G}$, we may assume, without loss of generality, that $\mathbf{G} = \mathbf{I}_r$ in the sequel. In case $p_0 < p_1$ and/or $q_0 < q_1$, note that $\mathbf{V}$ and $\mathbf{W}$ have only zeros in rows $p_0+1, \ldots, p_1$ and rows $p_1+q_0+1, \ldots, p_1+q_1$.

This class of affine-invariant null hypotheses covers a wide range of problems of practical interest. In the sequel, the following particular cases will be treated in details in Section 6.

(a) The multivariate Durbin-Watson problem, which corresponds to $\boldsymbol{\theta}_0 = \mathbf{0}$, $\boldsymbol{\Upsilon}_I = \mathbf{I}_{km}$, and $\boldsymbol{\Upsilon}_{II} = \emptyset$, where $\emptyset$ denotes the void matrix. This allows for testing serial independence of the error term in an unspecified linear model versus VARMA errors of orders less than or equal to p_1 and q_1 (the linear model structure of the trend plays the role of the nuisance).

(b) Testing the orders of VARMA errors. In the second example, we consider the problem of testing a VARMA(p_0, q_0) model versus a higher-order VARMA(p_1, q_1). This is obtained by letting $\boldsymbol{\theta}_0 = \mathbf{0}$, $\boldsymbol{\Upsilon}_I = \emptyset$, and

$$\boldsymbol{\Upsilon}_{II} = \begin{pmatrix} \mathbf{I}_{p_0} & \mathbf{0}_{p_0 \times q_0} \\ \mathbf{0}_{(p_1-p_0)\times p_0} & \mathbf{0}_{(p_1-p_0)\times q_0} \\ \mathbf{0}_{q_0 \times p_0} & \mathbf{I}_{q_0} \\ \mathbf{0}_{(q_1-q_0)\times p_0} & \mathbf{0}_{(q_1-q_0)\times q_0} \end{pmatrix}$$

(here again, the linear model structure of the trend plays the role of the nuisance). The particular case where $p_1 - p_0 = q_1 - q_0 = 1$ plays an important role in several model identification procedures (see, e.g., Pötscher 1983, 1985, or Garel and Hallin 1999 for the univariate case). For the sake of notational simplicity, we restrict to $p_1 - p_0 = 1$, $q_1 = q_0 = 0$ in the sequel.

(c) Testing against switching location regime. Let $(t_i^{(n)})$, $i = 1, \ldots, m-1$, be $(m-1)$-tuple of sequences such that $t_0^{(n)} := 0 < t_1^{(n)} < \ldots < t_{m-1}^{(n)} < t_m^{(n)} := n$ for all n. Denoting by $\mathbf{e}_i^{(m)}$ the ith vector of the canonical basis in $\mathbb{R}^m$, consider the design matrix defined by

$$\mathbf{x}_t^{(n)} = \mathbf{e}_i^{(m)}, \qquad \text{for } t_{i-1}^{(n)} < t \leq t_i^{(n)}.$$

The resulting model is a VARMA(p_1, q_1) one, with time-dependent trend (more precisely, with mean $\boldsymbol{\beta}_i$ between $t = t_{i-1}^{(n)} + 1$ and $t = t_i^{(n)}$). In this setup, the testing problem associated with $\boldsymbol{\Upsilon}_I = (1, \ldots, 1)' \otimes \mathbf{I}_k$, $\boldsymbol{\Upsilon}_{II} = \mathbf{I}_{k^2(p_1+q_1)}$ corresponds to the problem of testing the absence of different regimes, i.e., to the null hypothesis $\mathcal{H}_0 : \boldsymbol{\beta}_1 = \ldots = \boldsymbol{\beta}_m$. The coefficients of the VARMA operators here are nuisance parameters. Note that if there is no serial component in the model, then this reduces to the standard m-sample problem, i.e., to the most standard testing problem in analysis of variance.

2 Main assumptions.

In this section we collect, for convenient reference, all assumptions we need in the sequel. These assumptions are dealing with the design of the model, the innovation density, the score functions to be used in test statistics, and the estimators of unspecified and nuisance parameters.

We begin with some structural conditions on the trend part of the model. The following assumptions are standard in the context (see Garel and Hallin 1995).

Assumption (A1). Letting $\mathbf{C}_i^{(n)} := (n-i)^{-1}\sum_{t=i+1}^n \mathbf{x}_t^{(n)}\mathbf{x}_{t-i}^{(n)\prime}$, $i = 0, 1, \ldots, n-1$, denote by $\mathbf{D}^{(n)}$ the diagonal matrix with elements $(\mathbf{C}_0^{(n)})_{11}, \ldots, (\mathbf{C}_0^{(n)})_{mm}$.

(i) $(\mathbf{C}_0^{(n)})_{jj} > 0$ for all j.

(ii) Let $\mathbf{R}_i^{(n)} := (\mathbf{D}^{(n)})^{-1/2}\mathbf{C}_i^{(n)}(\mathbf{D}^{(n)})^{-1/2}$. The limits $\lim_{n\to\infty}\mathbf{R}_i^{(n)} =: \mathbf{R}_i$ exist for all i; $\mathbf{R}_0$ is positive definite, and therefore can be factorized into $\mathbf{R}_0 = (\mathbf{K}\mathbf{K}^{'})^{-1}$ for some full-rank $m \times m$ matrix $\mathbf{K}$. Letting $\mathbf{K}^{(n)} := (\mathbf{D}^{(n)})^{-1/2}\mathbf{K}$ (defining $\mathbf{K}^{(n)}$, note that $\mathbf{K}^{(n)}$ also has full rank).

(iii) The classical Noether conditions hold : the $(\mathbf{x}_t^{(n)})_j$, $t = 1, \ldots, n$, are not all equal, and, letting $\bar{x}_j^{(n)} := n^{-1}\sum_{t=1}^n (\mathbf{x}_t^{(n)})_j$,

$$\lim_{n\to\infty} \frac{\max_{1\le t\le n}\left((\mathbf{x}_t^{(n)})_j - \bar{x}_j^{(n)}\right)^2}{\sum_{t=1}^n \left((\mathbf{x}_t^{(n)})_j - \bar{x}_j^{(n)}\right)^2} = 0, \quad j = 1, \ldots, m.$$

The description of the asymptotic behaviour of the proposed test statistics under local alternatives will require the following reinforcement of (A1).

Assumption (A1′). Same as Assumption (A1), but we further assume that $\lim_{n\to\infty}[\mathbf{D}^{(n)}/\operatorname{tr}\mathbf{D}^{(n)}] =: \mathbf{D}^2$, where $\mathbf{D}$ is a finite, positive definite diagonal matrix.

For the serial part of the model, we essentially require the VARMA model (2) to be causal and invertible. The assumptions on the difference operators are actually the same as in Hallin and Paindaveine (2002d), where the problem of testing the adequacy of a specified VARMA model is considered.

Assumption (A2). All solutions of $\det(\mathbf{I}_k - \sum_{i=1}^{p_0}\mathbf{A}_i z^i) = 0$ and $\det(\mathbf{I}_k + \sum_{i=1}^{q_0}\mathbf{B}_i z^i) = 0$ $(|\mathbf{A}_{p_0}| \neq 0 \neq |\mathbf{B}_{q_0}|)$ lie outside the unit ball in $\mathbb{C}$. Moreover, the greatest common left divisor of $\mathbf{I}_k - \sum_{i=1}^{p_0}\mathbf{A}_i z^i$ and $\mathbf{I}_k + \sum_{i=1}^{q_0}\mathbf{B}_i z^i$ is the identity matrix $\mathbf{I}_k$.

In the sequel, we denote by $\mathbf{G}_u(\boldsymbol{\theta})$, $u \in \mathbb{N}$, the Green's matrices associated with the autoregressive difference operator $\mathbf{A}(L) = \mathbf{I}_k - \sum_{i=1}^{p_0}\mathbf{A}_i L^i$. These matrices can be defined recursively by $\mathbf{A}(L)\mathbf{G}_u = \mathbf{G}_u - \sum_{i=1}^{\min(p_0,u)}\mathbf{A}_i\mathbf{G}_{u-i} = \delta_{u0}\,\mathbf{I}_k$, where $\delta_{u0} = 1$ if $u = 0$, and $\delta_{u0} = 0$ otherwise.

Assumption (A2) also allows for defining $\mathbf{G}_u$ by means of

$$\sum_{u=0}^{+\infty} \mathbf{G}_u z^u := \left(\mathbf{I}_k - \sum_{i=1}^{p_0} \mathbf{A}_i z^i \right)^{-1}, \qquad z \in \mathbb{C},\ |z| < 1. \tag{3}$$

Similarly, we denote by $\mathbf{H}_u(\boldsymbol{\theta})$, $u \in \mathbb{N}$, the Green's matrices associated with the moving average difference operator $\mathbf{B}(L)$. Clearly, all these Green's matrices are continuous functions of $\boldsymbol{\theta}$. When no confusion is possible, we will not stress their dependence on $\boldsymbol{\theta}$.

The residuals $(\mathbf{Z}_1^{(n)}(\boldsymbol{\theta}), \ldots, \mathbf{Z}_n^{(n)}(\boldsymbol{\theta}))$ associated with a value $\boldsymbol{\theta}$ of the parameter then can be computed from a set of initial values $\boldsymbol{\varepsilon}_{-q_0+1} \ldots, \boldsymbol{\varepsilon}_0, \mathbf{Y}_{-p_0+1}^{(n)}, \ldots, \mathbf{Y}_0^{(n)}$ and the observed series $(\mathbf{Y}_1^{(n)}, \ldots, \mathbf{Y}_n^{(n)})$ via the recursion

$$\begin{aligned} \mathbf{Z}_t^{(n)}(\boldsymbol{\theta}) = & \sum_{i=0}^{t-1} \sum_{j=0}^{p_0} \mathbf{H}_i \mathbf{A}_j (\mathbf{Y}_{t-i-j}^{(n)} - \boldsymbol{\beta}' \mathbf{x}_{t-i-j}^{(n)}) \\ & + (\mathbf{H}_{t+q_0-1} \ldots \mathbf{H}_t) \begin{pmatrix} \mathbf{I}_k & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{B}_1 & \mathbf{I}_k & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{q_0-1} & \mathbf{B}_{q_0-2} & \ldots & \mathbf{I}_k \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_{-q_0+1} \\ \vdots \\ \boldsymbol{\varepsilon}_0 \end{pmatrix}. \end{aligned} \tag{4}$$

Assumption (A2) ensures that neither the (generally unobserved) values $(\boldsymbol{\varepsilon}_{-q_0+1}, \ldots, \boldsymbol{\varepsilon}_0)$ of the innovation, nor the initial values $(\mathbf{Y}_{-p_0+1}^{(n)}, \ldots, \mathbf{Y}_0^{(n)})$, have any influence on asymptotic results; they all safely can be put to zero in the sequel.

Under (A2), $\{\boldsymbol{\varepsilon}_t\}$ is $\{\mathbf{Y}_t\}$'s innovation process. Denote by $\boldsymbol{\Sigma}$ a symmetric positive definite $k \times k$ matrix, and let $f : \mathbb{R}_0^+ \to \mathbb{R}^+$ be such that $f > 0$ a.e. and $\int_0^\infty r^{k-1} f(r)\, dr < \infty$: we will assume throughout that $\{\boldsymbol{\varepsilon}_1^{(n)}, \ldots, \boldsymbol{\varepsilon}_n^{(n)}\}$ is a finite realization of an elliptic white noise process with shape matrix $\boldsymbol{\Sigma}$ and radial density f :

Assumption (B1). The innovation density is of the form $\prod_{t=1}^n \underline{f}(\mathbf{z}_t^{(n)}; \boldsymbol{\Sigma}, f)$, where

$$\underline{f}(\mathbf{z}_1; \boldsymbol{\Sigma}, f) := c_{k,f} \, (\det \boldsymbol{\Sigma})^{-1/2} f(\|\mathbf{z}_1\|_{\boldsymbol{\Sigma}}), \quad \mathbf{z}_1 \in \mathbb{R}^k. \tag{5}$$

As usual, $\|\mathbf{z}\|_{\boldsymbol{\Sigma}} := (\mathbf{z}' \boldsymbol{\Sigma}^{-1} \mathbf{z})^{1/2}$ denotes the norm of $\mathbf{z}$ in the metric associated with $\boldsymbol{\Sigma}$. The constant $c_{k,f}$ is the normalization factor $(\omega_k \, \mu_{k-1;f})^{-1}$, where ω_k stands for the $(k-1)$-dimensional Lebesgue measure of the unit sphere $\mathcal{S}^{k-1} \subset \mathbb{R}^k$, and $\mu_{l;f} := \int_0^\infty r^l f(r)\, dr$.

Denote by $\mathcal{H}_{\boldsymbol{\Upsilon}}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f)$ the hypothesis under which the observation $\mathbf{Y}^{(n)}$ is generated by (1) and (2), with a value $\boldsymbol{\theta}$ of the parameter of interest that satisfies $\boldsymbol{\theta} - \boldsymbol{\theta}_0 \in \mathcal{M}(\boldsymbol{\Upsilon})$, and with values $\boldsymbol{\Sigma}$ and f for the parameters of the underlying elliptical white noise. Denote by $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f)$ the hypothesis $\mathcal{H}_{\boldsymbol{\Upsilon}}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f)$, where $\boldsymbol{\Upsilon}$ is the void matrix (so that $\mathcal{M}(\boldsymbol{\Upsilon}) = \{\mathbf{0}\}$). The goal of this paper is to develop testing procedures for the null hypothesis $\mathcal{H}_{\boldsymbol{\Upsilon}}^{(n)}(\boldsymbol{\theta}_0) := \bigcup_{\boldsymbol{\Sigma}} \bigcup_f \mathcal{H}_{\boldsymbol{\Upsilon}}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f)$ against $\bigcup_{\boldsymbol{\theta} \neq \boldsymbol{\theta}_0 + \mathcal{M}(\boldsymbol{\Upsilon})} \bigcup_{\boldsymbol{\Sigma}} \bigcup_f \mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$, that

- are **non-parametric**, i.e., valid under the whole of $\mathcal{H}_{\boldsymbol{\Upsilon}}^{(n)}(\boldsymbol{\theta}_0)$ (under which the distribution of the noise is not specified beyond elliptical symmetry and possibly some moment constraints);

- are **locally and asymptotically optimal (LAO)** (locally asymptotically *most stringent*, in this case) at some fixed radial density $f_\star$, that is, against alternatives of the form $\bigcup_{\boldsymbol{\theta}\neq\boldsymbol{\theta}_0+\mathcal{M}(\boldsymbol{\Upsilon})}\bigcup_{\boldsymbol{\Sigma}}\mathcal{H}^{(n)}(\boldsymbol{\theta},\boldsymbol{\Sigma},f_\star)$; this of course requires local asymptotic normality (LAN) of the parametric submodel associated with $f_\star$;

- **comply with the invariance principle**: we restricted to null hypotheses that are invariant with respect to the group of affine transformations. The hypotheses considered are also invariant with respect to the group of continuous monotone radial transformations (acting on residuals); see Section 4.1 for a precise definition of this group. The proposed procedures should be (at least asymptotically) invariant with respect to these two groups.

Local asymptotic normality requires some further regularity assumptions on the innovation density. The set of assumptions (B) collects these assumptions.

Assumption (B1′). Same as Assumption (B1), but with $\mu_{k+1,f}<\infty$. Assumption (B2). The square root $f^{1/2}$ of the radial density f is in $W^{1,2}(\mathbb{R}_0^+,\mu_{k-1})$, where $W^{1,2}(\mathbb{R}_0^+,\mu_{k-1})$ denotes the subspace of $L^2(\mathbb{R}_0^+,\mu_{k-1})$ containing all functions admitting a weak derivative that also belongs to $L^2(\mathbb{R}_0^+,\mu_{k-1})$.

Assumption (B2) is strictly equivalent to the assumption that $\underline{f}^{1/2}$ is differentiable in quadratic mean (see Hallin and Paindaveine 2002a). Denoting by $(f^{1/2})'$ the weak derivative of $f^{1/2}$ in $L^2(\mathbb{R}_0^+,\mu_{k-1})$, let $\varphi_f:=-2\frac{(f^{1/2})'}{f^{1/2}}$. Under (B2), the *radial Fisher information* $\mathcal{I}_{k,f}:=\int_0^\infty[\varphi_f(r)]^2r^{k-1}f(r)\,dr$ is finite. In the pure location or purely serial problems considered in Hallin and Paindaveine (2002a, b, and d), this was sufficient for LAN. However, as pointed out by Garel and Hallin (1995), LAN, in this model where serial and nonserial features are mixed, requires the stronger assumption:

Assumption (B3). $\int_0^\infty[\varphi_f(r)]^4r^{k-1}f(r)\,dr<\infty$.

Assumptions (C) and (C′) impose some mild conditions on the score functions J_ℓ, $\ell=0,1,2$, to be used when building rank-based statistics.

Assumption (C). The score functions $J_\ell:]0,1[\to\mathbb{R}$, $\ell=0,1,2$, are continuous differences of two monotone increasing functions, and satisfy $\int_0^1[J_\ell(u)]^2\,du<\infty$ $(\ell=0,1,2)$.

The score functions yielding locally and asymptotically optimal procedures are of the form $J_0=J_1:=\varphi_{f_\star}\circ\tilde{F}_{\star k}^{-1}$ and $J_2:=\tilde{F}_{\star k}^{-1}$, for some radial density $f_\star$ (here $\tilde{F}_{\star k}$ stands for the cdf associated with the radial pdf $\tilde{f}_{\star k}(r)=(\mu_{k-1;f_\star})^{-1}\,r^{k-1}f_\star(r)\,I_{[r>0]}$, $r\in\mathbb{R}$). Assumption (C) then takes the form of an assumption on $f_\star$:

Assumption (C′). The radial density $f_\star$ is such that $\varphi_{f_\star}$ is the continuous difference of two monotone increasing functions, $\mu_{k+1;f_\star}<\infty$, and $\int_0^\infty[\varphi_{f_\star}(r)]^2r^{k-1}f_\star(r)\,dr<\infty$.

The shape matrix $\boldsymbol{\Sigma}$ in Assumption (B1) is unknown and has to be estimated. We assume the following.

Assumption (D1). A sequence $\widehat{\boldsymbol{\Sigma}}^{(n)}=\widehat{\boldsymbol{\Sigma}}^{(n)}(\mathbf{Z}_1,\ldots,\mathbf{Z}_n)$ of estimators of $\boldsymbol{\Sigma}$ exists, such that

(i) $\sqrt{n}(\widehat{\boldsymbol{\Sigma}}^{(n)}-a\,\boldsymbol{\Sigma})=O_{\mathrm{P}}(1)$ as $n\to\infty$ for some positive real a, and

(ii) $\widehat{\boldsymbol{\Sigma}}^{(n)}$ is invariant under permutations and reflections (with respect to the origin in $\mathbb{R}^k$) of the residuals $\mathbf{Z}_t$.

Assumption (D1) will be sufficient for the validity of the proposed procedures. However, their affine-invariance requires the following equivariance assumption on $\widehat{\mathbf{\Sigma}} := \widehat{\mathbf{\Sigma}}^{(n)}$.

ASSUMPTION (D2). The estimator $\widehat{\mathbf{\Sigma}}$ is quasi-affine-equivariant, in the sense that, for all $k \times k$ full-rank matrix $\mathbf{M}$, $\widehat{\mathbf{\Sigma}}(\mathbf{M}) = d\,\mathbf{M}\widehat{\mathbf{\Sigma}}\mathbf{M}'$, where $\widehat{\mathbf{\Sigma}}(\mathbf{M})$ stands for the statistic $\widehat{\mathbf{\Sigma}}$ computed from the n-tuple $(\mathbf{MZ}_1, \ldots, \mathbf{MZ}_n)$, and d denotes some positive scalar that may depend on $\mathbf{M}$ and $(\mathbf{Z}_1, \ldots, \mathbf{Z}_n)$.

Since the parameter of interest $\boldsymbol{\theta}$ remains partially unspecified under the null, we also need some preliminary estimate of $\boldsymbol{\theta}$. More precisely, we will assume the existence of an estimator $\hat{\boldsymbol{\theta}} := \hat{\boldsymbol{\theta}}^{(n)}$ for $\boldsymbol{\theta}$ satisfying Assumptions (E1) and (E2) below.

ASSUMPTION (E1). The sequence of estimator $(\hat{\boldsymbol{\theta}}^{(n)}, n \in \mathbb{N})$ is

(i) *constrained* : $\hat{\boldsymbol{\theta}}^{(n)} - \boldsymbol{\theta}_0 \in \mathcal{M}(\mathbf{\Upsilon})$ for all n,

(ii) *root-n consistent* : $\forall \boldsymbol{\theta} \in \boldsymbol{\theta}_0 + \mathcal{M}(\mathbf{\Upsilon})$, $n^{1/2}(\hat{\boldsymbol{\theta}}^{(n)} - \boldsymbol{\theta}) = O_{\mathrm{P}}(1)$, as $n \to \infty$, under $\bigcup_{\mathbf{\Sigma}} \bigcup_f \mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$, and

(iii) *locally asymptotically discrete* : $\forall \boldsymbol{\theta} \in \boldsymbol{\theta}_0 + \mathcal{M}(\mathbf{\Upsilon})$, $\forall c > 0$, there exists an $M(c) > 0$ such that the number of possible values of $\hat{\boldsymbol{\theta}}$ in balls of the form $\{\mathbf{t} \in \mathbb{R}^{k^2(p_1+q_1)} : \|n^{1/2}(\mathbf{t} - \boldsymbol{\theta})\| \leq c\}$ is bounded by M, uniformly as $n \to \infty$.

Assumptions (E1) (i), (ii) are satisfied by all classical estimators (Yule-Walker, least-squares, maximum likelihood, ...). The technical assumption (E1) (iii), which goes back to Le Cam (1960), is of little practical relevance, and it should be pointed out that, for fixed sample size, any estimate can be considered part of a locally asymptotically discrete sequence. While Assumption (E1) is classical for the univariate version of the testing problem under study, Assumption (E2) below is specific to the multivariate case (it is essentially void for $k = 1$), and is required if affine-invariance is to be achieved.

ASSUMPTION (E2). For any full-rank $k \times k$ matrix $\mathbf{M}$, denote by $\hat{\boldsymbol{\theta}}(\mathbf{M})$ the value of $\hat{\boldsymbol{\theta}}$ computed from the transformed sample $\mathbf{MY}_1, \ldots, \mathbf{MY}_n$. Then $\hat{\boldsymbol{\theta}}$ is affine-equivariant, meaning that $\hat{\boldsymbol{\theta}}(\mathbf{M}) = g_{\mathbf{M}}^{(m,p_1+q_1)}\, \hat{\boldsymbol{\theta}}$, for all $g_{\mathbf{M}}^{(m,p_1+q_1)} \in \mathcal{G}^m_{p_1+q_1}(k)$.

Equivalently, (E2) means that the estimators we consider are assumed to satisfy $\hat{\boldsymbol{\beta}}(\mathbf{M}) = \hat{\boldsymbol{\beta}}\mathbf{M}'$, $\hat{\mathbf{A}}_i(\mathbf{M}) = \mathbf{M}\hat{\mathbf{A}}_i\mathbf{M}^{-1}$ for all $i = 1, \ldots, p_0$, and $\hat{\mathbf{B}}_j(\mathbf{M}) = \mathbf{M}\hat{\mathbf{B}}_j\mathbf{M}^{-1}$ for all $j = 1, \ldots, q_0$. Note that the resulting Green's matrices then also are affine-equivariant, i.e., $\mathbf{G}_u(\hat{\boldsymbol{\theta}}(\mathbf{M})) = \mathbf{M}\,\mathbf{G}_u(\hat{\boldsymbol{\theta}})\mathbf{M}^{-1}$ and $\mathbf{H}_u(\hat{\boldsymbol{\theta}}(\mathbf{M})) = \mathbf{M}\,\mathbf{H}_u(\hat{\boldsymbol{\theta}})\mathbf{M}^{-1}$ for every integer u. In the sequel, we will write $\hat{\mathbf{G}}_u^{(n)}$ and $\hat{\mathbf{H}}_u^{(n)}$ for $\mathbf{G}_u(\hat{\boldsymbol{\theta}})$ and $\mathbf{H}_u(\hat{\boldsymbol{\theta}})$, respectively. Note that, for any constrained estimator $\hat{\boldsymbol{\theta}}$ satisfying (E2), $\hat{\boldsymbol{\theta}}(\mathbf{M})$ is also constrained, since $\hat{\boldsymbol{\theta}}(\mathbf{M}) = g_{\mathbf{M}}^{(m,p_1+q_1)}\, \hat{\boldsymbol{\theta}} \in g_{\mathbf{M}}^{(m,p_1+q_1)}(\boldsymbol{\theta}_0 + \mathcal{M}(\mathbf{\Upsilon})) = \boldsymbol{\theta}_0 + \mathcal{M}(\mathbf{\Upsilon})$ for all $\mathbf{M}$ (we restricted to pairs $(\boldsymbol{\theta}_0, \mathcal{M}(\mathbf{\Upsilon}))$ for which the null hypothesis is affine-invariant). In other words, affine-equivariance in (E2) and part (i) of (E1) are compatible, thanks to the affine-invariance of the null hypothesis.

3 Uniform local asymptotic normality (ULAN).

In this section, we briefly recall the ULAN (uniformly local asymptotic normality) result proved in Hallin and Paindaveine (2002f) for the model under study. The sequences of local alternatives to be considered for this property are associated with sequences of models of the form

$$\mathbf{Y}^{(n)} = \mathbf{X}^{(n)}\,\boldsymbol{\beta}^{(n)} + \mathbf{U}^{(n)}, \qquad \mathbf{A}^{(n)}(L)\,\mathbf{U}_t^{(n)} = \mathbf{B}^{(n)}(L)\,\boldsymbol{\varepsilon}_t^{(n)}, \quad t \in \mathbb{Z}, \tag{6}$$

where $\boldsymbol{\beta}^{(n)} := \boldsymbol{\beta} + n^{-1/2}\mathbf{K}^{(n)}\boldsymbol{\eta}^{(n)}$, $\mathbf{A}^{(n)}(L) := \mathbf{I}_k - \sum_{i=1}^{p_1}(\mathbf{A}_i + n^{-1/2}\boldsymbol{\gamma}_i^{(n)})\,L^i$, $\mathbf{B}^{(n)}(L) := \mathbf{I}_k + \sum_{i=1}^{q_1}(\mathbf{B}_i + n^{-1/2}\boldsymbol{\delta}_i^{(n)})\,L^i$, and the sequence

$$\boldsymbol{\tau}^{(n)} := \Big((\mathrm{vec}\,\boldsymbol{\eta}^{(n)\prime})', (\mathrm{vec}\,\boldsymbol{\gamma}_1^{(n)})', \ldots, (\mathrm{vec}\,\boldsymbol{\gamma}_{p_1}^{(n)})', (\mathrm{vec}\,\boldsymbol{\delta}_1^{(n)})', \ldots, (\mathrm{vec}\,\boldsymbol{\delta}_{q_1}^{(n)})'\Big)' \in \mathbb{R}^K$$

is bounded as $n \to \infty$. The perturbed parameter is thus

$$\boldsymbol{\theta}^{(n)} := \boldsymbol{\theta} + \boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)} := \boldsymbol{\theta} + n^{-1/2}\begin{pmatrix} \mathbf{K}^{(n)} \otimes \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{k^2(p_1+q_1)} \end{pmatrix}\boldsymbol{\tau}^{(n)}.$$

The corresponding sequence of local alternatives will be denoted by $\mathcal{H}^{(n)}(\boldsymbol{\theta} + \boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)}, \boldsymbol{\Sigma}, f)$.

Decompose $\mathbf{Z}_t(\boldsymbol{\theta}) := \mathbf{Z}_t^{(n)}(\boldsymbol{\theta})$ into $\mathbf{Z}_t(\boldsymbol{\theta}) = d_t(\boldsymbol{\theta}, \boldsymbol{\Sigma})\,\boldsymbol{\Sigma}^{1/2}\mathbf{U}_t(\boldsymbol{\theta}, \boldsymbol{\Sigma})$, where $d_t(\boldsymbol{\theta}, \boldsymbol{\Sigma}) = d_t^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}) := \|\mathbf{Z}_t(\boldsymbol{\theta})\|_{\boldsymbol{\Sigma}}$ and $\mathbf{U}_t(\boldsymbol{\theta}, \boldsymbol{\Sigma}) = \mathbf{U}_t^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}) := \boldsymbol{\Sigma}^{-1/2}\mathbf{Z}_t(\boldsymbol{\theta})/d_t(\boldsymbol{\theta}, \boldsymbol{\Sigma})$. As we will see, the central sequences involved in the ULAN result are linear combinations of (the entries of) the generalized cross-covariance matrices

$$\boldsymbol{\Gamma}_{i;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta}) := (n-i)^{-1}\,\boldsymbol{\Sigma}'^{-1/2}\left(\sum_{t=i+1}^{n} \varphi_f(d_t(\boldsymbol{\theta}, \boldsymbol{\Sigma}))\,d_{t-i}(\boldsymbol{\theta}, \boldsymbol{\Sigma})\ \mathbf{U}_t(\boldsymbol{\theta}, \boldsymbol{\Sigma})\mathbf{U}_{t-i}'(\boldsymbol{\theta}, \boldsymbol{\Sigma})\right)\boldsymbol{\Sigma}'^{1/2}, \tag{7}$$

and the matrices of nonserial statistics

$$\boldsymbol{\Lambda}_{i;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta}) := (n-i)^{-1}\,\boldsymbol{\Sigma}'^{-1/2}\sum_{t=i+1}^{n} \varphi_f(d_t(\boldsymbol{\theta}, \boldsymbol{\Sigma}))\,\mathbf{U}_t(\boldsymbol{\theta}, \boldsymbol{\Sigma})\,\mathbf{x}_{t-i}^{(n)\prime}\mathbf{K}^{(n)}, \tag{8}$$

which therefore contain all the relevant information (in the local and asymptotic sense). The coefficients of these linear combinations are rather complicated, though, and require some further notation, mainly connected with the algebra of linear difference equations.

Associated with any k-dimensional linear difference operator of the form $\mathbf{C}(L) := \sum_{i=0}^{\infty}\mathbf{C}_i\,L^i$ (letting $\mathbf{C}_i = \mathbf{0}$ for $i > s$, this includes, of course, the operators with finite order s), define, for any integers m and p, the $k^2m \times k^2p$ matrices

$$\mathbf{C}_{m,p}^{(l)} := \begin{pmatrix} \mathbf{C}_0 \otimes \mathbf{I}_k & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{C}_1 \otimes \mathbf{I}_k & \mathbf{C}_0 \otimes \mathbf{I}_k & \ldots & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{C}_{p-1} \otimes \mathbf{I}_k & \mathbf{C}_{p-2} \otimes \mathbf{I}_k & \ldots & \mathbf{C}_0 \otimes \mathbf{I}_k \\ \vdots & & & \vdots \\ \mathbf{C}_{m-1} \otimes \mathbf{I}_k & \mathbf{C}_{m-2} \otimes \mathbf{I}_k & \ldots & \mathbf{C}_{m-p} \otimes \mathbf{I}_k \end{pmatrix} \tag{9}$$

and

$$\mathbf{C}_{m,p}^{(r)} := \begin{pmatrix} \mathbf{I}_k \otimes \mathbf{C}_0 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{I}_k \otimes \mathbf{C}_1 & \mathbf{I}_k \otimes \mathbf{C}_0 & \ldots & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{I}_k \otimes \mathbf{C}_{p-1} & \mathbf{I}_k \otimes \mathbf{C}_{p-2} & \ldots & \mathbf{I}_k \otimes \mathbf{C}_0 \\ \vdots & & & \vdots \\ \mathbf{I}_k \otimes \mathbf{C}_{m-1} & \mathbf{I}_k \otimes \mathbf{C}_{m-2} & \ldots & \mathbf{I}_k \otimes \mathbf{C}_{m-p} \end{pmatrix}, \tag{10}$$

respectively; write $\mathbf{C}^{(l)}_m$ for $\mathbf{C}^{(l)}_{m,m}$ and $\mathbf{C}^{(r)}_m$ for $\mathbf{C}^{(r)}_{m,m}$. With this notation, note that $\mathbf{G}^{(l)}_m$, $\mathbf{G}^{(r)}_m$, $\mathbf{H}^{(l)}_m$, and $\mathbf{H}^{(r)}_m$ are the inverses of $\mathbf{A}^{(l)}_m$, $\mathbf{A}^{(r)}_m$, $\mathbf{B}^{(l)}_m$, and $\mathbf{B}^{(r)}_m$, respectively. Denoting by $\mathbf{C}^{'(l)}_{m,p}$ and $\mathbf{C}^{'(r)}_{m,p}$ the matrices associated with the transposed operator $\mathbf{C}^{'}(L) := \sum_{i=0}^{\infty} \mathbf{C}^{'}_i L^i$, we also have $\mathbf{G}^{'(l)}_m = (\mathbf{A}^{'(l)}_m)^{-1}$, $\mathbf{H}^{'(l)}_m = (\mathbf{B}^{'(l)}_m)^{-1}$, etc. We will use the notation $\bar{\mathbf{C}}^{(l)}_{m,p}, \bar{\mathbf{C}}^{(r)}_{m,p}, \bar{\mathbf{C}}^{(l)}_m$, etc. when the identity matrices involved in (9) and (10) are m-dimensional rather than k-dimensional.

Let $\pi := \max(p_1 - p_0, q_1 - q_0)$ and $\pi_0 := \pi + p_0 + q_0$, and define the $k^2\pi_0 \times k^2(p_1 + q_1)$ matrix

$$\mathbf{M}_{\boldsymbol{\theta}} := \left(\mathbf{G}^{'(l)}_{\pi_0,p_1} \vdots \mathbf{H}^{'(l)}_{\pi_0,q_1} \right); \tag{11}$$

under Assumption (A2), $\mathbf{M}_{\boldsymbol{\theta}}$ is of full rank.

Consider the operator $\mathbf{D}(L) := \mathbf{I}_k + \sum_{i=1}^{p_0+q_0} \mathbf{D}_i L^i$ (just as $\mathbf{M}_{\boldsymbol{\theta}}$, $\mathbf{D}(L)$ and most quantities defined below depend on $\boldsymbol{\theta}$, but, for simplicity, we are dropping this reference to $\boldsymbol{\theta}$), where, putting $\mathbf{G}_{-1} = \mathbf{G}_{-2} = ... = \mathbf{G}_{-p_0+1} = \mathbf{0} = \mathbf{H}_{-1} = \mathbf{H}_{-2} = ... = \mathbf{H}_{-q_0+1}$,

$$\begin{pmatrix} \mathbf{D}^{'}_1 \\ \vdots \\ \mathbf{D}^{'}_{p_0+q_0} \end{pmatrix} := - \begin{pmatrix} \mathbf{G}_{q_0} & \mathbf{G}_{q_0-1} & \dots & \mathbf{G}_{-p_0+1} \\ \mathbf{G}_{q_0+1} & \mathbf{G}_{q_0} & \dots & \mathbf{G}_{-p_0+2} \\ \vdots & & \ddots & \vdots \\ \mathbf{G}_{p_0+q_0-1} & \mathbf{G}_{p_0+q_0-2} & \dots & \mathbf{G}_0 \\ \mathbf{H}_{p_0} & \mathbf{H}_{p_0-1} & \dots & \mathbf{H}_{-q_0+1} \\ \mathbf{H}_{p_0+1} & \mathbf{H}_{p_0} & \dots & \mathbf{H}_{-q_0+2} \\ \vdots & & \ddots & \vdots \\ \mathbf{H}_{p_0+q_0-1} & \mathbf{H}_{p_0+q_0-2} & \dots & \mathbf{H}_0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{G}_{q_0+1} \\ \vdots \\ \mathbf{G}_{p_0+q_0} \\ \mathbf{H}_{p_0+1} \\ \vdots \\ \mathbf{H}_{p_0+q_0} \end{pmatrix}.$$

Note that $\mathbf{D}(L)\mathbf{G}^{'}_t = \mathbf{0}$ for $t = q_0 + 1, \dots, p_0 + q_0$, and $\mathbf{D}(L)\mathbf{H}^{'}_t = \mathbf{0}$ for $t = p_0 + 1, \dots, p_0 + q_0$.

Let $\{\boldsymbol{\Psi}^{(1)}_t, \dots, \boldsymbol{\Psi}^{(p_0+q_0)}_t\}$ be a set of $k \times k$ matrices forming a fundamental system of solutions of the homogeneous linear difference equation associated with $\mathbf{D}(L)$ (such a system can be obtained, for instance, from the Green's matrices of the operator $\mathbf{D}(L)$: see Hallin 1986). Define

$$\bar{\boldsymbol{\Psi}}_m(\boldsymbol{\theta}) := \begin{pmatrix} \boldsymbol{\Psi}^{(1)}_{\pi+1} & \dots & \boldsymbol{\Psi}^{(p_0+q_0)}_{\pi+1} \\ \boldsymbol{\Psi}^{(1)}_{\pi+2} & \dots & \boldsymbol{\Psi}^{(p_0+q_0)}_{\pi+2} \\ \vdots & & \vdots \\ \boldsymbol{\Psi}^{(1)}_{m} & \dots & \boldsymbol{\Psi}^{(p_0+q_0)}_{m} \end{pmatrix} \otimes \mathbf{I}_k \qquad (m > \pi),$$

$$\mathbf{P}_{\boldsymbol{\theta}} := \begin{pmatrix} \mathbf{I}_{k^2\pi} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{-1}_{\Psi} \end{pmatrix}, \quad \text{and} \quad \mathbf{Q}^{(n)}_{\boldsymbol{\theta}} := \mathbf{H}^{(r)}_{n-1} \mathbf{B}^{'(l)}_{n-1} \begin{pmatrix} \mathbf{I}_{k^2\pi} & \mathbf{0} \\ \mathbf{0} & \bar{\boldsymbol{\Psi}}_{n-1} \end{pmatrix}, \tag{12}$$

where $\mathbf{C}_\Psi$ is the Casorati matrix $\bar{\boldsymbol{\Psi}}_{\pi_0}$.

Finally, put

$$\mathbf{S}^{(n)}_{I;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}) := \left(n^{1/2} \, (\text{vec}\, \boldsymbol{\Lambda}^{(n)}_{0;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}))^{'}, \dots, (n-i)^{1/2} \, (\text{vec}\, \boldsymbol{\Lambda}^{(n)}_{i;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}))^{'}, \dots, (\text{vec}\, \boldsymbol{\Lambda}^{(n)}_{n-1;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}))^{'} \right)^{'},$$

$$n^{1/2} \mathbf{T}^{(n)}_{I;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}) := \mathbf{L}^{(n)'}_{\boldsymbol{\theta}} \mathbf{S}^{(n)}_{I;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}), \quad \text{and} \quad \mathbf{J}_{I;\boldsymbol{\theta},\boldsymbol{\Sigma}} := \lim_{n\to+\infty} \mathbf{L}^{(n)'}_{\boldsymbol{\theta}} (\boldsymbol{K}_n \otimes \boldsymbol{\Sigma}^{-1}) \, \mathbf{L}^{(n)}_{\boldsymbol{\theta}}, \tag{13}$$

where $\mathbf{L}^{(n)}_{\boldsymbol{\theta}} := \bar{\mathbf{H}}^{(r)}_n(\boldsymbol{\theta}) \bar{\mathbf{A}}^{(r)}_{n,1}(\boldsymbol{\theta})$, and where $\boldsymbol{K}_{l\tilde{l}}$ denotes the $lm \times \tilde{l}m$ matrix with block $\mathbf{K}^{'} \, \mathbf{R}_{|i-j|} \, \mathbf{K}$ in position (i, j) $(i = 1, \dots, l,\ j = 1, \dots, \tilde{l})$. We write $\boldsymbol{K}_l$ instead of $\boldsymbol{K}_{ll}$. Similarly, for the serial part, let

$$\mathbf{S}^{(n)}_{II;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}) := \Big((n-1)^{1/2}\,(\mathrm{vec}\,\boldsymbol{\Gamma}^{(n)}_{1;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}))', \ldots, (n-i)^{1/2}\,(\mathrm{vec}\,\boldsymbol{\Gamma}^{(n)}_{i;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}))', \ldots, (\mathrm{vec}\,\boldsymbol{\Gamma}^{(n)}_{n-1;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}))'\Big)',$$

$$n^{1/2}\mathbf{T}^{(n)}_{II;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}) := \mathbf{Q}^{(n)\prime}_{\boldsymbol{\theta}}\mathbf{S}^{(n)}_{II;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}), \quad \text{and} \quad \mathbf{J}_{II;\boldsymbol{\theta},\boldsymbol{\Sigma}} := \lim_{n\to+\infty}\mathbf{Q}^{(n)\prime}_{\boldsymbol{\theta}}\,[\mathbf{I}_{n-1}\otimes(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}^{-1})]\,\mathbf{Q}^{(n)}_{\boldsymbol{\theta}} \quad (14)$$

(convergence in (13) and (14) follows from the exponential decrease, as $u\to\infty$, under (A2), of the Green's matrices $\mathbf{G}_u$ and $\mathbf{H}_u$).

Eventually, we can state the ULAN proved in Hallin and Paindaveine (2002f).

Proposition 1 (ULAN) *Assume that Assumptions (A1), (A2), (B1′), (B2), and (B3) hold. Let $\boldsymbol{\theta}_n$ be such that $\boldsymbol{\theta}_n - \boldsymbol{\theta} = O(n^{-1/2})$. Then, the logarithm $L^{(n)}_{\boldsymbol{\theta}_n+\boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)}/\boldsymbol{\theta}_n;\boldsymbol{\Sigma},f}$ of the likelihood ratio associated with the sequence of local alternatives $\mathcal{H}^{(n)}(\boldsymbol{\theta}_n+\boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)},\boldsymbol{\Sigma},f)$ with respect to $\mathcal{H}^{(n)}(\boldsymbol{\theta}_n,\boldsymbol{\Sigma},f)$ is such that*

$$L^{(n)}_{\boldsymbol{\theta}_n+\boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)}/\boldsymbol{\theta}_n;\boldsymbol{\Sigma},f}(\mathbf{Y}^{(n)}) = (\boldsymbol{\tau}^{(n)})'\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\Sigma},f}(\boldsymbol{\theta}_n) - \frac{1}{2}(\boldsymbol{\tau}^{(n)})'\boldsymbol{\Gamma}_{\boldsymbol{\Sigma},f}(\boldsymbol{\theta})\,\boldsymbol{\tau}^{(n)} + o_{\mathrm{P}}(1),$$

as $n\to\infty$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_n,\boldsymbol{\Sigma},f)$, with the central sequence

$$\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\Sigma},f}(\boldsymbol{\theta}_n) := \begin{pmatrix}\boldsymbol{\Delta}^{(n)}_{I;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}_n)\\ \boldsymbol{\Delta}^{(n)}_{II;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}_n)\end{pmatrix} := n^{1/2}\begin{pmatrix}\mathbf{I}_{km} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}'_{\boldsymbol{\theta}_n}\mathbf{P}'_{\boldsymbol{\theta}_n}\end{pmatrix}\begin{pmatrix}\mathbf{T}^{(n)}_{I;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}_n)\\ \mathbf{T}^{(n)}_{II;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}_n)\end{pmatrix}, \quad (15)$$

and the information matrix

$$\boldsymbol{\Gamma}_{\boldsymbol{\Sigma},f}(\boldsymbol{\theta}) := \begin{pmatrix}\boldsymbol{\Gamma}_{I;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}) & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Gamma}_{II;\boldsymbol{\Sigma},f}(\boldsymbol{\theta})\end{pmatrix},$$

where $\boldsymbol{\Gamma}_{I;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}) := \frac{1}{k}\mathcal{I}_{k,f}\mathbf{J}_{I;\boldsymbol{\theta},\boldsymbol{\Sigma}}$ and $\boldsymbol{\Gamma}_{II;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}) := \frac{\mu_{k+1;f}\,\mathcal{I}_{k,f}}{k^2\,\mu_{k-1;f}}\mathbf{N}_{\boldsymbol{\theta},\boldsymbol{\Sigma}}$, with $\mathbf{N}_{\boldsymbol{\theta},\boldsymbol{\Sigma}} := \mathbf{M}'_{\boldsymbol{\theta}}\mathbf{P}'_{\boldsymbol{\theta}}\mathbf{J}_{II;\boldsymbol{\theta},\boldsymbol{\Sigma}}\mathbf{P}_{\boldsymbol{\theta}}\mathbf{M}_{\boldsymbol{\theta}}$. Moreover, $\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\Sigma},f}(\boldsymbol{\theta}_n)$, still under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_n,\boldsymbol{\Sigma},f)$, is asymptotically $\mathcal{N}_K(\mathbf{0},\boldsymbol{\Gamma}_{\boldsymbol{\Sigma},f}(\boldsymbol{\theta}))$.

Note that the asymptotic information matrix $\boldsymbol{\Gamma}_{\boldsymbol{\Sigma},f}(\boldsymbol{\theta})$ may be singular (such a singularity occurs as soon as $p_1 > p_0$ and $q_1 > q_0$). In such a case, a careful treatment, involving generalized inverses, will be required in the derivation of the asymptotic distributions of test statistics.

4 Multivariate signs and ranks, serial and nonserial signed rank statistics.

4.1 Multivariate signs and ranks.

The generalized cross-covariances (7) and nonserial statistics (8) are measurable with respect to the Mahalanobis distances $d_t(\boldsymbol{\theta},\boldsymbol{\Sigma}) = \|\boldsymbol{\Sigma}^{-1/2}\mathbf{Z}(\boldsymbol{\theta})\|$ between the residuals $\mathbf{Z}_t(\boldsymbol{\theta})$ and the origin in $\mathbb{R}^k$, and the "multivariate signs" $\mathbf{U}_t(\boldsymbol{\theta},\boldsymbol{\Sigma}) = \boldsymbol{\Sigma}^{-1/2}\mathbf{Z}_t(\boldsymbol{\theta})/\|\boldsymbol{\Sigma}^{-1/2}\mathbf{Z}_t(\boldsymbol{\theta})\|$.

For each $\boldsymbol{\Sigma}$ and n, the group of *continuous monotone radial transformations* $\mathcal{G}^{(n)}_{\boldsymbol{\Sigma}} = \{\mathcal{G}^{(n)}_g\}$, acting on $(\mathbb{R}^k)^n$ and characterized by

$$\mathcal{G}^{(n)}_g(\mathbf{Z}_1(\boldsymbol{\theta}),\ldots,\mathbf{Z}_n(\boldsymbol{\theta})) := \Big(g(d_1(\boldsymbol{\theta},\boldsymbol{\Sigma}))\,\boldsymbol{\Sigma}^{1/2}\mathbf{U}_1(\boldsymbol{\theta},\boldsymbol{\Sigma}),\ldots,g(d_n(\boldsymbol{\theta},\boldsymbol{\Sigma}))\,\boldsymbol{\Sigma}^{1/2}\mathbf{U}_n(\boldsymbol{\theta},\boldsymbol{\Sigma})\Big), \quad (16)$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous monotone increasing function such that $g(0) = 0$ and $\lim_{r\to\infty} g(r) = \infty$, is a generating group for $\bigcup_f \mathcal{H}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$. Along with the signs $(\mathbf{U}_1(\boldsymbol{\theta}, \boldsymbol{\Sigma}), \ldots, \mathbf{U}_n(\boldsymbol{\theta}, \boldsymbol{\Sigma}))$, the ranks $(R_1(\boldsymbol{\theta}, \boldsymbol{\Sigma}), \ldots, R_n(\boldsymbol{\theta}, \boldsymbol{\Sigma}))$ of the distances $d_t(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ constitute a maximal invariant for that group $\mathcal{G}^{(n)}_{\boldsymbol{\Sigma}}$ of radial transformations.

Because the true value of the shape matrix is unkown, the *genuine* ranks $R_t(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ and signs $\mathbf{U}_t(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ cannot be computed from the residuals $\mathbf{Z}_1(\boldsymbol{\theta}), \ldots, \mathbf{Z}_n(\boldsymbol{\theta})$, but the following alternative quantities can.

4.2 Pseudo-Mahalanobis signs and ranks.

The pseudo-Mahalanobis signs are defined as $\mathbf{W}_t(\boldsymbol{\theta}) = \mathbf{W}^{(n)}_t(\boldsymbol{\theta}) := \widehat{\boldsymbol{\Sigma}}^{-1/2}\mathbf{Z}_t(\boldsymbol{\theta})/\|\widehat{\boldsymbol{\Sigma}}^{-1/2}\mathbf{Z}_t(\boldsymbol{\theta})\|$, where $\widehat{\boldsymbol{\Sigma}}$ is the estimator in Assumptions (D1)-(D2). Similarly, the pseudo-Mahalanobis ranks $\widehat{R}_t(\boldsymbol{\theta}) := \widehat{R}^{(n)}_t(\boldsymbol{\theta})$ are defined as the ranks of the pseudo-Mahalanobis distances $d_t(\boldsymbol{\theta}, \hat{\boldsymbol{\Sigma}}) = \|\hat{\boldsymbol{\Sigma}}^{-1/2}\mathbf{Z}_t(\boldsymbol{\theta})\|$. The terminology *Mahalanobis signs and ranks* will be used when $\widehat{\boldsymbol{\Sigma}}$ is the empirical covariance matrix.

4.3 Hyperplane-based signs and ranks.

Pseudo-Mahalanobis signs and ranks are based on an estimation of the underlying shape matrix. A completely different approach can be based on counts of hyperplanes, and leads to a modification of Randles' s interdirections (namely, the *absolute interdirections*) for multivariate signs, and to Oja and Paindaveine (2002)'s concept of *lift interdirection ranks* for multivariate ranks.

Write $\mathcal{Q} := \{i_1, i_2, \ldots, i_{k-1}\}$ $(1 \le i_1 < i_2 < \ldots < i_{k-1} \le n)$ and $\mathcal{P} := \{j_1, j_2, \ldots, j_k\}$ $(1 \le j_1 < j_2 < \ldots < j_k \le n)$ for arbitrary ordered sets of indices with sizes $(k-1)$ and k, respectively. Denote by $\mathbf{e}_{\mathcal{Q}}$ and $(d_{0\mathcal{P}}, \mathbf{d}'_{\mathcal{P}})'$ the vectors whose components are the cofactors of the last column in the arrays

$$(\mathbf{Z}_{i_1}(\boldsymbol{\theta}), \ldots, \mathbf{Z}_{i_{k-1}}(\boldsymbol{\theta}), \mathbf{z}) \quad \text{and} \quad \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ \mathbf{Z}_{j_1}(\boldsymbol{\theta}) & \mathbf{Z}_{j_2}(\boldsymbol{\theta}) & \ldots & \mathbf{Z}_{j_k}(\boldsymbol{\theta}) & \mathbf{z} \end{pmatrix},$$

respectively. The vector $\mathbf{e}_{\mathcal{Q}}$ (resp. $\mathbf{d}_{\mathcal{P}}$) is orthogonal to the hyperplane $\Pi(\mathcal{Q})$ spanned by $\mathbf{Z}_{i_1}(\boldsymbol{\theta}), \ldots, \mathbf{Z}_{i_{k-1}}(\boldsymbol{\theta})$ (resp. the hyperplane $\Pi(\mathcal{P})$ going through $\mathbf{Z}_{i_1}(\boldsymbol{\theta}), \ldots, \mathbf{Z}_{i_k}(\boldsymbol{\theta})$), and the sign of $\mathbf{e}'_{\mathcal{Q}}\mathbf{z}$ (resp. of $d_{0\mathcal{P}} + \mathbf{d}'_{\mathcal{P}}\mathbf{z}$) indicates on which side of $\Pi(\mathcal{Q})$ (resp. of $\Pi(\mathcal{P})$) the point $\mathbf{z}$ lies.

The *absolute interdirection* associated with residual $\mathbf{Z}_i(\boldsymbol{\theta})$ in the n-tuple $(\mathbf{Z}_1(\boldsymbol{\theta}), \ldots, \mathbf{Z}_n(\boldsymbol{\theta}))$ is defined as

$$\mathbf{V}_i(\boldsymbol{\theta}) = \mathbf{V}^{(n)}_i(\boldsymbol{\theta}) := (\cos(\pi p^{(n)}_{i;1}(\boldsymbol{\theta})), \ldots, \cos(\pi p^{(n)}_{i;k}(\boldsymbol{\theta})))',$$

with $p^{(n)}_{i;l}(\boldsymbol{\theta}) := \binom{n}{k-1}^{-1} c(\widehat{\boldsymbol{\Sigma}}^{1/2}\mathbf{e}^{(k)}_l, \mathbf{Z}_i(\boldsymbol{\theta}))$, where $c(\mathbf{v}, \mathbf{w})$ denotes the hyperplane-based empirical angular distance

$$c(\mathbf{v}, \mathbf{w}) := \frac{1}{2}\sum_{\mathcal{Q}}\{1 - \text{sign}(\mathbf{e}'_{\mathcal{Q}}\mathbf{v})\ \text{sign}(\mathbf{e}'_{\mathcal{Q}}\mathbf{w})\}.$$

Note that the statistics $q^{(n)}_{ij}(\boldsymbol{\theta}) := c(\mathbf{Z}_i(\boldsymbol{\theta}), \mathbf{Z}_j(\boldsymbol{\theta}))$ are the so-called Randles' interdirections (Randles 1989); $q^{(n)}_{ij}$ is—up to a small-sample correction—the number of hyperplanes in $\mathbb{R}^k$ passing through the origin and $(k-1)$ out of the $(n-2)$ points $\mathbf{Z}_1(\boldsymbol{\theta}), \ldots, \mathbf{Z}_{i-1}(\boldsymbol{\theta}), \mathbf{Z}_{i+1}(\boldsymbol{\theta}), \ldots, \mathbf{Z}_{j-1}(\boldsymbol{\theta}), \mathbf{Z}_{j+1}(\boldsymbol{\theta}), \ldots, \mathbf{Z}_n(\boldsymbol{\theta})$ that separate $\mathbf{Z}_i(\boldsymbol{\theta})$ and $\mathbf{Z}_j(\boldsymbol{\theta})$.

A hyperplane-based empirical distance between a vector $\mathbf{v}$ and the origin in $\mathbb{R}^k$ can be defined as

$$l^{(n)}(\mathbf{v}) := \sum_{\mathcal{P}} \frac{1 - \text{sign}(d_{0\mathcal{P}} + \mathbf{d}'_{\mathcal{P}}\mathbf{v})\,\text{sign}(d_{0\mathcal{P}} - \mathbf{d}'_{\mathcal{P}}\mathbf{v})}{2},$$

i.e., as the number of hyperplanes in $\mathbb{R}^k$ passing through k out of the n points $\mathbf{Z}_1(\boldsymbol{\theta}), \ldots, \mathbf{Z}_n(\boldsymbol{\theta})$ that are separating $\mathbf{v}$ and its reflection $-\mathbf{v}$. For symmetry reasons, we rather consider the symmetrized distances

$$\underline{l}^{(n)}(\mathbf{v}) := \sum_{\mathcal{P}} \sum_{\mathbf{s}} \frac{1 - \text{sign}(d_{0\mathcal{P}}(\mathbf{s}) + \mathbf{d}_{\mathcal{P}}(\mathbf{s})'\mathbf{v})\,\text{sign}(d_{0\mathcal{P}}(\mathbf{s}) - \mathbf{d}_{\mathcal{P}}(\mathbf{s})'\mathbf{v})}{2},$$

where, for $\mathcal{P} = (j_1, \ldots, j_k)$ and $\mathbf{s} \in \{-1, 1\}^k$, $(d_{0\mathcal{P}}(\mathbf{s}), \mathbf{d}_{\mathcal{P}}(\mathbf{s})')'$ stands for the vector of cofactors associated with the last column in the array

$$\begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ s_1\mathbf{Z}_{j_1}(\boldsymbol{\theta}) & s_2\mathbf{Z}_{j_2}(\boldsymbol{\theta}) & \ldots & s_k\mathbf{Z}_{j_k}(\boldsymbol{\theta}) & \mathbf{z} \end{pmatrix}$$

(see Oja and Paindaveine 2002). The *lift interdirection ranks* are the ranks $\underline{R}_i := \underline{R}_i^{(n)}$ of the symmetrized lift-interdirections $\underline{l}_i^{(n)} := \underline{l}^{(n)}(\mathbf{Z}_i(\boldsymbol{\theta}))$, $i = 1, \ldots, n$.

4.4 Serial and nonserial signed rank statistics.

The nonparametric (signed rank) J-score versions of the serial and nonserial statistics (7) and (8) are, in the serial case,

$$\underset{\sim}{\boldsymbol{\Gamma}}{}^{(n)}_{i;J}(\boldsymbol{\theta}) := \widehat{\boldsymbol{\Sigma}}'^{-1/2} \left(\frac{1}{n-i} \sum_{t=i+1}^{n} J_1\Big(\frac{\widehat{R}_t(\boldsymbol{\theta})}{n+1}\Big) J_2\Big(\frac{\widehat{R}_{t-i}(\boldsymbol{\theta})}{n+1}\Big)\, \mathbf{W}_t(\boldsymbol{\theta})\, \mathbf{W}'_{t-i}(\boldsymbol{\theta}) \right) \widehat{\boldsymbol{\Sigma}}'^{1/2}, \tag{17}$$

and, in the nonserial case,

$$\underset{\sim}{\boldsymbol{\Lambda}}{}^{(n)}_{i;J}(\boldsymbol{\theta}) := (n-i)^{-1}\, \widehat{\boldsymbol{\Sigma}}'^{-1/2} \sum_{t=i+1}^{n} J_0\Big(\frac{\widehat{R}_t(\boldsymbol{\theta})}{n+1}\Big)\, \mathbf{W}_t(\boldsymbol{\theta})\, \mathbf{x}^{(n)\prime}_{t-i} \mathbf{K}^{(n)}, \tag{18}$$

where the score functions J_ℓ ($\ell = 0, 1, 2$) are as in Assumption (C). Here we used pseudo-Mahalanobis signs and ranks. But every combination of a concept of multivariate signs (either Mahalanobis signs, pseudo-Mahalanobis signs, or absolute interdirections) with a concept of multivariate ranks (Mahalanobis, pseudo-Mahalanobis, or lift-interdirection ranks) may be considered and actually yields the same asymptotic representation results, as shown by the following proposition (see Hallin and Paindaveine (2002f) for a proof). Note however that their equivariance properties may be different (see the next subsection).

Proposition 2 *Assume that Assumptions (A1), (A2), (B1), (C), and (D1) hold. Then, letting*

$$\widetilde{\boldsymbol{\Gamma}}^{(n)}_{i;J;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}) := \boldsymbol{\Sigma}'^{-1/2} \left(\frac{1}{n-i} \sum_{t=i+1}^{n} J_1(\tilde{F}_k(d_t(\boldsymbol{\theta},\boldsymbol{\Sigma}))) J_2(\tilde{F}_k(d_{t-i}(\boldsymbol{\theta},\boldsymbol{\Sigma})))\ \mathbf{U}_t(\boldsymbol{\theta},\boldsymbol{\Sigma}) \mathbf{U}'_{t-i}(\boldsymbol{\theta},\boldsymbol{\Sigma}) \right) \boldsymbol{\Sigma}'^{1/2} \tag{19}$$

and

$$\widetilde{\boldsymbol{\Lambda}}^{(n)}_{i;J;\boldsymbol{\Sigma},f}(\boldsymbol{\theta}) := (n-i)^{-1}\, \boldsymbol{\Sigma}'^{-1/2} \sum_{t=i+1}^{n} J_0(\tilde{F}_k(d_t(\boldsymbol{\theta},\boldsymbol{\Sigma})))\ \mathbf{U}_t(\boldsymbol{\theta},\boldsymbol{\Sigma})\, \mathbf{x}^{(n)\prime}_{t-i} \mathbf{K}^{(n)}, \tag{20}$$

(i) $\mathrm{vec}\,(\underset{\sim}{\mathbf{\Lambda}}{}^{(n)}_{i;J}(\boldsymbol{\theta}) - \widetilde{\mathbf{\Lambda}}^{(n)}_{i;J;\mathbf{\Sigma},f}(\boldsymbol{\theta}))$ *and* $\mathrm{vec}\,(\underset{\sim}{\mathbf{\Gamma}}{}^{(n)}_{i;J}(\boldsymbol{\theta}) - \widetilde{\mathbf{\Gamma}}^{(n)}_{i;J;\mathbf{\Sigma},f}(\boldsymbol{\theta}))$ *are* $o_{\mathrm{P}}(n^{-1/2})$ *for all* i*, as* $n \to \infty$*, and*

(ii) the same result still holds if in $\underset{\sim}{\mathbf{\Lambda}}{}^{(n)}_{i;J}(\boldsymbol{\theta})$ *and* $\underset{\sim}{\mathbf{\Gamma}}{}^{(n)}_{i;J}(\boldsymbol{\theta})$ *the pseudo-Mahalanobis signs* $\mathbf{W}_t(\boldsymbol{\theta})$ *are replaced by the corresponding absolute interdirections, and/or the pseudo-Mahalanobis ranks* $\widehat{R}_t(\boldsymbol{\theta})$ *are replaced by the lift-interdirection ranks* $\underline{R}_t(\boldsymbol{\theta})$*.*

Let $D_k(J;f) := \int_0^1 J(u)\,\tilde{F}_k^{-1}(u)\,du$ and $C_k(J;f) := \int_0^1 J(u)\,\varphi_f \circ \tilde{F}_k^{-1}(u)\,du$, where J denotes some score function defined over $]0,1[$. When J is a density over $\mathbb{R}_0^+$ rather than a score function, we write $D_k(f_1,f_2)$ and $C_k(f_1,f_2)$ for $D_k(\tilde{F}_{1k}^{-1};f_2)$ and $C_k(J_{k,f_1};f_2)$ respectively; for simplicity, we also write $C_k(f)$ and $D_k(f)$ instead of $C_k(f,f)$ and $D_k(f,f)$. The asymptotic behaviour of the nonparametric statistics (17) and (18) trivially results from Proposition 2 and the following lemma (see Hallin and Paindaveine 2002f).

Lemma 1 *For all integers* $l, \tilde{l}$*, the vector*

$$\Big(n^{1/2}\,(\mathrm{vec}\,\widetilde{\mathbf{\Lambda}}^{(n)}_{0;J;\mathbf{\Sigma},f}(\boldsymbol{\theta}))', \ldots, (n-l+1)^{1/2}\,(\mathrm{vec}\,\widetilde{\mathbf{\Lambda}}^{(n)}_{l-1;J;\mathbf{\Sigma},f}(\boldsymbol{\theta}))',$$
$$(n-1)^{1/2}\,(\mathrm{vec}\,\widetilde{\mathbf{\Gamma}}^{(n)}_{1;J;\mathbf{\Sigma},f}(\boldsymbol{\theta}))', \ldots, (n-\tilde{l})^{1/2}\,(\mathrm{vec}\,\widetilde{\mathbf{\Gamma}}^{(n)}_{\tilde{l};J;\mathbf{\Sigma},f}(\boldsymbol{\theta}))'\Big)'$$

is asymptotically normal under $\mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},f)$ *and under* $\mathcal{H}^{(n)}(\boldsymbol{\theta}+\boldsymbol{\nu}(n)\boldsymbol{\tau},\mathbf{\Sigma},f)$*, with mean* $\mathbf{0}$ *and mean*

$$\begin{pmatrix} \frac{1}{k}\,C_k(J_0;f)(\mathbf{I}_{lm}\otimes\mathbf{\Sigma}^{-1})[\lim_{n\to\infty}(\boldsymbol{K}_{ln}\otimes\mathbf{I}_k)\,\mathbf{L}^{(n)}_{\boldsymbol{\theta}}]\,(\mathrm{vec}\,\boldsymbol{\eta}') \\ \frac{1}{k^2}\,C_k(J_1;f)\,D_k(J_2;f)\,[\mathbf{I}_{\tilde{l}}\otimes(\mathbf{\Sigma}\otimes\mathbf{\Sigma}^{-1})]\,\mathbf{Q}^{(\tilde{l}+1)}_{\boldsymbol{\theta}}\,\mathbf{P}_{\boldsymbol{\theta}}\,\mathbf{M}_{\boldsymbol{\theta}}\,((\mathrm{vec}\,\boldsymbol{\gamma})',(\mathrm{vec}\,\boldsymbol{\delta})')' \end{pmatrix},$$

respectively, and covariance matrix

$$\begin{pmatrix} \frac{1}{k}\,\mathrm{E}[J_0^2(U)]\,(\boldsymbol{K}_l\otimes\mathbf{\Sigma}^{-1}) & \mathbf{0} \\ \mathbf{0} & \frac{1}{k^2}\,\mathrm{E}[J_1^2(U)]\,\mathrm{E}[J_2^2(U)]\;[\mathbf{I}_{\tilde{l}}\otimes(\mathbf{\Sigma}\otimes\mathbf{\Sigma}^{-1})] \end{pmatrix}.$$

Letting $\mathbf{h}_j = \mathbf{h}_j(\boldsymbol{\theta}) := \mathbf{H}_j(\boldsymbol{\theta}) - \sum_{i=1}^{\min(p_0,j)}\mathbf{H}_{j-i}(\boldsymbol{\theta})\mathbf{A}_i(\boldsymbol{\theta})$, $j = 0,1,2,\ldots$, note that

$$\lim_{n\to\infty}(\boldsymbol{K}_{ln}\otimes\mathbf{I}_k)\,\mathbf{L}^{(n)}_{\boldsymbol{\theta}} = \begin{pmatrix} \sum_{j=0}^{\infty}(\mathbf{K}'\,\mathbf{R}_{|j|}\,\mathbf{K})\otimes\mathbf{h}_j \\ \vdots \\ \sum_{j=0}^{\infty}(\mathbf{K}'\,\mathbf{R}_{|i-j|}\,\mathbf{K})\otimes\mathbf{h}_j \\ \vdots \\ \sum_{j=0}^{\infty}(\mathbf{K}'\,\mathbf{R}_{|l-j-1|}\,\mathbf{K})\otimes\mathbf{h}_j \end{pmatrix}.$$

Also, defining

$$\mathbf{a}_i(\boldsymbol{\tau};\boldsymbol{\theta}) := \sum_{j=1}^{\min(p_1,i)}\sum_{l=0}^{i-j}\sum_{k=0}^{\min(q_0,i-j-l)}(\mathbf{G}_{i-j-l-k}(\boldsymbol{\theta})\mathbf{B}_J(\boldsymbol{\theta})\otimes\mathbf{H}_l(\boldsymbol{\theta})')'\,\mathrm{vec}\,\boldsymbol{\gamma}_j,$$

and

$$\mathbf{b}_i(\boldsymbol{\tau};\boldsymbol{\theta}) := \sum_{j=1}^{\min(q_1,i)}(\mathbf{I}_J\otimes\mathbf{H}_{i-j}(\boldsymbol{\theta}))\,\mathrm{vec}\,\boldsymbol{\delta}_j,$$

one can easily check that

$$\begin{pmatrix} \mathbf{a}_1(\boldsymbol{\tau};\boldsymbol{\theta}) + \mathbf{b}_1(\boldsymbol{\tau};\boldsymbol{\theta}) \\ \vdots \\ \mathbf{a}_{\tilde{l}}(\boldsymbol{\tau};\boldsymbol{\theta}) + \mathbf{b}_{\tilde{l}}(\boldsymbol{\tau};\boldsymbol{\theta}) \end{pmatrix} = \mathbf{Q}_{\boldsymbol{\theta}}^{(\tilde{l}+1)} \, \mathbf{P}_{\boldsymbol{\theta}} \, \mathbf{M}_{\boldsymbol{\theta}} \begin{pmatrix} \mathrm{vec}\,\boldsymbol{\gamma} \\ \mathrm{vec}\,\boldsymbol{\delta} \end{pmatrix}.$$

This allows for a direct comparison between Lemma 1 and the corresponding univariate result (Proposition 4.3 in Hallin and Puri 1994).

4.5 Equivariance/invariance properties.

In this section, we use hats to indicate that all parameters involved are estimated. Consider the original sample $(\mathbf{Y}_1, \ldots, \mathbf{Y}_n)$ and the transformed sample $(\mathbf{MY}_1, \ldots, \mathbf{MY}_n)$, where $\mathbf{M}$ is a full-rank $k \times k$ matrix, and denote by $T(\mathbf{M})$ (resp. T) the value of a statistic T computed from the transformed (resp. original) sample. Assumption (E2) ensures that the residual sample of the $\hat{\mathbf{Z}}_i(\mathbf{M}) = \mathbf{Z}(\hat{\boldsymbol{\theta}}(\mathbf{M}))$'s is affine-equivariant, meaning that

$$(\hat{\mathbf{Z}}_1(\mathbf{M}), \ldots, \hat{\mathbf{Z}}_n(\mathbf{M})) = (\mathbf{M}\hat{\mathbf{Z}}_1, \ldots, \mathbf{M}\hat{\mathbf{Z}}_n).$$

Under Assumption (D2), $\widehat{\boldsymbol{\Sigma}}^{-1/2}$ enjoys the equivariance property

$$\widehat{\boldsymbol{\Sigma}}^{-1/2}(\mathbf{M}) = d^{-1/2}\mathbf{O}\widehat{\boldsymbol{\Sigma}}^{-1/2}\mathbf{M}^{-1}, \tag{21}$$

for some $k \times k$ orthogonal matrix $\mathbf{O}$ (recall that $\widehat{\boldsymbol{\Sigma}}(\mathbf{M})$ and $\widehat{\boldsymbol{\Sigma}}$ are computed from the residual samples $(\hat{\mathbf{Z}}_1(\mathbf{M}), \ldots, \hat{\mathbf{Z}}_n(\mathbf{M}))$ and $(\hat{\mathbf{Z}}_1, \ldots, \hat{\mathbf{Z}}_n)$, respectively). The affine-invariance/equivariance properties of pseudo-Mahalanobis signs and ranks easily follow. More precisely, denoting by $\hat{\mathbf{W}}_t(\mathbf{M})$ and $\widehat{R}_t(\mathbf{M})$ the pseudo-Mahalanobis signs and ranks computed from the transformed residuals $(\hat{\mathbf{Z}}_1(\mathbf{M}), \ldots, \hat{\mathbf{Z}}_n(\mathbf{M}))$, we have

$$\hat{\mathbf{W}}_t(\mathbf{M}) = \mathbf{O}\hat{\mathbf{W}}_t, \quad \widehat{R}_t(\mathbf{M}) = \widehat{R}_t,$$

where $\mathbf{O}$ is the orthogonal matrix in (21).

As for hyperplane-based signs and ranks, absolute interdirections are only asymptotically affine-equivariant, i.e., under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$,

$$\hat{\mathbf{V}}_t(\mathbf{M}) = \mathbf{O}\hat{\mathbf{V}}_t + o_{\mathrm{P}}(1), \qquad \text{as } n \to \infty, \tag{22}$$

still with the orthogonal matrix $\mathbf{O}$ in (21). Lift-interdirection ranks $\underline{\hat{R}}_t := \underline{R}_t(\hat{\boldsymbol{\theta}})$ are strictly affine-invariant (see Oja and Paindaveine 2002).

This entails, for the nonparametric statistics $\underset{\sim}{\hat{\boldsymbol{\Lambda}}}{}^{(n)}_{i;J}$ and $\underset{\sim}{\hat{\boldsymbol{\Gamma}}}{}^{(n)}_{i;J}$, the following equivariance properties.

Lemma 2 *Assume that Assumptions (D2) and (E2) hold. Denote by* $\underset{\sim}{\hat{\boldsymbol{\Lambda}}}{}^{(n)}_{i;J}(\mathbf{M})$ *and* $\underset{\sim}{\hat{\boldsymbol{\Gamma}}}{}^{(n)}_{i;J}(\mathbf{M})$ *the statistics* $\underset{\sim}{\hat{\boldsymbol{\Lambda}}}{}^{(n)}_{i;J}$ *and* $\underset{\sim}{\hat{\boldsymbol{\Gamma}}}{}^{(n)}_{i;J}$ *computed from the n-tuple* $(\mathbf{MY}_1, \ldots, \mathbf{MY}_n)$*, where* $\mathbf{M}$ *is a* $k \times k$ *full-rank matrix. Then,*

$$\underset{\sim}{\hat{\boldsymbol{\Lambda}}}{}^{(n)}_{i;J}(\mathbf{M}) = d^{-1/2}\mathbf{M}^{-1\prime}\,\underset{\sim}{\hat{\boldsymbol{\Lambda}}}{}^{(n)}_{i;J} \quad \textit{and} \quad \underset{\sim}{\hat{\boldsymbol{\Gamma}}}{}^{(n)}_{i;J}(\mathbf{M}) = \mathbf{M}^{-1\prime}\,\underset{\sim}{\hat{\boldsymbol{\Gamma}}}{}^{(n)}_{i;J}\mathbf{M}';$$

the same result still holds if in $\hat{\underset{\sim}{\mathbf{\Lambda}}}^{(n)}_{i;J}$ *and* $\hat{\underset{\sim}{\mathbf{\Gamma}}}^{(n)}_{i;J}$ *the pseudo-Mahalanobis ranks* $\widehat{R}_t(\hat{\boldsymbol{\theta}})$ *are replaced by the lift-interdirection ranks* $\underline{R}_t(\hat{\boldsymbol{\theta}})$*.*

Proof. The result directly follows from the equivariance properties of pseudo-Mahalanobis signs and ranks. □

If the pseudo-Mahalanobis signs $\mathbf{W}_t(\hat{\boldsymbol{\theta}})$ in $\hat{\underset{\sim}{\mathbf{\Lambda}}}^{(n)}_{i;J}$ and $\hat{\underset{\sim}{\mathbf{\Gamma}}}^{(n)}_{i;J}$ are replaced by the corresponding absolute interdirections (in combination with any type of ranks), then it is clear from (22) that $\hat{\underset{\sim}{\mathbf{\Lambda}}}^{(n)}_{i;J}$ and $\hat{\underset{\sim}{\mathbf{\Gamma}}}^{(n)}_{i;J}$ can only be asymptotically affine-equivariant. The resulting hyperplane-based test statistics will accordingly be only asymptotically affine-invariant (see Section 5 and the proof of Proposition 3).

5 Aligned rank tests.

5.1 The proposed rank-based procedures.

Let $n^{1/2}\underset{\sim}{\mathbf{T}}^{(n)}_J(\boldsymbol{\theta})$ be given by

$$\begin{pmatrix} n^{1/2}\underset{\sim}{\mathbf{T}}^{(n)}_{I;J}(\boldsymbol{\theta}) \\ n^{1/2}\underset{\sim}{\mathbf{T}}^{(n)}_{II;J}(\boldsymbol{\theta}) \end{pmatrix} := \begin{pmatrix} \mathbf{L}^{(n)\prime}_{\boldsymbol{\theta}}\,\underset{\sim}{\mathbf{S}}^{(n)}_{I;J}(\boldsymbol{\theta}) \\ \mathbf{Q}^{(n)\prime}_{\boldsymbol{\theta}}\,\underset{\sim}{\mathbf{S}}^{(n)}_{II;J}(\boldsymbol{\theta}) \end{pmatrix} := \begin{pmatrix} \mathbf{L}^{(n)\prime}_{\boldsymbol{\theta}}(n^{1/2}(\mathrm{vec}\,\underset{\sim}{\mathbf{\Lambda}}^{(n)}_{0;J}(\boldsymbol{\theta}))',\ldots,(\mathrm{vec}\,\underset{\sim}{\mathbf{\Lambda}}^{(n)}_{n-1;J}(\boldsymbol{\theta}))')' \\ \mathbf{Q}^{(n)\prime}_{\boldsymbol{\theta}}((n-1)^{1/2}\,(\mathrm{vec}\,\underset{\sim}{\mathbf{\Gamma}}^{(n)}_{1;J}(\boldsymbol{\theta}))',\ldots,(\mathrm{vec}\,\underset{\sim}{\mathbf{\Gamma}}^{(n)}_{n-1;J}(\boldsymbol{\theta}))')' \end{pmatrix},$$

and define

$$\mathbf{J}^{(n)}_{I;\boldsymbol{\theta},\mathbf{\Sigma}} := \mathbf{L}^{(n)\prime}_{\boldsymbol{\theta}}(\boldsymbol{K}_n\otimes\mathbf{\Sigma}^{-1})\,\mathbf{L}^{(n)}_{\boldsymbol{\theta}} \quad\text{and}\quad \mathbf{J}^{(n)}_{II;\boldsymbol{\theta},\mathbf{\Sigma}} := \mathbf{Q}^{(n)\prime}_{\boldsymbol{\theta}}\Big[\mathbf{I}_{n-1}\otimes(\mathbf{\Sigma}\otimes\mathbf{\Sigma}^{-1})\Big]\,\mathbf{Q}^{(n)}_{\boldsymbol{\theta}}.$$

Denote by $\dot{\mathbf{M}}_{\boldsymbol{\theta}}$ the full-rank $k^2\pi_0\times k^2(p_0+q_0)$ matrix resulting from $\mathbf{M}_{\boldsymbol{\theta}}$ by deleting columns $k^2p_0+1,\ldots,k^2p_1$ and $k^2(p_1+q_0)+1,\ldots,k^2(p_1+q_1)$. Similarly, let $\dot{\mathbf{\Upsilon}}_{II}$ be the $k^2(p_0+q_0)\times r_{II}$ array resulting from $\mathbf{\Upsilon}_{II}$ by deleting lines $k^2p_0+1,\ldots,k^2p_1$ and $k^2(p_1+q_0)+1,\ldots,k^2(p_1+q_1)$. Note that $\mathbf{M}_{\boldsymbol{\theta}}\,\mathbf{\Upsilon}_{II}=\dot{\mathbf{M}}_{\boldsymbol{\theta}}\,\dot{\mathbf{\Upsilon}}_{II}$. Finally, let

$$\bar{\mathbf{Q}}^{(n)}_{I;J;\mathbf{\Sigma}}(\boldsymbol{\theta}) := \frac{k}{\mathrm{E}[J_0^2(U)]}\Big[(\mathbf{J}^{(n)}_{I;\boldsymbol{\theta},\mathbf{\Sigma}})^{-1} - (\mathbf{K}^{(n)}\otimes\mathbf{I}_k)^{-1}\,\mathbf{\Upsilon}_I \Big(\mathbf{\Upsilon}_I'\,(\mathbf{K}^{(n)}\otimes\mathbf{I}_k)'^{-1}\,\mathbf{J}^{(n)}_{I;\boldsymbol{\theta},\mathbf{\Sigma}}\,(\mathbf{K}^{(n)}\otimes\mathbf{I}_k)^{-1}\,\mathbf{\Upsilon}_I\Big)^{-1}\mathbf{\Upsilon}_I'\,(\mathbf{K}^{(n)}\otimes\mathbf{I}_k)'^{-1}\Big],$$

and, denoting by $\mathbf{A}^-$ an arbitrary generalized inverse of $\mathbf{A}$,

$$\bar{\mathbf{Q}}^{(n)}_{II;J;\mathbf{\Sigma}}(\boldsymbol{\theta}) := \frac{k^2}{\mathrm{E}[J_1^2(U)]\mathrm{E}[J_2^2(U)]}\Big[(\mathbf{J}^{(n)}_{II;\boldsymbol{\theta},\mathbf{\Sigma}})^{-1} - \mathbf{P}_{\boldsymbol{\theta}}\dot{\mathbf{M}}_{\boldsymbol{\theta}}\,\dot{\mathbf{\Upsilon}}_{II} \Big(\dot{\mathbf{\Upsilon}}_{II}'\,\dot{\mathbf{M}}_{\boldsymbol{\theta}}'\,\mathbf{P}_{\boldsymbol{\theta}}'\mathbf{J}^{(n)}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}\,\mathbf{P}_{\boldsymbol{\theta}}\dot{\mathbf{M}}_{\boldsymbol{\theta}}\,\dot{\mathbf{\Upsilon}}_{II}\Big)^{-}\dot{\mathbf{\Upsilon}}_{II}'\,\dot{\mathbf{M}}_{\boldsymbol{\theta}}'\,\mathbf{P}_{\boldsymbol{\theta}}'\Big].$$

Then the J-score version of the proposed test statistics is

$$\mathcal{W}_J^{(n)} := n \left(\underset{\sim}{\mathbf{T}}_J^{(n)}(\hat{\boldsymbol{\theta}}) \right)' \begin{pmatrix} \bar{\mathbf{Q}}_{I;J;\widehat{\boldsymbol{\Sigma}}}(\hat{\boldsymbol{\theta}}) & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{Q}}_{II;J;\widehat{\boldsymbol{\Sigma}}}(\hat{\boldsymbol{\theta}}) \end{pmatrix} \underset{\sim}{\mathbf{T}}_J^{(n)}(\hat{\boldsymbol{\theta}}) =: \mathcal{W}_{I;J}^{(n)} + \mathcal{W}_{II;J}^{(n)}$$

$$:= n \left(\underset{\sim}{\mathbf{T}}_{I;J}^{(n)}(\hat{\boldsymbol{\theta}}) \right)' \bar{\mathbf{Q}}_{I;J;\widehat{\boldsymbol{\Sigma}}}^{(n)}(\hat{\boldsymbol{\theta}})\, \underset{\sim}{\mathbf{T}}_{I;J}^{(n)}(\hat{\boldsymbol{\theta}}) + n \left(\underset{\sim}{\mathbf{T}}_{II;J}^{(n)}(\hat{\boldsymbol{\theta}}) \right)' \bar{\mathbf{Q}}_{II;J;\widehat{\boldsymbol{\Sigma}}}^{(n)}(\hat{\boldsymbol{\theta}})\, \underset{\sim}{\mathbf{T}}_{II;J}^{(n)}(\hat{\boldsymbol{\theta}}),$$

where $\widehat{\boldsymbol{\Sigma}}$ denotes the shape matrix estimator in Assumptions (D1)-(D2). The version allowing for local asymptotic optimality at radial density $f_\star$ is associated with the scores $J_0 = J_1 := \varphi_{f_\star} \circ \tilde{F}_{\star k}^{-1}$ and $J_2 := \tilde{F}_{\star k}^{-1}$. The corresponding statistics will be denoted by $\mathcal{W}_{f_\star}^{(n)}$.

Finally, in order to describe the asymptotic behaviour of $\mathcal{W}_J^{(n)}$ under local alternatives, define

$$r_{\boldsymbol{\theta},\boldsymbol{\Sigma}}(\boldsymbol{\eta}) := \left(\mathrm{vec}\,\boldsymbol{\eta}'\right)' \Big[\mathbf{J}_{I;\boldsymbol{\theta},\boldsymbol{\Sigma}} - \mathbf{J}_{I;\boldsymbol{\theta},\boldsymbol{\Sigma}} \left(\mathbf{K}^{-1}\mathbf{D} \otimes \mathbf{I}_k\right) \boldsymbol{\Upsilon}_I$$

$$\left(\boldsymbol{\Upsilon}_I' \left(\mathbf{K}^{-1}\mathbf{D} \otimes \mathbf{I}_k\right)' \mathbf{J}_{I;\boldsymbol{\theta},\boldsymbol{\Sigma}} \left(\mathbf{K}^{-1}\mathbf{D} \otimes \mathbf{I}_k\right) \boldsymbol{\Upsilon}_I\right)^{-1} \boldsymbol{\Upsilon}_I' \left(\mathbf{K}^{-1}\mathbf{D} \otimes \mathbf{I}_k\right)' \mathbf{J}_{I;\boldsymbol{\theta},\boldsymbol{\Sigma}}\Big] \left(\mathrm{vec}\,\boldsymbol{\eta}'\right)$$

and

$$s_{\boldsymbol{\theta},\boldsymbol{\Sigma}}(\boldsymbol{\gamma},\boldsymbol{\delta}) := \begin{pmatrix} \mathrm{vec}\,\boldsymbol{\gamma} \\ \mathrm{vec}\,\boldsymbol{\delta} \end{pmatrix}' \left[\mathbf{N}_{\boldsymbol{\theta},\boldsymbol{\Sigma}} - \mathbf{N}_{\boldsymbol{\theta},\boldsymbol{\Sigma}} \boldsymbol{\Upsilon}_{II} \left(\boldsymbol{\Upsilon}_{II}' \mathbf{N}_{\boldsymbol{\theta},\boldsymbol{\Sigma}} \boldsymbol{\Upsilon}_{II}\right)^{-} \boldsymbol{\Upsilon}_{II}' \mathbf{N}_{\boldsymbol{\theta},\boldsymbol{\Sigma}}\right] \begin{pmatrix} \mathrm{vec}\,\boldsymbol{\gamma} \\ \mathrm{vec}\,\boldsymbol{\delta} \end{pmatrix},$$

where $\mathbf{D}$ is the array involved in Assumption (A1$'$) and $\mathbf{N}_{\boldsymbol{\theta},\boldsymbol{\Sigma}}$ is defined in Proposition 1. We can now state the main result of this paper.

Proposition 3 *Assume that Assumptions (A1), (A2), (B1$'$), (B2), (B3), (C), (D1), (D2), (E1), and (E2) hold. Consider the sequence of aligned rank tests $\phi_J^{(n)}$ (resp. $\phi_{f_\star}^{(n)}$) that reject the null hypothesis $\mathcal{H}_{\boldsymbol{\Upsilon}}^{(n)}(\boldsymbol{\theta}_0)$ whenever $\mathcal{W}_J^{(n)}$ (resp. $\mathcal{W}_{f_\star}^{(n)}$) exceeds the α-upper quantile $\chi^2_{km+k^2\pi_0-r,1-\alpha}$ of a chi-square distribution with $km + k^2\pi_0 - r$ degrees of freedom. Then,*

(i) $\mathcal{W}_J^{(n)}$ is strictly affine-invariant (only asymptotically, if absolute interdirections are used as multivariate signs), and asymptotically invariant with respect to the group of continuous monotone radial transformations;

(ii) $\mathcal{W}_J^{(n)}$ is asymptotically chi-square with $km + k^2\pi_0 - r$ degrees of freedom under $\mathcal{H}_{\boldsymbol{\Upsilon}}^{(n)}(\boldsymbol{\theta}_0)$ (so that $\phi_J^{(n)}$ has asymptotic level α);

(iii) $\mathcal{W}_J^{(n)}$ is asymptotically noncentral chi-square, still with $km + k^2\pi_0 - r$ degrees of freedom and noncentrality parameter

$$\frac{1}{k}\,\frac{C_k^2(J_0;f)}{\mathrm{E}[J_0^2(U)]}\, r_{\boldsymbol{\theta},\boldsymbol{\Sigma}}(\boldsymbol{\eta}) + \frac{1}{k^2}\,\frac{C_k^2(J_1;f)}{\mathrm{E}[J_1^2(U)]}\,\frac{D_k^2(J_2;f)}{\mathrm{E}[J_2^2(U)]}\, s_{\boldsymbol{\theta},\boldsymbol{\Sigma}}(\boldsymbol{\gamma},\boldsymbol{\delta})$$

under $\mathcal{H}^{(n)}(\boldsymbol{\theta}+\boldsymbol{\nu}(n)\boldsymbol{\tau},\boldsymbol{\Sigma},f)$, $\boldsymbol{\theta}-\boldsymbol{\theta}_0 \in \mathcal{M}(\boldsymbol{\Upsilon})$ and $\boldsymbol{\tau} \notin \mathcal{M}(\boldsymbol{\Upsilon})$, provided that (A1) is reinforced into (A1$'$);

(iv) for any $f_\star$ satisfying Assumptions (B1′), (B2), (B3) and (C′), the sequence of tests $\phi^{(n)}_{f_\star}$ is locally asymptotically most stringent for $\mathcal{H}^{(n)}_{\boldsymbol{\Upsilon}}(\boldsymbol{\theta}_0)$ against $\bigcup_{\boldsymbol{\theta}\neq\boldsymbol{\theta}_0+\mathcal{M}(\boldsymbol{\Upsilon})}\bigcup_{\boldsymbol{\Sigma}}\mathcal{H}^{(n)}(\boldsymbol{\theta},\boldsymbol{\Sigma},f_\star)$, at probability level α.

To prove this proposition, we will need the following asymptotic linearity result, which straightforwardly follows from the asymptotic linearity result proved in Hallin and Paindaveine (2002f); see the Appendix for a proof.

Lemma 3 *Assume that Assumptions (A1), (A2), (B1′), (B2), (B3), (C), (D1), and (E1) hold. Then*

$$n^{1/2}(\underset{\sim}{\mathbf{T}}^{(n)}_{I;J}(\hat{\boldsymbol{\theta}})-\underset{\sim}{\mathbf{T}}^{(n)}_{I;J}(\boldsymbol{\theta}))+\frac{1}{k}\,C_k(J_0;f)\,\mathbf{J}_{I;\boldsymbol{\theta},\boldsymbol{\Sigma}}\,(\mathbf{K}^{(n)}\otimes\mathbf{I}_k)^{-1}\,n^{1/2}(\hat{\boldsymbol{\theta}}^{(n)}_I-\boldsymbol{\theta}_I)$$

and

$$n^{1/2}(\underset{\sim}{\mathbf{T}}^{(n)}_{II;J}(\hat{\boldsymbol{\theta}})-\underset{\sim}{\mathbf{T}}^{(n)}_{II;J}(\boldsymbol{\theta}))+\frac{1}{k^2}\,D_k(J_2;f)\,C_k(J_1;f)\,\mathbf{J}_{II;\boldsymbol{\theta},\boldsymbol{\Sigma}}\,\mathbf{P}_{\boldsymbol{\theta}}\,\mathbf{M}_{\boldsymbol{\theta}}\,n^{1/2}(\hat{\boldsymbol{\theta}}^{(n)}_{II}-\boldsymbol{\theta}_{II})$$

are $o_{\mathrm{P}}(1)$ as $n\to\infty$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta},\boldsymbol{\Sigma},f)$.

Proof of Proposition 3. *(i)* We first prove that $\mathcal{W}^{(n)}_J$ is affine-invariant. Note that it is clear that the scalar factor $d^{-1/2}$ in the equivariance relation (21) has no influence on the affine-invariance of $\mathcal{W}^{(n)}_J$; consequently, we will assume, without loss of generality, that $d=1$. With this notation of Section 4.5, Lemma 2 yields $\underset{\sim}{\hat{\mathbf{S}}}^{(n)}_{I;J}(\mathbf{M})=g^{(mn,0)}_{\mathbf{M}'^{-1}}\,\underset{\sim}{\hat{\mathbf{S}}}^{(n)}_{I;J}$. From Assumption (E2), $\mathbf{L}^{(n)}_{\hat{\boldsymbol{\theta}}}(\mathbf{M})=g^{(mn,0)}_{\mathbf{M}}\mathbf{L}^{(n)}_{\hat{\boldsymbol{\theta}}}\,g^{(m,0)}_{\mathbf{M}^{-1}}$. Consequently, $\underset{\sim}{\hat{\mathbf{T}}}^{(n)}_{I;J}(\mathbf{M})=g^{(m,0)}_{\mathbf{M}'^{-1}}\,\underset{\sim}{\hat{\mathbf{T}}}^{(n)}_{I;J}$. Analogously, $\underset{\sim}{\hat{\mathbf{S}}}^{(n)}_{II;J}(\mathbf{M})=g^{(0,n-1)}_{\mathbf{M}'^{-1}}\,\underset{\sim}{\hat{\mathbf{S}}}^{(n)}_{II;J}$, $\mathbf{Q}^{(n)}_{\hat{\boldsymbol{\theta}}}(\mathbf{M})=g^{(0,n-1)}_{\mathbf{M}}\mathbf{Q}^{(n)}_{\hat{\boldsymbol{\theta}}}\,g^{(0,\pi_0)}_{\mathbf{M}^{-1}}$, and therefore, $\underset{\sim}{\hat{\mathbf{T}}}^{(n)}_{II;J}(\mathbf{M})=g^{(0,\pi_0)}_{\mathbf{M}'^{-1}}\,\underset{\sim}{\hat{\mathbf{T}}}^{(n)}_{II;J}$. This implies that $\underset{\sim}{\hat{\mathbf{T}}}^{(n)}_{J}(\mathbf{M})=g^{(m,\pi_0)}_{\mathbf{M}'^{-1}}\,\underset{\sim}{\hat{\mathbf{T}}}^{(n)}_{J}$.

For the variances, $\mathbf{J}^{(n)}_{I;\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\Sigma}}}(\mathbf{M})=g^{(m,0)}_{\mathbf{M}'^{-1}}\mathbf{J}^{(n)}_{I;\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\Sigma}}}\,g^{(m,0)}_{\mathbf{M}^{-1}}$ and $\mathbf{J}^{(n)}_{II;\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\Sigma}}}(\mathbf{M})=g^{(0,\pi_0)}_{\mathbf{M}'^{-1}}\mathbf{J}^{(n)}_{II;\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\Sigma}}}\,g^{(0,\pi_0)}_{\mathbf{M}^{-1}}$. Since, moreover, $\mathbf{P}_{\hat{\boldsymbol{\theta}}}(\mathbf{M})=g^{(0,\pi_0)}_{\mathbf{M}}\mathbf{P}_{\hat{\boldsymbol{\theta}}}\,g^{(0,\pi_0)}_{\mathbf{M}^{-1}}$ and $\mathbf{M}_{\hat{\boldsymbol{\theta}}}(\mathbf{M})=g^{(0,\pi_0)}_{\mathbf{M}}\mathbf{M}_{\hat{\boldsymbol{\theta}}}\,g^{(0,p_1+q_1)}_{\mathbf{M}^{-1}}$, standard algebra shows that $\mathcal{W}^{(n)}_J(\mathbf{M})=n\,(\underset{\sim}{\mathbf{T}}^{(n)}_J(\hat{\boldsymbol{\theta}}))'\boldsymbol{\Psi}'\boldsymbol{\Lambda}\boldsymbol{\Psi}\,\underset{\sim}{\mathbf{T}}^{(n)}_J(\hat{\boldsymbol{\theta}})$, where

$$\boldsymbol{\Psi}:=\begin{pmatrix}\frac{1}{k}\,\mathrm{E}[J_0^2(U)]\,(\mathbf{J}^{(n)}_{I;\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\Sigma}}})^{-1} & \mathbf{0}\\ \mathbf{0} & \frac{1}{k^2}\,\mathrm{E}[J_1^2(U)]\,\mathrm{E}[J_2^2(U)]\,(\mathbf{J}^{(n)}_{II;\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\Sigma}}})^{-1}\end{pmatrix}^{1/2}$$

and

$$\boldsymbol{\Lambda}:=\mathbf{I}_{km+k^2\pi_0}-\Pi\left(\begin{pmatrix}\mathbf{J}^{(n)}_{I;\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\Sigma}}} & \mathbf{0}\\ \mathbf{0} & \mathbf{J}^{(n)}_{II;\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\Sigma}}}\end{pmatrix}^{1/2}\begin{pmatrix}(\mathbf{K}^{(n)}\otimes\mathbf{I}_k)^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{P}_{\hat{\boldsymbol{\theta}}}\mathbf{M}_{\hat{\boldsymbol{\theta}}}\end{pmatrix}g^{(m,p_1+q_1)}_{\mathbf{M}^{-1}}\,\boldsymbol{\Upsilon}\right).$$

Now, recall that we restricted ourselves to affine-invariant null hypotheses, i.e., to couples $(\boldsymbol{\theta}_0,\boldsymbol{\Upsilon})$ for which $g^{(m,p_1+q_1)}_{\mathbf{M}}(\boldsymbol{\theta}_0+\mathcal{M}(\boldsymbol{\Upsilon}))=\boldsymbol{\theta}_0+\mathcal{M}(\boldsymbol{\Upsilon})$ for any full-rank matrix $\mathbf{M}$. This implies that $\mathcal{M}(g^{(m,p_1+q_1)}_{\mathbf{M}}\boldsymbol{\Upsilon})=\mathcal{M}(\boldsymbol{\Upsilon})$ for all such $\mathbf{M}$ (see the proof of Proposition 2 in Hallin and Paindaveine 2002c). The affine-invariance of $\mathcal{W}^{(n)}_J$ follows.

The asymptotic representation result of Proposition 2 will be sufficient (see the proof of (ii), (iii) below) to prove that all versions of $\mathcal{W}_J^{(n)}$ (based on any type of signs and ranks) have the same asymptotic representation, and thus are asymptotically equivalent; the asymptotic affine-invariance of the absolute-interdirection-based version of $\mathcal{W}_J^{(n)}$ follows since we showed above that the pseudo-Mahalanobis version of $\mathcal{W}_J^{(n)}$ is strictly affine-invariant.

Let us now prove that $\mathcal{W}_J^{(n)}$ is asymptotically invariant with respect to the group of continuous monotone radial transformations. Let $n^{1/2}\widetilde{\mathbf{T}}^{(n)}_{J;\mathbf{\Sigma}}(\boldsymbol{\theta})$ be given by

$$\begin{pmatrix} n^{1/2}\widetilde{\mathbf{T}}^{(n)}_{I;J;\mathbf{\Sigma}}(\boldsymbol{\theta}) \\ n^{1/2}\widetilde{\mathbf{T}}^{(n)}_{II;J;\mathbf{\Sigma}}(\boldsymbol{\theta}) \end{pmatrix} := \begin{pmatrix} \mathbf{L}^{(n)\prime}_{\boldsymbol{\theta}}(n^{1/2}(\mathrm{vec}\,\widetilde{\mathbf{\Lambda}}^{(n)}_{0;J;\mathbf{\Sigma}}(\boldsymbol{\theta}))',\ldots,(\mathrm{vec}\,\widetilde{\mathbf{\Lambda}}^{(n)}_{n-1;J;\mathbf{\Sigma}}(\boldsymbol{\theta}))')' \\ \mathbf{Q}^{(n)\prime}_{\boldsymbol{\theta}}((n-1)^{1/2}\,(\mathrm{vec}\,\widetilde{\mathbf{\Gamma}}^{(n)}_{1;J;\mathbf{\Sigma}}(\boldsymbol{\theta}))',\ldots,(\mathrm{vec}\,\widetilde{\mathbf{\Gamma}}^{(n)}_{n-1;J;\mathbf{\Sigma}}(\boldsymbol{\theta}))')' \end{pmatrix},$$

where

$$\widetilde{\mathbf{\Lambda}}^{(n)}_{i;J;\mathbf{\Sigma}}(\boldsymbol{\theta}) := (n-i)^{-1}\,\mathbf{\Sigma}'^{-1/2}\sum_{t=i+1}^{n} J_0\Big(\frac{R_t(\boldsymbol{\theta},\mathbf{\Sigma})}{n+1}\Big)\,\mathbf{U}_t(\boldsymbol{\theta},\mathbf{\Sigma})\,\mathbf{x}^{(n)\prime}_{t-i}\mathbf{K}^{(n)},$$

and

$$\widetilde{\mathbf{\Gamma}}^{(n)}_{i;J;\mathbf{\Sigma}}(\boldsymbol{\theta}) := \mathbf{\Sigma}'^{-1/2}\left(\frac{1}{n-i}\sum_{t=i+1}^{n} J_1\Big(\frac{R^{(n)}_t(\boldsymbol{\theta},\mathbf{\Sigma})}{n+1}\Big)J_2\Big(\frac{R_{t-i}(\boldsymbol{\theta},\mathbf{\Sigma})}{n+1}\Big)\,\mathbf{U}_t(\boldsymbol{\theta},\mathbf{\Sigma})\,\mathbf{U}'_{t-i}(\boldsymbol{\theta},\mathbf{\Sigma})\right)\mathbf{\Sigma}'^{1/2}.$$

Proceeding as in the proof of Proposition 2, one can verify that $\underset{\sim}{\mathbf{T}}{}^{(n)}_{J}(\boldsymbol{\theta}) - \widetilde{\mathbf{T}}^{(n)}_{J;\mathbf{\Sigma}}(\boldsymbol{\theta})$ is $o_{\mathrm{P}}(n^{-1/2})$ under $\cup_f \mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},f)$. Using Lemma 3, this yields

$$n^{1/2}\,\underset{\sim}{\mathbf{T}}{}^{(n)}_{J}(\hat{\boldsymbol{\theta}}) = n^{1/2}\widetilde{\mathbf{T}}^{(n)}_{J;\mathbf{\Sigma}}(\boldsymbol{\theta}) -$$

$$\frac{C_k(J_0;f)}{k}\begin{pmatrix} \mathbf{J}_{I;\boldsymbol{\theta},\mathbf{\Sigma}}\,(\mathbf{K}^{(n)}\otimes\mathbf{I}_k)^{-1} & \mathbf{0} \\ \mathbf{0} & \frac{D_k(J_2;f)}{k}\,\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}\,\mathbf{P}_{\boldsymbol{\theta}}\,\mathbf{M}_{\boldsymbol{\theta}} \end{pmatrix} n^{1/2}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}) + o_{\mathrm{P}}(1),$$

under $\mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},f)$. On the other hand, using the continuity of $\bar{\mathbf{Q}}_{I;J;\mathbf{\Sigma}}(\boldsymbol{\theta})$ and $\bar{\mathbf{Q}}_{II;J;\mathbf{\Sigma}}(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ and $\mathbf{\Sigma}$, we obtain that, under $\cup_f \mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},f)$,

$$\mathcal{W}_J^{(n)} = \left(n^{1/2}\,\underset{\sim}{\mathbf{T}}{}^{(n)}_{J}(\hat{\boldsymbol{\theta}})\right)'\begin{pmatrix} \bar{\mathbf{Q}}_{I;J;\mathbf{\Sigma}}(\boldsymbol{\theta}) & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{Q}}_{II;J;\mathbf{\Sigma}}(\boldsymbol{\theta}) \end{pmatrix}\left(n^{1/2}\,\underset{\sim}{\mathbf{T}}{}^{(n)}_{J}(\hat{\boldsymbol{\theta}})\right) + o_{\mathrm{P}}(1)$$

(here, and in the sequel, $\bar{\mathbf{Q}}_{I;J;\mathbf{\Sigma}}(\boldsymbol{\theta})$ (resp. $\bar{\mathbf{Q}}_{II;J;\mathbf{\Sigma}}(\boldsymbol{\theta})$) denotes the array obtained by replacing $\mathbf{J}^{(n)}_{I;\boldsymbol{\theta},\mathbf{\Sigma}}$ by $\mathbf{J}_{I;\boldsymbol{\theta},\mathbf{\Sigma}}$ (resp. $\mathbf{J}^{(n)}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}$ by $\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}$) in $\bar{\mathbf{Q}}^{(n)}_{I;J;\mathbf{\Sigma}}(\boldsymbol{\theta})$ (resp. in $\bar{\mathbf{Q}}^{(n)}_{II;J;\mathbf{\Sigma}}(\boldsymbol{\theta})$)). Writing $\bar{\mathbf{K}}$ for $\mathbf{K}^{(n)}\otimes\mathbf{I}_k$, and using Lemma 2.2.6 (c) in Rao and Mitra (1971),

$$\begin{pmatrix} \bar{\mathbf{Q}}_{I;J;\mathbf{\Sigma}}(\boldsymbol{\theta}) & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{Q}}_{II;J;\mathbf{\Sigma}}(\boldsymbol{\theta}) \end{pmatrix}\begin{pmatrix} \mathbf{J}_{I;\boldsymbol{\theta},\mathbf{\Sigma}}\,\bar{\mathbf{K}}^{-1} & \mathbf{0} \\ \mathbf{0} & \frac{D_k(J_2;f)}{k}\,\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}\,\mathbf{P}_{\boldsymbol{\theta}}\,\mathbf{M}_{\boldsymbol{\theta}} \end{pmatrix}\mathbf{\Upsilon} =$$

$$\begin{pmatrix} c_I(\bar{\mathbf{K}}^{-1}\mathbf{\Upsilon}_I - \bar{\mathbf{K}}^{-1}\,\mathbf{\Upsilon}_I(\mathbf{\Upsilon}'_I\,\bar{\mathbf{K}}'^{-1}\,\mathbf{J}_{I;\boldsymbol{\theta},\mathbf{\Sigma}}\,\bar{\mathbf{K}}^{-1}\,\mathbf{\Upsilon}_I)^{-1}\mathbf{\Upsilon}'_I\,\bar{\mathbf{K}}'^{-1}\,\mathbf{J}_{I;\boldsymbol{\theta},\mathbf{\Sigma}}\,\bar{\mathbf{K}}^{-1}\mathbf{\Upsilon}_I) \\ c_{II}(\mathbf{P}_{\boldsymbol{\theta}}\mathbf{M}_{\boldsymbol{\theta}}\,\mathbf{\Upsilon}_{II} - \mathbf{P}_{\boldsymbol{\theta}}\mathbf{M}_{\boldsymbol{\theta}}\,\mathbf{\Upsilon}_{II}(\mathbf{\Upsilon}'_{II}\mathbf{M}'_{\boldsymbol{\theta}}\mathbf{P}'_{\boldsymbol{\theta}}\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}\,\mathbf{P}_{\boldsymbol{\theta}}\mathbf{M}_{\boldsymbol{\theta}}\,\mathbf{\Upsilon}_{II})^{-}\mathbf{\Upsilon}'_{II}\mathbf{M}'_{\boldsymbol{\theta}}\mathbf{P}'_{\boldsymbol{\theta}}\,\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}\,\mathbf{P}_{\boldsymbol{\theta}}\mathbf{M}_{\boldsymbol{\theta}}\,\mathbf{\Upsilon}_{II}) \end{pmatrix} = \mathbf{0},$$

for some constants c_I, c_{II}. This and the constraints on $\hat{\boldsymbol{\theta}}$ jointly entail that, under $\cup_f \mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},f)$, with $\boldsymbol{\theta}-\boldsymbol{\theta}_0 \in \mathcal{M}(\mathbf{\Upsilon})$,

$$\mathcal{W}_J^{(n)} = \left(n^{1/2}\widetilde{\mathbf{T}}^{(n)}_{J;\mathbf{\Sigma}}(\boldsymbol{\theta})\right)'\begin{pmatrix} \bar{\mathbf{Q}}_{I;J;\mathbf{\Sigma}}(\boldsymbol{\theta}) & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{Q}}_{II;J;\mathbf{\Sigma}}(\boldsymbol{\theta}) \end{pmatrix}\left(n^{1/2}\widetilde{\mathbf{T}}^{(n)}_{J;\mathbf{\Sigma}}(\boldsymbol{\theta})\right) + o_{\mathrm{P}}(1),$$

which proves that $\mathcal{W}_J^{(n)}$ is indeed asymptotically invariant with respect to the group $\mathcal{G}_{\mathbf{\Sigma}}^{(n)}$, since $n^{1/2}\widetilde{\mathbf{T}}_{J;\mathbf{\Sigma}}^{(n)}(\boldsymbol{\theta})$ is strictly invariant with respect to that group.

(ii), (iii) Part *(i)* of the Proposition, and the continuity of $\bar{\mathbf{Q}}_{I;J;\mathbf{\Sigma}}(\boldsymbol{\theta})$ and $\bar{\mathbf{Q}}_{II;J;\mathbf{\Sigma}}(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ and $\mathbf{\Sigma}$ again, imply that $\mathcal{W}_J^{(n)}$ has the same asymptotic behaviour as

$$n\,(\underset{\sim}{\mathbf{T}}_{I;J}^{(n)}(\boldsymbol{\theta}))'\bar{\mathbf{Q}}_{I;J;\mathbf{\Sigma}}(\boldsymbol{\theta})\,\underset{\sim}{\mathbf{T}}_{I;J}^{(n)}(\boldsymbol{\theta})+n\,(\underset{\sim}{\mathbf{T}}_{II;J}^{(n)}(\boldsymbol{\theta}))'\bar{\mathbf{Q}}_{II;J;\mathbf{\Sigma}}(\boldsymbol{\theta})\,\underset{\sim}{\mathbf{T}}_{II;J}^{(n)}(\boldsymbol{\theta}) \qquad (23)$$

(under $\mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},f)$, $\boldsymbol{\theta}-\boldsymbol{\theta}_0\in\mathcal{M}(\mathbf{\Upsilon})$ as under the sequence of local alternatives $\mathcal{H}^{(n)}(\boldsymbol{\theta}+\boldsymbol{\nu}(n)\boldsymbol{\tau},\mathbf{\Sigma},f)$, with $\boldsymbol{\theta}-\boldsymbol{\theta}_0\in\mathcal{M}(\mathbf{\Upsilon})$, $\boldsymbol{\tau}\notin\mathcal{M}(\mathbf{\Upsilon})$). On the other hand, Proposition 2 implies that (23) behaves as

$$n\,\left(\widetilde{\mathbf{T}}_{I;J;\mathbf{\Sigma},f}^{(n)}(\boldsymbol{\theta})\right)'\bar{\mathbf{Q}}_{I;J;\mathbf{\Sigma}}(\boldsymbol{\theta})\,\widetilde{\mathbf{T}}_{I;J;\mathbf{\Sigma},f}^{(n)}(\boldsymbol{\theta})+n\,\left(\widetilde{\mathbf{T}}_{I;J;\mathbf{\Sigma},f}^{(n)}(\boldsymbol{\theta})\right)'\bar{\mathbf{Q}}_{II;J;\mathbf{\Sigma}}(\boldsymbol{\theta})\,\widetilde{\mathbf{T}}_{II;J;\mathbf{\Sigma},f}^{(n)}(\boldsymbol{\theta}), \qquad (24)$$

where we let $n^{1/2}\widetilde{\mathbf{T}}_{I;J;\mathbf{\Sigma},f}^{(n)}(\boldsymbol{\theta}) := \mathbf{L}_{\boldsymbol{\theta}}^{(n)'}(n^{1/2}\,(\mathrm{vec}\,\widetilde{\mathbf{\Lambda}}_{0;J;\mathbf{\Sigma},f}^{(n)}(\boldsymbol{\theta}))',\ldots,(\mathrm{vec}\,\widetilde{\mathbf{\Lambda}}_{n-1;J;\mathbf{\Sigma},f}^{(n)}(\boldsymbol{\theta}))')'$ and $n^{1/2}\widetilde{\mathbf{T}}_{II;J;\mathbf{\Sigma},f}^{(n)}(\boldsymbol{\theta}) := \mathbf{Q}_{\boldsymbol{\theta}}^{(n)'}((n-1)^{1/2}\,(\mathrm{vec}\,\widetilde{\mathbf{\Gamma}}_{1;J;\mathbf{\Sigma},f}^{(n)}(\boldsymbol{\theta}))',\ldots,(\mathrm{vec}\,\widetilde{\mathbf{\Gamma}}_{n-1;J;\mathbf{\Sigma},f}^{(n)}(\boldsymbol{\theta}))')'$.

Now, $n^{1/2}\widetilde{\mathbf{T}}_{I;J;\mathbf{\Sigma},f}^{(n)}(\boldsymbol{\theta})$ is asymptotically km-normal, with mean $\mathbf{0}$ under $\mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},f)$ (with $\boldsymbol{\theta}-\boldsymbol{\theta}_0\in\mathcal{M}(\mathbf{\Upsilon})$), mean $(C_k(J_0;f)/k)\,\mathbf{J}_{I;\boldsymbol{\theta},\mathbf{\Sigma}}\,(\mathrm{vec}\,\boldsymbol{\eta}')$ under the sequence of local alternatives under consideration, and variance $(\mathrm{E}[J_0^2(U)]/k)\,\mathbf{J}_{I;\boldsymbol{\theta},\mathbf{\Sigma}}$ under both. Since $(\mathrm{E}[J_0^2(U)]/k)\,\mathbf{J}_{I;\boldsymbol{\theta},\mathbf{\Sigma}}^{1/2}\,\bar{\mathbf{Q}}_{I;J;\mathbf{\Sigma}}(\boldsymbol{\theta})\,\mathbf{J}_{I;\boldsymbol{\theta},\mathbf{\Sigma}}^{1/2}$ is a symmetric idempotent matrix with rank $km-r_I$, this implies that the first term in (24) is asymptotically chi-square with $km-r_I$ degrees of freedom under $\mathcal{H}_{\mathbf{\Upsilon}}^{(n)}(\boldsymbol{\theta}_0)$, and asymptotically noncentral chi-square, still with $km-r_I$ degrees of freedom but with noncentrality parameter $(C_k^2(J_0;f)/(k\,\mathrm{E}[J_0^2(U)]))\,r_{\boldsymbol{\theta},\mathbf{\Sigma}}(\boldsymbol{\eta})$ under $\mathcal{H}^{(n)}(\boldsymbol{\theta}+\boldsymbol{\nu}(n)\boldsymbol{\tau},\mathbf{\Sigma},f)$ with $\boldsymbol{\theta}-\boldsymbol{\theta}_0\in\mathcal{M}(\mathbf{\Upsilon})$, and $\boldsymbol{\tau}\notin\mathcal{M}(\mathbf{\Upsilon})$.

For the serial part in (24), $n^{1/2}\widetilde{\mathbf{T}}_{II;J;\mathbf{\Sigma},f}^{(n)}(\boldsymbol{\theta})$ is asymptotically $k^2\pi_0$-normal, with mean $\mathbf{0}$ under $\mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},f)$, $\boldsymbol{\theta}-\boldsymbol{\theta}_0\in\mathcal{M}(\mathbf{\Upsilon})$, mean

$$\frac{1}{k^2}\,D_k(J_2;f)\,C_k(J_1;f)\,\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}\,\mathbf{P}_{\boldsymbol{\theta}}\,\mathbf{M}_{\boldsymbol{\theta}}\begin{pmatrix}\mathrm{vec}\,\boldsymbol{\gamma}\\ \mathrm{vec}\,\boldsymbol{\delta}\end{pmatrix}$$

under the sequence of alternatives under consideration, and variance $(\mathrm{E}[J_1^2(U)]\mathrm{E}[J_2^2(U)]/k^2)\,\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}$ under both. The result follows from the fact that $(\mathrm{E}[J_1^2(U)]\mathrm{E}[J_2^2(U)]/k^2)\,\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}^{1/2}\,\bar{\mathbf{Q}}_{II;J;\mathbf{\Sigma}}(\boldsymbol{\theta})\,\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}^{1/2}$ is a symmetric idempotent matrix with rank

$$\begin{aligned}
&\mathrm{tr}\,(\mathbf{I}_{k^2\pi_0}-\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}^{1/2}\mathbf{P}_{\boldsymbol{\theta}}\mathbf{M}_{\boldsymbol{\theta}}\,\mathbf{\Upsilon}_{II}(\mathbf{\Upsilon}_{II}'\mathbf{M}_{\boldsymbol{\theta}}'\mathbf{P}_{\boldsymbol{\theta}}'\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}\,\mathbf{P}_{\boldsymbol{\theta}}\mathbf{M}_{\boldsymbol{\theta}}\,\mathbf{\Upsilon}_{II})^{-}\mathbf{\Upsilon}_{II}'\mathbf{M}_{\boldsymbol{\theta}}'\mathbf{P}_{\boldsymbol{\theta}}'\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}^{1/2})\\
&\quad= k^2\pi_0-\mathrm{tr}\,((\mathbf{\Upsilon}_{II}'\mathbf{M}_{\boldsymbol{\theta}}'\mathbf{P}_{\boldsymbol{\theta}}'\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}\mathbf{P}_{\boldsymbol{\theta}}\mathbf{M}_{\boldsymbol{\theta}}\,\mathbf{\Upsilon}_{II})(\mathbf{\Upsilon}_{II}'\mathbf{M}_{\boldsymbol{\theta}}'\mathbf{P}_{\boldsymbol{\theta}}'\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}\,\mathbf{P}_{\boldsymbol{\theta}}\mathbf{M}_{\boldsymbol{\theta}}\,\mathbf{\Upsilon}_{II})^{-})\\
&\quad= k^2\pi_0-\mathrm{rank}(\mathbf{\Upsilon}_{II}'\mathbf{M}_{\boldsymbol{\theta}}'\mathbf{P}_{\boldsymbol{\theta}}'\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}\mathbf{P}_{\boldsymbol{\theta}}\mathbf{M}_{\boldsymbol{\theta}}\,\mathbf{\Upsilon}_{II})\\
&\quad= k^2\pi_0-\mathrm{rank}(\dot{\mathbf{M}}_{\boldsymbol{\theta}}\,\dot{\mathbf{\Upsilon}}_{II})=k^2\pi_0-\min(k^2(p_0+q_0),r_{II})=k^2\pi_0-r_{II},
\end{aligned}$$

since $\mathbf{M}_{\boldsymbol{\theta}}\,\mathbf{\Upsilon}_{II}=\dot{\mathbf{M}}_{\boldsymbol{\theta}}\,\dot{\mathbf{\Upsilon}}_{II}$ is the product of two full rank matrices.

(iv) Applied to the current problem, Hallin and Puri (1994)'s general Lemma 5.12 shows that the test $\underline{\phi}_{\mathbf{\Sigma},f_\star}^{(n)}$ that rejects the null hypothesis whenever

$$\begin{aligned}
&\mathbf{\Delta}_{\mathbf{\Sigma},f_\star}^{(n)'}(\boldsymbol{\theta})(\mathbf{I}-\mathbf{\Gamma}_{\mathbf{\Sigma},f_\star}(\boldsymbol{\theta})\mathbf{\Upsilon}(\mathbf{\Upsilon}'\mathbf{\Gamma}_{\mathbf{\Sigma},f_\star}(\boldsymbol{\theta})\mathbf{\Upsilon})^{-}\mathbf{\Upsilon}'\mathbf{\Gamma}_{\mathbf{\Sigma},f_\star}(\boldsymbol{\theta})(\mathbf{\Gamma}_{\mathbf{\Sigma},f_\star}(\boldsymbol{\theta}))^{-})'(\mathbf{\Gamma}_{\mathbf{\Sigma},f_\star}(\boldsymbol{\theta}))^{-}\\
&\qquad(\mathbf{I}-\mathbf{\Gamma}_{\mathbf{\Sigma},f_\star}(\boldsymbol{\theta})\mathbf{\Upsilon}(\mathbf{\Upsilon}'\mathbf{\Gamma}_{\mathbf{\Sigma},f_\star}(\boldsymbol{\theta})\mathbf{\Upsilon})^{-}\mathbf{\Upsilon}'\mathbf{\Gamma}_{\mathbf{\Sigma},f_\star}(\boldsymbol{\theta})(\mathbf{\Gamma}_{\mathbf{\Sigma},f_\star}(\boldsymbol{\theta}))^{-})\,\mathbf{\Delta}_{\mathbf{\Sigma},f_\star}^{(n)}(\boldsymbol{\theta})>\chi_{s,1-\alpha}^2,
\end{aligned}$$

where $s := \mathrm{rank}(\boldsymbol{\Gamma}_{\boldsymbol{\Sigma},f_\star}(\boldsymbol{\theta})) - \mathrm{rank}(\boldsymbol{\Upsilon}'\boldsymbol{\Gamma}_{\boldsymbol{\Sigma},f_\star}(\boldsymbol{\theta})\boldsymbol{\Upsilon})$, is locally and asymptotically most stringent for $\mathcal{H}_{\boldsymbol{\Upsilon}}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f_\star)$ against $\bigcup_{\boldsymbol{\theta}\neq\boldsymbol{\theta}_0+\mathcal{M}(\boldsymbol{\Upsilon})} \bigcup_{\boldsymbol{\Sigma}} \mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f_\star)$, at asymptotic probability level α. Of course, the same optimality property holds for the asymptotically equivalent (under $\mathcal{H}_{\boldsymbol{\Upsilon}}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f_\star)$, and under contiguous alternatives) test $\underline{\phi}_{f_\star}^{(n)}$ that rejects the null hypothesis whenever

$$\underline{\mathcal{W}}_{f_\star}^{(n)} := \underset{\sim}{\boldsymbol{\Delta}}_{f_\star}^{(n)\prime}(\hat{\boldsymbol{\theta}})(\mathbf{I} - \hat{\boldsymbol{\Gamma}}_{f_\star}^{(n)}(\hat{\boldsymbol{\theta}})\boldsymbol{\Upsilon}(\boldsymbol{\Upsilon}'\hat{\boldsymbol{\Gamma}}_{f_\star}^{(n)}(\hat{\boldsymbol{\theta}})\boldsymbol{\Upsilon})^{-}\boldsymbol{\Upsilon}'\hat{\boldsymbol{\Gamma}}_{f_\star}^{(n)}(\hat{\boldsymbol{\theta}})(\hat{\boldsymbol{\Gamma}}_{f_\star}^{(n)}(\hat{\boldsymbol{\theta}}))^{-})'(\hat{\boldsymbol{\Gamma}}_{f_\star}^{(n)}(\hat{\boldsymbol{\theta}}))^{-} \tag{25}$$
$$(\mathbf{I} - \hat{\boldsymbol{\Gamma}}_{f_\star}^{(n)}(\hat{\boldsymbol{\theta}})\boldsymbol{\Upsilon}(\boldsymbol{\Upsilon}'\hat{\boldsymbol{\Gamma}}_{f_\star}^{(n)}(\hat{\boldsymbol{\theta}})\boldsymbol{\Upsilon})^{-}\boldsymbol{\Upsilon}'\hat{\boldsymbol{\Gamma}}_{f_\star}^{(n)}(\hat{\boldsymbol{\theta}})(\hat{\boldsymbol{\Gamma}}_{f_\star}^{(n)}(\hat{\boldsymbol{\theta}}))^{-})\, \underset{\sim}{\boldsymbol{\Delta}}_{f_\star}^{(n)}(\hat{\boldsymbol{\theta}}) > \chi^2_{s,1-\alpha},$$

where

$$\underset{\sim}{\boldsymbol{\Delta}}_{f_\star}^{(n)}(\boldsymbol{\theta}) := n^{1/2} \begin{pmatrix} \mathbf{I}_{km} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\boldsymbol{\theta}}'\mathbf{P}_{\boldsymbol{\theta}}' \end{pmatrix} \underset{\sim}{\mathbf{T}}_{J}^{(n)}(\boldsymbol{\theta}),$$

with $J_0 = J_1 := \varphi_{f_\star} \circ \tilde{F}_{\star k}^{-1}$ and $J_2 = \tilde{F}_{\star k}^{-1}$, and

$$\hat{\boldsymbol{\Gamma}}_{f_\star}^{(n)}(\boldsymbol{\theta}) := \begin{pmatrix} \frac{1}{k}\mathcal{I}_{k,f_\star}\mathbf{J}_{I;\boldsymbol{\theta},\hat{\boldsymbol{\Sigma}}}^{(n)} & \mathbf{0} \\ \mathbf{0} & \frac{\mu_{k+1;f_\star}\mathcal{I}_{k,f_\star}}{k^2\mu_{k-1;f_\star}}\mathbf{M}_{\boldsymbol{\theta}}'\mathbf{P}_{\boldsymbol{\theta}}'\mathbf{J}_{II;\boldsymbol{\theta},\hat{\boldsymbol{\Sigma}}}^{(n)}\mathbf{P}_{\boldsymbol{\theta}}\mathbf{M}_{\boldsymbol{\theta}} \end{pmatrix} = \boldsymbol{\Gamma}_{\boldsymbol{\Sigma},f_\star}(\boldsymbol{\theta}) + o_{\mathrm{P}}(1)$$

under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f_\star)$.

If we can assume that $\underline{\mathcal{W}}_{f_\star}^{(n)} = \mathcal{W}_{f_\star}^{(n)}$, then, by *(ii)*, we have $s = km + k^2\pi_0 - r$, so that $\underline{\phi}_{f_\star}^{(n)}$ and $\phi_{f_\star}^{(n)}$ actually coincide. The result then follows from the invariance properties of $\phi_{f_\star}^{(n)}$. In order to complete the proof, it is thus sufficient to show that indeed $\underline{\mathcal{W}}_{f_\star}^{(n)} = \mathcal{W}_{f_\star}^{(n)}$. The block-diagonal structure of the quadratic form in the definition of $\underline{\mathcal{W}}_{f_\star}^{(n)}$ allows for a decomposition of the form $\underline{\mathcal{W}}_{I;f_\star}^{(n)} + \underline{\mathcal{W}}_{II;f_\star}^{(n)}$, where $\underline{\mathcal{W}}_{I;f_\star}^{(n)}$ (resp. $\underline{\mathcal{W}}_{II;f_\star}^{(n)}$) deals with the trend part (resp. the serial part). While routine algebra yields $\underline{\mathcal{W}}_{I;f_\star}^{(n)} = \mathcal{W}_{I;f_\star}^{(n)}$, the situation for the serial part is more intricate, mainly due to the presence of generalized inverses. Write $\hat{\mathbf{P}}$, $\hat{\mathbf{M}}$, $\hat{\mathbf{J}}_{II}$, and $\hat{\mathbf{N}}$ for $\mathbf{P}_{\hat{\boldsymbol{\theta}}}$, $\mathbf{M}_{\hat{\boldsymbol{\theta}}}$, $\mathbf{J}_{II;\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\Sigma}}}^{(n)}$, and $\mathbf{M}_{\hat{\boldsymbol{\theta}}}'\mathbf{P}_{\hat{\boldsymbol{\theta}}}'\mathbf{J}_{II;\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\Sigma}}}^{(n)}\mathbf{P}_{\hat{\boldsymbol{\theta}}}\mathbf{M}_{\hat{\boldsymbol{\theta}}}$, respectively. Standard calculation yields

$$\underline{\mathcal{W}}_{II;f_\star}^{(n)} = \frac{nk^2\mu_{k-1;f_\star}}{\mu_{k+1;f_\star}\mathcal{I}_{k,f_\star}} \underset{\sim}{\mathbf{T}}_{II;J}^{(n)\prime}(\hat{\boldsymbol{\theta}})\,\hat{\mathbf{P}}\,\hat{\mathbf{M}}\left\{\hat{\mathbf{N}}^{-}\left[\mathbf{I} - \hat{\mathbf{N}}\,\boldsymbol{\Upsilon}_{II}\left(\boldsymbol{\Upsilon}_{II}'\,\hat{\mathbf{N}}\,\boldsymbol{\Upsilon}_{II}\right)^{-}\boldsymbol{\Upsilon}_{II}'\,\hat{\mathbf{N}}\,\hat{\mathbf{N}}^{-}\right]\right\}\hat{\mathbf{M}}'\,\hat{\mathbf{P}}'\,\underset{\sim}{\mathbf{T}}_{II;J}^{(n)}(\hat{\boldsymbol{\theta}}),$$

that is, in view of Lemma 2.2.6 (c) in Rao and Mitra (1971),

$$\underline{\mathcal{W}}_{II;f_\star}^{(n)} = \frac{nk^2\mu_{k-1;f_\star}}{\mu_{k+1;f_\star}\mathcal{I}_{k,f_\star}} \underset{\sim}{\mathbf{T}}_{II;J}^{(n)\prime}(\hat{\boldsymbol{\theta}})\left[\hat{\mathbf{P}}\,\hat{\mathbf{M}}\,\hat{\mathbf{N}}^{-}\,\hat{\mathbf{M}}'\,\hat{\mathbf{P}}' - \hat{\mathbf{P}}\,\hat{\mathbf{M}}\,\boldsymbol{\Upsilon}_{II}\left(\boldsymbol{\Upsilon}_{II}'\,\hat{\mathbf{N}}\,\boldsymbol{\Upsilon}_{II}\right)^{-}\boldsymbol{\Upsilon}_{II}'\,\hat{\mathbf{M}}'\,\hat{\mathbf{P}}'\right]\underset{\sim}{\mathbf{T}}_{II;J}^{(n)}(\hat{\boldsymbol{\theta}}).$$

This implies that $\underline{\mathcal{W}}_{II;f_\star}^{(n)} = \mathcal{W}_{II;f_\star}^{(n)}$, since $\hat{\mathbf{M}}\,\boldsymbol{\Upsilon}_{II} = \dot{\hat{\mathbf{M}}}\,\dot{\boldsymbol{\Upsilon}}_{II}$, and since, from Lemma 2.2.5 (c) in Rao and Mitra (1971), $\hat{\mathbf{P}}\,\hat{\mathbf{M}}\,(\hat{\mathbf{M}}'\hat{\mathbf{P}}'\hat{\mathbf{J}}_{II}\,\hat{\mathbf{P}}\,\hat{\mathbf{M}})^{-}\,\hat{\mathbf{M}}'\hat{\mathbf{P}}' = \hat{\mathbf{J}}_{II}^{-1}$. Consequently, $\underline{\mathcal{W}}_{f_\star}^{(n)} = \mathcal{W}_{f_\star}^{(n)}$. $\square$

Note that, for given values of p_0 and q_0, $\underset{\sim}{\mathbf{T}}_{J}^{(n)}(\hat{\boldsymbol{\theta}})$, $\dot{\mathbf{M}}_{\hat{\boldsymbol{\theta}}}$, $\dot{\boldsymbol{\Upsilon}}_{II}$, and π_0 depend on p_1 and q_1 only through $\pi = \max(p_1 - p_0, q_1 - q_0)$. Consequently, $\phi_J^{(n)}$ also depends on p_1 and q_1 through π only.

We conclude this section by stressing the fact that the value of the test statistic $\mathcal{W}_J^{(n)}$ does not depend on the particular choice of the fundamental system $\hat{\boldsymbol{\Psi}} := \{\boldsymbol{\Psi}_t^{(1)}(\hat{\boldsymbol{\theta}}), \ldots, \boldsymbol{\Psi}_t^{(p_0+q_0)}(\hat{\boldsymbol{\theta}})\}$ in $\mathbf{P}_{\hat{\boldsymbol{\theta}}}$

and $\mathbf{Q}_{\hat{\boldsymbol{\theta}}}^{(n)}$. Of course, only the serial part $\mathcal{W}_{II;J}^{(n)}$ possibly may depend on the fundamental system. Now, for any fundamental system $\hat{\boldsymbol{\Phi}} := \{\boldsymbol{\Phi}_t^{(1)}(\hat{\boldsymbol{\theta}}), \ldots, \boldsymbol{\Phi}_t^{(p_0+q_0)}(\hat{\boldsymbol{\theta}})\}$, there exists an invertible matrix $\bar{\boldsymbol{\Lambda}}$ such that $\mathbf{Q}_{\hat{\boldsymbol{\theta}};\hat{\boldsymbol{\Phi}}} = \mathbf{Q}_{\hat{\boldsymbol{\theta}};\hat{\boldsymbol{\Psi}}}\bar{\boldsymbol{\Lambda}}$ (see the proof of Proposition 4(i) in Hallin and Paindaveine (2002d)). It easily follows that $\underset{\sim}{\mathbf{T}}{}_{II;J;\hat{\boldsymbol{\Phi}}}^{(n)}(\hat{\boldsymbol{\theta}}) := \bar{\boldsymbol{\Lambda}}'\,\underset{\sim}{\mathbf{T}}{}_{II;J;\hat{\boldsymbol{\Psi}}}^{(n)}(\hat{\boldsymbol{\theta}})$, $\mathbf{J}_{II;\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\Sigma}};\hat{\boldsymbol{\Phi}}}^{(n)} = \bar{\boldsymbol{\Lambda}}'\,\mathbf{J}_{II;\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\Sigma}};\hat{\boldsymbol{\Psi}}}^{(n)}\bar{\boldsymbol{\Lambda}}$ and $\mathbf{P}_{\hat{\boldsymbol{\theta}};\hat{\boldsymbol{\Phi}}} = \bar{\boldsymbol{\Lambda}}^{-1}\,\mathbf{P}_{\hat{\boldsymbol{\theta}};\hat{\boldsymbol{\Phi}}}$. Consequently, we obtain that $\mathcal{W}_{K;\hat{\boldsymbol{\Phi}}}^{(n)} = \mathcal{W}_{K;\hat{\boldsymbol{\Psi}}}^{(n)}$.

5.2 The Gaussian procedure.

Define

$$\mathbf{J}_{I;\mathcal{N};\boldsymbol{\theta}}^{(n)} := \mathbf{L}_{\boldsymbol{\theta}}^{(n)\prime}\left(\boldsymbol{K}_n \otimes (\mathbf{S}_{\boldsymbol{\theta}}^{(n)})^{-1}\right)\mathbf{L}_{\boldsymbol{\theta}}^{(n)} \quad \text{and} \quad \mathbf{J}_{II;\mathcal{N};\boldsymbol{\theta}}^{(n)} := \mathbf{Q}_{\boldsymbol{\theta}}^{(n)\prime}\left[\mathbf{I}_{n-1} \otimes \hat{\boldsymbol{\Gamma}}_{\boldsymbol{\theta}}^{(n)}\right]\mathbf{Q}_{\boldsymbol{\theta}}^{(n)},$$

where $\mathbf{S}_{\boldsymbol{\theta}}^{(n)} := n^{-1}\sum_{t=1}^{n}\mathbf{Z}_t(\boldsymbol{\theta})\mathbf{Z}_t'(\boldsymbol{\theta})$ is a consistent estimator, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$, of the innovation covariance $(\mathrm{E}[(\tilde{F}_k^{-1}(U))^2]/k)\,\boldsymbol{\Sigma}$, and

$$\hat{\boldsymbol{\Gamma}}_{\boldsymbol{\theta}}^{(n)} := (n-1)^{-1}\sum_{t=2}^{n}\mathrm{vec}\left(\mathbf{Z}_t(\boldsymbol{\theta})\mathbf{Z}_{t-1}'(\boldsymbol{\theta})\right)\left(\mathrm{vec}\left(\mathbf{Z}_t(\boldsymbol{\theta})\mathbf{Z}_{t-1}'(\boldsymbol{\theta})\right)\right)'$$

is consistent for $(\mathrm{E}[(\tilde{F}_k^{-1}(U))^2]/k)^2\,\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}^{-1}$ under the same sequence of hypotheses. Let

$$\bar{\mathbf{Q}}_{I;\mathcal{N}}^{(n)}(\boldsymbol{\theta}) := (\mathbf{J}_{I;\mathcal{N};\boldsymbol{\theta}}^{(n)})^{-1} - (\mathbf{K}^{(n)}\otimes\mathbf{I}_k)^{-1}\,\boldsymbol{\Upsilon}_I \left(\boldsymbol{\Upsilon}_I'\,(\mathbf{K}^{(n)}\otimes\mathbf{I}_k)'^{-1}\,\mathbf{J}_{I;\mathcal{N};\boldsymbol{\theta}}^{(n)}\,(\mathbf{K}^{(n)}\otimes\mathbf{I}_k)^{-1}\,\boldsymbol{\Upsilon}_I\right)^{-1}\boldsymbol{\Upsilon}_I'\,(\mathbf{K}^{(n)}\otimes\mathbf{I}_k)'^{-1},$$

and

$$\bar{\mathbf{Q}}_{II;\mathcal{N}}^{(n)}(\boldsymbol{\theta}) := (\mathbf{J}_{II;\mathcal{N};\boldsymbol{\theta}}^{(n)})^{-1} - \mathbf{P}_{\boldsymbol{\theta}}\dot{\mathbf{M}}_{\boldsymbol{\theta}}\,\dot{\boldsymbol{\Upsilon}}_{II}\left(\dot{\boldsymbol{\Upsilon}}_{II}'\,\dot{\mathbf{M}}_{\boldsymbol{\theta}}'\,\mathbf{P}_{\boldsymbol{\theta}}'\mathbf{J}_{II;\mathcal{N};\boldsymbol{\theta}}^{(n)}\,\mathbf{P}_{\boldsymbol{\theta}}\dot{\mathbf{M}}_{\boldsymbol{\theta}}\,\dot{\boldsymbol{\Upsilon}}_{II}\right)^{-}\dot{\boldsymbol{\Upsilon}}_{II}'\,\dot{\mathbf{M}}_{\boldsymbol{\theta}}'\,\mathbf{P}_{\boldsymbol{\theta}}'.$$

Then the Gaussian parametric test statistic is

$$\mathcal{W}_{\mathcal{N}}^{(n)} := n\left(\mathbf{T}_{I;\mathbf{S},\phi}^{(n)}(\hat{\boldsymbol{\theta}})\right)'\bar{\mathbf{Q}}_{I;\mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}})\,\mathbf{T}_{I;\mathbf{S},\phi}^{(n)}(\hat{\boldsymbol{\theta}}) + n\left(\mathbf{T}_{II;\mathbf{S},\phi}^{(n)\prime}(\hat{\boldsymbol{\theta}})\right)'\bar{\mathbf{Q}}_{II;\mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}})\,\mathbf{T}_{II;\mathbf{S},\phi}^{(n)}(\hat{\boldsymbol{\theta}}),$$

where $\mathbf{T}_{I;\mathbf{S},\phi}^{(n)}(\boldsymbol{\theta})$ and $\mathbf{T}_{II;\mathbf{S},\phi}^{(n)}(\boldsymbol{\theta})$ are defined in (13) and (14) respectively, $\mathbf{S} := \mathbf{S}_{\boldsymbol{\theta}}^{(n)}$, and $\phi(r) := \exp(-r^2/2)$ stands for the Gaussian radial density. Note that $\mathbf{T}_{I;\mathbf{S},\phi}^{(n)}(\boldsymbol{\theta})$ and $\mathbf{T}_{II;\mathbf{S},\phi}^{(n)}(\boldsymbol{\theta})$ are based on Gaussian statistics of the form

$$\boldsymbol{\Lambda}_{i;\mathbf{S},\phi}^{(n)}(\boldsymbol{\theta}) = (\mathbf{S}_{\boldsymbol{\theta}}^{(n)})^{-1}\left(\frac{1}{n-i}\sum_{t=i+1}^{n}\mathbf{Z}_t(\boldsymbol{\theta})\,\mathbf{x}_{t-i}^{(n)\prime}\mathbf{K}^{(n)}\right) \text{ and } \boldsymbol{\Gamma}_{i;\mathbf{S},\phi}^{(n)}(\boldsymbol{\theta}) = (\mathbf{S}_{\boldsymbol{\theta}}^{(n)})^{-1}\left(\frac{1}{n-i}\sum_{t=i+1}^{n}\mathbf{Z}_t(\boldsymbol{\theta})\,\mathbf{Z}_{t-i}'(\boldsymbol{\theta})\right), \tag{26}$$

respectively.

Proposition 4 *Assume that Assumptions (A1), (A2), (B1′), (B2), (B3), (E1) and (E2) hold. Consider the sequence of parametric Gaussian tests $\phi_{\mathcal{N}}^{(n)}$ that reject the null hypothesis $\mathcal{H}_{\boldsymbol{\Upsilon}}^{(n)}(\boldsymbol{\theta}_0)$ whenever $\mathcal{W}_{\mathcal{N}}^{(n)}$ exceeds the α-upper quantile $\chi^2_{km+k^2\pi_0-r,1-\alpha}$ of a chi-square distribution with $km+k^2\pi_0-r$ degrees of freedom. Then,* *(i)* $\mathcal{W}_{\mathcal{N}}^{(n)}$ *is strictly affine-invariant;*

(ii) $\mathcal{W}_{\mathcal{N}}^{(n)}$ *is asymptotically chi-square with* $km + k^2\pi_0 - r$ *degrees of freedom under* $\mathcal{H}_{\mathbf{\Upsilon}}^{(n)}(\boldsymbol{\theta}_0)$ *(so that* $\phi_{\mathcal{N}}^{(n)}$ *has asymptotic level* α*);*

(iii) $\mathcal{W}_{\mathcal{N}}^{(n)}$ *is asymptotically noncentral chi-square, still with* $km + k^2\pi_0 - r$ *degrees of freedom but with noncentrality parameter*

$$\frac{k}{D_k(f)}\, r_{\boldsymbol{\theta},\mathbf{\Sigma}}(\boldsymbol{\eta}) + s_{\boldsymbol{\theta},\mathbf{\Sigma}}(\boldsymbol{\gamma},\boldsymbol{\delta}),$$

under $\mathcal{H}^{(n)}(\boldsymbol{\theta}+\boldsymbol{\nu}(n)\boldsymbol{\tau},\mathbf{\Sigma},f)$, $\boldsymbol{\theta}-\boldsymbol{\theta}_0 \in \mathcal{M}(\mathbf{\Upsilon})$ *and* $\boldsymbol{\tau} \notin \mathcal{M}(\mathbf{\Upsilon})$, *provided that (A1) is reinforced into (A1′);*

(iv) The sequence of tests $\phi_{\mathcal{N}}^{(n)}$ *is locally asymptotically most stringent for* $\mathcal{H}_{\mathbf{\Upsilon}}^{(n)}(\boldsymbol{\theta}_0)$ *against* $\bigcup_{\boldsymbol{\theta}\neq\boldsymbol{\theta}_0+\mathcal{M}(\mathbf{\Upsilon})}\bigcup_{\mathbf{\Sigma}}\mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},\phi)$, *at probability level* α.

Again, the tests statistics $\mathcal{W}_{\mathcal{N}}^{(n)}$ do not depend on the particular choice of the fundamental system $\{\mathbf{\Psi}_t^{(1)}(\hat{\boldsymbol{\theta}}),\ldots,\mathbf{\Psi}_t^{(p_0+q_0)}(\hat{\boldsymbol{\theta}})\}$, and, for given values of p_0 and q_0, depend on p_1 and q_1 through $\pi = \max(p_1 - p_0, q_1 - q_0)$ only.

The proof of Proposition 4 follows along the same lines as for Proposition 3. The key ingredient is again an asymptotic linearity result, which in this parametric Gaussian context takes the following form (the proof of Lemma 3 readily extends to this situation).

Proposition 5 *Assume that Assumptions (A1), (A2), (B1′), (B2), (B3) and (D1) hold. Then*

$$n^{1/2}(\mathbf{T}_{I;\mathbf{S},\phi}^{(n)}(\hat{\boldsymbol{\theta}}) - \mathbf{T}_{I;\mathbf{S},\phi}^{(n)}(\boldsymbol{\theta})) + \frac{k}{D_k(f)}\,\mathbf{J}_{I;\boldsymbol{\theta},\mathbf{\Sigma}}\,(\mathbf{K}^{(n)}\otimes\mathbf{I}_k)^{-1}\,n^{1/2}(\hat{\boldsymbol{\theta}}_I^{(n)} - \boldsymbol{\theta}_I)$$

and

$$n^{1/2}(\mathbf{T}_{II;\mathbf{S},\phi}^{(n)}(\hat{\boldsymbol{\theta}}) - \mathbf{T}_{II;\mathbf{S},\phi}^{(n)}(\boldsymbol{\theta})) + \mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}\,\mathbf{P}_{\boldsymbol{\theta}}\,\mathbf{M}_{\boldsymbol{\theta}}\,n^{1/2}(\hat{\boldsymbol{\theta}}_{II}^{(n)} - \boldsymbol{\theta}_{II})$$

are $o_{\mathrm{P}}(1)$ *as* $n\to\infty$, *under* $\mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},f)$.

5.3 Asymptotic performances.

We finally turn to asymptotic relative efficiencies of the tests $\phi_J^{(n)}$ with respect to the Gaussian test $\phi_{\mathcal{N}}^{(n)}$. The ARE values in the following proposition directly follow as the ratios of the noncentrality parameters in the asymptotic distributions of the various test statistics under local alternatives (see Propositions 3 and 4).

Proposition 6 *Assume that Assumptions (A1′), (A2), (B1′), (B2), (B3), (C), (D1), (D2), (E1) and (E2) hold. Then, the asymptotic relative efficiency of* $\phi_J^{(n)}$ *with respect to the Gaussian test* $\phi_{\mathcal{N}}^{(n)}$, *under radial density* f, *is*

$$\mathrm{ARE}_{k,f}(\phi_J^{(n)}/\phi_{\mathcal{N}}^{(n)}) = (1-\lambda_{\boldsymbol{\theta},\mathbf{\Sigma},f}(\boldsymbol{\tau}))\,\frac{1}{k^2}\,D_k(f)\,\frac{C_k^2(J_0;f)}{\mathrm{E}[J_0^2(U)]} + \lambda_{\boldsymbol{\theta},\mathbf{\Sigma},f}(\boldsymbol{\tau})\,\frac{1}{k^2}\,\frac{D_k^2(J_2;f)}{\mathrm{E}[J_1^2(U)]}\,\frac{C_k^2(J_1;f)}{\mathrm{E}[J_2^2(U)]},$$

where $\lambda_{\boldsymbol{\theta},\mathbf{\Sigma},f}(\boldsymbol{\tau}) := (D_k(f)s_{\boldsymbol{\theta},\mathbf{\Sigma}}(\boldsymbol{\gamma},\boldsymbol{\delta}))/(k\,r_{\boldsymbol{\theta},\mathbf{\Sigma}}(\boldsymbol{\eta}) + D_k(f)\,s_{\boldsymbol{\theta},\mathbf{\Sigma}}(\boldsymbol{\gamma},\boldsymbol{\delta})) \in [0,1]$.

Denoting by $\text{ARE}^{(\text{loc})}_{k,f}(\phi^{(n)}_J/\phi^{(n)}_{\mathcal{N}})$ and $\text{ARE}^{(\text{ser})}_{k,f}(\phi^{(n)}_J/\phi^{(n)}_{\mathcal{N}})$ the AREs achieved in the pure location and purely serial problems, respectively, we have

$$\text{ARE}_{k,f}(\phi^{(n)}_J/\phi^{(n)}_{\mathcal{N}}) = (1-\lambda_{\boldsymbol{\theta},\boldsymbol{\Sigma},f}(\boldsymbol{\tau}))\,\text{ARE}^{(\text{loc})}_{k,f}(\phi^{(n)}_J/\phi^{(n)}_{\mathcal{N}}) + \lambda_{\boldsymbol{\theta},\boldsymbol{\Sigma},f}(\boldsymbol{\tau})\,\text{ARE}^{(\text{ser})}_{k,f}(\phi^{(n)}_J/\phi^{(n)}_{\mathcal{N}}).$$

This shows that the asymptotic relative efficiencies of the proposed procedures with respect to the parametric Gaussian procedure are convex linear combinations of the corresponding asymptotic relative efficiencies in the pure location and purely serial models (see Hallin and Paindaveine 2002a and 2002b, respectively).

6 Examples.

6.1 A multivariate Durbin-Watson test.

The generalized Durbin-Watson testing problem corresponds to $\boldsymbol{\theta}_0 = \mathbf{0}$, $\boldsymbol{\Upsilon}_I = \mathbf{I}_{km}$, and $\boldsymbol{\Upsilon}_{II} := \emptyset$. Letting $\pi = \max(p_1, q_1)$, one easily checks that $\mathcal{W}^{(n)}_{I;J} = 0$, $n^{1/2}\,\underset{\sim}{\mathbf{T}}^{(n)}_{II;J}(\boldsymbol{\theta}) = \underset{\sim}{\mathbf{S}}^{(n)}_{II;\pi+1}(\boldsymbol{\theta})$, and $\mathbf{J}^{(n)}_{II;\boldsymbol{\theta},\boldsymbol{\Sigma}} = \mathbf{I}_\pi \otimes (\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}^{-1})$, so that

$$\mathcal{W}^{(n)}_J = \mathcal{W}^{(n)}_{II;J} = \frac{k^2}{\text{E}[J_1^2(U)]\text{E}[J_2^2(U)]}\sum_{i=1}^{\pi}(n-i)^{-1} \times\sum_{s,t=i+1}^{n} J_1\Big(\frac{\hat{R}_s(\hat{\boldsymbol{\beta}})}{n+1}\Big)J_1\Big(\frac{\hat{R}_t(\hat{\boldsymbol{\beta}})}{n+1}\Big)J_2\Big(\frac{\hat{R}_{s-i}(\hat{\boldsymbol{\beta}})}{n+1}\Big)J_2\Big(\frac{\hat{R}_{t-i}(\hat{\boldsymbol{\beta}})}{n+1}\Big)\ \mathbf{W}'_{s-i}(\hat{\boldsymbol{\beta}})\mathbf{W}_{t-i}(\hat{\boldsymbol{\beta}})\ \mathbf{W}'_{s}(\hat{\boldsymbol{\beta}})\mathbf{W}_{t}(\hat{\boldsymbol{\beta}}) \qquad (27)$$

(If there is no trend part in the model, the test statistic (27) is the purely pseudo-Mahalanobis version of the test statistic based on pseudo-Mahalanobis ranks and interdirections proposed in Hallin and Paindaveine (2002b) in the problem of testing for serial randomness). The resulting Durbin-Watson test consists (at level α) in rejecting the null hypothesis of independent noise as soon as $\mathcal{W}^{(n)}_J$ exceeds the α-upper quantile of a chi-square distribution with $k^2\pi$ degrees of freedom. One could also obtain purely hyperplane-based Durbin-Watson tests (that are *strictly* affine-invariant in this case) by replacing the pseudo-Mahalanobis ranks $\hat{R}_t(\hat{\boldsymbol{\beta}})$ and the pseudo-Mahalanobis angles $\mathbf{W}'_s(\hat{\boldsymbol{\beta}})\mathbf{W}_t(\hat{\boldsymbol{\beta}})$ by lift-interdirection ranks $\underline{R}_t(\hat{\boldsymbol{\beta}})$ and Randles' interdirections $q_{st}(\hat{\boldsymbol{\beta}})$, respectively.

6.2 Testing the order of a VAR model.

For the problem of testing AR(p_0) dependence against AR(p_0+1) dependence, the proposed tests consist (at level α) in rejecting the null hypothesis as soon as

$$\mathcal{W}^{(n)}_J = \mathcal{W}^{(n)}_{II;J} = n\left(\underset{\sim}{\mathbf{T}}^{(n)}_{II;J}(\hat{\boldsymbol{\theta}})\right)'\bar{\mathbf{Q}}^{(n)}_{II;J;\widehat{\boldsymbol{\Sigma}}}(\hat{\boldsymbol{\theta}})\ \underset{\sim}{\mathbf{T}}^{(n)}_{II;J}(\hat{\boldsymbol{\theta}}) \qquad (28)$$

exceeds the α-upper quantile of a chi-square distribution with k^2 degrees of freedom, where, letting $\mathbf{v}^{(n)}(\boldsymbol{\theta}) := (\mathbf{v}^{(n)\prime}_1(\boldsymbol{\theta}),\ldots,\mathbf{v}^{(n)\prime}_{p_0}(\boldsymbol{\theta}))'$ and $\mathbf{v}^{(n)}_i(\boldsymbol{\theta}) := \sum_{t=\max(i,2)}^{n-1}(n-t)^{1/2}\mathbf{G}_{t-i}(\boldsymbol{\theta})(\text{vec}\,\underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{t;J}(\boldsymbol{\theta}))$,

$$n^{1/2}\,\underset{\sim}{\mathbf{T}}^{(n)}_{II;J}(\boldsymbol{\theta}) := \begin{pmatrix} (n-1)^{1/2}\,(\text{vec}\,\underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{1;J}(\boldsymbol{\theta})) \\ \mathbf{v}^{(n)}(\boldsymbol{\theta}) \end{pmatrix},$$

and

$$\bar{\mathbf{Q}}^{(n)}_{II;J;\mathbf{\Sigma}}(\boldsymbol{\theta}) := \frac{k^2}{\mathrm{E}[J_1^2(U)]\mathrm{E}[J_2^2(U)]}\left[\begin{pmatrix} \mathbf{\Sigma}\otimes\mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{w}^2 \end{pmatrix}^{-1} - \begin{pmatrix} \mathbf{I}_{k^2} & \mathbf{0} \\ & \mathbf{I}_{k^2p_0} \end{pmatrix}\mathbf{W}^{-2}\begin{pmatrix} \mathbf{I}_{k^2} & \mathbf{0} \\ & \mathbf{I}_{k^2p_0} \end{pmatrix}'\right].$$

Above, $\mathbf{w}^2$ and $\mathbf{W}^2$ stand for the $k^2p_0 \times k^2p_0$ arrays with block $\sum_{t=\max(i,j,2)}^{n-1} \mathbf{G}_{t-i}(\boldsymbol{\theta})\mathbf{\Sigma}\mathbf{G}'_{t-j}(\boldsymbol{\theta}) \otimes \mathbf{\Sigma}^{-1}$ and $\sum_{t=\max(i,j)}^{n-1} \mathbf{G}_{t-i}(\boldsymbol{\theta})\mathbf{\Sigma}\mathbf{G}'_{t-j}(\boldsymbol{\theta}) \otimes \mathbf{\Sigma}^{-1}$, respectively, in position (i,j) $(i,j = 1,\ldots,p_0)$. Note that $\mathbf{W}^2 = \mathbf{w}^2 + \mathbf{e}_1^{(p_0)}\mathbf{e}_1^{(p_0)\prime} \otimes (\mathbf{\Sigma}\otimes\mathbf{\Sigma}^{-1})$ only differ from $\mathbf{w}^2$ through the block in position $(1,1)$.

The test statistic (28) has the same algebraic structure than in the univariate case (see Hallin and Puri (1994), or Garel and Hallin (1999)). However, it should be pointed out that the test statistic associated with the problem of testing MA(q_0) dependence versus MA(q_0+1) dependence is much more complex than in the univariate case. This is due to the presence of the factors $\mathbf{H}^{(r)}_{n-1}$ and $\mathbf{B}'^{(l)}_{n-1}$ in $\mathbf{Q}^{(n)}_{\boldsymbol{\theta}}$ which cancel each other in the univariate case *only*. In the multivariate case, they do not, yielding in $n^{1/2}\,\underset{\sim}{\mathbf{T}}^{(n)}_{II;J}(\boldsymbol{\theta})$ quite intricate linear combinations of the cross-covariance matrices $\underset{\sim}{\mathbf{\Gamma}}^{(n)}_{t;J}(\boldsymbol{\theta})$.

6.3 Detecting switching location regimes.

We finally consider the problem of detecting the presence of different "location regimes" in a VAR(1) series with a time-dependent trend (with mean $\boldsymbol{\beta}_i$ in $C_i := \{t^{(n)}_{i-1}+1,\ldots,t^{(n)}_i\}$). More precisely, the null hypothesis $\mathcal{H}_0 : \boldsymbol{\beta}_1 = \ldots = \boldsymbol{\beta}_m$ we are considering here is associated with $\mathbf{\Upsilon}_I = (1,\ldots,1)' \otimes \mathbf{I}_k$, $\mathbf{\Upsilon}_{II} = \mathbf{I}_{k^2}$. Letting $\boldsymbol{\lambda}^{(n)} := ((\lambda_1^{(n)})^{-1/2},\ldots,(\lambda_m^{(n)})^{-1/2})'$, with $\lambda_j^{(n)} := n_j/n$, the test statistic is

$$\mathcal{W}^{(n)}_J = \mathcal{W}^{(n)}_{I;J} = n\left(\underset{\sim}{\mathbf{T}}^{(n)}_{I;J}(\hat{\boldsymbol{\theta}})\right)'\bar{\mathbf{Q}}^{(n)}_{I;J;\hat{\mathbf{\Sigma}}}(\hat{\boldsymbol{\theta}})\;\underset{\sim}{\mathbf{T}}^{(n)}_{I;J}(\hat{\boldsymbol{\theta}}),$$

where

$$n^{1/2}\,\underset{\sim}{\mathbf{T}}^{(n)}_{I;J}(\boldsymbol{\theta}) = \left[\mathbf{I}_{km} + \sqrt{\frac{n}{n-1}}(\mathbf{I}_m\otimes\mathbf{A}')L^{-1}\right]\begin{pmatrix} \frac{1}{\sqrt{n_1}}\sum_{t\in C_1} J_0\Big(\frac{\hat{R}_t(\boldsymbol{\theta})}{n+1}\Big)\hat{\mathbf{\Sigma}}^{-1/2\prime}\mathbf{W}_t(\boldsymbol{\theta}) \\ \frac{1}{\sqrt{n_2}}\sum_{t\in C_2} J_0\Big(\frac{\hat{R}_t(\boldsymbol{\theta})}{n+1}\Big)\hat{\mathbf{\Sigma}}^{-1/2\prime}\mathbf{W}_t(\boldsymbol{\theta}) \\ \vdots \\ \frac{1}{\sqrt{n_m}}\sum_{t\in C_m} J_0\Big(\frac{\hat{R}_t(\boldsymbol{\theta})}{n+1}\Big)\hat{\mathbf{\Sigma}}^{-1/2\prime}\mathbf{W}_t(\boldsymbol{\theta}) \end{pmatrix},$$

and

$$\bar{\mathbf{Q}}^{(n)}_{I;J;\mathbf{\Sigma}}(\boldsymbol{\theta}) := \frac{k}{\mathrm{E}[J_0^2(U)]}\left[(\mathbf{I}_m - \boldsymbol{\lambda}^{(n)}\boldsymbol{\lambda}^{(n)\prime})\otimes[\mathbf{\Sigma}^{-1}+\mathbf{A}'\mathbf{\Sigma}^{-1}\mathbf{A}]^{-1}\right].$$

If there is no serial part in the model (i.e., when the errors are independent white noise), the problem reduces to the m-sample location problem (classical MANOVA), and the test statistic takes the form (just put $\mathbf{A} = \mathbf{0}$)

$$\mathcal{W}_J^{(n)} = \frac{k}{\mathrm{E}[J_0^2(U)]} \left[\sum_{j=1}^m \frac{1}{n_j} \sum_{i,\tilde{\imath} \in C_j} J_0\Big(\frac{\hat{R}_i(\hat{\boldsymbol\beta})}{n+1}\Big) J_0\Big(\frac{\hat{R}_{\tilde{\imath}}(\hat{\boldsymbol\beta})}{n+1}\Big) \mathbf{W}_i'(\hat{\boldsymbol\beta}) \mathbf{W}_{\tilde{\imath}}(\hat{\boldsymbol\beta}) \right.$$

$$\left. - \sum_{j,\tilde{\jmath}=1}^m \frac{n}{\sqrt{n_j}\sqrt{n_{\tilde{\jmath}}}} \sum_{i,\tilde{\imath} \in C_j} J_0\Big(\frac{\hat{R}_i(\hat{\boldsymbol\beta})}{n+1}\Big) J_0\Big(\frac{\hat{R}_{\tilde{\imath}}(\hat{\boldsymbol\beta})}{n+1}\Big) \mathbf{W}_i'(\hat{\boldsymbol\beta}) \mathbf{W}_{\tilde{\imath}}(\hat{\boldsymbol\beta}) \right],$$

i.e., a purely pseudo-Mahalanobis version of Randles and Um (1998)'s test statistic. Again a strictly affine-invariant purely hyperplane-based version of $\mathcal{W}_J^{(n)}$ can be obtained in the same way as for the Durbin-Watson tests, just by plugging in lift-interdirection ranks and Randles' interdirections.

7 Appendix: proof of Lemma 3.

The proof of Lemma 3 is based on the following asymptotic linearity result for the individual nonserial and serial statistics $\underset{\sim}{\boldsymbol\Lambda}_{i;J}^{(n)}$ and $\underset{\sim}{\boldsymbol\Gamma}_{i;J}^{(n)}$ (see Hallin and Paindaveine 2002f).

Proposition 7 *Assume that Assumptions (A1), (A2), (B1′), (B2), (B3), (C), and (D1) hold. Then*

$$(n-i)^{1/2} \operatorname{vec}(\underset{\sim}{\boldsymbol\Lambda}_{i;J}^{(n)}(\boldsymbol\theta + \boldsymbol\nu(n)\boldsymbol\tau^{(n)}) - \underset{\sim}{\boldsymbol\Lambda}_{i;J}^{(n)}(\boldsymbol\theta)) + \frac{1}{k} C_k(J_0; f)$$
$$(\mathbf{I}_m \otimes \boldsymbol\Sigma^{-1}) \left(\sum_{j=0}^{\infty} (\mathbf{K}' \, \mathbf{R}_{|i-j|} \, \mathbf{K}) \otimes \mathbf{h}_j \right) \left(\operatorname{vec} \boldsymbol\eta^{(n)'} \right) = o_{\mathrm{P}}(1), \qquad (29)$$

and

$$(n-i)^{1/2} \operatorname{vec}(\underset{\sim}{\boldsymbol\Gamma}_{i;J}^{(n)}(\boldsymbol\theta + \boldsymbol\nu(n)\boldsymbol\tau^{(n)}) - \underset{\sim}{\boldsymbol\Gamma}_{i;J}^{(n)}(\boldsymbol\theta)) + \frac{1}{k^2} D_k(J_2; f)\, C_k(J_1; f)$$
$$(\boldsymbol\Sigma \otimes \boldsymbol\Sigma^{-1}) \left[\mathbf{a}_i(\boldsymbol\tau^{(n)}; \boldsymbol\theta) + \mathbf{b}_i(\boldsymbol\tau^{(n)}; \boldsymbol\theta) \right] = o_{\mathrm{P}}(1), \qquad (30)$$

as $n \to \infty$, under $\mathcal{H}^{(n)}(\boldsymbol\theta, \boldsymbol\Sigma, f)$.

Proof of Lemma 3. Let us first prove the first statement in Lemma 3. Clearly,

$$\begin{aligned} n^{1/2}(\underset{\sim}{\mathbf{T}}_{I;J}^{(n)}(\hat{\boldsymbol\theta}) - \underset{\sim}{\mathbf{T}}_{I;J}^{(n)}(\boldsymbol\theta)) &= \mathbf{L}_{\hat{\boldsymbol\theta}}^{(n)'} \underset{\sim}{\mathbf{S}}_{I;J}^{(n)}(\hat{\boldsymbol\theta}) - \mathbf{L}_{\boldsymbol\theta}^{(n)'} \underset{\sim}{\mathbf{S}}_{I;J}^{(n)}(\boldsymbol\theta) \\ &= \sum_{i=0}^{n-1} (n-i)^{1/2} \left[(\mathbf{I}_m \otimes \hat{\mathbf{h}}_i') \operatorname{vec} \underset{\sim}{\boldsymbol\Lambda}_{i;J}^{(n)}(\hat{\boldsymbol\theta}) - (\mathbf{I}_m \otimes \mathbf{h}_i') \operatorname{vec} \underset{\sim}{\boldsymbol\Lambda}_{i;J}^{(n)}(\boldsymbol\theta) \right]. \end{aligned}$$

Now, for some fixed integer s (and $n > s+1$),

$$n^{1/2}(\underset{\sim}{\mathbf{T}}^{(n)}_{I;J}(\hat{\boldsymbol{\theta}}) - \underset{\sim}{\mathbf{T}}^{(n)}_{I;J}(\boldsymbol{\theta})) = \sum_{i=0}^{s}(n-i)^{1/2}\left[\left(\mathbf{I}_m \otimes (\hat{\mathbf{h}}_i^{'} - \mathbf{h}_i^{'})\right) \text{ vec } \underset{\sim}{\mathbf{\Lambda}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}})\right] \tag{31}$$

$$+\sum_{i=0}^{s}\left[(\mathbf{I}_m \otimes \mathbf{h}_i^{'})\ \left((n-i)^{1/2}\text{vec}\,(\underset{\sim}{\mathbf{\Lambda}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}}) - \underset{\sim}{\mathbf{\Lambda}}^{(n)}_{i;J}(\boldsymbol{\theta}))\right)\right]$$

$$+\sum_{i=s+1}^{n-1}(n-i)^{1/2}\left[(\mathbf{I}_m \otimes \hat{\mathbf{h}}_i^{'}) \text{ vec } \underset{\sim}{\mathbf{\Lambda}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}}) - (\mathbf{I}_m \otimes \mathbf{h}_i^{'}) \text{ vec } \underset{\sim}{\mathbf{\Lambda}}^{(n)}_{i;J}(\boldsymbol{\theta})\right].$$

Next, the local discreteness of $\hat{\boldsymbol{\theta}}^{(n)}$ (see Assumption (D1)(iii)) allows to replace $\boldsymbol{\theta}^{(n)} = \boldsymbol{\theta}+\boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)}$ with $\hat{\boldsymbol{\theta}}^{(n)}$ in (29) (see Kreiss 1987, Lemma 4.4). Since $\boldsymbol{\beta}^{(n)} = \boldsymbol{\beta} + n^{-1/2}\mathbf{K}^{(n)}\boldsymbol{\eta}^{(n)}$ can be written under the form $n^{1/2}\text{vec}\,(\boldsymbol{\beta}^{(n)'} - \boldsymbol{\beta}^{'}) = (\mathbf{K}^{(n)} \otimes \mathbf{I}_k)(\text{vec}\,\boldsymbol{\eta}^{(n)'})$, this yields

$$(n-i)^{1/2} \text{ vec } \left(\underset{\sim}{\mathbf{\Lambda}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}}^{(n)}) - \underset{\sim}{\mathbf{\Lambda}}^{(n)}_{i;J}(\boldsymbol{\theta})\right) = -\frac{1}{k}\,C_k(J_0;f) \tag{32}$$

$$(\mathbf{I}_m \otimes \mathbf{\Sigma}^{-1})\left(\sum_{j=0}^{\infty}(\mathbf{K}^{'}\,\mathbf{R}_{|i-j|}\,\mathbf{K}) \otimes \mathbf{h}_j\right)(\mathbf{K}^{(n)} \otimes \mathbf{I}_k)^{-1}n^{1/2}(\hat{\boldsymbol{\theta}}^{(n)}_I - \boldsymbol{\theta}_I) + \mathbf{R}^{(n)}_i,$$

where $\mathbf{R}^{(n)}_i$ is $o_{\mathrm{P}}(1)$ as $n \to \infty$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$. Substituting in (31), we obtain

$$n^{1/2}(\underset{\sim}{\mathbf{T}}^{(n)}_{I;J}(\hat{\boldsymbol{\theta}}) - \underset{\sim}{\mathbf{T}}^{(n)}_{I;J}(\boldsymbol{\theta})) = \sum_{i=0}^{s}(n-i)^{1/2}\left[\left(\mathbf{I}_m \otimes (\hat{\mathbf{h}}_i - \mathbf{h}_i)^{'}\right) \text{ vec } \underset{\sim}{\mathbf{\Lambda}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}})\right]$$

$$-\frac{1}{k}\,C_k(J_0;f)\sum_{i=0}^{s}\left[(\mathbf{I}_m \otimes \mathbf{h}_i^{'}\mathbf{\Sigma}^{-1})\left(\sum_{j=0}^{\infty}(\mathbf{K}^{'}\,\mathbf{R}_{|i-j|}\,\mathbf{K}) \otimes \mathbf{h}_j\right)(\mathbf{K}^{(n)} \otimes \mathbf{I}_k)^{-1}n^{1/2}(\hat{\boldsymbol{\theta}}^{(n)}_I - \boldsymbol{\theta}_I)\right]$$

$$+\sum_{i=s+1}^{n-1}(n-i)^{1/2}\left[(\mathbf{I}_m \otimes \hat{\mathbf{h}}_i^{'}) \text{ vec } \underset{\sim}{\mathbf{\Lambda}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}}) - (\mathbf{I}_m \otimes \mathbf{h}_i^{'}) \text{ vec } \underset{\sim}{\mathbf{\Lambda}}^{(n)}_{i;J}(\boldsymbol{\theta})\right] + \sum_{i=0}^{s}\mathbf{R}^{(n)}_i.$$

Finally, this yields the decomposition

$$n^{1/2}(\underset{\sim}{\mathbf{T}}^{(n)}_{I;J}(\hat{\boldsymbol{\theta}}) - \underset{\sim}{\mathbf{T}}^{(n)}_{I;J}(\boldsymbol{\theta})) + \frac{1}{k}\,C_k(J_0;f)$$

$$\times\Big[\sum_{i,j=0}^{\infty}(\mathbf{I}_m \otimes \mathbf{h}_i^{'})\left((\mathbf{K}^{'}\,\mathbf{R}_{|i-j|}\,\mathbf{K}) \otimes \mathbf{\Sigma}^{-1}\right)(\mathbf{I}_m \otimes \mathbf{h}_j)\Big](\mathbf{K}^{(n)} \otimes \mathbf{I}_k)^{-1}n^{1/2}(\hat{\boldsymbol{\theta}}^{(n)}_I - \boldsymbol{\theta}_I)$$

$$= \mathbf{T}^{(n,s)}_1 + \mathbf{T}^{(n,s)}_2,$$

where

$$\mathbf{T}^{(n,s)}_1 := \sum_{i=0}^{s}(n-i)^{1/2}\left[\left(\mathbf{I}_m \otimes (\hat{\mathbf{h}}_i - \mathbf{h}_i)^{'}\right) \text{ vec } \underset{\sim}{\mathbf{\Lambda}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}})\right] + \sum_{i=0}^{s}\mathbf{R}^{(n)}_i,$$

and

$$\mathbf{T}^{(n,s)}_2 := \sum_{i=s+1}^{n-1}(n-i)^{1/2}\left[(\mathbf{I}_m \otimes \hat{\mathbf{h}}_i^{'}) \text{ vec } \underset{\sim}{\mathbf{\Lambda}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}}) - (\mathbf{I}_m \otimes \mathbf{h}_i^{'}) \text{ vec } \underset{\sim}{\mathbf{\Lambda}}^{(n)}_{i;J}(\boldsymbol{\theta})\right] +$$

$$\frac{1}{k}\,C_k(J_0;f)\sum_{i=s+1}^{\infty}\left[(\mathbf{I}_m \otimes \mathbf{h}_i^{'}\mathbf{\Sigma}^{-1})\left(\sum_{j=0}^{\infty}(\mathbf{K}^{'}\,\mathbf{R}_{|i-j|}\,\mathbf{K}) \otimes \mathbf{h}_j\right)\right](\mathbf{K}^{(n)} \otimes \mathbf{I}_k)^{-1}n^{1/2}(\hat{\boldsymbol{\theta}}^{(n)}_I - \boldsymbol{\theta}_I).$$

Since $\sum_{i,j=0}^{\infty} (\mathbf{I}_m \otimes \mathbf{h}_i^{'}) \left((\mathbf{K}^{'}\, \mathbf{R}_{|i-j|}\, \mathbf{K}) \otimes \mathbf{\Sigma}^{-1} \right) (\mathbf{I}_m \otimes \mathbf{h}_j) \; = \mathbf{J}_{I;\boldsymbol{\theta},\mathbf{\Sigma}}$, the first statement in Lemma 3 takes the form

$$n^{1/2}(\underset{\sim}{\mathbf{T}}_{I;J}^{(n)}(\hat{\boldsymbol{\theta}}) - \underset{\sim}{\mathbf{T}}_{I;J}^{(n)}(\boldsymbol{\theta})) + \frac{1}{k} C_k(J_0; f)$$
$$\Big[\sum_{i,j=0}^{\infty} (\mathbf{I}_m \otimes \mathbf{h}_i^{'}) \left((\mathbf{K}^{'}\, \mathbf{R}_{|i-j|}\, \mathbf{K}) \otimes \mathbf{\Sigma}^{-1} \right) (\mathbf{I}_m \otimes \mathbf{h}_j) \Big] (\mathbf{K}^{(n)} \otimes \mathbf{I}_k)^{-1} n^{1/2} (\hat{\boldsymbol{\theta}}_I^{(n)} - \boldsymbol{\theta}_I) = o_{\mathrm{P}}(1)$$

as $n \to \infty$ under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$. Now, it follows, from the continuity of $\boldsymbol{\theta} \mapsto \mathbf{h}_i(\boldsymbol{\theta})$ and the boundedness (in probability, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$; see (32)) of $(n-i)^{1/2}\mathrm{vec}\, \underset{\sim}{\mathbf{\Lambda}}_{i;J}^{(n)}(\hat{\boldsymbol{\theta}})$, that $\mathbf{T}_1^{(n,s)}$ is $o_{\mathrm{P}}(1)$ under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$, for any fixed s, as $n \to \infty$. On the other hand, the exponential decrease in i of the $\mathbf{h}_i$'s and the root-n consistency of $\hat{\boldsymbol{\theta}}$ imply that $\mathbf{T}_2^{(n,s)}$ is $o_{\mathrm{P}}(1)$ under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$, as $s \to \infty$, uniformly in n.

Now, $P\left[\|\mathbf{T}_1^{(n,s)} + \mathbf{T}_2^{(n,s)}\| > \delta \right] \leq P\left[\|\mathbf{T}_1^{(n,s)}\| > \delta/2 \right] + P\left[\|\mathbf{T}_2^{(n,s)}\| > \delta/2 \right]$, for all s and n. For any $\varepsilon > 0$, one can always choose $s = S$ sufficiently large so that $P\left[\|\mathbf{T}_2^{(n,S)}\| > \delta/2 \right] < \varepsilon$ uniformly in n. Since $\mathbf{T}_1^{(n,S)}$ is $o_P(1)$ as $n \to \infty$, it is possible to find a integer $N = N(\varepsilon)$ such that $P\left[\|\mathbf{T}_1^{(n,S)}\| > \delta/2 \right] < \varepsilon$ for all $n \geq N$. Consequently, for all $\varepsilon > 0$, $N = N(\varepsilon)$ is such that

$$P\Bigg[\Big\| n^{1/2}(\underset{\sim}{\mathbf{T}}_{I;J}^{(n)}(\hat{\boldsymbol{\theta}}) - \underset{\sim}{\mathbf{T}}_{I;J}^{(n)}(\boldsymbol{\theta})) + \frac{1}{k} C_k(J_0; f)$$
$$\Big[\sum_{i,j=0}^{\infty} (\mathbf{I}_m \otimes \mathbf{h}_i^{'}) \left((\mathbf{K}^{'}\, \mathbf{R}_{|i-j|}\, \mathbf{K}) \otimes \mathbf{\Sigma}^{-1} \right) (\mathbf{I}_m \otimes \mathbf{h}_j) \Big] (\mathbf{K}^{(n)} \otimes \mathbf{I}_k)^{-1} n^{1/2} (\hat{\boldsymbol{\theta}}_I^{(n)} - \boldsymbol{\theta}_I) \Big\| > \delta \Bigg] < 2\varepsilon$$

for all $n \geq N$. The result follows.

The proof of the serial part of Lemma 3 is quite similar. Denoting by $\mathbf{Q}_{i,j} = \mathbf{Q}_{i,j}^{(n)}$ (resp. $\hat{\mathbf{Q}}_{i,j} = \hat{\mathbf{Q}}_{i,j}^{(n)}$) the $k^2 \times k^2$ block in position (i, j) $(i = 1, \ldots, n-1,\ j = 1, \ldots, \pi_0)$ in $\mathbf{Q}_{\boldsymbol{\theta}}^{(n)}$ (resp. in $\mathbf{Q}_{\hat{\boldsymbol{\theta}}}^{(n)}$), we have

$$n^{1/2}(\underset{\sim}{\mathbf{T}}_{II;J}^{(n)}(\hat{\boldsymbol{\theta}}) - \underset{\sim}{\mathbf{T}}_{II;J}^{(n)}(\boldsymbol{\theta})) = \mathbf{Q}_{\hat{\boldsymbol{\theta}}}^{(n)'} \underset{\sim}{\mathbf{S}}_{II;J}^{(n)}(\hat{\boldsymbol{\theta}}) - \mathbf{Q}_{\boldsymbol{\theta}}^{(n)'} \underset{\sim}{\mathbf{S}}_{II;J}^{(n)}(\boldsymbol{\theta})$$
$$= \sum_{i=1}^{n-1} (n-i)^{1/2} \left[\begin{pmatrix} \hat{\mathbf{Q}}_{i,1}^{'} \\ \vdots \\ \hat{\mathbf{Q}}_{i,\pi_0}^{'} \end{pmatrix} \mathrm{vec}\, \underset{\sim}{\mathbf{\Gamma}}_{i;J}^{(n)}(\hat{\boldsymbol{\theta}}) - \begin{pmatrix} \mathbf{Q}_{i,1}^{'} \\ \vdots \\ \mathbf{Q}_{i,\pi_0}^{'} \end{pmatrix} \mathrm{vec}\, \underset{\sim}{\mathbf{\Gamma}}_{i;J}^{(n)}(\boldsymbol{\theta}) \right].$$

The same decomposition as for the trend part then yields, for some fixed integer s (and still for $n > s+1$),

$$n^{1/2}(\underset{\sim}{\mathbf{T}}^{(n)}_{II;J}(\hat{\boldsymbol{\theta}}) - \underset{\sim}{\mathbf{T}}^{(n)}_{II;J}(\boldsymbol{\theta})) = \sum_{i=1}^{s}(n-i)^{1/2}\left[\begin{pmatrix}\hat{\mathbf{Q}}'_{i,1}\\ \vdots \\ \hat{\mathbf{Q}}'_{i,\pi_0}\end{pmatrix} - \begin{pmatrix}\mathbf{Q}'_{i,1}\\ \vdots \\ \mathbf{Q}'_{i,\pi_0}\end{pmatrix}\right]\text{vec}\,\underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}}) \tag{33}$$

$$+\sum_{i=1}^{s}\left[\begin{pmatrix}\mathbf{Q}'_{i,1}\\ \vdots \\ \mathbf{Q}'_{i,\pi_0}\end{pmatrix}\Big((n-i)^{1/2}\text{vec}\,(\underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}}) - \underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i;J}(\boldsymbol{\theta}))\Big)\right]$$

$$+\sum_{i=s+1}^{n-1}(n-i)^{1/2}\left[\begin{pmatrix}\hat{\mathbf{Q}}'_{i,1}\\ \vdots \\ \hat{\mathbf{Q}}'_{i,\pi_0}\end{pmatrix}\text{vec}\,\underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}}) - \begin{pmatrix}\mathbf{Q}'_{i,1}\\ \vdots \\ \mathbf{Q}'_{i,\pi_0}\end{pmatrix}\text{vec}\,\underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i;J}(\boldsymbol{\theta})\right].$$

Again, the local discreteness of $\hat{\boldsymbol{\theta}}^{(n)}$ and (30) yield

$$(n-i)^{1/2}\,\text{vec}\,\Big(\underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}}^{(n)}) - \underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i;J}(\boldsymbol{\theta})\Big) \tag{34}$$

$$= -\frac{1}{k^2}\,D_k(J_2;f)\,C_k(J_1;f)\,(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}^{-1})\,\Big[\mathbf{a}_i(n^{1/2}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta});\boldsymbol{\theta}) + \mathbf{b}_i(n^{1/2}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta});\boldsymbol{\theta})\Big] + \mathbf{R}_i^{(n)},$$

$$= -\frac{1}{k^2}\,D_k(J_2;f)\,C_k(J_1;f)\,(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}^{-1})\,(\mathbf{Q}_{i,1}\ldots\mathbf{Q}_{i,\pi_0})\,\mathbf{P}_{\boldsymbol{\theta}}\,\mathbf{M}_{\boldsymbol{\theta}}\,n^{1/2}(\hat{\boldsymbol{\theta}}_{II} - \boldsymbol{\theta}_{II}) + \mathbf{R}_i^{(n)},$$

where $\mathbf{R}_i^{(n)}$ is $o_{\mathrm{P}}(1)$ (as $n\to\infty$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta},\boldsymbol{\Sigma},f)$), so that (33) becomes

$$n^{1/2}(\underset{\sim}{\mathbf{T}}^{(n)}_{II;J}(\hat{\boldsymbol{\theta}}) - \underset{\sim}{\mathbf{T}}^{(n)}_{II;J}(\boldsymbol{\theta})) = \sum_{i=1}^{s}(n-i)^{1/2}\left[\begin{pmatrix}\hat{\mathbf{Q}}'_{i,1}\\ \vdots \\ \hat{\mathbf{Q}}'_{i,\pi_0}\end{pmatrix} - \begin{pmatrix}\mathbf{Q}'_{i,1}\\ \vdots \\ \mathbf{Q}'_{i,\pi_0}\end{pmatrix}\right]\text{vec}\,\underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}})$$

$$-\frac{1}{k^2}\,D_k(J_2;f)\,C_k(J_1;f)\,\sum_{i=1}^{s}\left[\begin{pmatrix}\mathbf{Q}'_{i,1}\\ \vdots \\ \mathbf{Q}'_{i,\pi_0}\end{pmatrix}(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}^{-1})\,(\mathbf{Q}_{i,1}\ldots\mathbf{Q}_{i,\pi_0})\right]\mathbf{P}_{\boldsymbol{\theta}}\,\mathbf{M}_{\boldsymbol{\theta}}\,n^{1/2}(\hat{\boldsymbol{\theta}}^{(n)}_{II} - \boldsymbol{\theta}_{II})$$

$$+\sum_{i=s+1}^{n-1}(n-i)^{1/2}\left[\begin{pmatrix}\hat{\mathbf{Q}}'_{i,1}\\ \vdots \\ \hat{\mathbf{Q}}'_{i,\pi_0}\end{pmatrix}\text{vec}\,\underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}}) - \begin{pmatrix}\mathbf{Q}'_{i,1}\\ \vdots \\ \mathbf{Q}'_{i,\pi_0}\end{pmatrix}\text{vec}\,\underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i;J}(\boldsymbol{\theta})\right] + \sum_{i=1}^{s}\mathbf{R}_i^{(n)}.$$

Noting that

$$\sum_{i=1}^{s}\left[\begin{pmatrix}\mathbf{Q}'_{i,1}\\ \vdots \\ \mathbf{Q}'_{i,\pi_0}\end{pmatrix}(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}^{-1})\,(\mathbf{Q}_{i,1}\ldots\mathbf{Q}_{i,\pi_0})\right] = \mathbf{Q}_{\boldsymbol{\theta}}^{(s+1)'}\Big[\mathbf{I}_s\otimes(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}^{-1})\Big]\,\mathbf{Q}_{\boldsymbol{\theta}}^{(s+1)},$$

we finally decompose

$$n^{1/2}(\underset{\sim}{\mathbf{T}}^{(n)}_{II;J}(\hat{\boldsymbol{\theta}}) - \underset{\sim}{\mathbf{T}}^{(n)}_{II;J}(\boldsymbol{\theta})) + \frac{1}{k^2}\, D_k(J_2;f)\, C_k(J_1;f)\, \mathbf{J}_{II;\boldsymbol{\theta},\boldsymbol{\Sigma}}\, \mathbf{P}_{\boldsymbol{\theta}}\, \mathbf{M}_{\boldsymbol{\theta}}\, n^{1/2}(\hat{\boldsymbol{\theta}}^{(n)}_{II} - \boldsymbol{\theta}_{II})$$

into $\mathbf{T}_1^{(n,s)} + \mathbf{T}_2^{(n,s)}$, where

$$\mathbf{T}_1^{(n,s)} := \sum_{i=1}^{s} (n-i)^{1/2} \left[\begin{pmatrix} \hat{\mathbf{Q}}'_{i,1} \\ \vdots \\ \hat{\mathbf{Q}}'_{i,\pi_0} \end{pmatrix} - \begin{pmatrix} \mathbf{Q}'_{i,1} \\ \vdots \\ \mathbf{Q}'_{i,\pi_0} \end{pmatrix} \right] \mathrm{vec}\, \underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}}) + \sum_{i=1}^{s} \mathbf{R}_i^{(n)}$$

and

$$\mathbf{T}_2^{(n,s)} := \frac{1}{k^2}\, D_k(J_2;f)\, C_k(J_1;f)\, \left[\mathbf{J}_{II;\boldsymbol{\theta},\boldsymbol{\Sigma}} - \mathbf{Q}_{\boldsymbol{\theta}}^{(s+1)'} [\mathbf{I}_s \otimes (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1})]\, \mathbf{Q}_{\boldsymbol{\theta}}^{(s+1)} \right] \mathbf{P}_{\boldsymbol{\theta}}\, \mathbf{M}_{\boldsymbol{\theta}}$$

$$n^{1/2}(\hat{\boldsymbol{\theta}}^{(n)}_{II} - \boldsymbol{\theta}_{II}) + \sum_{i=s+1}^{n-1} (n-i)^{1/2} \left[\begin{pmatrix} \hat{\mathbf{Q}}'_{i,1} \\ \vdots \\ \hat{\mathbf{Q}}'_{i,\pi_0} \end{pmatrix} \mathrm{vec}\, \underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}}) - \begin{pmatrix} \mathbf{Q}'_{i,1} \\ \vdots \\ \mathbf{Q}'_{i,\pi_0} \end{pmatrix} \mathrm{vec}\, \underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i;J}(\boldsymbol{\theta}) \right].$$

As for the trend part, the continuity in $\boldsymbol{\theta}$ of the Green's matrices, the fact that $(n-i)^{1/2}\mathrm{vec}\, \underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i;J}(\hat{\boldsymbol{\theta}})$ is $O_{\mathrm{P}}(1)$ (as $n \to \infty$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$), and the root-$n$ consistency of $\hat{\boldsymbol{\theta}}$, allow to show that $\mathbf{T}_1^{(n,s)}$ vanishes in probability under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$, for fixed s, as $n \to \infty$, and that $\mathbf{T}_2^{(n,s)}$ is $o_{\mathrm{P}}(1)$ under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$, as $s \to \infty$, uniformly in n. The result follows. □

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