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Asymptotic Linearity of Serial and Nonserial Multivariate Signed Rank Statistics

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Abstract

Asymptotic linearity plays a key role in estimation and testing in the presence of nuisance parameters. This property is established, in the very general context of a multivariate general linear model with elliptical VARMA errors, for the serial and nonserial multivariate rank statistics considered in Hallin and Paindaveine (2002a and b) and Oja and Paindaveine (2002). These statistics, which are multivariate versions of classical signed rank statistics, involve *(i) multivariate signs* based either on (pseudo-)Mahalanobis residuals, or on a modified version (*absolute* interdirections) of Randles' s *interdirections*, and *(ii)* a concept of ranks based either on (pseudo-)Mahalanobis distances or on *lift-interdirections*.

1 Introduction.

1.1 Rank-based inference for multivariate observations.

Whereas the classical univariate theory of rank-based inference (rank tests and R-estimation) presents a pretty complete and coherent body of methods applicable to a variety of models, ranging from simple location and scale problems to general linear and time series models, the corresponding multivariate theory is much less systematic and elaborate. The reason for this relative underdevelopment certainly lies in the difficulty of defining an adequate multivariate concept of ranks. And indeed, except for the theory of componentwise ranks (Puri and Sen 1971), which suffers a severe lack of affine invariance, the results in the area are rather piecemeal, scattered, and incomplete.

Recently, however, an important activity has been developed in this area. Hettmansperger et al. (1994, 1997), Möttönen et al. (1995, 1997, 1998), Oja (1999), Ollila et al. (2001), and Visuri et al. (2002) are proposing estimation and testing methods based either on *spatial signs and ranks*, or on an affine-equivariant concept of signs and ranks related with the well-known Oja (1983) median. Randles (1989), Peters and Randles (1990), Randles and Peters (1991), Jan and Randles (1994), Randles and Um (1998) are proposing affine-invariant multivariate signed rank procedures based on Randles (1989)'s concept of *interdirections* (a multivariate

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sign concept) and the ranks of Mahalanobis distances. Their procedures require elliptically symmetric errors, while Oja's are valid under a more general assumption of central symmetry.

Invariance, in this strand of literature, is mainly considered in connection with robustness (as opposed to efficiency). Moreover, all methods are restricted to location and regression models with independent observations.

Inspired by Le Cam's asymptotic theory of statistical experiments, a different point of view is taken in a series of papers by Hallin and Paindaveine (2002a, b, d, and e) and Oja and Paindaveine (2002), where, based on the same concepts of multivariate signs and ranks as above, locally asymptotically optimal procedures are developed for a broader class of models, including multivariate time series ones.

These results however only address those testing problems for which exact residuals can be computed under the null hypothesis—essentially, thus, null hypotheses of the form $\boldsymbol{\theta} = \boldsymbol{\theta}_0$, under which the parameter of interest $\boldsymbol{\theta}$ is completely specified.

In practice, null hypotheses of interest seldom are of that type, and usually consist in imposing some limited number of constraints under which $\boldsymbol{\theta}$ still remains partially unspecified. The univariate literature on ranks then usually proposes tests based on the so-called *aligned ranks*, computed from estimated residuals. The key result in the study of the asymptotic behaviour of these aligned ranks is an *asymptotic linearity property* of the test statistics under consideration : see Jurečková (1969), van Eeden (1972), Heiler and Willers (1988), Koul (1992), Hallin and Puri (1994), and many others for univariate rank and signed rank results of the same type.

The purpose of this paper is to derive an asymptotic linearity property in the multivariate case, for the serial and nonserial statistics proposed in Hallin and Paindaveine (2002 a, b, d, and e) and Oja and Paindaveine (2002), in the very broad context of linear models with VARMA errors. The resulting multivariate aligned rank tests are studied in a companion paper (Hallin and Paindaveine 2002f).

2 Multivariate ranks, multivariate signs, and rank-based statistics.

2.1 Serial and nonserial statistics.

Let $\mathbf{Z} := (\mathbf{Z}_1, \ldots, \mathbf{Z}_n)$ be an n-tuple of i.i.d. k-variate random vectors. Denoting by $\boldsymbol{\Sigma}$ a symmetric positive definite $k \times k$ matrix (the *scatter matrix*), and by $f : \mathbb{R}_0^+ \to \mathbb{R}^+$ a nonnegative function (the *radial density*) such that $f > 0$ a.e. and $\int_0^\infty r^{k-1} f(r)\, dr < \infty$, we assume throughout that $\mathbf{Z}$ has an elliptical density. More precisely, we make the following assumption.

Assumption (A1). $\mathbf{Z}$ has an elliptical density, of the form $\prod_{t=1}^n \underline{f}(\mathbf{z}_t; \boldsymbol{\Sigma}, f)$, $(\mathbf{z}_1, \ldots, \mathbf{z}_n) \in \mathbb{R}^{nk}$, where

$$\underline{f}(\mathbf{z}; \boldsymbol{\Sigma}, f) := c_{k,f}\, (\det \boldsymbol{\Sigma})^{-1/2} f(\|\mathbf{z}\|_{\boldsymbol{\Sigma}}), \quad \mathbf{z} \in \mathbb{R}^k. \tag{1}$$

As usual, $\|\mathbf{z}\|_{\boldsymbol{\Sigma}} := (\mathbf{z}' \boldsymbol{\Sigma}^{-1} \mathbf{z})^{1/2}$ denotes the norm of $\mathbf{z}$ in the metric associated with $\boldsymbol{\Sigma}$. The constant $c_{k,f}$ is the normalization factor $(\omega_k\, \mu_{k-1;f})^{-1}$, where ω_k stands for the $(k-1)$-dimensional Lebesgue measure of the unit sphere $\mathcal{S}^{k-1} \subset \mathbb{R}^k$, and $\mu_{l;f} := \int_0^\infty r^l f(r)\, dr$.

Here and in the sequel, we write $\boldsymbol{\Sigma}^{-1/2}$ for the unique upper-triangular $k \times k$ array with positive diagonal elements satisfying $\boldsymbol{\Sigma}^{-1} = (\boldsymbol{\Sigma}^{-1/2})' \boldsymbol{\Sigma}^{-1/2}$. Each vector $\mathbf{Z}_t$ decomposes into $\mathbf{Z}_t = d_t(\boldsymbol{\Sigma})\, \boldsymbol{\Sigma}^{1/2} \mathbf{U}_t(\boldsymbol{\theta}, \boldsymbol{\Sigma})$, where $d_t(\boldsymbol{\Sigma}) := \|\mathbf{Z}_t\|_{\boldsymbol{\Sigma}}$, and $\mathbf{U}_t(\boldsymbol{\Sigma}) := \boldsymbol{\Sigma}^{-1/2} \mathbf{Z}_t / d_t(\boldsymbol{\Sigma})$. Note that

$\mathbf{U}_1(\mathbf{\Sigma}), \ldots, \mathbf{U}_n(\mathbf{\Sigma})$ are i.i.d., and uniformly distributed over $\mathcal{S}^{k-1}$, hence generalizing the traditional concept of signs : we henceforth call them *multivariate signs*. Similarly, $d_1(\mathbf{\Sigma}), \ldots, d_n(\mathbf{\Sigma})$ are i.i.d. with probability density function

$$\tilde{f}_k(r) := (\mu_{k-1;f})^{-1}\, r^{k-1} f(r)\, I_{[r>0]}, \quad r \in \mathbb{R}. \tag{2}$$

Denote by $\tilde{F}_k$ the corresponding distribution function.

The ULAN property of the multiresponse linear model with elliptical VARMA errors and the structure of the corresponding central sequence (see the Appendix) imply that all the relevant information (in this very general framework) about the serial component of the model is contained in generalized cross-covariance matrices of the form

$$\mathbf{\Gamma}^{(n)}_{i;\mathbf{\Sigma},K} := (n-i)^{-1}\mathbf{\Sigma}'^{-1/2}\left(\sum_{t=i+1}^{n} K_1(d_t(\mathbf{\Sigma}))K_2(d_{t-i}(\mathbf{\Sigma}))\ \mathbf{U}_t(\mathbf{\Sigma})\mathbf{U}'_{t-i}(\mathbf{\Sigma})\right)\mathbf{\Sigma}'^{1/2}, \quad i = 1, \ldots, n-1, \tag{3}$$

where K_1 and K_2 are adequate real-valued score functions. For the trend part of the model, this information is contained in nonserial statistics of the form

$$\mathbf{\Lambda}^{(n)}_{i;\mathbf{\Sigma},K} := (n-i)^{-1}\ \mathbf{\Sigma}'^{-1/2} \sum_{t=i+1}^{n} K_0(d_t(\mathbf{\Sigma}))\,\mathbf{U}_t(\mathbf{\Sigma})\,\mathbf{x}'_{t-i}\mathbf{K}^{(n)}, \quad i = 1, \ldots, n-1, \tag{4}$$

where K_0 again is an adequate score function, whereas $\mathbf{x}'_{t-i}\mathbf{K}^{(n)}$ are nonrandom weights related with the regression constants in the model; see Section 3.1 for details.

2.2 Pseudo-Mahalanobis signs and ranks.

Both the serial statistics in (3) and the nonserial ones in (4) are measurable with respect to

(a) the distances $d_t(\mathbf{\Sigma})$ between the *sphericized* vectors $\mathbf{\Sigma}^{-1/2}\mathbf{Z}_t$ and the origin in $\mathbb{R}^k$ which, under the assumptions made, are i.i.d. over the positive real line, so that their ranks have the same distribution-freeness and maximal invariance properties as those of the absolute values of any univariate symmetrically distributed univariate n-tuple, and

(b) the *multivariate signs* $\mathbf{U}_t(\mathbf{\Sigma}) := \mathbf{\Sigma}^{-1/2}\mathbf{Z}_t/d_t(\mathbf{\Sigma})$ which, under the same conditions, are uniformly distributed over the unit sphere.

These statistics however both involve the (generally unknown) scatter matrix $\mathbf{\Sigma}$. If finite second-order moments exist, a "natural" root-n consistent candidate for estimating $\mathbf{\Sigma}$ is the empirical covariance matrix $n^{-1}\sum_{t=1}^{n}\mathbf{Z}_t\mathbf{Z}'_t$. The robustness properties of empirical covariances however are rather poor, and finite second order moments need not exist. More generally, we thus assume the following.

Assumption (B1). A sequence $\widehat{\mathbf{\Sigma}}^{(n)}$ of estimators of $\mathbf{\Sigma}$ exists, such that

(i) $\sqrt{n}(\widehat{\mathbf{\Sigma}}^{(n)} - a\,\mathbf{\Sigma}) = O_{\mathrm{P}}(1)$ as $n \to \infty$ for some positive real a, and

(ii) $\widehat{\mathbf{\Sigma}}^{(n)}$ is invariant under permutations and reflections (with respect to the origin in $\mathbb{R}^k$) of the vectors $\mathbf{Z}_t$.

Assumption (B1) will be sufficient for the asymptotic linearity result in Section 4. However, the affine-equivariance of the proposed nonparametric versions of (3) and (4) requires the following equivariance assumption on $\widehat{\mathbf{\Sigma}}^{(n)}$.

ASSUMPTION (B2). The estimator $\widehat{\mathbf{\Sigma}} := \widehat{\mathbf{\Sigma}}^{(n)}$ is quasi-affine-equivariant, in the sense that, for all $k \times k$ full-rank matrix $\mathbf{M}$, $\widehat{\mathbf{\Sigma}}(\mathbf{M}) = d\,\mathbf{M}\widehat{\mathbf{\Sigma}}\mathbf{M}'$, where $\widehat{\mathbf{\Sigma}}(\mathbf{M})$ stands for the statistic $\widehat{\mathbf{\Sigma}}$ computed from the n-tuple $(\mathbf{M}\mathbf{Z}_1, \ldots, \mathbf{M}\mathbf{Z}_n)$, and d denotes some positive scalar that may depend on $\mathbf{M}$ and on the sample $(\mathbf{Z}_t,\, t = 1, \ldots, n)$, but not on t.

Note that, under Assumption (B2), the ranks $\widehat{R}_t$, $t = 1, \ldots, n$, of the pseudo-Mahalanobis distances $(\mathbf{Z}_t'\widehat{\mathbf{\Sigma}}^{-1}\mathbf{Z}_t)^{1/2}$, $t = 1, \ldots, n$, are strictly affine-invariant. Call these ranks the *pseudo-Mahalanobis ranks*; the corresponding multivariate signs $\mathbf{W}_t := \mathbf{U}_t(\widehat{\mathbf{\Sigma}})$ will be referred to as *pseudo-Mahalanobis signs*. The terminology *Mahalanobis signs* and *Mahalanobis ranks* will be used in case $\widehat{\mathbf{\Sigma}}$ is the classical covariance matrix.

Denoting, by $\widehat{\mathbf{\Sigma}}^{-1/2}(\mathbf{M})$ the statistic $\widehat{\mathbf{\Sigma}}^{-1/2}$ computed from the n-tuple $(\mathbf{M}\mathbf{Z}_1, \ldots, \mathbf{M}\mathbf{Z}_n)$, $\widehat{\mathbf{\Sigma}}^{-1/2}$ under Assumption (B2) enjoys the equivariance property

$$\widehat{\mathbf{\Sigma}}^{-1/2}(\mathbf{M}) = d^{-1/2}\mathbf{O}\widehat{\mathbf{\Sigma}}^{-1/2}\mathbf{M}^{-1}, \tag{5}$$

where $\mathbf{O}$ is some $k \times k$ orthogonal matrix.

For each $\mathbf{\Sigma}$ and n, the group of *continuous monotone radial transformations* $\mathcal{G}_{\mathbf{\Sigma}}^{(n)} = \{\mathcal{G}_g^{(n)}\}$, acting on $(\mathbb{R}^k)^n$ and characterized by

$$\mathcal{G}_g^{(n)}(\mathbf{Z}_1, \ldots, \mathbf{Z}_n) := \Big(g(d_1(\mathbf{\Sigma}))\,\mathbf{\Sigma}^{1/2}\mathbf{U}_1(\mathbf{\Sigma}), \ldots, g(d_n(\mathbf{\Sigma}))\,\mathbf{\Sigma}^{1/2}\mathbf{U}_n(\mathbf{\Sigma})\Big), \tag{6}$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous monotone increasing function such that $g(0) = 0$ and $\lim_{r\to\infty} g(r) = \infty$, is a generating group for the family of elliptical densities $\bigcup_f \left\{\prod_{t=1}^n \underline{f}(.\,; \mathbf{\Sigma}, f)\right\}$. Along with the signs $(\mathbf{U}_1(\mathbf{\Sigma}), \ldots, \mathbf{U}_n(\mathbf{\Sigma}))$, the ranks $(R_1^{(n)}(\mathbf{\Sigma}), \ldots, R_n^{(n)}(\mathbf{\Sigma}))$ of the distances $d_t^{(n)}(\mathbf{\Sigma})$ constitute a maximal invariant for that group $\mathcal{G}_{\mathbf{\Sigma}}^{(n)}$ of radial transformations. These *genuine* ranks cannot be computed from $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$. However, they can be consistently recovered by considering the pseudo-Mahalanobis ranks $\widehat{R}_t^{(n)}$, as shown by the following result (see Peters and Randles 1990 for a proof).

Lemma 1 *Assume that Assumptions (A1) and (B1) hold. Then, for all* t, $\left(\widehat{R}_t^{(n)} - R_t^{(n)}(\mathbf{\Sigma})\right)$ *is* $o_{\mathrm{P}}(n)$ *as* $n \to \infty$.

The pseudo-Mahalanobis signs $\mathbf{W}_t^{(n)} := \mathbf{U}_t(\widehat{\mathbf{\Sigma}}^{(n)})$ are obviously invariant under $\mathcal{G}_{\mathbf{\Sigma}}^{(n)}$, irrespective of the true value of $\mathbf{\Sigma}$. They also are affine-equivariant in the following sense : if $\mathbf{W}_t^{(n)}(\mathbf{M})$ denotes a sign computed from $(\mathbf{M}\mathbf{Z}_1, \ldots, \mathbf{M}\mathbf{Z}_n)$, then $\mathbf{W}_t^{(n)}(\mathbf{M}) = \mathbf{O}\mathbf{W}_t^{(n)}$, where $\mathbf{O}$ is the orthogonal matrix involved in (5). Finally, the following consistency result is proved in Hallin and Paindaveine (2002d).

Lemma 2 *Assume that Assumptions (A1) and (B1) hold. Then, for all* t, $\mathbf{W}_t^{(n)} = \mathbf{U}_t^{(n)}(\mathbf{\Sigma}) + O_{\mathrm{P}}(n^{-1/2})$ *as* $n \to \infty$.

For $k = 1$, pseudo-Mahalanobis ranks and pseudo-Mahalanobis signs reduce to the ranks of absolute values and traditional signs, respectively.

2.3 Hyperplane-based signs and ranks.

Pseudo-Mahalanobis signs and ranks were entirely based on an estimation of the underlying scatter matrix. A completely different approach can based on counts of hyperplanes, and leads to a modification of Randles' s interdirections (namely, the *absolute interdirections*) for multivariate signs, to Oja and Paindaveine (2002)'s concept of *lift interdirection ranks* for the ranks.

Writing $\mathcal{Q} := \{i_1, i_2, \ldots, i_{k-1}\}$ $(1 \leq i_1 < i_2 < \ldots < i_{k-1} \leq n)$ for an arbitrary ordered set of indices with size $(k-1)$, let $\mathbf{Z}_\mathcal{Q} := (\mathbf{Z}_{i_1}, \ldots, \mathbf{Z}_{i_{k-1}})$. Denote by $\mathbf{e}_\mathcal{Q}$ the vector whose components are the cofactors of the last column in the array $(\mathbf{Z}_\mathcal{Q} \vdots \mathbf{z})$. This vector $\mathbf{e}_\mathcal{Q}$ is orthogonal to the hyperplane spanned by the $k-1$ columns of $\mathbf{Z}_\mathcal{Q}$. Writing sign(.) for the sign function $x \mapsto \text{sign}(x) := I_{[x>0]} - I_{[x<0]}$, the quantity $\text{sign}(\mathbf{e}'_\mathcal{Q}\mathbf{z})$ provides a precise meaning for the statement "$\mathbf{z}$ lies *above*, *on*, or *below* the hyperplane with equation $\mathbf{e}'_\mathcal{Q}\mathbf{z} = 0$" (the ordering in $\mathcal{Q}$ determines what is meant by "above", as opposed to "below").

A hyperplane-based empirical angular distance between two vectors $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^k$ then can be defined as

$$c(\mathbf{v}, \mathbf{w}) := \frac{1}{2} \sum_\mathcal{Q} \{1 - \text{sign}(\mathbf{e}'_\mathcal{Q}\mathbf{v}) \ \text{sign}(\mathbf{e}'_\mathcal{Q}\mathbf{w})\}.$$

The statistics $q_{ij}^{(n)} := c(\mathbf{Z}_i, \mathbf{Z}_j)$ are the so-called Randles interdirections (see Randles 1989); $q_{ij}^{(n)}$ is—up to a small-sample correction—the number of hyperplanes in $\mathbb{R}^k$ passing through the origin and $(k-1)$ out of the $(n-2)$ points $\mathbf{Z}_1, \ldots, \mathbf{Z}_{i-1}, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{j-1}, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_n$ that separate $\mathbf{Z}_i$ and $\mathbf{Z}_j$. Interdirections provide affine-invariant estimations of the Euclidean angles between the sphericized vectors $\boldsymbol{\Sigma}^{-1/2}\mathbf{Z}_i$, that is, they estimate the scalar products between the corresponding spatial signs $\mathbf{U}_i(\boldsymbol{\Sigma})$ defined in Section 2.1. More precisely, one can show the following (see Hallin and Paindaveine (2002a) for a proof based on U-statistics).

Lemma 3 *Assume that Assumption (A1) holds. Then,* $\binom{n}{k-1}^{-1} c^{(n)}(\mathbf{v}, \mathbf{w})$ *is an unbiased and consistent estimator for*

$$\pi^{-1} \arccos \left(\left(\frac{\boldsymbol{\Sigma}^{-1/2}\mathbf{v}}{\|\boldsymbol{\Sigma}^{-1/2}\mathbf{v}\|} \right)' \left(\frac{\boldsymbol{\Sigma}^{-1/2}\mathbf{w}}{\|\boldsymbol{\Sigma}^{-1/2}\mathbf{w}\|} \right) \right).$$

Lemma 3 implies that Randles' interdirections allow for an estimation of the cosines $\mathbf{U}'_i(\boldsymbol{\Sigma})\mathbf{U}_j(\boldsymbol{\Sigma})$. These cosines (the signs $\mathbf{U}_i(\boldsymbol{\Sigma})$ themselves are not required) are sufficient in some important particular cases (such as one-way analysis of variance), since the parametric versions of locally asymptotically optimal test statistics involve the $\mathbf{U}_i(\boldsymbol{\Sigma})$'s only through their mutual cosines. In such cases, Randles' interdirections can be used with the same success as in Hallin and Paindaveine (2002a) and (2002b), or Randles and Um (1998).

For more sophisticated testing problems however, such as the problem of testing for the adequacy of a VARMA model (see Hallin and Paindaveine 2002d), locally asymptotically optimal parametric procedures involve the $\mathbf{U}_i(\boldsymbol{\Sigma})$'s through quantities of the form $\mathbf{U}'_i(\boldsymbol{\Sigma})\mathbf{N}\mathbf{U}_j(\boldsymbol{\Sigma})$, where $\mathbf{N}$ is some symmetric positive definite matrix (which often depends on the shape matrix $\boldsymbol{\Sigma}$, and therefore has to be estimated). In such cases, Randles' interdirections are not sufficient anymore, as they cannot estimate the scalar products $(\mathbf{N}^{1/2}\mathbf{U}_i(\boldsymbol{\Sigma}))'(\mathbf{N}^{1/2}\mathbf{U}_j(\boldsymbol{\Sigma}))$. We therefore introduce the following concept of *absolute interdirections*. Denoting by $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ the canonical basis in $\mathbb{R}^k$, consider the interdirection $c_{i;l}^{(n)} := c(\widehat{\boldsymbol{\Sigma}}^{1/2}\mathbf{e}_l, \mathbf{Z}_i)$ associated with the pair $(\widehat{\boldsymbol{\Sigma}}^{1/2}\mathbf{e}_l, \mathbf{Z}_i)$ in the sample $(\mathbf{Z}_1, \ldots, \mathbf{Z}_n)$, and let $\mathbf{V}_i^{(n)} := (\cos(\pi p_{i;1}^{(n)}), \ldots, \cos(\pi p_{i;k}^{(n)}))'$, where $p_{i;l}^{(n)} := \binom{n}{k-1}^{-1} c_{i;l}^{(n)}$.

Call $\mathbf{V}_i^{(n)}$ the *absolute interdirection* associated with residual $\mathbf{Z}_i^{(n)}$. Absolute interdirections enjoy the following consistency and equivariance properties.

Lemma 4 (i) $|c(\mathbf{u},\mathbf{w}) - c(\mathbf{v},\mathbf{w})| \leq c(\mathbf{u},\mathbf{v})$, *for all* $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}^k$.
Assume that Assumptions (A1) and (B1) hold. Then,

(ii) $\binom{n}{k-1}^{-1} c(\widehat{\mathbf{\Sigma}}^{1/2}\mathbf{v},\mathbf{w}) = \binom{n}{k-1}^{-1} c(\mathbf{\Sigma}^{1/2}\mathbf{v},\mathbf{w}) + o_{L_1}(1)$, *as* $n \to \infty$, *for all* $\mathbf{v},\mathbf{w} \in \mathbb{R}^k$, *and*

(iii) $\mathbf{V}_i^{(n)} = \mathbf{U}_i^{(n)}(\mathbf{\Sigma}) + o_{\mathrm{P}}(1)$, *as* $n \to \infty$.

Assume moreover that Assumption (B2) holds. Then, denoting by $\mathbf{V}_i(\mathbf{M})$ *the statistic* $\mathbf{V}_i$ *computed from the* n*-tuple* $(\mathbf{M}\mathbf{Z}_1, \ldots, \mathbf{M}\mathbf{Z}_n)$, *where* $\mathbf{M}$ *is a* $k \times k$ *full-rank matrix,*

(iv) $\mathbf{V}_i(\mathbf{M}) = \mathbf{O}\mathbf{U}_i(\mathbf{\Sigma}) + o_{\mathrm{P}}(1)$ *as* $n \to \infty$ *(so that* $\mathbf{V}_i^{(n)}(\mathbf{M}) = \mathbf{O}\mathbf{V}_i^{(n)} + o_{\mathrm{P}}(1)$ *as* $n \to \infty$*), where* $\mathbf{O}$ *is the orthogonal matrix involved in the equivariance relation (5).*

Proof. (i) Clearly,

$$\begin{aligned}
|c(\mathbf{u},\mathbf{w}) - c(\mathbf{v},\mathbf{w})| &= (1/2)\Big|\sum_{\mathcal{Q}}[\mathrm{sign}(\mathbf{e}_{\mathcal{Q}}{}'\mathbf{u}) - \mathrm{sign}(\mathbf{e}_{\mathcal{Q}}{}'\mathbf{v})]\,\mathrm{sign}(\mathbf{e}_{\mathcal{Q}}{}'\mathbf{w})\Big| \\
&\leq (1/2)\sum_{\mathcal{Q}}|\mathrm{sign}(\mathbf{e}_{\mathcal{Q}}{}'\mathbf{u}) - \mathrm{sign}(\mathbf{e}_{\mathcal{Q}}{}'\mathbf{v})| \\
&= (1/2)\sum_{\mathcal{Q}}(1 - \mathrm{sign}(\mathbf{e}_{\mathcal{Q}}{}'\mathbf{u})\mathrm{sign}(\mathbf{e}_{\mathcal{Q}}{}'\mathbf{v})) = c(\mathbf{u},\mathbf{v}).
\end{aligned}$$

(ii) In view of (i) and Lemma 3, we get

$$\begin{aligned}
\mathrm{E}\left[\binom{n}{k-1}^{-1}|c(\widehat{\mathbf{\Sigma}}^{1/2}\mathbf{v},\mathbf{w}) - c(\mathbf{\Sigma}^{1/2}\mathbf{v},\mathbf{w})|\,\Big|\,\widehat{\mathbf{\Sigma}}\right] &\leq \mathrm{E}\left[\binom{n}{k-1}^{-1}c(\widehat{\mathbf{\Sigma}}^{1/2}\mathbf{v},\mathbf{\Sigma}^{1/2}\mathbf{v})\,\Big|\,\widehat{\mathbf{\Sigma}}\right] \\
&= \pi^{-1}\arccos\left(\left(\frac{\mathbf{\Sigma}^{-1/2}\widehat{\mathbf{\Sigma}}^{1/2}\mathbf{v}}{\|\mathbf{\Sigma}^{-1/2}\widehat{\mathbf{\Sigma}}^{1/2}\mathbf{v}\|}\right)'\frac{\mathbf{v}}{\|\mathbf{v}\|}\right),
\end{aligned}$$

which vanishes in probability as $n \to \infty$. Since the corresponding conditional expectation is bounded, this convergence also holds in the L_1-sense, so that the unconditional expectation goes to zero, as $n \to \infty$.

(iii) The mean value theorem implies that

$$\begin{aligned}
\mathrm{E}\left[|\mathbf{e}_l'(\mathbf{V}_i^{(n)} - \mathbf{U}_i^{(n)}(\mathbf{\Sigma}))|\,\Big|\,\mathbf{Z}_i\right] &\leq \pi\mathrm{E}\left[|p_{i;l} - \pi^{-1}\arccos(\mathbf{e}_l'\mathbf{U}_i^{(n)}(\mathbf{\Sigma}))|\,\Big|\,\mathbf{Z}_i\right] \\
&\leq \pi\mathrm{E}\left[\binom{n}{k-1}^{-1}|c(\widehat{\mathbf{\Sigma}}^{1/2}\mathbf{e}_l,\mathbf{Z}_i) - c(\mathbf{\Sigma}^{1/2}\mathbf{e}_l,\mathbf{Z}_i)|\,\Big|\,\mathbf{Z}_i\right] \\
&\quad +\pi\mathrm{E}\left[|\binom{n}{k-1}^{-1}c(\mathbf{\Sigma}^{1/2}\mathbf{e}_l,\mathbf{Z}_i) - \pi^{-1}\arccos(\mathbf{e}_l'\mathbf{U}_i^{(n)}(\mathbf{\Sigma}))|\,\Big|\,\mathbf{Z}_i\right].
\end{aligned}$$

The result then follows from (ii) and from Lemma 3.

(iv) Denote by $c(\mathbf{v},\mathbf{w}) = c(\mathbf{v},\mathbf{w};\mathbf{Z})$ (resp. by $c(\mathbf{v},\mathbf{w};\mathbf{MZ})$) the interdirection associated with $(\mathbf{v},\mathbf{w})$ in the sample $\mathbf{Z}_1,\ldots,\mathbf{Z}_n$ (resp. in the sample $\mathbf{MZ}_1,\ldots,\mathbf{MZ}_n$). Then

$$\begin{aligned}\binom{n}{k-1}^{-1} c\Big(\widehat{\boldsymbol{\Sigma}}^{1/2}(\mathbf{M})\mathbf{e}_l, \mathbf{MZ}_i;\mathbf{MZ}\Big) &= \binom{n}{k-1}^{-1} c\Big(d^{1/2}\mathbf{M}\,\widehat{\boldsymbol{\Sigma}}^{1/2}\,\mathbf{O}'\mathbf{e}_l, \mathbf{MZ}_i;\mathbf{MZ}\Big)\\ &= \binom{n}{k-1}^{-1} c\Big(\widehat{\boldsymbol{\Sigma}}^{1/2}\,\mathbf{O}'\mathbf{e}_l, \mathbf{Z}_i;\mathbf{Z}\Big),\end{aligned}$$

so that (working as in (iii)), we obtain

$$\mathbf{e}_l'\mathbf{V}_i^{(n)}(\mathbf{M}) = \cos(\pi p_{i;l}^{(n)}(\mathbf{M})) = (\mathbf{O}'\mathbf{e}_l)'\mathbf{U}_i^{(n)}(\boldsymbol{\Sigma}) + o_{\mathrm{P}}(1),$$

as $n\to\infty$. □

Lemma 4 shows that absolute interdirections allow to estimate any function of the standardized residuals $\mathbf{U}_i$, and in particular quantities of the form $\mathbf{U}_i'\mathbf{N}\mathbf{U}_j$. In case $\boldsymbol{\Sigma}\mapsto\mathbf{N}(\boldsymbol{\Sigma})$ is continuous, and provided that $\mathbf{N}(a\boldsymbol{\Sigma}) = \mathbf{N}(\boldsymbol{\Sigma})$ for any $a\in\mathbb{R}^+$, the estimator $\widehat{\boldsymbol{\Sigma}}^{(n)}$ can be plugged in without affecting asymptotic results. Note that, unlike (pseudo-)Mahalanobis signs $\mathbf{W}_t := \widehat{\boldsymbol{\Sigma}}^{-1/2}\mathbf{Z}_t/\|\widehat{\boldsymbol{\Sigma}}^{-1/2}\mathbf{Z}_t\|$, absolute interdirections are only *asymptotically* affine-equivariant.

We now consider hyperplane-based ranks. Write $\mathcal{P} := \{j_1, j_2,\ldots,j_k\}$ $(1\le j_1<j_2<\ldots<j_k\le n)$ for an arbitrary ordered set of indices with size k. Denote by $(d_{0\mathcal{P}},\mathbf{d}_{\mathcal{P}}')'$ the vector whose components are the cofactors of the last column in the array

$$\begin{pmatrix} 1 & 1 & \ldots & 1 & 1\\ \mathbf{Z}_{j_1} & \mathbf{Z}_{j_2} & \ldots & \mathbf{Z}_{j_k} & \mathbf{z}\end{pmatrix}.$$

The vector $\mathbf{d}_{\mathcal{P}}$ is orthogonal to the hyperplane going through $\mathbf{Z}_{j_1},\ldots,\mathbf{Z}_{j_k}$, hence has equation $d_{0\mathcal{P}}+\mathbf{d}_{\mathcal{P}}'\mathbf{z}=0$. Again, the sign of $d_{0\mathcal{P}}+\mathbf{d}_{\mathcal{P}}'\mathbf{z}$ allows to determine on which side of that hyperplane the point z lies. A hyperplane-based empirical distance between some vector $\mathbf{v}$ and the origin in $\mathbb{R}^k$ then can be defined as

$$l^{(n)}(\mathbf{v}) := \sum_{\mathcal{P}} \frac{1-\mathrm{sign}(d_{0\mathcal{P}}+\mathbf{d}_{\mathcal{P}}'\mathbf{v})\,\mathrm{sign}(d_{0\mathcal{P}}-\mathbf{d}_{\mathcal{P}}'\mathbf{v})}{2},$$

i.e., as the number of hyperplanes in $\mathbb{R}^k$ passing through k out of the n points $\mathbf{Z}_1,\ldots,\mathbf{Z}_n$, that are separating $\mathbf{v}$ and its reflection $-\mathbf{v}$. This concept of distance from the origin introduced by Oja and Paindaveine (2002) however suffers a lack of symmetry; that is why they rather considered the symmetrized distances

$$\underline{l}^{(n)}(\mathbf{v}) := \sum_{\mathcal{P}}\sum_{\mathbf{s}} \frac{1-\mathrm{sign}(d_{0\mathcal{P}}(\mathbf{s})+\mathbf{d}_{\mathcal{P}}(\mathbf{s})'\mathbf{v})\,\mathrm{sign}(d_{0\mathcal{P}}(\mathbf{s})-\mathbf{d}_{\mathcal{P}}(\mathbf{s})'\mathbf{v})}{2},$$

where, for some $\mathcal{P}=(j_1,\ldots,j_k)$ and some $\mathbf{s}\in\{-1,1\}^k$, $(d_{0\mathcal{P}}(\mathbf{s}),\mathbf{d}_{\mathcal{P}}(\mathbf{s})')'$ stands for the vector of cofactors associated with the last column in the array

$$\begin{pmatrix} 1 & 1 & \ldots & 1 & 1\\ s_1\mathbf{Z}_{j_1} & s_2\mathbf{Z}_{j_2} & \ldots & s_k\mathbf{Z}_{j_k} & \mathbf{z}\end{pmatrix}.$$

The resulting (symmetrized) *lift-interdirections* $\underline{l}_i^{(n)} := \underline{l}^{(n)}(\mathbf{Z}_i)$, $i=1,\ldots,n$, are invariant under reflections (w.r.t. the origin in $\mathbb{R}^k$) of the $\mathbf{Z}_i$'s. As shown by the following result, their ranks $\underline{R}_i^{(n)}$ are asymptotically equivalent to the ranks of the *genuine* distances $d_i(\boldsymbol{\Sigma})$.

Lemma 5 *Assume that Assumptions (A1) holds. Then, for all* t, $\left(\underline{R}_t^{(n)} - R_t^{(n)}(\mathbf{\Sigma})\right)$ *is* $o_{\mathrm{P}}(n)$ *as* $n \to \infty$.

This asymptotic equivalence result between the true ranks and the ranks of (symmetrized) lift-interdirections (along with the invariance of the latter under permutations and reflections of the observations) allows for building multivariate signed-rank procedures based on interdirections and the ranks of lift-interdirections for a broad class of location and serial problems (see Oja and Paindaveine 2002).

2.4 Serial and nonserial multivariate signed rank statistics.

Several rank-based versions of the serial and nonserial statistics (3) and (4) will be considered in the sequel, each of them based on the combination of a concept of multivariate signs (either Mahalanobis signs, pseudo-Mahalanobis signs, or absolute interdirections) with a concept of multivariate ranks (Mahalanobis, pseudo-Mahalanobis, or lift-interdirection ranks).

The versions based on Mahalanobis or pseudo-Mahalanobis signs and ranks are, in the serial case,

$$\underset{\sim}{\mathbf{\Gamma}}{}_{i;J}^{(n)} := \widehat{\mathbf{\Sigma}}^{\prime -1/2} \left(\frac{1}{n-i} \sum_{t=i+1}^{n} J_1\Big(\frac{\widehat{R}_t^{(n)}}{n+1}\Big) J_2\Big(\frac{\widehat{R}_{t-i}^{(n)}}{n+1}\Big) \, \mathbf{W}_t^{(n)} \, \mathbf{W}_{t-i}^{(n)\prime} \right) \widehat{\mathbf{\Sigma}}^{\prime 1/2}, \tag{7}$$

and, in the nonserial case,

$$\underset{\sim}{\mathbf{\Lambda}}{}_{i;J}^{(n)} := (n-i)^{-1} \, \widehat{\mathbf{\Sigma}}^{\prime -1/2} \sum_{t=i+1}^{n} J_0\Big(\frac{\widehat{R}_t^{(n)}}{n+1}\Big) \, \mathbf{W}_t^{(n)} \, \mathbf{x}_{t-i}^{(n)\prime} \mathbf{K}^{(n)}. \tag{8}$$

These versions will serve as reference versions, in the sense that, in order to avoid unnecessary additional notation, asymptotic linearity will be stated formally for (7) and (8) only (part (i) of Proposition 2), then extended (part (ii) of the same proposition) to the other versions (based on the other concepts of signs and ranks). Note that, contrary to the score functions K_0, K_1, and K_2 appearing in (3) and (4), the score functions J_0, J_1, and J_2 in (7) and (8) are defined over the open unit interval $]0,1[$. The relation between the J scores and the K scores (which depends on the underlying density) will be clarified in the asymptotic representation results of Proposition 1.

These asymptotic representation results require some technical assumptions on the score functions J_0, J_1, and J_2. More precisely, we will assume the following.

ASSUMPTION (C). The score functions $J_\ell :]0,1[\to \mathbb{R}$, $\ell = 0,1,2$, are continuous differences of two monotone increasing functions, and satisfy $\int_0^1 [J_\ell(u)]^2 \, du < \infty$ ($\ell = 0,1,2$).

We now can state the asymptotic representation results for $\underset{\sim}{\mathbf{\Gamma}}{}_{i;J}^{(n)}$ and $\underset{\sim}{\mathbf{\Lambda}}{}_{i;J}^{(n)}$. Letting

$$\widetilde{\mathbf{\Gamma}}_{i;J;\mathbf{\Sigma},f}^{(n)} := \mathbf{\Sigma}^{\prime -1/2} \left(\frac{1}{n-i} \sum_{t=i+1}^{n} J_1(\tilde{F}_k(d_t^{(n)}(\mathbf{\Sigma}))) J_2(\tilde{F}_k(d_{t-i}^{(n)}(\mathbf{\Sigma}))) \, \mathbf{U}_t^{(n)}(\mathbf{\Sigma}) \mathbf{U}_{t-i}^{(n)\prime}(\mathbf{\Sigma}) \right) \mathbf{\Sigma}^{\prime 1/2}, \tag{9}$$

and

$$\widetilde{\mathbf{\Lambda}}_{i;J;\mathbf{\Sigma},f}^{(n)} := (n-i)^{-1} \, \mathbf{\Sigma}^{\prime -1/2} \sum_{t=i+1}^{n} J_0(\tilde{F}_k(d_t^{(n)}(\mathbf{\Sigma}))) \, \mathbf{U}_t^{(n)}(\mathbf{\Sigma}) \, \mathbf{x}_{t-i}^{(n)\prime} \mathbf{K}^{(n)}, \tag{10}$$

we have the following.

Proposition 1 *Assume that Assumptions (A1), (B1), and (C) hold. Then,*

(i) $\mathrm{vec}\,(\underset{\sim}{\mathbf{\Lambda}}{}^{(n)}_{i;J} - \widetilde{\mathbf{\Lambda}}^{(n)}_{i;J;\mathbf{\Sigma},f})$ *and* $\mathrm{vec}\,(\underset{\sim}{\mathbf{\Gamma}}{}^{(n)}_{i;J} - \widetilde{\mathbf{\Gamma}}^{(n)}_{i;J;\mathbf{\Sigma},f})$ *are* $o_{\mathrm{P}}(n^{-1/2})$ *for all* i*, as* $n \to \infty$*, and*

(ii) the same result still holds if the pseudo-Mahalanobis signs $\mathbf{W}^{(n)}_t$ *in* $\underset{\sim}{\mathbf{\Lambda}}{}^{(n)}_{i;J}$ *and* $\underset{\sim}{\mathbf{\Gamma}}{}^{(n)}_{i;J}$ *are replaced by the corresponding absolute interdirections, and/or if the pseudo-Mahalanobis ranks* $\widehat{R}^{(n)}_t$ *are replaced by the lift-interdirection ranks* $\underline{R}^{(n)}_t$*.*

Proof. (i) The result for the serial part is established in Proposition 2 of Hallin and Paindaveine (2002d), where, however, Tyler's estimator of scatter is used for $\widehat{\mathbf{\Sigma}}$; one can easily check that the same proof holds for any estimate satisfying Assumption (B1). The proof for the trend part follows along similar lines, and is left to the reader.

(ii) A closer look at the proof of (i) (see Proposition 2 of Hallin and Paindaveine (2002d)) shows that it only requires that the estimated ranks $\widehat{R}^{(n)}_t$ are

(a) invariant under permutations and reflections (with respect to the origin in $\mathbb{R}^k$) of the residuals, and

(b) asymptotically equivalent to the "true" ranks, meaning that, for all t,

$$\widehat{R}^{(n)}_t/(n+1) = \left[R^{(n)}_t(\mathbf{\Sigma})/(n+1)\right] + o_{\mathrm{P}}(1) \quad \text{as } n \to \infty.$$

Similarly, all estimators $\mathbf{W}^{(n)}_t$ (for the signs) that

(c) satisfy $\mathbf{W}^{(n)}_t(s_1\mathbf{Z}_1, \ldots, s_n\mathbf{Z}_n) = s_t\mathbf{W}^{(n)}_t(\mathbf{Z}_1, \ldots, \mathbf{Z}_n)$ for all $(s_1, \ldots, s_n) \in \{-1, 1\}^n$, and

(d) are asymptotically equivalent to the "true" signs, meaning that, for all t,

$$\mathbf{W}^{(n)}_t = \mathbf{U}^{(n)}_t(\mathbf{\Sigma}) + o_{\mathrm{P}}(1) \quad \text{as } n \to \infty,$$

successfully can be substituted for the pseudo-Mahalanobis signs in the proof of (i). This yields the desired result since, from Section 2.3, it is clear that lift-interdirection ranks and absolute interdirections do satisfy (a), (b), (c), and (d). □

3 The linear model with VARMA error terms.

3.1 The model.

Asymptotic linearity properties are characterizing the impact of a "small" perturbation of underlying parameters on the asymptotic behaviour of the statistics under study. Such properties thus are intimately related to some underlying model. The model considered throughout this paper is the very general multivariate linear model with VARMA error terms

$$\mathbf{Y}^{(n)} = \mathbf{X}^{(n)}\boldsymbol{\beta} + \mathbf{U}^{(n)}, \tag{11}$$

where

$$\mathbf{X}^{(n)} := \begin{pmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,m} \\ \vdots & \vdots & & \vdots \\ x_{n,1} & x_{n,2} & \ldots & x_{n,m} \end{pmatrix} := \begin{pmatrix} \mathbf{x}'_1 \\ \vdots \\ \mathbf{x}'_n \end{pmatrix} \quad \text{and} \quad \boldsymbol{\beta} := \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \ldots & \beta_{1,k} \\ \vdots & \vdots & & \vdots \\ \beta_{m,1} & \beta_{m,2} & \ldots & \beta_{m,k} \end{pmatrix}$$

denote an $n \times m$ matrix of constants (the design matrix), and $\boldsymbol{\beta}$ the $m \times k$ regression parameter, respectively. Instead of the traditional assumption that the error term

$$\mathbf{U}^{(n)} := \begin{pmatrix} U_{1,1} & U_{1,2} & \dots & U_{1,k} \\ \vdots & \vdots & & \vdots \\ U_{n,1} & U_{n,2} & \dots & U_{n,k} \end{pmatrix} := \begin{pmatrix} \mathbf{U}_1' \\ \vdots \\ \mathbf{U}_n' \end{pmatrix}$$

is white noise, we rather assume $\mathbf{U}_t,\ t = 1, \dots, n$ to be a finite realization (of length n) of a solution of the multivariate linear stochastic difference equation (a VARMA(p_0, q_0) model)

$$\mathbf{A}(L)\,\mathbf{U}_t = \mathbf{B}(L)\,\boldsymbol{\varepsilon}_t, \qquad t \in \mathbb{Z}, \tag{12}$$

where $\mathbf{A}(L) := \mathbf{I}_k - \sum_{i=1}^{p_0} \mathbf{A}_i L^i$ and $\mathbf{B}(L) := \mathbf{I}_k + \sum_{i=1}^{q_0} \mathbf{B}_i L^i$ for some $(p_0 + q_0)$-tuple of $k \times k$ real matrices $(\mathbf{A}_1, \dots, \mathbf{A}_{p_0}, \mathbf{B}_1, \dots, \mathbf{B}_{q_0})$, $\{\boldsymbol{\varepsilon}_t \,|\, t \in \mathbb{Z}\}$ is a k-dimensional white-noise process, and L stands for the lag operator. Under this model, the observation

$$\mathbf{Y}^{(n)} := \begin{pmatrix} Y_{1,1} & Y_{1,2} & \dots & Y_{1,k} \\ \vdots & \vdots & & \vdots \\ Y_{n,1} & Y_{n,2} & \dots & Y_{n,k} \end{pmatrix} := \begin{pmatrix} \mathbf{Y}_1' \\ \vdots \\ \mathbf{Y}_n' \end{pmatrix}$$

is the realization of a k-variate VARMA process $\{\mathbf{Y}_t,\ t \in \mathbb{Z}\}$ with trend $\boldsymbol{\beta}'\mathbf{x}_t$.

Of course, asymptotic linearity requires some regularity assumptions. These assumptions deal with the asymptotic behaviour of the design matrices $\mathbf{X}^{(n)}$, the coefficients and the (elliptical) innovation density of the VARMA model (12), and the score functions involved in the statistics under study. For convenient reference, all these assumptions are listed here.

Let us begin with some structural conditions on the trend part of the model. The following assumptions are standard in the context (see Garel and Hallin 1995).

Assumption (D1). Let $\mathbf{C}_i^{(n)} := (n-i)^{-1} \sum_{t=i+1}^{n} \mathbf{x}_t^{(n)} \mathbf{x}_{t-i}^{(n)'}$, $i = 0, 1, \dots, n-1$, and denote by $\mathbf{D}^{(n)}$ the diagonal matrix with elements $(\mathbf{C}_0^{(n)})_{11}, \dots, (\mathbf{C}_0^{(n)})_{mm}$.

(i) $(\mathbf{C}_0^{(n)})_{jj} > 0$ for all j.

(ii) Let $\mathbf{R}_i^{(n)} := (\mathbf{D}^{(n)})^{-1/2} \mathbf{C}_i^{(n)} (\mathbf{D}^{(n)})^{-1/2}$. The limits $\lim_{n\to\infty} \mathbf{R}_i^{(n)} =: \mathbf{R}_i$ exist for all i; $\mathbf{R}_0$ is positive definite, and therefore can be factorized into $\mathbf{R}_0 = (\mathbf{K}\,\mathbf{K}')^{-1}$ for some full-rank $m \times m$ matrix $\mathbf{K}$. Letting $\mathbf{K}^{(n)} := (\mathbf{D}^{(n)})^{-1/2}\mathbf{K}$, note that $\mathbf{K}^{(n)}$ is also of full rank.

(iii) The classical Noether conditions hold : the $(\mathbf{x}_t^{(n)})_j$, $t = 1, \dots, n$, are not all equal, and, letting $\bar{x}_j^{(n)} := n^{-1} \sum_{t=1}^{n} (\mathbf{x}_t^{(n)})_j$,

$$\lim_{n\to\infty} \frac{\max_{1\le t\le n} \left((\mathbf{x}_t^{(n)})_j - \bar{x}_j^{(n)} \right)^2}{\sum_{t=1}^{n} \left((\mathbf{x}_t^{(n)})_j - \bar{x}_j^{(n)} \right)^2} = 0, \quad j = 1, \dots, m.$$

Note that the Noether conditions also imply that

$$\lim_{n\to\infty} \frac{\max_{1\le t\le n} \left(\mathbf{x}_t^{(n)} \right)_j^2}{\sum_{t=1}^{n} \left(\mathbf{x}_t^{(n)} \right)_j^2} = 0, \quad j = 1, \dots, m. \tag{13}$$

For the serial part of the model, we essentially require the VARMA model (12) to be causal and invertible. The assumptions on the difference operators are actually the same as in Hallin and Paindaveine (2002d), where the problem of testing the adequacy of a specified VARMA model is considered.

ASSUMPTION (D2). All solutions of $\det(\mathbf{I}_k - \sum_{i=1}^{p_0} \mathbf{A}_i z^i) = 0$ and $\det(\mathbf{I}_k + \sum_{i=1}^{q_0} \mathbf{B}_i z^i) = 0$ ($|\mathbf{A}_{p_0}| \neq 0 \neq |\mathbf{B}_{q_0}|$) lie outside the unit ball in $\mathbb{C}$. Moreover, the greatest common left divisor of $\mathbf{I}_k - \sum_{i=1}^{p_0} \mathbf{A}_i z^i$ and $\mathbf{I}_k + \sum_{i=1}^{q_0} \mathbf{B}_i z^i$ is the identity matrix $\mathbf{I}_k$.

Under Assumption (D2), $\{\boldsymbol{\varepsilon}_t\}$ is $\{\mathbf{U}_t\}$'s (hence also $\{\mathbf{Y}_t\}$'s) innovation process. The set of assumptions (A) deals with the density of this innovation. For local asymptotic normality, the assumption of elliptical symmerty (Assumption (A1)) is to be reinforced into

ASSUMPTION (A1$'$). Same as Assumption (A1), but with $\mu_{k+1,f} < \infty$.

Moreover, $f^{1/2}$ is also required to satisfy a quadratic mean differentiability property:

ASSUMPTION (A2). The square root $f^{1/2}$ of the radial density f is in $W^{1,2}(\mathbb{R}_0^+, \mu_{k-1})$, where $W^{1,2}(\mathbb{R}_0^+, \mu_{k-1})$ denotes the subspace of $L^2(\mathbb{R}_0^+, \mu_{k-1})$ containing all functions admitting a weak derivative that also belongs to $L^2(\mathbb{R}_0^+, \mu_{k-1})$.

Assumption (A2) is strictly equivalent to the assumption that $f^{1/2}$ is differentiable in quadratic mean (see Hallin and Paindaveine 2002a). Denoting by $(f^{1/2})'$ the weak derivative of $f^{1/2}$ in $L^2(\mathbb{R}_0^+, \mu_{k-1})$, let $\varphi_f := -2\frac{(f^{1/2})'}{f^{1/2}}$. Under (A2), the *radial Fisher information* $\mathcal{I}_{k,f} := \int_0^\infty [\varphi_f(r)]^2 r^{k-1} f(r)\, dr$ is finite. In the pure location or purely serial problems considered in Hallin and Paindaveine (2002a, b, and d), this was sufficient for LAN. However, as pointed out by Garel and Hallin (1995), LAN, in this model where serial and nonserial features are mixed, requires the stronger assumption:

ASSUMPTION (A3). $\int_0^\infty [\varphi_f(r)]^4 r^{k-1} f(r)\, dr < \infty$.

Finally, the score functions yielding locally and asymptotically optimal procedures are of the form $J_0 = J_1 := \varphi_{f_\star} \circ \tilde{F}_{\star k}^{-1}$ and $J_2 := \tilde{F}_{\star k}^{-1}$, for some radial density $f_\star$ (with obvious notation $\varphi_{f_\star}$ and $\tilde{F}_{\star k}$). Assumption (C) then takes the form of an assumption on $f_\star$:

ASSUMPTION (C$'$). The radial density $f_\star$ is such that $\varphi_{f_\star}$ is the continuous difference of two monotone increasing functions, $\mu_{k+1;f_\star} < \infty$, and $\int_0^\infty [\varphi_{f_\star}(r)]^2 r^{k-1} f_\star(r)\, dr < \infty$.

3.2 Uniform local asymptotic normality (ULAN).

Under the assumptions made, the model described in Section 3.1 is uniformly asymptotically normal (ULAN : see Appendix). Letting $\mathbf{A}_i := \mathbf{0}$ for $p_0 < i \leq p_1$ and $\mathbf{B}_i := \mathbf{0}$ for $q_0 < i \leq q_1$, denote by

$$\boldsymbol{\theta} := \Big((\mathrm{vec}\, \boldsymbol{\beta}')', (\mathrm{vec}\, \mathbf{A}_1)', \ldots, (\mathrm{vec}\, \mathbf{A}_{p_1})', (\mathrm{vec}\, \mathbf{B}_1)', \ldots, (\mathrm{vec}\, \mathbf{B}_{q_1})' \Big)'$$

the vector of parameters indexing the model. The orders $p_1 \geq p_0$ and $q_1 \geq q_0$ are taken into account in order to allow for testing against higher order VARMA dependencies (namely, testing VARMA(p_0, q_0) against VARMA(p_1, q_1)). The hypothesis under which the observation has been generated by model (11)-(12) with parameter value $\boldsymbol{\theta}$, scatter matrix $\boldsymbol{\Sigma}$, and radial density f will be denoted as $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$.

The sequences of local alternatives to be considered for this property are associated with sequences of models of the form

$$\mathbf{Y}^{(n)} = \mathbf{X}^{(n)}\boldsymbol{\beta}^{(n)} + \mathbf{U}^{(n)}, \qquad \mathbf{A}^{(n)}(L)\,\mathbf{U}_t^{(n)} = \mathbf{B}^{(n)}(L)\,\boldsymbol{\varepsilon}_t^{(n)}, \quad t \in \mathbb{Z}, \tag{14}$$

where $\boldsymbol{\beta}^{(n)} = \boldsymbol{\beta} + n^{-1/2}\mathbf{K}^{(n)}\boldsymbol{\eta}^{(n)}$, $\mathbf{A}^{(n)}(L) := \mathbf{I}_k - \sum_{i=1}^{p_1}(\mathbf{A}_i + n^{-1/2}\boldsymbol{\gamma}_i^{(n)})\,L^i$ and $\mathbf{B}^{(n)}(L) := \mathbf{I}_k + \sum_{i=1}^{q_1}(\mathbf{B}_i + n^{-1/2}\boldsymbol{\delta}_i^{(n)})\,L^i$, and the sequence

$$\boldsymbol{\tau}^{(n)} := \Big((\mathrm{vec}\,\boldsymbol{\eta}^{(n)\prime})', (\mathrm{vec}\,\boldsymbol{\gamma}_1^{(n)})', \ldots, (\mathrm{vec}\,\boldsymbol{\gamma}_{p_1}^{(n)})', (\mathrm{vec}\,\boldsymbol{\delta}_1^{(n)})', \ldots, (\mathrm{vec}\,\boldsymbol{\delta}_{q_1}^{(n)})'\Big)' \in \mathbb{R}^K = \mathbb{R}^{km+k^2(p_1+q_1)}$$

is bounded as $n \to \infty$: $\sup_n(\boldsymbol{\tau}^{(n)})'\boldsymbol{\tau}^{(n)} < \infty$. The perturbed parameter is thus

$$\boldsymbol{\theta}^{(n)} := \boldsymbol{\theta} + \boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)} := \boldsymbol{\theta} + n^{-1/2}\begin{pmatrix} \mathbf{K}^{(n)} \otimes \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{k^2(p_1+q_1)} \end{pmatrix}\boldsymbol{\tau}^{(n)}.$$

The corresponding sequence of local alternatives will be denoted by $\mathcal{H}^{(n)}(\boldsymbol{\theta} + \boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)}, \boldsymbol{\Sigma}, f)$.

Denote by $\mathbf{G}_u(\boldsymbol{\theta})$, $u \in \mathbb{N}$, the Green's matrices associated with the autoregressive difference operator $\mathbf{A}(L) = \mathbf{I}_k - \sum_{i=1}^{p_0}\mathbf{A}_iL^i$. These matrices can be defined recursively by $\mathbf{A}(L)\mathbf{G}_u = \mathbf{G}_u - \sum_{i=1}^{\min(p_0,u)}\mathbf{A}_i\mathbf{G}_{u-i} = \delta_{u0}\,\mathbf{I}_k$, where $\delta_{u0} = 1$ if $u = 0$, and $\delta_{u0} = 0$ otherwise. Assumption (D2) also allows for defining $\mathbf{G}_u$ by means of

$$\sum_{u=0}^{+\infty}\mathbf{G}_u z^u := \left(\mathbf{I}_k - \sum_{i=1}^{p_0}\mathbf{A}_i z^i\right)^{-1}, \qquad z \in \mathbb{C},\ |z| < 1; \tag{15}$$

Similarly, we denote by $\mathbf{H}_u(\boldsymbol{\theta})$, $u \in \mathbb{N}$, the Green's matrices associated with the moving average difference operators $\mathbf{B}(L)$. Clearly, all these Green's matrices are continuous functions of $\boldsymbol{\theta}$. When no confusion is possible, we will not stress their dependence on $\boldsymbol{\theta}$. The residuals $(\mathbf{Z}_1^{(n)}(\boldsymbol{\theta}), \ldots, \mathbf{Z}_n^{(n)}(\boldsymbol{\theta}))$ associated with a value $\boldsymbol{\theta}$ of the parameter then can be computed from the initial values $\boldsymbol{\varepsilon}_{-q_0+1}\ldots, \boldsymbol{\varepsilon}_0, \mathbf{Y}_{-p_0+1}^{(n)}, \ldots, \mathbf{Y}_0^{(n)}$ and the observed series $(\mathbf{Y}_1^{(n)}, \ldots, \mathbf{Y}_n^{(n)})$ via the recursion

$$\mathbf{Z}_t^{(n)}(\boldsymbol{\theta}) = \sum_{i=0}^{t-1}\sum_{j=0}^{p_0}\mathbf{H}_i\mathbf{A}_j(\mathbf{Y}_{t-i-j}^{(n)} - \boldsymbol{\beta}'\mathbf{x}_{t-i-j}^{(n)}) \tag{16}$$
$$+(\mathbf{H}_{t+q_0-1}\ldots\mathbf{H}_t)\begin{pmatrix} \mathbf{I}_k & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{B}_1 & \mathbf{I}_k & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{q_0-1} & \mathbf{B}_{q_0-2} & \ldots & \mathbf{I}_k \end{pmatrix}\begin{pmatrix} \boldsymbol{\varepsilon}_{-q_0+1} \\ \vdots \\ \boldsymbol{\varepsilon}_0 \end{pmatrix}.$$

Assumption (D2) ensures that neither the (generally unobserved) values $(\boldsymbol{\varepsilon}_{-q_0+1}, \ldots, \boldsymbol{\varepsilon}_0)$ of the innovation, nor the initial values $(\mathbf{Y}_{-p_0+1}^{(n)}, \ldots, \mathbf{Y}_0^{(n)})$, have an influence on asymptotic results; they all safely can be put to zero in the sequel.

The ULAN property of the model and the structure of the central sequence (see Appendix) imply that all the relevant information (in this elliptical context) is contained in the generalized cross-covariance matrices $\boldsymbol{\Gamma}_{i;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta})$ and the nonserial statistics $\boldsymbol{\Lambda}_{i;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta})$ (see (3) and (4)) computed from the residuals $\mathbf{Z}_t^{(n)}(\boldsymbol{\theta})$, with the score functions $K_0(d) = K_1(d) = \varphi_f(d)$ and $K_2(d) = d$. We refer to the Appendix for details.

4 Asymptotic linearity.

We now can state and prove the main result of this paper. Define

$$\mathbf{h}_j = \mathbf{h}_j(\boldsymbol{\theta}) := \mathbf{H}_j(\boldsymbol{\theta}) - \sum_{i=1}^{\min(p_0,j)} \mathbf{H}_{j-i}(\boldsymbol{\theta})\mathbf{A}_i(\boldsymbol{\theta}), \quad j = 0, 1, 2, \ldots,$$

$$\mathbf{a}_i(\boldsymbol{\tau};\boldsymbol{\theta}) := \sum_{j=1}^{\min(p_1,i)} \sum_{l=0}^{i-j} \sum_{k=0}^{\min(q_0,i-j-l)} (\mathbf{G}_{i-j-l-k}(\boldsymbol{\theta})\mathbf{B}_k(\boldsymbol{\theta}) \otimes \mathbf{H}_l(\boldsymbol{\theta})')' \operatorname{vec}\boldsymbol{\gamma}_j, \tag{17}$$

and

$$\mathbf{b}_i(\boldsymbol{\tau};\boldsymbol{\theta}) := \sum_{j=1}^{\min(q_1,i)} (\mathbf{I}_k \otimes \mathbf{H}_{i-j}(\boldsymbol{\theta})) \operatorname{vec}\boldsymbol{\delta}_j. \tag{18}$$

Let further $D_k(J;f) := \int_0^1 J(u)\, \tilde{F}_k^{-1}(u)\, du$ and $C_k(J;f) := \int_0^1 J(u)\, \varphi_f \circ \tilde{F}_k^{-1}(u)\, du$.

Proposition 2 *Assume that Assumptions (A1′), (A2), (A3), (B1), (C) (or (C′)), (D1), and (D2) hold. Then,*

(i)
$$(n-i)^{1/2} \left\{ \operatorname{vec} \underset{\sim}{\boldsymbol{\Lambda}}{}^{(n)}_{i;J}(\boldsymbol{\theta} + \boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)}) - \operatorname{vec} \underset{\sim}{\boldsymbol{\Lambda}}{}^{(n)}_{i;J}(\boldsymbol{\theta}) \right\}$$
$$+ \frac{1}{k} C_k(J_0;f)(\mathbf{I}_m \otimes \boldsymbol{\Sigma}^{-1}) \left(\sum_{j=0}^{\infty} (\mathbf{K}' \, \mathbf{R}_{|i-j|} \, \mathbf{K}) \otimes \mathbf{h}_j \right) \left(\operatorname{vec} \boldsymbol{\eta}^{(n)\prime} \right) = o_{\mathrm{P}}(1), \tag{19}$$

and

$$(n-i)^{1/2} \left\{ \operatorname{vec} \underset{\sim}{\boldsymbol{\Gamma}}{}^{(n)}_{i;J}(\boldsymbol{\theta} + \boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)}) - \operatorname{vec} \underset{\sim}{\boldsymbol{\Gamma}}{}^{(n)}_{i;J}(\boldsymbol{\theta}) \right\}$$
$$+ \frac{1}{k^2} D_k(J_2;f)\, C_k(J_1;f)(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \left[\mathbf{a}_i(\boldsymbol{\tau}^{(n)};\boldsymbol{\theta}) + \mathbf{b}_i(\boldsymbol{\tau}^{(n)};\boldsymbol{\theta}) \right] = o_{\mathrm{P}}(1), \tag{20}$$

as $n \to \infty$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$, and

(ii) the same result still holds if the pseudo-Mahalanobis signs $\mathbf{W}_t^{(n)}$ in $\underset{\sim}{\boldsymbol{\Lambda}}{}^{(n)}_{i;J}$ and $\underset{\sim}{\boldsymbol{\Gamma}}{}^{(n)}_{i;J}$ are replaced by the corresponding absolute interdirections, and/or if the pseudo-Mahalanobis ranks $\widehat{R}_t^{(n)}$ are replaced by the lift-interdirection ranks $\underline{R}_t^{(n)}$.

The proof of Proposition 2 relies on a series of lemmas. In the remaining of this section, we will write $\mathbf{Z}_t^0$ and $\mathbf{Z}_t^n$ for $\mathbf{Z}_t^{(n)}(\boldsymbol{\theta})$ and $\mathbf{Z}_t^{(n)}(\boldsymbol{\theta} + \boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)})$, respectively. Accordingly, let $d_t^0 := \|\boldsymbol{\Sigma}^{-1/2}\mathbf{Z}_t^0\|$, $\mathbf{U}_t^0 := \boldsymbol{\Sigma}^{-1/2}\mathbf{Z}_t^0/d_t^0$, $d_t^n := \|\boldsymbol{\Sigma}^{-1/2}\mathbf{Z}_t^n\|$, and $\mathbf{U}_t^n := \boldsymbol{\Sigma}^{-1/2}\mathbf{Z}_t^n/d_t^n$. We begin by two preliminary results.

Lemma 6 *Under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$,*

(i) $\max_{1 \le t \le n} \|\mathbf{Z}_t^n - \mathbf{Z}_t^0\| = o_{\mathrm{P}}(1)$ *as $n \to \infty$;*

(ii) $\max_{1 \le t \le n} |d_t^n - d_t^0| = o_{\mathrm{P}}(1)$ *as $n \to \infty$;*

(iii) *denoting by* I_A *the indicator function of the set* A*,* $\max_{1\le t\le n}\left(\|\mathbf{U}_t^n-\mathbf{U}_t^0\|I_{[d_t^0>\varepsilon]}\right)=o_{\mathrm{P}}(1)$ *as* $n\to\infty$*, for all* $\varepsilon>0$*. Moreover,* $\|\mathbf{U}_t^n-\mathbf{U}_t^0\|=o_{\mathrm{P}}(1)$ *as* $n\to\infty$*, for all* t*.*

Lemma 7 *Under* $\mathcal{H}^{(n)}(\boldsymbol{\theta},\boldsymbol{\Sigma},f)$ *and for sufficiently large* n*,* $\{\mathbf{Z}_t^n,t\in\mathbb{Z}\}$ *is an absolutely regular process, with mixing rates* $\beta^{(n)}(j)$*,* $j\in\mathbb{N}$*, satisfying* $\beta^{(n)}(j)\le\beta(j)$*, where* $\beta(j)$ *is exponentially decreasing (to zero) as* $j\to\infty$*.*

Proof of Lemma 6. (i) Writing $\mathbf{H}_i^n$ for $\mathbf{H}_i(\boldsymbol{\theta}+\boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)})$, we may write, in view of (16),

$$\begin{aligned}
\mathbf{Z}_t^n-\mathbf{Z}_t^0 &= \sum_{i=0}^{t-1}\mathbf{H}_i^n\mathbf{A}^{(n)}(L)[\mathbf{Y}_{t-i}^{(n)}-(\boldsymbol{\beta}+n^{-1/2}\mathbf{K}^{(n)}\boldsymbol{\eta}^{(n)})'\mathbf{x}_{t-i}^{(n)}]\\
&\qquad -\sum_{i=0}^{t-1}\mathbf{H}_i\mathbf{A}(L)[\mathbf{Y}_{t-i}^{(n)}-\boldsymbol{\beta}'\mathbf{x}_{t-i}^{(n)}]\\
&= n^{-1/2}\left[n^{1/2}\sum_{i=0}^{t-1}(\mathbf{H}_i^n-\mathbf{H}_i)\mathbf{A}(L)-\sum_{i=0}^{t-1}\mathbf{H}_i^n\boldsymbol{\gamma}^{(n)}(L)\right](\mathbf{Y}_{t-i}^{(n)}-\boldsymbol{\beta}'\mathbf{x}_{t-i}^{(n)})\\
&\qquad -n^{-1/2}\sum_{i=0}^{t-1}\mathbf{H}_i^n\mathbf{A}^{(n)}(L)\,\boldsymbol{\eta}^{(n)\prime}\mathbf{K}^{(n)\prime}\mathbf{x}_{t-i}^{(n)},
\end{aligned}\tag{21}$$

where $\boldsymbol{\gamma}^{(n)}(L):=\sum_{i=1}^{p_1}\boldsymbol{\gamma}_i^{(n)}L^i$. Using the fact that $n^{1/2}\sum_{i=0}^{\infty}\|\mathbf{H}_i^n-\mathbf{H}_i\|$ is bounded as $n\to\infty$ (see Lemma 4.3 in Garel and Hallin 1995), it can be easily checked that the sums of the norms of the matrix coefficients of $[n^{1/2}\sum_{i=0}^{t-1}(\mathbf{H}_i^n-\mathbf{H}_i)\mathbf{A}(L)-\sum_{i=0}^{t-1}\mathbf{H}_i^n\boldsymbol{\gamma}^{(n)}(L)]$ are uniformly bounded (for n sufficiently large). Consequently,

$$\mathbf{e}_t^{(n)}:=[n^{1/2}\sum_{i=0}^{t-1}(\mathbf{H}_i^n-\mathbf{H}_i)\mathbf{A}(L)-\sum_{i=0}^{t-1}\mathbf{H}_i^n\boldsymbol{\gamma}^{(n)}(L)](\mathbf{Y}_{t-i}^{(n)}-\boldsymbol{\beta}'\mathbf{x}_{t-i}^{(n)})$$

is a stationary process with finite variance. Therefore, $\max_{1\le t\le n}\|\mathbf{e}_t^{(n)}\|$ is $o_{\mathrm{P}}(n^{1/2})$.

For the non-random term in (21), using the same type of arguments as above, it is easily seen that

$$\max_{1\le t\le n}\left\|n^{-1/2}\sum_{i=0}^{t-1}\mathbf{H}_i^n\mathbf{A}^{(n)}(L)\,\boldsymbol{\eta}^{(n)\prime}\mathbf{K}^{(n)\prime}\mathbf{x}_{t-i}^{(n)}\right\|\le Cn^{-1/2}\max_{1\le t\le n}\|\mathbf{K}^{(n)\prime}\mathbf{x}_{t-i}^{(n)}\|.$$

Now, note that

$$\begin{aligned}
\|\mathbf{K}^{(n)\prime}\mathbf{x}_{t-i}^{(n)}\| &= \|\mathbf{K}'(\mathbf{D}^{(n)})^{-1/2}\mathbf{x}_{t-i}^{(n)}\|\\
&\le \|\mathbf{K}\|\,[\mathbf{x}_{t-i}^{(n)\prime}(\mathbf{D}^{(n)})^{-1}\mathbf{x}_{t-i}^{(n)}]^{1/2}\\
&< n^{1/2}\|\mathbf{K}\|\left[\sum_{j=1}^{m}\frac{\left(\mathbf{x}_{t-i}^{(n)}\right)_j^2}{\sum_{t=1}^n\left(\mathbf{x}_t^{(n)}\right)_j^2}\right]^{1/2},
\end{aligned}$$

which, in view of (13), is $o(n^{1/2})$ as $n\to\infty$, uniformly in t. The result follows.

(ii) This trivially results from (i), and from the chain of inequalities (for all $t=1,\ldots,n$)

$$|d_t^n-d_t^0|\le\|\boldsymbol{\Sigma}^{-1/2}(\mathbf{Z}_t^n-\mathbf{Z}_t^0)\|\le\|\boldsymbol{\Sigma}^{-1/2}\|\|\mathbf{Z}_t^n-\mathbf{Z}_t^0\|\le\|\boldsymbol{\Sigma}^{-1/2}\|\max_{1\le t\le n}\|\mathbf{Z}_t^n-\mathbf{Z}_t^0\|.$$

(iii) Working along the same lines as in the proof of Lemma 2 of Hallin and Paindaveine (2002d), we obtain that $\|\mathbf{U}_t^n - \mathbf{U}_t^0\| I_{[d_t^0 > \varepsilon]} \leq (2/\varepsilon)\, \|\boldsymbol{\Sigma}^{-1/2}\|\, \|\mathbf{Z}_t^n - \mathbf{Z}_t^0\|$, which, in view of (i), yields the first statement. To establish the second one, note that

$$\mathrm{P}[\|\mathbf{U}_t^n - \mathbf{U}_t^0\| > \delta] \leq \mathrm{P}\left[\|\mathbf{U}_t^n - \mathbf{U}_t^0\| I_{[d_t^0 > \varepsilon]} > \delta\right] + \mathrm{P}[d_t^0 \leq \varepsilon].$$

Since the second term can be made as small as possible by choosing a suitable ε, the result follows from the first part of (iii). □

Proof of Lemma 7. Letting $\mathbf{H}^{(n)}(L) := \sum_{i=0}^{\infty} \mathbf{H}_i^n L^i$, we have, from equation (16), $\mathbf{Z}_t^n = \mathbf{H}^{(n)}(L)\mathbf{A}^{(n)}(L)[\mathbf{Y}_t^{(n)} - (\boldsymbol{\beta} + n^{-1/2}\mathbf{K}^{(n)}\boldsymbol{\eta}^{(n)})'\mathbf{x}_t^{(n)}]$ and $\mathbf{Z}_t^0 = \mathbf{H}(L)\mathbf{A}(L)[\mathbf{Y}_t^{(n)} - \boldsymbol{\beta}'\mathbf{x}_t^{(n)}] = \boldsymbol{\varepsilon}_t$, so that

$$\mathbf{Z}_t^n = \mathbf{H}^{(n)}(L)\mathbf{A}^{(n)}(L)\mathbf{G}(L)\mathbf{B}(L)\,\boldsymbol{\varepsilon}_t - n^{-1/2}\mathbf{H}^{(n)}(L)\mathbf{A}^{(n)}(L)\,\boldsymbol{\eta}^{(n)\prime}\mathbf{K}^{(n)\prime}\mathbf{x}_t^{(n)},$$

where the $\boldsymbol{\varepsilon}_t$'s are i.i.d. with the probability density function $\underline{f}$ given in (1). Consequently, the process $\{\tilde{\mathbf{Z}}_t^n := \mathbf{Z}_t^n - \mathrm{E}_0[\mathbf{Z}_t^n], t \in \mathbb{Z}\}$ (here, and in the sequel, expectation E_0 is taken under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$) satisfies the infinite order linear difference equation

$$\tilde{\mathbf{Z}}_t^n = \mathbf{H}^{(n)}(L)\mathbf{A}^{(n)}(L)\mathbf{G}(L)\mathbf{B}(L)\,\boldsymbol{\varepsilon}_t =: \sum_{j=0}^{\infty} \mathbf{E}_j^{(n)} \boldsymbol{\varepsilon}_{t-j}.$$

Let $\alpha_l^{(n)} := \sum_{j=l}^{\infty} \|\mathbf{E}_j^{(n)}\|$. It follows from Theorem 2.1 in Pham and Tran (1985) that, if

(i) $\int |\underline{f}(\mathbf{x} + \boldsymbol{\Delta}) - \underline{f}(\mathbf{x})|\, d\mathbf{x} \leq K\|\boldsymbol{\Delta}\|$,

(ii) $\int \|\mathbf{x}\|^{\delta} \underline{f}(\mathbf{x})\, d\mathbf{x} < \infty$, for some $\delta > 0$,

(iii) $\sum_{j=0}^{\infty} \|\mathbf{E}_j^{(n)}\| < \infty$, $\sum_{j=0}^{\infty} \mathbf{E}_j^{(n)} z^j \neq \mathbf{0}$, for all $|z| \leq 1$, and

(iv) $\sum_{l=1}^{\infty} (\alpha_l^{(n)})^{\delta/(1+\delta)} < \infty$,

then, $\{\tilde{\mathbf{Z}}_t^n, t \in \mathbb{Z}\}$ is absolutely regular, with mixing rates $\beta^{(n)}(j) \leq K \sum_{l=j}^{\infty} (\alpha_l^{(n)})^{\delta/(1+\delta)}$.

We check that Conditions (i)-(iv) hold here. Denoting by $\|.\|_2$ the L^2-norm and by $\mathbf{D}\underline{f}^{1/2}$ the quadratic mean gradient of $\underline{f}^{1/2}$, we have

$$\begin{aligned}
\int |\underline{f}(\mathbf{x} + \boldsymbol{\Delta}) - \underline{f}(\mathbf{x})|\, d\mathbf{x} &\leq \|\underline{f}^{1/2}(. + \boldsymbol{\Delta}) - \underline{f}^{1/2}(.)\|_2 \|\underline{f}^{1/2}(. + \boldsymbol{\Delta}) + \underline{f}^{1/2}(.)\|_2 \\
&\leq 2\, \|\underline{f}^{1/2}(. + \boldsymbol{\Delta}) - \underline{f}^{1/2}(.)\|_2 \\
&\leq 2\, \|\underline{f}^{1/2}(. + \boldsymbol{\Delta}) - \underline{f}^{1/2}(.) - \boldsymbol{\Delta}'\mathbf{D}\underline{f}^{1/2}(.)\|_2 + 2\, \|\boldsymbol{\Delta}'\mathbf{D}\underline{f}^{1/2}(.)\|_2 \\
&\leq 2\left((1/k)\, \mathcal{I}_{k,f}\, \boldsymbol{\Delta}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}\right)^{1/2} + 2\, \|\boldsymbol{\Delta}\| \|\mathbf{D}\underline{f}^{1/2}(.)\|_2,
\end{aligned}$$

where we used Lemma 2.2(i) in Garel and Hallin (1995) to bound the first term. Since the quadratic mean gradient is in $L^2(\mathbb{R}^k)$, Condition (i) is satisfied. Of course, Assumption (A1$'$) implies that Condition (ii) is satisfied with $\delta = 2$.

It follows from Assumption (D2) that $(\|\mathbf{E}_j^{(n)}\|)$ is exponentially decreasing to zero in j (for fixed n), so that the first part of Condition (iii) clearly holds (note that the second part of

Condition (iii) directly follows from Assumption (D2)). It is then a simple exercise to check that the sequence $(\alpha_l^{(n)})$ is also exponentially decreasing to zero in j (still for fixed n). Consequently, Condition (iv) is satisfied, and Pham and Tran (1985)'s Theorem 2.1 applies.

As above, the exponential decrease in l of the $(\alpha_l^{(n)})$'s implies the exponential decrease in l of the mixing rates $\beta^{(n)}(j)$ of the associated absolutely regular process. The uniformity in n of the exponential decrease of $\beta^{(n)}(j)$ is obtained, as in the univariate case, by showing (as in Kreiss (1987), Lemma 6.1) that the above bounds on the norms $\|\mathbf{E}_j^{(n)}\|$ hold uniformly in n (for sufficiently large n). $\square$

Proof of Proposition 2. We now prove the asymptotic linearity result (20). One can check that the proof of (19) follows along the same lines, and is actually simpler. We first consider the following truncation of the score functions J_ℓ, $\ell = 1, 2$. For all $m \in \mathbb{N}_0$, define

$$J_\ell^{(m)}(u) := \begin{cases} 0 & \text{if} \quad u \leq \frac{1}{m} \\ J_\ell\left(\frac{2}{m}\right) m \left(u - \frac{1}{m}\right) & \text{if} \quad \frac{1}{m} < u \leq \frac{2}{m} \\ J_\ell(u) & \text{if} \quad \frac{2}{m} < u \leq 1 - \frac{2}{m} \\ J_\ell\left(1 - \frac{2}{m}\right) m \left(\left(1 - \frac{1}{m}\right) - u\right) & \text{if} \quad 1 - \frac{2}{m} < u \leq 1 - \frac{1}{m} \\ 0 & \text{if} \quad u > 1 - \frac{1}{m}. \end{cases}$$

Since J_ℓ is continuous (see Assumption (C)), the function $J_\ell^{(m)}$ is also continuous on $]0, 1[$. Clearly, $J_\ell^{(m)}$ is compactly supported in $]0, 1[$ for all m; consequently, it is bounded for all m. As already mentioned, it can safely be assumed that J_ℓ is a monotone increasing function (rather than the difference of two monotone inscreasing functions), so that (at least for m sufficiently large) $|J_\ell^{(m)}|$ is bounded by $|J_\ell|$ uniformly in m and u, i.e., there exists some M such that $|J_\ell^{(m)}(u)| \leq |J_\ell(u)|$ for all $u \in]0, 1[$ and all $m \geq M$.

We have to prove that, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$, as $n \to \infty$,

$$(n-i)^{1/2}\mathrm{vec}\,(\underset{\sim}{\boldsymbol{\Gamma}}{}_{i;J}^{(n)}(\boldsymbol{\theta} + \boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)}) - \underset{\sim}{\boldsymbol{\Gamma}}{}_{i;J}^{(n)}(\boldsymbol{\theta}))$$
$$+ \frac{1}{k^2} D_k(J_2; f)\, C_k(J_1; f)(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \left[\mathbf{a}_i(\boldsymbol{\tau}^{(n)}; \boldsymbol{\theta}) + \mathbf{b}_i(\boldsymbol{\tau}^{(n)}; \boldsymbol{\theta})\right] \quad (22)$$

is $o_{\mathrm{P}}(1)$. Proposition 1 shows that $(n-i)^{1/2}\mathrm{vec}\,(\underset{\sim}{\boldsymbol{\Gamma}}{}_{i;J}^{(n)}(\boldsymbol{\theta})) - (n-i)^{1/2}\,\mathrm{vec}\,(\widetilde{\boldsymbol{\Gamma}}_{i;J;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta}))$ is $o_{\mathrm{P}}(1)$, as $n \to \infty$, under the same sequence of hypotheses. Similarly,

$$(n-i)^{1/2}\,\mathrm{vec}\,(\underset{\sim}{\boldsymbol{\Gamma}}{}_{i;J}^{(n)}(\boldsymbol{\theta} + \boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)})) - (n-i)^{1/2}\,\mathrm{vec}\,(\widetilde{\boldsymbol{\Gamma}}_{i;J;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta} + \boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)})) \quad (23)$$

is $o_{\mathrm{P}}(1)$ as $n \to \infty$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta} + \boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)}, \boldsymbol{\Sigma}, f)$. It follows from contiguity that (23) is also $o_{\mathrm{P}}(1)$ under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$, as $n \to \infty$. Consequently, (22) is asymptotically equivalent, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$, to

$$(n-i)^{1/2}\,\mathrm{vec}\,(\widetilde{\boldsymbol{\Gamma}}_{i;J;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta} + \boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)})) - (n-i)^{1/2}\,\mathrm{vec}\,(\widetilde{\boldsymbol{\Gamma}}_{i;J;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta}))$$
$$+ \frac{1}{k^2} D_k(J_2; f)\, C_k(J_1; f)\, (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \left[\mathbf{a}_i(\boldsymbol{\tau}^{(n)}; \boldsymbol{\theta}) + \mathbf{b}_i(\boldsymbol{\tau}^{(n)}; \boldsymbol{\theta})\right]. \quad (24)$$

Using the fact that $\mathrm{vec}\,(\mathbf{A}_1\mathbf{B}\mathbf{A}_2) = (\mathbf{A}_2' \otimes \mathbf{A}_1)\,\mathrm{vec}\,\mathbf{B}$, (24) can be written as $(\mathbf{\Sigma}^{1/2} \otimes \mathbf{\Sigma}'^{-1/2})\,\mathbf{C}^{(n)}$, where

$$\begin{aligned}
\mathbf{C}^{(n)} := {} & (n-i)^{-1/2}\,\mathrm{vec}\left[\sum_{t=i+1}^{n} J_1(\tilde{F}_k(d_t^n))\,J_2(\tilde{F}_k(d_{t-i}^n))\,\mathbf{U}_t^n\mathbf{U}_{t-i}^{n'}\right] \\
& -(n-i)^{-1/2}\,\mathrm{vec}\left[\sum_{t=i+1}^{n} J_1(\tilde{F}_k(d_t^0))\,J_2(\tilde{F}_k(d_{t-i}^0))\,\mathbf{U}_t^0\mathbf{U}_{t-i}^{0'}\right] \\
& +\frac{1}{k^2}\,D_k(J_2;f)\,C_k(J_1;f)\,(\mathbf{\Sigma}'^{1/2} \otimes \mathbf{\Sigma}^{-1/2})\left[\mathbf{a}_i(\boldsymbol{\tau}^{(n)};\boldsymbol{\theta}) + \mathbf{b}_i(\boldsymbol{\tau}^{(n)};\boldsymbol{\theta})\right]. \qquad (25)
\end{aligned}$$

Clearly, it is sufficient to show that $\mathbf{C}^{(n)} = o_{\mathrm{P}}(1)$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$, as $n \to \infty$. Now, decompose $\mathbf{C}^{(n)}$ into $\mathbf{C}^{(n)} = \mathbf{D}_1^{(n;m)} + \mathbf{D}_2^{(n;m)} + \mathbf{R}_1^{(n;m)} + \mathbf{R}_2^{(n;m)} + \mathbf{R}_3^{(n;m)}$, where, denoting by E_0 the expectation under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$,

$$\begin{aligned}
\mathbf{D}_1^{(n;m)} := {} & (n-i)^{-1/2}\,\mathrm{vec}\left[\sum_{t=i+1}^{n} J_1^{(m)}(\tilde{F}_k(d_t^n))\,J_2(\tilde{F}_k(d_{t-i}^n))\,\mathbf{U}_t^n\mathbf{U}_{t-i}^{n'}\right] \\
& -(n-i)^{-1/2}\,\mathrm{vec}\left[\sum_{t=i+1}^{n} J_1^{(m)}(\tilde{F}_k(d_t^0))\,J_2^{(m)}(\tilde{F}_k(d_{t-i}^0))\,\mathbf{U}_t^0\mathbf{U}_{t-i}^{0'}\right] \\
& -(n-i)^{-1/2}\mathrm{E}_0\left[\mathrm{vec}\left[\sum_{t=i+1}^{n} J_1^{(m)}(\tilde{F}_k(d_t^n))\,J_2^{(m)}(\tilde{F}_k(d_{t-i}^n))\mathbf{U}_t^n\mathbf{U}_{t-i}^{n'}\right]\right],
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}_2^{(n;m)} := {} & (n-i)^{-1/2}\mathrm{E}_0\left[\mathrm{vec}\left[\sum_{t=i+1}^{n} J_1^{(m)}(\tilde{F}_k(d_t^n))\,J_2^{(m)}(\tilde{F}_k(d_{t-i}^n))\mathbf{U}_t^n\mathbf{U}_{t-i}^{n'}\right]\right] \\
& +\frac{1}{k^2}\,D_k(J_2^{(m)};f)\,C_k(J_1^{(m)};f)\,(\mathbf{\Sigma}'^{1/2} \otimes \mathbf{\Sigma}^{-1/2})\left[\mathbf{a}_i(\boldsymbol{\tau}^{(n)};\boldsymbol{\theta}) + \mathbf{b}_i(\boldsymbol{\tau}^{(n)};\boldsymbol{\theta})\right],
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_1^{(n;m)} := {} & (n-i)^{-1/2}\,\mathrm{vec}\left[\sum_{t=i+1}^{n}\left[J_1(\tilde{F}_k(d_t^0))\,J_2(\tilde{F}_k(d_{t-i}^0))\right.\right. \\
& \left.\left. -J_1^{(m)}(\tilde{F}_k(d_t^0))\,J_2^{(m)}(\tilde{F}_k(d_{t-i}^0))\right]\mathbf{U}_t^0\mathbf{U}_{t-i}^{0'}\right],
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_2^{(n;m)} := {} & (n-i)^{-1/2}\,\mathrm{vec}\left[\sum_{t=i+1}^{n}\left[J_1(\tilde{F}_k(d_t^n))\,J_2(\tilde{F}_k(d_{t-i}^n))\right.\right. \\
& \left.\left. -J_1^{(m)}(\tilde{F}_k(d_t^n))\,J_2^{(m)}(\tilde{F}_k(d_{t-i}^n))\right]\mathbf{U}_t^n\mathbf{U}_{t-i}^{n'}\right],
\end{aligned}$$

and

$$\mathbf{R}_3^{(n;m)} := \frac{1}{k^2}\left[D_k(J_2;f)\,C_k(J_1;f) - D_k(J_2^{(m)};f)\,C_k(J_1^{(m)};f)\right]$$
$$(\boldsymbol{\Sigma}'^{1/2}\otimes\boldsymbol{\Sigma}^{-1/2})\left[\mathbf{a}_i(\boldsymbol{\tau}^{(n)};\boldsymbol{\theta}) + \mathbf{b}_i(\boldsymbol{\tau}^{(n)};\boldsymbol{\theta})\right].$$

We prove that $\mathbf{C}^{(n)} = o_{\mathrm{P}}(1)$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta},\boldsymbol{\Sigma},f)$, as $n\to\infty$ (thus completing the proof of (20)) by establishing that $\mathbf{D}_1^{(n;m)}$ and $\mathbf{D}_2^{(n;m)}$ are $o_{\mathrm{P}}(1)$ under $\mathcal{H}^{(n)}(\boldsymbol{\theta},\boldsymbol{\Sigma},f)$, as $n\to\infty$, for fixed m, and that $\mathbf{R}_1^{(n;m)}$, $\mathbf{R}_2^{(n;m)}$ and $\mathbf{R}_3^{(n;m)}$ are $o_{\mathrm{P}}(1)$ under the same sequence of hypotheses, as $m\to\infty$, uniformly in n. For the sake of convenience, these three results are treated as separate lemmas (Lemmas 8 and 9, and Lemma 10, respectively).

Decompose $\mathbf{D}_1^{(n;m)}$ into $\mathbf{D}_{1,1}^{(n;m)} + \mathbf{D}_{1,2}^{(n;m)} - \mathrm{E}_0[\mathbf{D}_{1,1}^{(n;m)}]$, where

$$\mathbf{D}_{1,1}^{(n;m)} := (n-i)^{-1/2}\mathrm{vec}\left[\sum_{t=i+1}^{n}\left(J_1^{(m)}(\tilde{F}_k(d_t^n))\,\mathbf{U}_t^n - J_1^{(m)}(\tilde{F}_k(d_t^0))\,\mathbf{U}_t^0\right)J_2^{(m)}(\tilde{F}_k(d_{t-i}^n))\,\mathbf{U}_{t-i}^{n\prime}\right]$$

and

$$\mathbf{D}_{1,2}^{(n;m)} := (n-i)^{-1/2}\mathrm{vec}\left[\sum_{t=i+1}^{n}J_1^{(m)}(\tilde{F}_k(d_t^0))\,\mathbf{U}_t^0\left(J_2^{(m)}(\tilde{F}_k(d_{t-i}^n))\,\mathbf{U}_{t-i}^n - J_2^{(m)}(\tilde{F}_k(d_{t-i}^0))\mathbf{U}_{t-i}^0\right)'\right].$$

(taking into account the independence between $\mathbf{Z}_t^0$ and $\mathbf{Z}_{t-i}^n$ under $\mathcal{H}^{(n)}(\boldsymbol{\theta},\boldsymbol{\Sigma},f)$). We then have the following.

Lemma 8 *For any fixed m,*

(i) $\mathrm{E}_0\left[\left\|\mathbf{D}_{1,1}^{(n;m)} - \mathrm{E}_0[\mathbf{D}_{1,1}^{(n;m)}]\right\|^2\right] = o(1)$, *as $n\to\infty$;*

(ii) $\mathrm{E}_0\left[\left\|\mathbf{D}_{1,2}^{(n;m)}\right\|^2\right] = o(1)$, *as $n\to\infty$;*

(iii) $\mathbf{D}_1^{(n;m)} = o_{\mathrm{P}}(1)$, *as $n\to\infty$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta},\boldsymbol{\Sigma},f)$.*

Lemma 9 *For any fixed m, $\mathbf{D}_2^{(n;m)} = o(1)$, as $n\to\infty$.*

Lemma 10 (i) *Under $\mathcal{H}^{(n)}(\boldsymbol{\theta},\boldsymbol{\Sigma},f)$, $\mathbf{R}_1^{(n;m)}$ is $o_{\mathrm{P}}(1)$, as $m\to\infty$, uniformly in n.*

(ii) *Under $\mathcal{H}^{(n)}(\boldsymbol{\theta},\boldsymbol{\Sigma},f)$, $\mathbf{R}_2^{(n;m)}$ is $o_{\mathrm{P}}(1)$, as $m\to\infty$, uniformly in n (for n sufficiently large).*

(iii) *$\mathbf{R}_3^{(n;m)}$ is $o(1)$, as $m\to\infty$, uniformly in n.*

Proof of Lemma 8. Let us begin with the second part of Lemma 8.

Part (ii). Since $(\mathrm{vec}\,(\mathbf{u}\mathbf{v}'))'\mathrm{vec}\,(\mathbf{x}\mathbf{y}') = \mathrm{tr}\,[(\mathbf{u}\mathbf{v}')'\mathbf{x}\mathbf{y}'] = (\mathbf{u}'\mathbf{x})(\mathbf{v}'\mathbf{y})$, for any k-vectors $\mathbf{u},\mathbf{v},\mathbf{x},\mathbf{y}$, we obtain

$$\mathrm{E}_0[(\mathbf{D}_{1,2}^{(n;m)})'(\mathbf{D}_{1,2}^{(n;m)})] = (n-i)^{-1}\sum_{s,t=i+1}^{n}\mathrm{E}_0\Big[J_1^{(m)}(\tilde{F}_k(d_s^0))\,\mathbf{U}_s^{0\prime}\,J_1^{(m)}(\tilde{F}_k(d_t^0))\,\mathbf{U}_t^0$$
$$\times\left(J_2^{(m)}(\tilde{F}_k(d_{s-i}^n))\,\mathbf{U}_{s-i}^n - J_2^{(m)}(\tilde{F}_k(d_{s-i}^0))\,\mathbf{U}_{s-i}^0\right)'\left(J_2^{(m)}(\tilde{F}_k(d_{t-i}^n))\,\mathbf{U}_{t-i}^n - J_2^{(m)}(\tilde{F}_k(d_{t-i}^0))\,\mathbf{U}_{t-i}^0\right)\Big].$$

Due to the independence, for $s \neq t$, between $\mathbf{Z}^0_{\max(s,t)}$ and $(\mathbf{Z}^0_{\min(s,t)}, \mathbf{Z}^0_{s-i}, \mathbf{Z}^0_{t-i}, \mathbf{Z}^n_{s-i}, \mathbf{Z}^n_{t-i})$ (note that, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$, $\{\mathbf{Z}^0_t, t \in \mathbb{Z}\}$ is the innovation process of $\{\mathbf{Z}^n_t, t \in \mathbb{Z}\}$), this is equal to

$$(n-i)^{-1} \sum_{t=i+1}^{n} \mathrm{E}_0 \left[\left(J_1^{(m)}(\tilde{F}_k(d^0_t)) \right)^2 \| J_2^{(m)}(\tilde{F}_k(d^n_{t-i}))\, \mathbf{U}^n_{t-i} - J_2^{(m)}(\tilde{F}_k(d^0_{t-i}))\, \mathbf{U}^0_{t-i} \|^2 \right].$$

Since $J_1^{(m)}$ is bounded, it is sufficient to show that

$$\mathrm{E}_0[\| J_2^{(m)}(\tilde{F}_k(d^n_{t-i}))\, \mathbf{U}^n_{t-i} - J_2^{(m)}(\tilde{F}_k(d^0_{t-i}))\, \mathbf{U}^0_{t-i} \|^2] = o(1), \qquad \text{as } n \to \infty, \tag{26}$$

uniformly in t. Now, with $\eta > 0$ such that $\tilde{F}_k(\eta) < 1/m$, we have $J_2^{(m)}(\tilde{F}_k(d^0_{t-i})) I_{[d^0_{t-i} \leq \eta]} = 0$ (note that $\tilde{F}_k$ is a continuous strictly monotone increasing function that maps $\mathbb{R}_0^+$ onto $]0,1[$). This yields

$$\begin{aligned} &\| J_2^{(m)}(\tilde{F}_k(d^n_{t-i}))\, \mathbf{U}^n_{t-i} - J_2^{(m)}(\tilde{F}_k(d^0_{t-i}))\, \mathbf{U}^0_{t-i} \| \\ &\qquad \leq | J_2^{(m)}(\tilde{F}_k(d^n_{t-i})) - J_2^{(m)}(\tilde{F}_k(d^0_{t-i}))|\, \|\mathbf{U}^n_{t-i}\| + |J_2^{(m)}(\tilde{F}_k(d^0_{t-i}))|\, \|\mathbf{U}^n_{t-i} - \mathbf{U}^0_{t-i}\| \\ &\qquad \leq | J_2^{(m)}(\tilde{F}_k(d^n_{t-i})) - J_2^{(m)}(\tilde{F}_k(d^0_{t-i}))| + |J_2^{(m)}(\tilde{F}_k(d^0_{t-i}))|\, \|\mathbf{U}^n_{t-i} - \mathbf{U}^0_{t-i}\| I_{[d^0_{t-i} > \eta]}, \end{aligned}$$

so that

$$\begin{aligned} &\| J_2^{(m)}(\tilde{F}_k(d^n_{t-i}))\, \mathbf{U}^n_{t-i} - J_2^{(m)}(\tilde{F}_k(d^0_{t-i}))\, \mathbf{U}^0_{t-i} \|^2 \\ &\qquad \leq C\, | J_2^{(m)}(\tilde{F}_k(d^n_{t-i})) - J_2^{(m)}(\tilde{F}_k(d^0_{t-i}))|^2 + C\, \|\mathbf{U}^n_{t-i} - \mathbf{U}^0_{t-i}\|^2 I_{[d^0_{t-i} > \eta]}, \end{aligned}$$

for some constant C. Lemma 6(ii) and the continuity of $J_2^{(m)} \circ \tilde{F}_k$ imply that $J_2^{(m)}(\tilde{F}_k(d^n_{t-i})) - J_2^{(m)}(\tilde{F}_k(d^0_{t-i})) = o_{\mathrm{P}}(1)$ as $n \to \infty$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$. Since $J_2^{(m)}$ is bounded, this convergence to zero also holds in quadratic mean. Similarly, using Lemma 6(iii) and the boundedness of $\mathbf{U}^0_{t-i}$ and $\mathbf{U}^n_{t-i}$, we obtain that $\|\mathbf{U}^n_{t-i} - \mathbf{U}^0_{t-i}\|^2 I_{[d^0_{t-i} > \eta]}$ is $o(1)$ in quadratic mean, as $n \to \infty$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$. The convergence in (26) follows.

Part (i). Letting $\mathbf{T}_{t;i} := \mathrm{vec}\left[\left(J_1^{(m)}(\tilde{F}_k(d^n_t))\, \mathbf{U}^n_t - J_1^{(m)}(\tilde{F}_k(d^0_t))\, \mathbf{U}^0_t \right) J_2^{(m)}(\tilde{F}_k(d^n_{t-i}))\, \mathbf{U}^{n\prime}_{t-i} \right]$, we have

$$\begin{aligned} \mathrm{E}_0\left[\left\| \mathbf{D}^{(n;m)}_{1,1} - \mathrm{E}_0[\mathbf{D}^{(n;m)}_{1,1}] \right\|^2 \right] &= \mathrm{E}_0\left[\left(\mathbf{D}^{(n;m)}_{1} - \mathrm{E}_0[\mathbf{D}^{(n;m)}_{1,1}] \right)' \left(\mathbf{D}^{(n;m)}_{1,1} - \mathrm{E}_0[\mathbf{D}^{(n;m)}_{1,1}] \right) \right] \\ &= \mathrm{tr}\left[\mathrm{Var}_0\left[\mathbf{D}^{(n;m)}_{1,1} \right] \right] = (n-i)^{-1} \mathrm{tr}\left[\mathrm{Var}_0\left[\sum_{t=i+1}^{n} \mathbf{T}_{t;i} \right] \right] \\ &= \mathrm{tr}\left[\mathrm{Var}_0\left[\mathbf{T}_{t;i} \right] \right] + \sum_{j=1}^{n-i-1} \frac{n-j-i}{n-i}\, \mathrm{tr}\left[\mathrm{Cov}_0\left[\mathbf{T}_{t;i}, \mathbf{T}_{t-j;i} \right] \right]. \end{aligned} \tag{27}$$

First note that

$$\mathrm{tr}\,[\mathrm{Var}_0\,[\mathbf{T}_{t;i}]] = \mathrm{E}_0[\,(\mathbf{T}_{t;i} - \mathrm{E}_0[\mathbf{T}_{t;i}])'\,(\mathbf{T}_{t;i} - \mathrm{E}_0[\mathbf{T}_{t;i}])] \leq \mathrm{E}_0[\|\mathbf{T}_{t;i}\|^2\,],$$

where, using again $(\text{vec}\,(\mathbf{u}\mathbf{v}'))'\text{vec}\,(\mathbf{x}\mathbf{y}') = (\mathbf{u}'\mathbf{x})(\mathbf{v}'\mathbf{y})$ and the boundedness of $J_2^{(m)}$,

$$\begin{aligned}
\mathrm{E}_0\left[\|\mathbf{T}_{t;i}\|^2\right] &= \mathrm{E}_0\left[\left\|J_1^{(m)}(\tilde{F}_k(d_t^n))\,\mathbf{U}_t^n - J_1^{(m)}(\tilde{F}_k(d_t^0))\,\mathbf{U}_t^0\right\|^2\left(J_2^{(m)}(\tilde{F}_k(d_{t-i}^n))\right)^2\right] \\
&\leq C\,\mathrm{E}_0\left[\left\|J_1^{(m)}(\tilde{F}_k(d_t^n))\,\mathbf{U}_t^n - J_1^{(m)}(\tilde{F}_k(d_t^0))\,\mathbf{U}_t^0\right\|^2\right],
\end{aligned}$$

which — compare with (26) — is $o(1)$, as $n \to \infty$, uniformly in t. On the other hand, the absolute regularity of $\{\mathbf{Z}_t^n, t \in \mathbb{Z}\}$ (Lemma 7) and the fact that $\{\mathbf{Z}_t^0, t \in \mathbb{Z}\}$ is (under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$) the innovation process of $\{\mathbf{Z}_t^n, t \in \mathbb{Z}\}$ imply that the process $\{(\mathbf{Z}_t^n, \mathbf{Z}_t^0), t \in \mathbb{Z}\}$ is also absolutely regular with the same mixing rates as $\{\mathbf{Z}_t^n, t \in \mathbb{Z}\}$. Using Lemma 1 of Yoshihara (1976) (with $p := k$, $k := 2$, $\delta := 1$, and $h(\mathbf{x}_1, \mathbf{x}_2) := \text{tr}\,(\mathbf{x}_1\mathbf{x}_2') = \mathbf{x}_1'\mathbf{x}_2$), we obtain

$$\begin{aligned}
\left|\text{tr}\ [\text{Cov}_0\,[\mathbf{T}_{t;i}, \mathbf{T}_{t-j;i}]]\right| &= \left|\mathrm{E}_0[\mathbf{T}_{t;i}'\mathbf{T}_{t-j;i}] - \mathrm{E}_0[\mathbf{T}_{t;i}']\mathrm{E}_0\,[\mathbf{T}_{t-k;i}]\right| \\
&\leq 4\,\mathrm{E}_0[\|\mathbf{T}_{t;i}\|^2]\,(\beta^{(n)}(j))^{1/2} \leq 4\,\mathrm{E}_0[\|\mathbf{T}_{t;i}\|^2]\,(\beta(j))^{1/2},
\end{aligned}$$

where the sequence $(\beta(j))$ is as in Lemma 7. Consequently,

$$\begin{aligned}
\left|\sum_{j=1}^{n-i-1}\frac{n-j-i}{n-i}\,\text{tr}\,[\text{Cov}_0[\mathbf{T}_{t;i}, \mathbf{T}_{t-j;i}]]\right| &\leq \sum_{j=1}^{\infty}\left|\text{tr}\ [\text{Cov}_0\,[\mathbf{T}_{t;i}, \mathbf{T}_{t-j;i}]]\right| \\
&\leq 4\,\mathrm{E}_0[\|\mathbf{T}_{t;i}\|^2]\sum_{j=1}^{\infty}(\beta(j))^{1/2} \\
&\leq C\,\mathrm{E}_0[\|\mathbf{T}_{t;i}\|^2],
\end{aligned}$$

since the series converges (due to the exponential decrease of the $\beta(j)$'s; see Lemma 7 again). This entails that both terms in (27) are bounded by (a constant multiple of) $\mathrm{E}_0[\|\mathbf{T}_{t;i}\|^2]$, a quantity which, as we showed above, is $o(1)$ as $n \to \infty$. The result follows.

Part (iii) trivially follows from Parts (i) and (ii), and the fact that convergence in quadratic mean implies convergence in probability. □

Proof of Lemma 9. Let

$$\mathbf{B}_1^{(n;m)} := (n-i)^{-1/2}\text{vec}\left[\sum_{t=i+1}^{n} J_1^{(m)}(\tilde{F}_k(d_t^0))\,J_2^{(m)}(\tilde{F}_k(d_{t-i}^0))\,\mathbf{U}_t^0\,\mathbf{U}_{t-i}^{0\prime}\right].$$

Proceeding as in Lemma 11, one can show that

$$\mathbf{B}_1^{(n;m)} \xrightarrow{\mathcal{L}} \mathcal{N}_{k^2}\left(\mathbf{0}, \frac{1}{k^2}\,\mathrm{E}[(J_1^{(m)}(U))^2]\,\mathrm{E}[(J_2^{(m)}(U))^2]\ \mathbf{I}_{k^2}\right), \tag{28}$$

as $n \to \infty$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$. Under the sequence of local alternatives $\mathcal{H}^{(n)}(\boldsymbol{\theta}+\boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)}, \boldsymbol{\Sigma}, f)$,

$$\begin{aligned}
\mathbf{B}_1^{(n;m)} &- \frac{1}{k^2}\,C_k(J_1^{(m)}; f)\,D_k(J_2^{(m)}; f)\,(\boldsymbol{\Sigma}^{1/2\prime}\otimes\boldsymbol{\Sigma}^{-1/2})\left[\mathbf{a}_i(\boldsymbol{\tau}^{(n)}; \boldsymbol{\theta}) + \mathbf{b}_i(\boldsymbol{\tau}^{(n)}; \boldsymbol{\theta})\right] \\
&\xrightarrow{\mathcal{L}} \mathcal{N}_{k^2}\left(\mathbf{0}, \frac{1}{k^2}\,\mathrm{E}[(J_1^{(m)}(U))^2]\,\mathrm{E}[(J_2^{(m)}(U))^2]\ \mathbf{I}_{k^2}\right),
\end{aligned}$$

as $n \to \infty$. Letting

$$\mathbf{B}_2^{(n;m)} := (n-i)^{-1/2}\mathrm{vec}\left[\sum_{t=i+1}^{n} J_1^{(m)}(\tilde{F}_k(d_t^n))\, J_2^{(m)}(\tilde{F}_k(d_{t-i}^n))\, \mathbf{U}_t^n\, \mathbf{U}_{t-i}^{n\prime}\right],$$

it follows from uniform local asymptotic normality that

$$\mathbf{B}_2^{(n;m)} + \frac{1}{k^2}\, C_k(J_1^{(m)}; f)\, D_k(J_2^{(m)}; f)\, (\mathbf{\Sigma}^{1/2\prime} \otimes \mathbf{\Sigma}^{-1/2}) \left[\mathbf{a}_i(\boldsymbol{\tau}^{(n)}; \boldsymbol{\theta}) + \mathbf{b}_i(\boldsymbol{\tau}^{(n)}; \boldsymbol{\theta})\right]$$
$$\xrightarrow{\mathcal{L}} \mathcal{N}_{k^2}\left(\mathbf{0}, \frac{1}{k^2}\,\mathrm{E}[(J_1^{(m)}(U))^2]\,\mathrm{E}[(J_2^{(m)}(U))^2]\ \mathbf{I}_{k^2}\right), \tag{29}$$

as $n \to \infty$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$.

Now, Lemma 8(iii) yields that $\mathbf{D}_1^{(n;m)} = \mathbf{B}_2^{(n;m)} - \mathbf{B}_1^{(n;m)} - \mathrm{E}_0[\mathbf{B}_2^{(n;m)}] = o_{\mathrm{P}}(1)$, as $n \to \infty$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$. Using this and (28), we obtain that

$$\mathbf{B}_2^{(n;m)} - \mathrm{E}_0[\mathbf{B}_2^{(n;m)}] \xrightarrow{\mathcal{L}} \mathcal{N}_{k^2}\left(\mathbf{0}, \frac{1}{k^2}\,\mathrm{E}[(J_1^{(m)}(U))^2]\,\mathrm{E}[(J_2^{(m)}(U))^2]\ \mathbf{I}_{k^2}\right),$$

as $n \to \infty$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$. Comparing with (29), it follows that

$$\mathbf{D}_2^{(n;m)} = \mathrm{E}_0\left[\mathbf{B}_2^{(n;m)}\right] + \frac{1}{k^2}\, C_k(J_1^{(m)}; f)\, D_k(J_2^{(m)}; f)\, (\mathbf{\Sigma}^{1/2\prime} \otimes \mathbf{\Sigma}^{-1/2}) \left[\mathbf{a}_i(\boldsymbol{\tau}^{(n)}; \boldsymbol{\theta}) + \mathbf{b}_i(\boldsymbol{\tau}^{(n)}; \boldsymbol{\theta})\right]$$

is $o(1)$, as $n \to \infty$, as was to be proved. □

We now complete the proof of (20) by proving Lemma 10.

Proof of Lemma 10. (i) In view of the independence between the d_t^0's and the $\mathbf{U}_t^0$'s under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$, we obtain

$$\mathrm{E}_0[\|\mathbf{R}_1^{(n;m)}\|^2] = \frac{1}{n-i} \sum_{s,t=i+1}^{n} \mathrm{E}_0\Big[[J_1(\tilde{F}_k(d_s^0))\, J_2(\tilde{F}_k(d_{s-i}^0)) - J_1^{(m)}(\tilde{F}_k(d_s^0))\, J_2^{(m)}(\tilde{F}_k(d_{s-i}^0))]$$
$$[J_1(\tilde{F}_k(d_t^0))\, J_2(\tilde{F}_k(d_{t-i}^0)) - J_1^{(m)}(\tilde{F}_k(d_t^0))\, J_2^{(m)}(\tilde{F}_k(d_{t-i}^0))]\Big]\, \mathrm{E}_0\Big[(\mathrm{vec}\,(\mathbf{U}_s^0 \mathbf{U}_{s-i}^{0\prime}))'\, \mathrm{vec}\,(\mathbf{U}_t^0 \mathbf{U}_{t-i}^{0\prime})\Big]$$
$$= \frac{1}{n-i} \sum_{t=i+1}^{n} \mathrm{E}_0\Big[[J_1(\tilde{F}_k(d_t^0))\, J_2(\tilde{F}_k(d_{t-i}^0)) - J_1^{(m)}(\tilde{F}_k(d_t^0))\, J_2^{(m)}(\tilde{F}_k(d_{t-i}^0))]^2\Big]$$
$$= \int_0^1 \int_0^1 [J_1(u)\, J_2(v) - J_1^{(m)}(u)\, J_2^{(m)}(v)]^2\, du\, dv. \tag{30}$$

Now, $J_1^{(m)}(u)\, J_2^{(m)}(v)$ converges to $J_1(u)\, J_2(v)$, for all $(u, v) \in\ ]0,1[\times]0,1[$. Also, since $|J_\ell^{(m)}(u)| \leq |J_\ell(u)|$, $\ell = 1, 2$, for all $m \geq M$, the integrand in (30) is bounded (uniformly in m) by $4\,|J_1(u)|^2 |J_2(v)|^2$, which is integrable on $]0,1[\times]0,1[$ (see Assumption (C)). Consequently, the Lebesgue dominated convergence theorem yields that $\mathrm{E}_0[\|\mathbf{R}_1^{(n;m)}\|^2] = o(1)$, as $m \to \infty$. This convergence is of course uniform in n, since $\mathrm{E}_0[\|\mathbf{R}_1^{(n;m)}\|^2]$ does not depend on n.

(ii) The claim in (ii) is the same as in (i), except that d_t^n and $\mathbf{U}_t^n$ replace d_t^0 and $\mathbf{U}_t^0$, respectively. Accordingly, it holds under $\mathcal{H}^{(n)}(\boldsymbol{\theta} + \boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)}, \mathbf{\Sigma}, f)$. That it also holds under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$ follows from Lemma 3.5 in Jurečková (1969).

(iii) Note that

$$\begin{aligned}
|D_k(J_2;f) - D_k(J_2^{(m)};f)|^2 &= \left|\int_0^1 \left(J_2(u) - J_2^{(m)}(u)\right) \tilde{F}_k^{-1}(u)\, du\right|^2 \\
&\leq \frac{\mu_{k+1;f}}{\mu_{k-1;f}} \int_0^1 \left|J_2(u) - J_2^{(m)}(u)\right|^2 du.
\end{aligned}$$

Again, $|J_2^{(m)}(u) - J_2(u)|^2 \leq 4|J_2(u)|^2$, with $\int_0^1 |J_2(u)|^2\, du < \infty$. Consequently, the pointwise convergence of $(J_2^{(m)})$ to J_2 implies that $D_k(J_2;f) - D_k(J_2^{(m)};f) = o(1)$ as $m \to \infty$. We similarly obtain that $C_k(J_1;f) - C_k(J_1^{(m)};f) = o(1)$, as $m \to \infty$.

Using the fact that the sequence $(\boldsymbol{\tau}^{(n)})$ is bounded (and the definitions of $\mathbf{a}_i(\boldsymbol{\tau}^{(n)};\boldsymbol{\theta})$, $\mathbf{b}_i(\boldsymbol{\tau}^{(n)};\boldsymbol{\theta})$ in (17), (18)), this implies that, for some real constant C,

$$\begin{aligned}
\left\|\mathbf{R}_3^{(n;m)}\right\| &\leq \frac{1}{k^2} \left|D_k(J_2;f)\, C_k(J_1;f) - D_k(J_2^{(m)};f)\, C_k(J_1^{(m)};f)\right| \\
&\qquad \times \left\|\boldsymbol{\Sigma}'^{1/2} \otimes \boldsymbol{\Sigma}^{-1/2}\right\| \left\|\mathbf{a}_i(\boldsymbol{\tau}^{(n)};\boldsymbol{\theta}) + \mathbf{b}_i(\boldsymbol{\tau}^{(n)};\boldsymbol{\theta})\right\| \\
&\leq C \left|D_k(J_2;f)\, C_k(J_1;f) - D_k(J_2^{(m)};f)\, C_k(J_1^{(m)};f)\right|,
\end{aligned}$$

which is $o(1)$, as $m \to \infty$, uniformly in n. □

5 Appendix : ULAN.

Associated with any k-dimensional linear difference operator of the form $\mathbf{C}(L) := \sum_{i=0}^{\infty} \mathbf{C}_i L^i$ (letting $\mathbf{C}_i = \mathbf{0}$ for $i > s$, this includes, of course, the operators with finite order s), define, for any integers m and p, the $k^2m \times k^2p$ matrices

$$\mathbf{C}_{m,p}^{(l)} := \begin{pmatrix}
\mathbf{C}_0 \otimes \mathbf{I}_k & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{C}_1 \otimes \mathbf{I}_k & \mathbf{C}_0 \otimes \mathbf{I}_k & \dots & \mathbf{0} \\
\vdots & & \ddots & \vdots \\
\mathbf{C}_{p-1} \otimes \mathbf{I}_k & \mathbf{C}_{p-2} \otimes \mathbf{I}_k & \dots & \mathbf{C}_0 \otimes \mathbf{I}_k \\
\vdots & & & \vdots \\
\mathbf{C}_{m-1} \otimes \mathbf{I}_k & \mathbf{C}_{m-2} \otimes \mathbf{I}_k & \dots & \mathbf{C}_{m-p} \otimes \mathbf{I}_k
\end{pmatrix} \tag{31}$$

and

$$\mathbf{C}_{m,p}^{(r)} := \begin{pmatrix}
\mathbf{I}_k \otimes \mathbf{C}_0 & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{I}_k \otimes \mathbf{C}_1 & \mathbf{I}_k \otimes \mathbf{C}_0 & \dots & \mathbf{0} \\
\vdots & & \ddots & \vdots \\
\mathbf{I}_k \otimes \mathbf{C}_{p-1} & \mathbf{I}_k \otimes \mathbf{C}_{p-2} & \dots & \mathbf{I}_k \otimes \mathbf{C}_0 \\
\vdots & & & \vdots \\
\mathbf{I}_k \otimes \mathbf{C}_{m-1} & \mathbf{I}_k \otimes \mathbf{C}_{m-2} & \dots & \mathbf{I}_k \otimes \mathbf{C}_{m-p}
\end{pmatrix}, \tag{32}$$

respectively; write $\mathbf{C}_m^{(l)}$ for $\mathbf{C}_{m,m}^{(l)}$ and $\mathbf{C}_m^{(r)}$ for $\mathbf{C}_{m,m}^{(r)}$. With this notation, note that $\mathbf{G}_m^{(l)}$, $\mathbf{G}_m^{(r)}$, $\mathbf{H}_m^{(l)}$, and $\mathbf{H}_m^{(r)}$ are the inverses of $\mathbf{A}_m^{(l)}$, $\mathbf{A}_m^{(r)}$, $\mathbf{B}_m^{(l)}$, and $\mathbf{B}_m^{(r)}$, respectively. Denoting by $\mathbf{C}_{m,p}^{'(l)}$ and $\mathbf{C}_{m,p}^{'(r)}$ the matrices associated with the transposed operator $\mathbf{C}'(L) := \sum_{i=0}^{\infty} \mathbf{C}_i' L^i$, we also have

$\mathbf{G}_m^{'(l)} = (\mathbf{A}_m^{'(l)})^{-1}$, $\mathbf{H}_m^{'(l)} = (\mathbf{B}_m^{'(l)})^{-1}$, etc. We will use the notation $\bar{\mathbf{C}}_{m,p}^{(l)}, \bar{\mathbf{C}}_{m,p}^{(r)}, \bar{\mathbf{C}}_m^{(l)}$, etc, when the identity matrices involved in (31) and (32) are m-dimensional rather than k-dimensional.

Let $\pi := \max(p_1 - p_0, q_1 - q_0)$ and $\pi_0 := \pi + p_0 + q_0$, and define the $k^2\pi_0 \times k^2(p_1 + q_1)$ matrix

$$\mathbf{M}_{\boldsymbol{\theta}} := \left(\mathbf{G}_{\pi_0,p_1}^{'(l)} \vdots \mathbf{H}_{\pi_0,q_1}^{'(l)} \right); \tag{33}$$

under Assumption (D2), $\mathbf{M}_{\boldsymbol{\theta}}$ is of full rank.

Consider the operator $\mathbf{D}(L) := \mathbf{I}_k + \sum_{i=1}^{p_0+q_0} \mathbf{D}_i\, L^i$ (just as $\mathbf{M}_{\boldsymbol{\theta}}$, $\mathbf{D}(L)$ and most quantities defined below depend on $\boldsymbol{\theta}$; for simplicity, however, we are dropping this reference to $\boldsymbol{\theta}$), where, putting $\mathbf{G}_{-1} = \mathbf{G}_{-2} = ... = \mathbf{G}_{-p_0+1} = \mathbf{0} = \mathbf{H}_{-1} = \mathbf{H}_{-2} = ... = \mathbf{H}_{-q_0+1}$,

$$\begin{pmatrix} \mathbf{D}_1^{'} \\ \vdots \\ \mathbf{D}_{p_0+q_0}^{'} \end{pmatrix} := - \begin{pmatrix} \mathbf{G}_{q_0} & \mathbf{G}_{q_0-1} & \cdots & \mathbf{G}_{-p_0+1} \\ \mathbf{G}_{q_0+1} & \mathbf{G}_{q_0} & \cdots & \mathbf{G}_{-p_0+2} \\ \vdots & & \ddots & \vdots \\ \mathbf{G}_{p_0+q_0-1} & \mathbf{G}_{p_0+q_0-2} & \cdots & \mathbf{G}_0 \\ \mathbf{H}_{p_0} & \mathbf{H}_{p_0-1} & \cdots & \mathbf{H}_{-q_0+1} \\ \mathbf{H}_{p_0+1} & \mathbf{H}_{p_0} & \cdots & \mathbf{H}_{-q_0+2} \\ \vdots & & \ddots & \vdots \\ \mathbf{H}_{p_0+q_0-1} & \mathbf{H}_{p_0+q_0-2} & \cdots & \mathbf{H}_0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{G}_{q_0+1} \\ \vdots \\ \mathbf{G}_{p_0+q_0} \\ \mathbf{H}_{p_0+1} \\ \vdots \\ \mathbf{H}_{p_0+q_0} \end{pmatrix}.$$

Note that $\mathbf{D}(L)\mathbf{G}_t^{'} = \mathbf{0}$ for $t = q_0 + 1, \ldots, p_0 + q_0$, and $\mathbf{D}(L)\mathbf{H}_t^{'} = \mathbf{0}$ for $t = p_0 + 1, \ldots, p_0 + q_0$.

Let $\{\boldsymbol{\Psi}_t^{(1)}, \ldots, \boldsymbol{\Psi}_t^{(p_0+q_0)}\}$ be a set of $k \times k$ matrices forming a fundamental system of solutions of the homogeneous linear difference equation associated with $\mathbf{D}(L)$ (such a system can be obtained, for instance, from the Green's matrices of the operator $\mathbf{D}(L)$: see Hallin 1986). Define

$$\bar{\boldsymbol{\Psi}}_m(\boldsymbol{\theta}) := \begin{pmatrix} \boldsymbol{\Psi}_{\pi+1}^{(1)} & \cdots & \boldsymbol{\Psi}_{\pi+1}^{(p_0+q_0)} \\ \boldsymbol{\Psi}_{\pi+2}^{(1)} & \cdots & \boldsymbol{\Psi}_{\pi+2}^{(p_0+q_0)} \\ \vdots & & \vdots \\ \boldsymbol{\Psi}_m^{(1)} & \cdots & \boldsymbol{\Psi}_m^{(p_0+q_0)} \end{pmatrix} \otimes \mathbf{I}_k \qquad (m > \pi),$$

$$\mathbf{P}_{\boldsymbol{\theta}} := \begin{pmatrix} \mathbf{I}_{k^2\pi} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_\Psi^{-1} \end{pmatrix}, \quad \text{and} \quad \mathbf{Q}_{\boldsymbol{\theta}}^{(n)} := \mathbf{H}_{n-1}^{(r)}\, \mathbf{B}_{n-1}^{'(l)} \begin{pmatrix} \mathbf{I}_{k^2\pi} & \mathbf{0} \\ \mathbf{0} & \bar{\boldsymbol{\Psi}}_{n-1} \end{pmatrix}, \tag{34}$$

where $\mathbf{C}_\Psi$ is the Casorati matrix $\bar{\boldsymbol{\Psi}}_{\pi_0}$.

Considering the matrices $\boldsymbol{\Lambda}_{i;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta})$ associated with the scores $K_0 = \varphi_f$, put

$$\mathbf{S}_{I;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta}) := \left(n^{1/2}\, (\text{vec}\, \boldsymbol{\Lambda}_{0;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta}))^{'}, \ldots, (n-i)^{1/2}\, (\text{vec}\, \boldsymbol{\Lambda}_{i;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta}))^{'}, \ldots, (\text{vec}\, \boldsymbol{\Lambda}_{n-1;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta}))^{'}\right)^{'},$$

$$n^{1/2}\mathbf{T}_{I;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta}) := \mathbf{L}_{\boldsymbol{\theta}}^{(n)'} \mathbf{S}_{I;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta}), \quad \text{and} \quad \mathbf{J}_{I;\boldsymbol{\theta},\boldsymbol{\Sigma}} := \lim_{n\to+\infty} \mathbf{L}_{\boldsymbol{\theta}}^{(n)'} (\boldsymbol{K}_n \otimes \boldsymbol{\Sigma}^{-1})\, \mathbf{L}_{\boldsymbol{\theta}}^{(n)}, \tag{35}$$

where $\mathbf{L}_{\boldsymbol{\theta}}^{(n)} := \bar{\mathbf{H}}_n^{(r)}(\boldsymbol{\theta}) \bar{\mathbf{A}}_{n,1}^{(r)}(\boldsymbol{\theta})$, and where $\boldsymbol{K}_{l\tilde{l}}$ denotes the $lm \times \tilde{l}m$ matrix whose $m \times m$ block in position (i, j) $(i = 1, \ldots, l,\ j = 1, \ldots, \tilde{l})$ is $\mathbf{K}^{'}\, \mathbf{R}_{|i-j|}\, \mathbf{K}$ (we write $\boldsymbol{K}_l$ instead of $\boldsymbol{K}_{ll}$). Similarly, for the serial part and the $\boldsymbol{\Gamma}_{i;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta})$ matrices associated with the score functions $K_1 = \varphi_f$ and $K_2 : d \mapsto d$, let

$$\mathbf{S}_{II;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta}) := \left((n-1)^{1/2}\, (\text{vec}\, \boldsymbol{\Gamma}_{1;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta}))^{'}, \ldots, (n-i)^{1/2}\, (\text{vec}\, \boldsymbol{\Gamma}_{i;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta}))^{'}, \ldots, (\text{vec}\, \boldsymbol{\Gamma}_{n-1;\boldsymbol{\Sigma},f}^{(n)}(\boldsymbol{\theta}))^{'}\right)^{'},$$

$$n^{1/2}\mathbf{T}^{(n)}_{II;\mathbf{\Sigma},f}(\boldsymbol{\theta}) := \mathbf{Q}^{(n)\prime}_{\boldsymbol{\theta}}\mathbf{S}^{(n)}_{II;\mathbf{\Sigma},f}(\boldsymbol{\theta}), \quad \text{and} \quad \mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}} := \lim_{n\to+\infty}\mathbf{Q}^{(n)\prime}_{\boldsymbol{\theta}}\,[\mathbf{I}_{n-1}\otimes(\mathbf{\Sigma}\otimes\mathbf{\Sigma}^{-1})]\,\mathbf{Q}^{(n)}_{\boldsymbol{\theta}} \tag{36}$$

(convergence in (35) and (36) follows from the exponential decrease, as $u\to\infty$, of the Green's matrices $\mathbf{G}_u$ and $\mathbf{H}_u$).

The following ULAN reinforcement of Garel and Hallin (1995)'s Proposition 3.1 then follows along the same steps as in Section 3 of Hallin and Paindaveine (2002d).

Proposition 3 (ULAN) *Assume that Assumptions (A1′), (A2), (A3), (D1), and (D2) hold. Let $\boldsymbol{\theta}_n$ be such that $\boldsymbol{\theta}_n-\boldsymbol{\theta}=O(n^{-1/2})$. Then, the logarithm $L^{(n)}_{\boldsymbol{\theta}_n+\boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)}/\boldsymbol{\theta}_n;\mathbf{\Sigma},f}$ of the likelihood ratio associated with the sequence of local alternatives $\mathcal{H}^{(n)}(\boldsymbol{\theta}_n+\boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)},\mathbf{\Sigma},f)$ with respect to $\mathcal{H}^{(n)}(\boldsymbol{\theta}_n,\mathbf{\Sigma},f)$ is such that*

$$L^{(n)}_{\boldsymbol{\theta}_n+\boldsymbol{\nu}(n)\boldsymbol{\tau}^{(n)}/\boldsymbol{\theta}_n;\mathbf{\Sigma},f}(\mathbf{Y}^{(n)}) = (\boldsymbol{\tau}^{(n)})'\mathbf{\Delta}^{(n)}_{\mathbf{\Sigma},f}(\boldsymbol{\theta}_n)-\frac{1}{2}(\boldsymbol{\tau}^{(n)})'\,\mathbf{\Gamma}_{\mathbf{\Sigma},f}(\boldsymbol{\theta})\,\boldsymbol{\tau}^{(n)}+o_{\mathrm{P}}(1),$$

as $n\to\infty$, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_n,\mathbf{\Sigma},f)$, with the central sequence

$$\mathbf{\Delta}^{(n)}_{\mathbf{\Sigma},f}(\boldsymbol{\theta}_n) := \begin{pmatrix}\mathbf{\Delta}^{(n)}_{I;\mathbf{\Sigma},f}(\boldsymbol{\theta}_n)\\ \mathbf{\Delta}^{(n)}_{II;\mathbf{\Sigma},f}(\boldsymbol{\theta}_n)\end{pmatrix} := n^{1/2}\begin{pmatrix}\mathbf{I}_{km} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}'_{\boldsymbol{\theta}_n}\mathbf{P}'_{\boldsymbol{\theta}_n}\end{pmatrix}\begin{pmatrix}\mathbf{T}^{(n)}_{I;\mathbf{\Sigma},f}(\boldsymbol{\theta}_n)\\ \mathbf{T}^{(n)}_{II;\mathbf{\Sigma},f}(\boldsymbol{\theta}_n)\end{pmatrix}, \tag{37}$$

and the information matrix

$$\mathbf{\Gamma}_{\mathbf{\Sigma},f}(\boldsymbol{\theta}) := \begin{pmatrix}\mathbf{\Gamma}_{I;\mathbf{\Sigma},f}(\boldsymbol{\theta}) & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}_{II;\mathbf{\Sigma},f}(\boldsymbol{\theta})\end{pmatrix},$$

where $\mathbf{\Gamma}_{I;\mathbf{\Sigma},f}(\boldsymbol{\theta}) := \frac{1}{k}\mathcal{I}_{k,f}\mathbf{J}_{I;\boldsymbol{\theta},\mathbf{\Sigma}}$ and $\mathbf{\Gamma}_{II;\mathbf{\Sigma},f}(\boldsymbol{\theta}) := \frac{\mu_{k+1;f}\,\mathcal{I}_{k,f}}{k^2\,\mu_{k-1;f}}\mathbf{N}_{\boldsymbol{\theta},\mathbf{\Sigma}}$, with $\mathbf{N}_{\boldsymbol{\theta},\mathbf{\Sigma}} := \mathbf{M}'_{\boldsymbol{\theta}}\mathbf{P}'_{\boldsymbol{\theta}}\mathbf{J}_{II;\boldsymbol{\theta},\mathbf{\Sigma}}\mathbf{P}_{\boldsymbol{\theta}}\mathbf{M}_{\boldsymbol{\theta}}$. Moreover, $\mathbf{\Delta}^{(n)}_{\mathbf{\Sigma},f}(\boldsymbol{\theta}_n)$, still under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_n,\mathbf{\Sigma},f)$, is asymptotically $\mathcal{N}_K(\mathbf{0},\mathbf{\Gamma}_{\mathbf{\Sigma},f}(\boldsymbol{\theta}))$.

Le Cam's third Lemma then yields, for the serial and nonserial statistics (9) and (10), the following asymptotic normality result under local alternatives.

Lemma 11 *Assume that (C) and the assumptions of Proposition 3 hold. Then, for all integers $l,\tilde{l}$, the vector*

$$\Big(n^{1/2}\,(\mathrm{vec}\,\widetilde{\mathbf{\Lambda}}^{(n)}_{0;J;\mathbf{\Sigma},f}(\boldsymbol{\theta}))',\ldots,(n-l+1)^{1/2}\,(\mathrm{vec}\,\widetilde{\mathbf{\Lambda}}^{(n)}_{l-1;J;\mathbf{\Sigma},f}(\boldsymbol{\theta}))',$$
$$(n-1)^{1/2}\,(\mathrm{vec}\,\widetilde{\mathbf{\Gamma}}^{(n)}_{1;J;\mathbf{\Sigma},f}(\boldsymbol{\theta}))',\ldots,(n-\tilde{l})^{1/2}\,(\mathrm{vec}\,\widetilde{\mathbf{\Gamma}}^{(n)}_{\tilde{l};J;\mathbf{\Sigma},f}(\boldsymbol{\theta}))'\Big)'$$

is asymptotically normal as $n\to\infty$, with mean $\mathbf{0}$ under $\mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},f)$ and mean

$$\begin{pmatrix}\frac{1}{k}\,C_k(J_0;f)(\mathbf{I}_{lm}\otimes\mathbf{\Sigma}^{-1})[\lim_{n\to\infty}(\boldsymbol{K}_{ln}\otimes\mathbf{I}_k)\,\mathbf{L}^{(n)}_{\boldsymbol{\theta}}]\,(\mathrm{vec}\,\boldsymbol{\eta}')\\ \frac{1}{k^2}\,C_k(J_1;f)\,D_k(J_2;f)\,[\mathbf{I}_{\tilde{l}}\otimes(\mathbf{\Sigma}\otimes\mathbf{\Sigma}^{-1})]\,\mathbf{Q}^{(\tilde{l}+1)}_{\boldsymbol{\theta}}\,\mathbf{P}_{\boldsymbol{\theta}}\,\mathbf{M}_{\boldsymbol{\theta}}\,((\mathrm{vec}\,\boldsymbol{\gamma})',(\mathrm{vec}\,\boldsymbol{\delta})')'\end{pmatrix},$$

under $\mathcal{H}^{(n)}(\boldsymbol{\theta}+\boldsymbol{\nu}(n)\boldsymbol{\tau},\mathbf{\Sigma},f)$, and covariance matrix

$$\begin{pmatrix}\frac{1}{k}\,\mathrm{E}[J_0^2(U)]\,(\boldsymbol{K}_l\otimes\mathbf{\Sigma}^{-1}) & \mathbf{0}\\ \mathbf{0} & \frac{1}{k^2}\,\mathrm{E}[J_1^2(U)]\,\mathrm{E}[J_2^2(U)]\;[\mathbf{I}_{\tilde{l}}\otimes(\mathbf{\Sigma}\otimes\mathbf{\Sigma}^{-1})]\end{pmatrix}.$$

under both.

Proof. The proof follows along the same argument as in Lemma 4.1 in Hallin and Garel (1995).□

Note that

$$\lim_{n\to\infty} (\boldsymbol{K}_{ln} \otimes \mathbf{I}_k)\, \mathbf{L}_{\boldsymbol{\theta}}^{(n)} = \begin{pmatrix} \sum_{j=0}^{\infty} (\mathbf{K}'\, \mathbf{R}_{|j|}\, \mathbf{K}) \otimes \mathbf{h}_j \\ \vdots \\ \sum_{j=0}^{\infty} (\mathbf{K}'\, \mathbf{R}_{|i-j|}\, \mathbf{K}) \otimes \mathbf{h}_j \\ \vdots \\ \sum_{j=0}^{\infty} (\mathbf{K}'\, \mathbf{R}_{|l-j-1|}\, \mathbf{K}) \otimes \mathbf{h}_j \end{pmatrix},$$

and that

$$\begin{pmatrix} \mathbf{a}_1(\boldsymbol{\tau};\boldsymbol{\theta}) + \mathbf{b}_1(\boldsymbol{\tau};\boldsymbol{\theta}) \\ \vdots \\ \mathbf{a}_{\tilde{l}}(\boldsymbol{\tau};\boldsymbol{\theta}) + \mathbf{b}_{\tilde{l}}(\boldsymbol{\tau};\boldsymbol{\theta}) \end{pmatrix} = \mathbf{Q}_{\boldsymbol{\theta}}^{(\tilde{l}+1)}\, \mathbf{P}_{\boldsymbol{\theta}}\, \mathbf{M}_{\boldsymbol{\theta}} \begin{pmatrix} \mathrm{vec}\,\boldsymbol{\gamma} \\ \mathrm{vec}\,\boldsymbol{\delta} \end{pmatrix}$$

(see Section 4 for the definitions of $\mathbf{h}_j$, $\mathbf{a}_j$, and $\mathbf{b}_j$). This allows for a direct comparison between Lemma 11 and the corresponding univariate result (Proposition 4.3) in Hallin and Puri (1994).

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