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AND COMMON PREFERENCES OF 
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INDEPENDENT UTILITIES 

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BIVARIATE STOCHASTIC DOMINANCE AND COMMON PREFERENCES OF DECISION-MAKERS WITH RISK INDEPENDENT UTILITIES

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Abstract

The close link between bivariate stochastic dominance relations and the common preferences of the decision-makers with independent multiattribute utility functions is discussed. Specifically, the common preferences of all the decision-makers with a utility function expressing risk independence are shown to coincide with bivariate stochastic dominance expressing correlation aversion. As an application, portfolios are compared to assess the possible hedging effect between two outcomes.

Key words and phrases: Bivariate ($s_1, s_2$)-increasing concave orders, bivariate stochastic dominance, conditional risk aversion, portfolios.

JEL code: D81.
1 Introduction and motivation

In this paper, we consider pairs \((X_1, X_2)\) of commodities, often called attributes. We assume that the first attribute is valued in the interval \([a_1, b_1]\) and the second one is valued in the interval \([a_2, b_2]\). For instance, \(X_1\) may represent wealth and \(X_2\) health as in ECKHOUTD, REY & SCHLESINGER (2007), or \(X_1\) and \(X_2\) may represent consumption at two different times. Sometimes, bivariate outcomes \((X_1, X_2)\) and \((Y_1, Y_2)\) representing financial consequences are compared after aggregation, that is, on the basis of the respective final wealths \(X_1 + X_2\) and \(Y_1 + Y_2\). Univariate expected utility theory can be used for that purpose. Aggregating the two outcomes is nevertheless not always optimal because preferences may not depend only on final wealths, but on the separate outcomes.

We assume that decision-makers act as if they maximize their respective expected utility. Specifically, a decision-maker with utility function \(u\) prefers \((Y_1, Y_2)\) over \((X_1, X_2)\) if \(\mathbb{E}[u(X_1, X_2)] \leq \mathbb{E}[u(Y_1, Y_2)]\), that is, if the expected utility for \((Y_1, Y_2)\) exceeds that for \((X_1, X_2)\). Faced with the difficulty to deal with general functions \(u\), many specific forms of utility functions have been suggested in the literature.

The easiest approach to evaluate bivariate consequences \((X_1, X_2)\) consists in postulating an additive form for the utility function \(u\), that is, \(u(x_1, x_2) = u_1(x_1) + u_2(x_2)\), where \(u_i\) is the utility function for the \(i\)th attribute, \(i = 1, 2\). Note that instead of adding the outcomes \(X_1\) and \(X_2\), their utilities \(u_1(X_1)\) and \(u_2(X_2)\) are added to compare the bivariate attributes. A decision-maker with such an additive utility function only considers the marginal distribution of the \(X_i\)’s, and not their actual dependence structure. The main advantage of the additive utility function is its simplicity: its assessment reduces to the elicitation of two one-dimensional utility functions \(u_1\) and \(u_2\). A major shortcoming of it is that it ignores the dependence structure of the \(X_i\)’s. Hedging effects cannot be appraised with such a utility function as the dependence structure does not enter the comparison. Specifically, if \((X_1, X_2)\) and \((Y_1, Y_2)\) have identical univariate marginals, that is,

\[
\Pr[X_1 \leq t] = \Pr[Y_1 \leq t] \text{ for all } t \quad \text{and} \quad \Pr[X_2 \leq t] = \Pr[Y_2 \leq t] \text{ for all } t, \quad (1.1)
\]

then

\[
\mathbb{E}[u(X_1, X_2)] = \mathbb{E}[u_1(X_1)] + \mathbb{E}[u_2(X_2)] = \mathbb{E}[u_1(Y_1)] + \mathbb{E}[u_2(Y_2)] = \mathbb{E}[u(Y_1, Y_2)]
\]

so that all the bivariate outcomes with the same marginals are considered as equivalent when additive utility functions are used.

Risk independence can be postulated to account for possible hedging effects, keeping the mathematical tractability. Let us denote as \(u^{(i,j)}\) the \((i, j)\)th mixed partial derivative of \(u\) with respect to \(x_1\) and \(x_2\), that is, \(u^{(i,j)} = \frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} u\). The conditional risk aversion for \(X_2\) given \(X_1 = x_1\) is defined as

\[
r_2(x_2|x_1) = -\frac{u^{(0,2)}(x_1, x_2)}{u^{(0,1)}(x_1, x_2)}.
\]

Similarly, the conditional risk aversion for \(X_1\) given \(X_2 = x_2\) is defined as

\[
r_1(x_1|x_2) = -\frac{u^{(2,0)}(x_1, x_2)}{u^{(1,0)}(x_1, x_2)}.
\]
Then, according to Keeney (1973), $X_2$ is said to be risk independent of $X_1$ if $r_2(x_2|x_1)$ does not depend on $x_1$. Similarly, $X_1$ is said to be risk independent of $X_2$ if $r_1(x_1|x_2)$ does not depend on $x_2$. If all the risk is associated with only one attribute and the other attribute is fixed, risk independence ensures that the decision-maker’s attitude toward risk only depends on that attribute involving the risk. This is also termed as utility independence since the decision-maker’s relative preferences over one attribute, when the other attribute is held fixed at some specified level, are the same regardless of that level.

Given $X_2$ is risk independent of $X_1$ and $X_1$ is risk independent of $X_2$ then the utility function $u$ can be expressed as a sum of univariate utility functions supplemented with their product. Specifically, we know from Theorem 1 in Keeney (1973) that such a utility function $u$ is either of the form

$$u(x_1, x_2) = u_1(x_1) + u_2(x_2) - \gamma(-u_1(x_1))(-u_2(x_2)), \quad \gamma > 0, \quad (1.2)$$

where both $u_1$ and $u_2$ are non-decreasing non-positive utility functions, or of the form

$$u(x_1, x_2) = u_1(x_1) + u_2(x_2) + \gamma u_1(x_1)u_2(x_2), \quad \gamma > 0, \quad (1.3)$$

where both $u_1$ and $u_2$ are non-decreasing non-negative utility functions. If the random couples $(X_1, X_2)$ and $(Y_1, Y_2)$ have the same univariate marginal distributions (that is, (1.1) is fulfilled), the first two terms in (1.2)-(1.3) do not matter and we are essentially interested in utility functions factoring into the product of two univariate utilities. This is why we also consider bivariate utilities of the form

$$u(x_1, x_2) = \alpha - \beta(-u_1(x_1))(-u_2(x_2)), \quad \alpha \in \mathbb{R}, \quad \beta > 0, \quad (1.4)$$

where both $u_1$ and $u_2$ are non-decreasing non-positive utility functions. Decision-makers with utilities of the form (1.4) consider the attributes as mutually utility independent and substitutional and tend to “diversify” between pairs of attributes. Diversification here means that gaining either a higher level of the first attribute and a lower level of the second one, or vice-versa is more desired than gaining either a lower or a higher level of both attributes. We know from Theorem 2 in Mosler (1984) that all the decision-makers with utilities of the form (1.4) prefer $(Y_1, Y_2)$ over $(X_1, X_2)$ if, and only if, the inequality

$$\Pr[X_1 \leq t_1, X_2 \leq t_2] \geq \Pr[Y_1 \leq t_1, Y_2 \leq t_2] \quad (1.5)$$

holds true for any thresholds $t_1$ and $t_2$.

It is easily seen from (1.4) that all the decision-makers with utility functions (1.4) prefer $(Y_1, Y_2)$ over $(X_1, X_2)$ if, and only if, the inequality

$$\mathbb{E}[h_1(X_1)h_2(X_2)] \geq \mathbb{E}[h_1(Y_1)h_2(Y_2)]$$

is valid for any non-negative non-increasing functions such that the expectations exist. Taking indicator of left-open intervals $(-\infty, t_i], \ i = 1, 2$, gives (1.5). In particular, this ensures that $Y_i$ dominates $X_i$ in the first-degree stochastic dominance, $i = 1, 2$. Provided both outcomes are non-negative, that is $a_i = 0$ for $i = 1, 2$, it can also be shown that all the decision-makers with utility functions (1.4) prefer $(Y_1, Y_2)$ over $(X_1, X_2)$ if, and only if,
\( \max \{w_1 Y_1, w_2 Y_2\} \) is larger than \( \max \{w_1 X_1, w_2 X_2\} \) in the first-degree stochastic dominance, whatever the weights \( w_1 > 0 \) and \( w_2 > 0 \), that is, the inequality
\[
\Pr[\max \{w_1 X_1, w_2 X_2\} > t] \leq \Pr[\max \{w_1 Y_1, w_2 Y_2\} > t]
\]
is valid for any threshold \( t \). Decision-makers with utilities (1.4) thus base their choice on the performance of the maximum of the two weighted outcomes.

We also consider utility functions of the form
\[
u(x_1, x_2) = \alpha + \beta u_1(x_1) u_2(x_2), \quad \alpha \in \mathbb{R}, \quad \beta > 0,
\]
(1.6)
where both \( u_1 \) and \( u_2 \) are non-decreasing non-negative utility functions. Decision-makers with utility functions of the form (1.6) consider the attributes as mutually utility independent and complementary: such a decision-maker prefers gaining either a lower or a higher level of both attributes rather than gaining either a higher level of the first attribute and a lower level of the second, or vice-versa. We know from Theorem 3 in Mosler (1984) that decision-makers with utilities of the form (1.6) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\) if, and only if, the inequality
\[
\Pr[X_1 > t_1, X_2 > t_2] \leq \Pr[Y_1 > t_1, Y_2 > t_2]
\]
holds true for any thresholds \( t_1 \) and \( t_2 \).

It is easily seen from (1.6) that all the decision-makers with utility functions (1.6) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\) if, and only if, the inequality
\[
\mathbb{E}[g_1(X_1) g_2(X_2)] \leq \mathbb{E}[g_1(Y_1) g_2(Y_2)]
\]
is valid for any non-negative non-decreasing functions \( g_1 \) and \( g_2 \) such that the expectations exist. Taking indicator of the right-open intervals \((t_i, +\infty), \ i = 1, 2\), gives (1.7). In particular, this ensures that \( Y_i \) dominates \( X_i \) in the first-degree stochastic dominance, \( i = 1, 2 \). Provided both outcomes are non-negative, that is, \( a_i = 0 \) for \( i = 1, 2 \), it can also be shown that all the decision-makers with utility functions (1.6) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\) if, and only if, \( \min \{w_1 Y_1, w_2 Y_2\} \) is larger than \( \min \{w_1 X_1, w_2 X_2\} \) in the first-degree stochastic dominance, whatever the weights \( w_1 > 0 \) and \( w_2 > 0 \), that is, the inequality
\[
\Pr[\min \{w_1 X_1, w_2 X_2\} > t] \leq \Pr[\min \{w_1 Y_1, w_2 Y_2\} > t]
\]
is valid for any threshold \( t \). Decision-makers with utilities (1.6) thus base their choice on the performance of the minimum of the two weighted outcomes.

Now, if \((X_1, X_2)\) and \((Y_1, Y_2)\) have the same univariate marginal distributions (that is, (1.1) is fulfilled) then the identity
\[
\Pr[Y_1 \leq t_1, Y_2 \leq t_2] - \Pr[X_1 \leq t_1, X_2 \leq t_2] = \Pr[Y_1 > t_1, Y_2 > t_2] - \Pr[X_1 > t_1, X_2 > t_2]
\]
is valid for any thresholds \( t_1 \) and \( t_2 \). We then deduce from (1.5)-(1.7) that all the decision-makers with utility function (1.4) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\) if, and only if, all the decision-makers with utility function (1.6) prefer \((X_1, X_2)\) over \((Y_1, Y_2)\). Once the marginal distributions are fixed, the only difference between \((X_1, X_2)\) and \((Y_1, Y_2)\) lies in the respective dependence structures. We then see that the dependence structure of \((Y_1, Y_2)\) is preferred.
over the dependence structure of \((X_1, X_2)\) by all the decision-makers with utility function (1.4) if, and only if, the dependence structure of \((X_1, X_2)\) is preferred over the dependence structure of \((Y_1, Y_2)\) by all the decision-makers with utility function (1.6).

Mosler (1984) also established that if all the decision-makers with utility functions of the form (1.4) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\) then the same conclusion is reached by a much broader class of economic agents with utility dependent preferences, namely those with a non-decreasing utility function such that \(u^{(1,1)} \leq 0\). Similarly, all the decision-makers with utility functions of the form (1.6) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\) if, and only if, all the economic agents with a non-decreasing utility function such that \(u^{(1,1)} \geq 0\) exhibit the same preferences. This means that the results derived under the assumptions of utility independence and given substitutional structure carry over to more general situations where utility independence disappears and only the signs of certain partial derivatives remain given (which determine the substitutional structure).

In this paper, we concentrate on decision-makers with utility function (1.4). Our aim is to extend the results obtained by Mosler (1984) to products of univariate utilities \(u_i\)’s expressing some desirable properties such as risk aversion, prudence and temperance. We will show that these common preferences agree with the preferences expressed by a much larger class of decision-makers, expressing some correlation aversion, recently studied in Denuit, Eeckhoudt & Rey (2008).

The text is organized as follows. Section 2 briefly recalls the basic results when random variables are considered (univariate stochastic dominance), whereas Section 3 discusses a bivariate extension where pairs of random variables with identical marginals are considered. Section 4 establishes that the bivariate stochastic dominance rules introduced in Section 3 agree with the common preferences of decision-makers with multiplicative utilities. The close link to correlation aversion is discussed in Section 5. As an application, portfolios are compared in Section 6 to assess the hedging effects existing between the two outcomes. The final Section 7 concludes.

2 Univariate stochastic dominance: Common preferences of decision-makers with \(s\)-increasing concave utility functions

2.1 Classes of differentiable \(s\)-increasing concave utility functions

For \(s = 1, 2, \ldots\), let us define the class \(U_{s-icv}\) of the differentiable \(s\)-increasing concave utility functions as the class containing all the functions \(u\) defined on the real line such that \((-1)^{k+1}u^{(k)} \geq 0\) for \(k = 1, 2, \ldots, s\), where \(u^{(k)}\) stands for the \(k\)th derivative of \(u\). The class \(U_{s-icv}\) thus contains the non-decreasing functions with derivatives of degrees 1 to \(s\) with alternating signs. All the commonly used utility functions belong to \(U_{s-icv}\) for all \(s\) (as their derivatives alternate in sign, beginning with positive marginal utility). For instance, all the completely monotone utility functions, including the logarithmic, exponential and power utility functions belong to \(U_{s-icv}\) for all \(s\).
2.2 Univariate $s$-increasing concave orders

Consider two random variables $X$ and $Y$, many stochastic orderings $\preceq_s$ can be defined by reference to some class $\mathcal{U}_s$ of measurable functions $u$ by

$$X \preceq_s Y \Leftrightarrow \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \text{ for all } u \in \mathcal{U}_s,$$  \hspace{1cm} (2.1)

provided that the expectations exist. If $X \preceq_s Y$ holds then $Y$ is said to dominate $X$ with respect to $\mathcal{U}_s$. Note that whether $X \preceq_s Y$ holds or not depends on the marginal distributions of $X$ and $Y$ only, and not on their actual dependence structure. Orderings defined by (2.1) are particularly appealing in expected utility theory, considering $\mathcal{U}_s$ as a set of utility functions $u$. Then, $\preceq_s$ allows to analyze risky decisions for all the agents whose utility function belongs to $\mathcal{U}_s$, without a complete elicitation of the individual utilities.

Recall from Denuit, Lefèvre & Shaked (1998) that the $s$-increasing concave order is defined from (2.1), taking for $\mathcal{U}_s$ the class $\mathcal{U}_{s-icv}$. More precisely, given two random variables $X$ and $Y$, $X$ is said to be smaller than $Y$ in the $s$-increasing concave order, denoted by $X \preceq_{s-icv} Y$ when $\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ for all the functions $u$ in $\mathcal{U}_{s-icv}$, provided the expectations exist. When the first $s-1$ moments are equal, the $s$-increasing concave orders are termed as the $s$-concave orders and coincide with the increasing $s$th degree risk of Ekern (1980). The convex counterparts of the $\preceq_{s-icv}$ orders have been thoroughly studied by Denuit, Lefèvre & Shaked (1998) and Denuit, De Vijlder & Lefèvre (1999).

For $s = 1$ to 4, the order relations $\preceq_{s-icv}$ express the common preferences of well-known classes of decision-makers. For $s = 1$, $\preceq_{1-icv}$ expresses the common preferences of all the profit-seeking decision-makers. For $s = 2$, $\preceq_{2-icv}$ reflects the common preferences of all the profit-seeking risk-averse decision-makers, i.e. those preferring $\mathbb{E}[X]$ over $X$, whatever $X$. For $s = 3$, $\preceq_{3-icv}$ expresses the common preferences of all the profit-seeking, risk-averse and prudent decision-makers. Finally, for $s = 4$, $\preceq_{4-icv}$ expresses the common preferences of all the profit-seeking, risk-averse, prudent and temperant decision-makers.

2.3 Maximal generator of $\preceq_{s-icv}$

Clearly, the properties of the integral orderings $\preceq_{s-icv}$ are inherited from the structures of the generating classes of functions $\mathcal{U}_{s-icv}$. The closure $\overline{\mathcal{U}}_{s-icv}$ of $\mathcal{U}_{s-icv}$ (with respect to the topology of pointwise convergence) contains all the utility functions $u$ such that $u^{(s-2)}$ exists, $(-1)^{k+1}u^{(k)} \geq 0$ for $k = 1, \ldots, s-2$, and $(-1)^su^{(s-2)}$ is concave (with the convention that $\overline{\mathcal{U}}_{1-icv}$ contains all the non-decreasing functions and $\overline{\mathcal{U}}_{2-icv}$ contains all the non-decreasing concave functions). The class $\overline{\mathcal{U}}_{s-icv}$ can be characterized by sign properties of divided differences. Recall that the $k$th divided difference, $k \geq 1$, of the function $u$ at distinct points $x_0, x_1, \ldots, x_k$, denoted by $[x_0, x_1, \ldots, x_k]u$, is defined recursively by

$$[x_0, x_1, \ldots, x_k]u = \frac{[x_1, x_2, \ldots, x_k]u - [x_0, x_1, \ldots, x_{k-1}]u}{x_k - x_0},$$  \hspace{1cm} (2.2)

starting from $[x_i]u = u(x_i), \ i = 0, 1, \ldots, k$. These divided differences extend derivatives to less regular functions. Then, $u \in \overline{\mathcal{U}}_{s-icv}$ if, and only if, $(-1)^{k+1}[x_0, x_1, \ldots, x_k]u \geq 0$ for any $x_1, x_1, \ldots, x_k, \ k = 1, 2, \ldots, s$.
Then, the class \( \mathcal{U}_{s-icv} \) of the \( s \)-increasing concave functions is the largest class of functions for which the implication \( X \preceq_{s-icv} Y \Rightarrow \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \) holds true for every ordered pair \((X, Y)\). For this reason, \( \mathcal{U}_{s-icv} \) is often called the maximal generator of the order \( \preceq_{s-icv} \). This means that \( \mathcal{U}_{s-icv} \) corresponds to the largest class of decision-makers whose preferences are in accordance with \( \preceq_{s-icv} \).

\[ \text{Characterization 2.1.} \]  
\[ \text{Given two random variables } X \text{ and } Y \text{ valued in } [a, b], \text{ we have } X \preceq_{s-icv} Y \text{ if, and only if, } \mathbb{E}[(b - X)^k] \geq \mathbb{E}[(b - Y)^k] \text{ for } k = 1, \ldots, s - 1 \text{ and } \mathbb{E}[(t - X)^{s-1}] \geq \mathbb{E}[(t - Y)^{s-1}] \text{ for all } t. \]

**Proof.** As pointed out in Denuit, Lefèvre & Mesfioui (1999, equivalence (2.9)), \( X \preceq_{s-icv} Y \) if, and only if, the inequality \( \mathbb{E}[g(b - X)] \geq \mathbb{E}[g(b - Y)] \) holds true for any function \( g \) with positive derivatives of degree 1 to \( s \), that is, \( g^{(k)}(b) \geq 0 \) for \( k = 1 \) to \( s \). Now, whatever the random variable \( Z \) valued in the interval \([0, c]\), the following expansion is valid:

\[
\mathbb{E}[g(Z)] = \sum_{k=0}^{s-1} \frac{g^{(k)}(0)}{k!} \mathbb{E}[Z^k] + \int_{t=0}^{c} \mathbb{E}[(Z - t)^{s-1}] \frac{g^{(s)}(t)}{(s-1)!} dt. \tag{2.3}
\]

Replacing \( Z \) with \( b - X \) and \( c \) with \( b - a \) gives the announced result. \( \Box \)

Let us denote as \( \mathcal{U}_{s-icv} \) the class of functions \( x \mapsto -(b - x)^k \) for \( k = 1, \ldots, s - 1 \), and \( x \mapsto -(t - x)^{s-1} \) for some real \( t \). Then, we see that \( X \preceq_{s-icv} Y \) holds true if, and only if, (2.1) is satisfied with \( \mathcal{U}_s = \mathcal{U}_{s-icv} \). This means that \( \mathcal{U}_{s-icv} \) corresponds to the decision-makers leading the common preferences of those in \( \mathcal{U}_{s-icv} \). A utility function of the form \( x \mapsto -(t - x)^{s-1} \) has a relatively simple interpretation. The economic agent has a threshold \( t \) and wishes to attain it. Once \( t \) is attained, the decision-maker is satisfied and does not value any exceedances over \( t \). However, if the wealth \( x \) does not attain \( t \) then the decision-maker is not fully satisfied and values the situation as \( -(t - x)^{s-1} \).

Characterization 2.1 allows us to relate \( \preceq_{s-icv} \) to the integrated left tails of the underlying distributions. Recall that the integrated left tails \( F_X^{[k]} \), \( k = 1, 2, \ldots \), of the distribution function \( F_X \) for the random variable \( X \) are defined by \( F_X^{[k]}(t) = \int_{x=-\infty}^{t} F_X^{[k-1]}(x) dx, k = 2, 3, \ldots \), starting from \( F_X^{[1]}(t) = F_X(t) \). It can be shown that, for each \( k \), either \( F_X^{[k]}(t) \) is non-negative and finite for all \( t \) or else \( F_X^{[k]}(t) = +\infty \) for all \( t \). The finiteness of \( F_X^{[k]} \) depends only on the left tail of \( F_X \). Moreover, \( F_X^{[k]} \) is finite if \( \mathbb{E}[X^k] \) exists.

It can be shown by induction and Fubini’s Theorem that

\[
F_X^{[s]}(t) = \frac{\mathbb{E}[(t - X)^{s-1}]}{(s-1)!}. \tag{2.4}
\]

Hence, Characterization 2.1 can be rephrased as follows.
Characterization 2.2. Given two random variables $X$ and $Y$, we have $X \preceq_{s-icv} Y$ if, and only if, $F_X^{[k]}(b) \geq F_Y^{[k]}(b)$ for $k = 1, \ldots, s - 1$ and $F_X^{[s]}(t) \geq F_Y^{[s]}(t)$ for all $t$.

Characterization 2.2 allows us to compare $\preceq_{s-icv}$ to the stochastic dominance relations introduced in Rolski (1976) and Fishburn (1976, 1980). Compared to $\preceq_{s-icv}$, these orderings are defined by pointwise comparison of $F_X^{[s]}$ and $F_Y^{[s]}$, without imposing any constraints on $F_X^{[k]}(b)$ and $F_Y^{[k]}(b)$ for $k = 1, \ldots, s - 1$. This in turn implies additional requirements on the tails of the utility functions agreeing with these orderings, which are avoided with $\preceq_{s-icv}$.

Considering the three sets of utility functions $\mathcal{U}_{s-icv}$, $\mathcal{U}_{s-icv}$, and $\mathcal{U}_{s-icv}$, we clearly have $\mathcal{U}_{s-icv} \subset \mathcal{U}_{s-icv}$ and $\mathcal{U}_{s-icv} \subset \mathcal{U}_{s-icv}$. Decision-makers with utilities in $\mathcal{U}_{s-icv}$ lead the common preferences of those with utilities in $\mathcal{U}_{s-icv}$, in the sense that

$$ X \preceq_{s-icv} Y \iff \text{(2.1) holds with } \mathcal{U}_* = \mathcal{U}_{s-icv} $$

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3 Bivariate stochastic dominance: Common preferences of decision-makers with $(s_1, s_2)$-increasing concave utility functions

3.1 Classes of differentiable $(s_1, s_2)$-increasing concave functions

Let us now consider utility functions $u$ defined on the real plane. This allows to account for bidimensional consequences. The class $\mathcal{U}_{(s_1, s_2)-icv}$ of the differentiable $(s_1, s_2)$-increasing concave functions is defined as the class of all the functions $u$ such that $(-1)^{k_1 + k_2 + 1} u^{(k_1, k_2)}(t) \geq 0$ for all $k_1 = 0, \ldots, s_1$, $k_2 = 0, \ldots, s_2$, such that $k_1 + k_2 \geq 1$.

Eeckhoudt, Rey & Schlesinger (2007) provided equivalence between the signs of the cross-derivatives $u^{(k_1, k_2)}$ and individual preferences within a particular class of simple lotteries. This leads to the concepts of cross-prudence and cross-temperance, exhibited by the elements of $\mathcal{U}_{(s_1, s_2)-icv}$ for $s_1$ and $s_2$ large enough. Specifically, an individual is said to be correlation averse if, and only if, $u^{(1,1)}(t) \leq 0$, cross-prudent in the second outcome if, and only if, $u^{(2,1)}(t) \geq 0$, cross-prudent in the first outcome if, and only if, $u^{(1,2)}(t) \geq 0$, and cross-temperant if, and only if, $u^{(2,2)}(t) \leq 0$. We come back to these issues in Section 5, where the meaning of utilities in $\mathcal{U}_{(s_1, s_2)-icv}$ is discussed.

3.2 Bivariate $(s_1, s_2)$-increasing concave orders

Univariate orderings (2.1) naturally extend to dimension 2. Specifically, let $\mathcal{U}_*$ be a class of measurable functions and let $(X_1, X_2)$ and $(Y_1, Y_2)$ be a pair of bivariate random vectors. Then, $(X_1, X_2)$ is said to be smaller than $(Y_1, Y_2)$ for the integral stochastic ordering $\preceq_*$ generated by $\mathcal{U}_*$ when

$$ \mathbb{E}[u(X_1, X_2)] \leq \mathbb{E}[u(Y_1, Y_2)] \text{ for all } u \in \mathcal{U}_*, $$

(3.1)
for which the expectations exist. As in the univariate case, these orderings are particularly appealing in expected utility theory, considering $U_*$ as a set of bivariate utility functions $u$.

Recall from Denuit, Lefèvre & Mesfioui (1999) that the bivariate $(s_1, s_2)$-increasing concave order is defined by (3.1) taking $U_* = \mathcal{U}(s_1, s_2)$-icv. Specifically, $(X_1, X_2)$ is said to be smaller than $(Y_1, Y_2)$ in the $(s_1, s_2)$-increasing concave ordering, denoted by $(X_1, X_2) \preceq_{(s_1, s_2)} \text{icv} (Y_1, Y_2)$, when $\mathbb{E}[u(X_1, X_2)] \leq \mathbb{E}[u(Y_1, Y_2)]$ for all the utility functions $u$ in $\mathcal{U}(s_1, s_2)$-icv, provided the expectations exist. It is easily seen that

$$(X_1, X_2) \preceq_{(s_1, s_2)} \text{icv} (Y_1, Y_2) \Rightarrow X_1 \preceq_{s_1} Y_1 \text{ and } X_2 \preceq_{s_2} \text{icv} Y_2$$

so that each component can be compared through univariate $s_1$- and $s_2$-increasing concave order.

We indicate that some special cases of the orderings $\preceq_{(s_1, s_2)} \text{icv}$ have been considered before in economics, e.g., in Atkinson & Bourguignon (1982) and Mosler (1984). The $(1,1)$-increasing concave order corresponds to the $\preceq_m$ order relation in Mosler (1984), and the $(2,2)$-increasing concave order corresponds to the $\preceq_{m3}$ order relation in Mosler (1984). In applied probability, the $(1,1)$-increasing concave order is known as the lower orthant order, while the $(2,2)$-increasing concave order is termed as the orthant concave order.

### 3.3 Maximal generator of $\preceq_{(s_1, s_2)} \text{icv}$

The maximal generator of $\preceq_{(s_1, s_2)} \text{icv}$ corresponds to the class $\mathcal{U}_{(s_1, s_2)} \text{icv}$ of bivariate $(s_1, s_2)$-increasing concave utility functions, defined by means of bivariate divided differences. Recall that the bivariate divided difference operator $[:,:]$ is defined by

$$
\begin{bmatrix}
x_0, \ldots, x_{s_1} \\
y_0, \ldots, y_{s_2}
\end{bmatrix} u = [x_0, \ldots, x_{s_1}] \circ [y_0, \ldots, y_{s_2}] u = [y_0, \ldots, y_{s_2}] \circ [x_0, \ldots, x_{s_1}] u.
$$

Now, the class $\mathcal{U}_{(s_1, s_2)} \text{icv}$ contains all the bivariate $(s_1, s_2)$-increasing concave utility functions, that is, the utilities $u$ such that the inequality

$$(-1)^{k_1+k_2+1} \begin{bmatrix} x_0, \ldots, x_{k_1} \\
y_0, \ldots, y_{k_2}
\end{bmatrix} u \geq 0$$

holds for all $x_0, \ldots, x_{k_1}, y_0, \ldots, y_{k_2}$, $k_1 = 0, \ldots, s_1$, $k_2 = 0, \ldots, s_2$ such that $k_1 + k_2 \geq 1$.

### 3.4 Characterization through integrated left tails

Let us first characterize the $(s_1, s_2)$-increasing concave order. Characterizing such an integral stochastic orderings generally consists in substituting for the generating cone of functions $\mathcal{U}_{(s_1, s_2)} \text{icv}$ either a dense subclass contained in $\mathcal{U}_{(s_1, s_2)} \text{icv}$, or a larger cone corresponding to the closure of $\mathcal{U}_{(s_1, s_2)} \text{icv}$ in some suitable topology. In the latter case, we get the maximal generator $\overline{\mathcal{U}}_{(s_1, s_2)} \text{icv}$ of $\preceq_{(s_1, s_2)} \text{icv}$. The next result identifies a subclass of the $(s_1, s_2)$-increasing concave functions which can be used as a test set to establish whether a $\preceq_{(s_1, s_2)} \text{icv}$ relation holds in a particular situation.
Characterization 3.1. Let \((X_1, X_2)\) and \((Y_1, Y_2)\) be two random vectors valued in \([a_1, b_1] \times [a_2, b_2]\). Then, \((X_1, X_2) \preceq_{(s_1, s_2) - \text{icv}} (Y_1, Y_2)\) if, and only if,

\[
\begin{align*}
\mathbb{E}[(b_1 - X_1)^{i_1}(b_2 - X_2)^{i_2}] & \geq \mathbb{E}[(b_1 - Y_1)^{i_1}(b_2 - Y_2)^{i_2}]
\text{for } i_1 = 0, \ldots, s_1 - 1 \text{ and } i_2 = 0, \ldots, s_2 - 1 \\
\text{and every } t_2 \geq 0
\end{align*}
\]

Proof. Recall from Property 2.3 in Denuit, Lefèvre & Mesfioui (1999) that \((X_1, X_2) \preceq_{(s_1, s_2) - \text{icv}} (Y_1, Y_2)\) holds if, and only if,

\[
\mathbb{E}[g(b_1 - Y_1, b_2 - Y_2)] \leq \mathbb{E}[g(b_1 - X_1, b_2 - X_2)]
\]

for any function \(g\) such that \(g^{(k_1, k_2)} \geq 0\) for all \(k_1 = 0, \ldots, s_1, k_2 = 0, \ldots, s_2\), such that \(k_1 + k_2 \geq 1\). Now, whatever the random variables \(Z_1\) and \(Z_2\) valued in \([0, c_1]\) and \([0, c_2]\), the expectation \(\mathbb{E}[g(Z_1, Z_2)]\) can be represented as

\[
\begin{align*}
\mathbb{E}[g(Z_1, Z_2)] & = \sum_{i_1=0}^{s_1-1} \sum_{i_2=0}^{s_2-1} g^{(i_1, i_2)}(0, 0) \frac{\mathbb{E}[Z_1^{i_1}Z_2^{i_2}]}{i_1! i_2!} \\
& + \sum_{i_1=0}^{s_1-1} \int_{t_2=0}^{c_2} \frac{\mathbb{E}[(Z_2 - t_2)^{s_2-1}Z_1^{i_1}]}{(s_2-1)! i_1!} g^{(i_1, i_2)}(0, t_2)dt_2 \\
& + \sum_{i_2=0}^{s_2-1} \int_{t_1=0}^{c_1} \frac{\mathbb{E}[(Z_1 - t_1)^{s_1-1}Z_2^{i_2}]}{(s_1-1)! i_2!} g^{(i_1, i_2)}(t_1, 0)dt_1 \\
& + \int_{t_1=0}^{c_1} \int_{t_2=0}^{c_2} \frac{\mathbb{E}[(Z_1 - t_1)^{s_1-1}(Z_2 - t_2)^{s_2-1}]}{(s_1-1)! (s_2-1)!} g^{(i_1, i_2)}(t_1, t_2)dt_2dt_1.
\end{align*}
\]

Now, replacing \(Z_i\) with \(b_i - X_i\) and \(c_i\) with \(b_i - a_i\), \(i = 1, 2\), gives the announced result. \(\square\)

If we define \(\mathcal{U}_{(s_1, s_2) - \text{icv}}\) as the set of utilities \((x_1, x_2) \mapsto - (b_1 - x_1)^{i_1}(b_2 - x_2)^{i_2}\) for \(i_1 = 0, \ldots, s_1 - 1\) and \(i_2 = 0, \ldots, s_2 - 1\), \((x_1, x_2) \mapsto - (b_1 - x_1)^{i_1}(t_2 - x_2)^{s_2-1}\) for \(i_1 = 0, \ldots, s_1 - 1\) and fixed \(t_2\), \((x_1, x_2) \mapsto - (t_1 - x_1)^{s_1-1}(b_2 - x_2)^{i_2}\) for \(i_2 = 0, \ldots, s_2 - 1\) and fixed \(t_1\), and \((x_1, x_2) \mapsto - (t_1 - x_1)^{s_1-1}(t_2 - x_2)^{s_2-1}\) for some fixed \(t_1\) and \(t_2\), we have that

\[
(X_1, X_2) \preceq_{(s_1, s_2) - \text{icv}} (Y_1, Y_2) \iff (3.1)\ \text{holds with } \mathcal{U}_c = \mathcal{U}_{(s_1, s_2) - \text{icv}}
\]

It can be shown that the expectations of elements of \(\mathcal{U}_{(s_1, s_2) - \text{icv}}\) are related to the bivariate integrated left tails. Specifically, let \((X_1, X_2)\) be a random vector valued \([a_1, b_1] \times [a_2, b_2]\),


with distribution function \( F(x_1, x_2) \). For \((x_1, x_2) \in [a_1, b_1] \times [a_2, b_2]\), we put \( F^{(1,1)}(x_1, x_2) \equiv F(x_1, x_2)(x_1, x_2) \), and we define recursively for \( s_1 \) and \( s_2 \geq 1 \),

\[
F^{(s_1+1,s_2)}(t_1, t_2) = \int_{y_1=a_1}^{t_1} F^{(s_1,s_2)}(y_1, t_2) dy_1, \tag{3.3}
\]

\[
F^{(s_1,s_2+1)}(t_1, t_2) = \int_{y_2=a_2}^{t_2} F^{(s_1,s_2)}(t_1, y_2) dy_2. \tag{3.4}
\]

It is easily shown that

\[
F^{(k_1,k_2)}(t_1, t_2) = \mathbb{E} \left[ (t_1 - X_1)^{k_1-1} (t_2 - X_2)^{k_2-1} \right] \frac{1}{(k_1-1)!(k_2-1)!}, \tag{3.5}
\]

which extends Lemma 1 in Scarsini (1985) to the case of unequal \( k_1 \) and \( k_2 \). Characterization 3.1 then allows us to state the next result.

**Characterization 3.2.** Let \((X_1, X_2)\) and \((Y_1, Y_2)\) be two random vectors valued in \([a_1, b_1] \times [a_2, b_2]\). Then

\[
(X_1, X_2) \preceq_{(s_1,s_2)-icv} (Y_1, Y_2) \iff \begin{cases} 
F^{(k_1,k_2)}(t_1, b_2) \geq F^{(k_1,k_2)}(b_1, b_2), \\
& \text{for } k_1 = 0, \ldots, s_1, \text{ } k_2 = 0, \ldots, s_2, \\
F^{(k_1,s_2)}(b_1, t_2) \geq F^{(k_1,s_2)}(b_1, t_2), \\
& \text{for all } t_2 \text{ and } k_1 = 1, \ldots, s_1, \\
F^{(s_1,k_2)}(t_1, b_2) \geq F^{(s_1,k_2)}(t_1, b_2), \\
& \text{for all } t_1 \text{ and } k_2 = 1, \ldots, s_2, \\
F^{(s_1,s_2)}(t_1, t_2) \geq F^{(s_1,s_2)}(t_1, t_2), \\
& \text{for all } t_1 \text{ and } t_2.
\end{cases}
\]

Scarsini (1985) developed a multivariate extension of the higher degree stochastic dominance introduced by Rolski (1976) and Fishburn (1976, 1980). The orderings by Scarsini (1985) however differ from the \( \preceq_{(s_1,s_2)-icv} \) orderings examined here in that they take \( s_1 = s_2 = s \) and impose only the fourth condition in Characterization 3.2.

### 4 Conditional risk aversion, risk independence, and bivariate stochastic dominance

Characterization 3.1 shows that the \((s_1,s_2)\)-increasing concave order is a product stochastic ordering, as it is precisely stated next.

**Proposition 4.1.** Let \((X_1, X_2)\) and \((Y_1, Y_2)\) be two random vectors valued in \([a_1, b_1] \times [a_2, b_2]\). Then,

\[
(X_1, X_2) \preceq_{(s_1,s_2)-icv} (Y_1, Y_2) \iff \mathbb{E}[u_1(X_1)u_2(X_2)] \geq \mathbb{E}[u_1(Y_1)u_2(Y_2)]
\]

for all the non-positive \( u_1 \in \mathcal{U}_{s_1-icv} \) and the non-positive \( u_2 \in \mathcal{U}_{s_2-icv} \) or, equivalently, for all the non-positive \( u_1 \in \mathcal{U}_{s_1-icv} \) and the non-positive \( u_2 \in \mathcal{U}_{s_2-icv} \).
This is a direct consequence of Characterization 3.1, noting that if \( u_1 \in U_{s_1-icv} \) and \( u_2 \in U_{s_2-icv} \) and both \( u_1 \) and \( u_2 \) are non-positive, then \( u = -u_1u_2 \in U_{(s_1,s_2)-icv} \). Note that we have to control the sign of both \( u_1 \) and \( u_2 \), to fix the sign of \( u^{(k_1,0)} = -u_1^{(k_1)}u_2 \) and of \( u^{(0,k_2)} = -u_1u_2^{(k_2)} \). We are now ready to establish that the common preferences of all the decision-makers with risk-independent utility function are equivalent with those of all the decision-makers with utility function in \( \overline{U}_{(s_1,s_2)-icv} \).

**Corollary 4.2.** Let \((X_1, X_2)\) and \((Y_1, Y_2)\) be two random vectors valued in \([a_1, b_1] \times [a_2, b_2]\). Then, \((X_1, X_2) \preceq_{(s_1,s_2)-icv} (Y_1, Y_2)\) holds if, and only if, all the decision-makers whose utility function is of the form (1.4) with \( u_1 \in U_{s_1-icv} \) and \( u_2 \in U_{s_2-icv} \) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\).

The proof of Corollary 4.2 is a direct application of Proposition 4.1.

## 5 Correlation aversion

### 5.1 Risk aversion

There are several concepts of risk aversion in the multivariate case. In this section, we discuss some of them. We refer the reader interested in a detailed study to DORFLEITNER & KRAPP (2007).

When the decision-maker faces bivariate outcomes, RICHARD (1975) relates risk aversion to substitutability of goods. More precisely, considering any \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \), and defining the lotteries

\[
\mathcal{L}_1 = \begin{cases} 
(x_1, y_1) \text{ with probability 0.5} \\
(x_2, y_2) \text{ with probability 0.5}
\end{cases} \quad \text{and} \quad \mathcal{L}_2 = \begin{cases} 
(x_1, y_2) \text{ with probability 0.5} \\
(x_2, y_1) \text{ with probability 0.5}
\end{cases}
\]

RICHARD (1975) required preference of \( \mathcal{L}_2 \) over \( \mathcal{L}_1 \) for risk aversion, which means that the bivariate utility function has to be submodular, i.e. the inequality

\[
u(x_1, y_1) + \nu(x_2, y_2) \leq \nu(x_1, y_2) + \nu(x_2, y_1)
\]

has to hold for any \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \). If \( \nu \) is twice differentiable, it is easily seen that this is equivalent to \( \nu^{(1,1)} \leq 0 \). The utilities in \( \overline{U}_{(1,1)-icv} \) are submodular and, thus, exhibit risk aversion in the sense of RICHARD (1975).

Note that the bivariate risk aversion defined by RICHARD (1975) does not require any condition on marginal risk aversion. It is not hard to find utility functions \( \nu \) which simultaneously verify \( \nu^{(1,1)} \leq 0 \) and \( \nu^{(2,0)} \geq 0 \) as well as \( \nu^{(0,2)} \geq 0 \) so that a decision-maker can be bivariate risk averse but still be risk-seeking for one attribute.

There are other ways to define risk aversion for bivariate outcomes. KIHLSTROM & MIRMAN (1974) introduced a second approach, in line with the univariate definition of risk-aversion. Precisely, an economic agent is bivariate risk-averse according to KIHLSTROM & MIRMAN (1974) if given the choice between a random vector \((X_1, X_2)\) and its expected value \((E[X_1], E[X_2])\), he always prefers the second to the first. From Jensen’s inequality, this holds if the Hessian matrix of \( \nu \) is negative semidefinite, that is, if \( \nu \) is concave. Note that even if \( u_1 \) and \( u_2 \) involved in (1.4) are concave, the bivariate utility function (1.4) is not necessarily
concave (it is only concave in each argument, or componentwise concave). This is in contrast with the additive case where \( u = u_1 + u_2 \) is concave if, and only if, each \( u_1 \) and \( u_2 \) is concave.

**Finkelstein, Kella & Scarsini** (1999) proved that provided \( \mathbb{E}[X_1|X_2] \) is non-decreasing in \( X_2 \), the inequality

\[
\mathbb{E}[u(X_1, X_2)] \leq \mathbb{E}[u(\mathbb{E}[X_1], X_2)]
\]

holds true provided \( u \) is submodular and concave in its first argument. This relates submodularity to the standard notion of risk aversion (defined as the preference of the expectation of a random variable over the random variable itself) provided the outcomes \( X_1 \) and \( X_2 \) are positively dependent. All the utilities in \( \mathcal{U}_{(s_1, s_2) - icv} \) exhibit risk aversion as defined in **Richard** (1975). If \( s_1 \geq 2 \) then (5.1) also holds true provided \( X_1 \) and \( X_2 \) are positively dependent.

### 5.2 Aversion to correlation increasing transformation

Another interpretation of the \((s_1, s_2)\)-increasing concavity relates to the concept of increasing correlation transformation due to **Epstein & Tanny** (1980) and exploited by **Denuit, Eeckhoudt & Rey** (2008). Specifically, let \( I_1 \) and \( I_2 \) be a couple of binary random variables such that \( \text{Pr}[I_i = 0] = 1 - \text{Pr}[I_i = 1] = p_i, i = 1, 2 \). Without loss of generality, we assume that \( p_1 \leq p_2 \). Let us now consider \( \rho \) such that \(-p_1 p_2 \leq \rho \leq p_1(1 - p_2)\) and define the joint distribution of \((I_1, I_2)\) as

\[
\begin{align*}
\text{Pr}[I_1 = 0, I_2 = 0] &= p_1 p_2 + \rho \\
\text{Pr}[I_1 = 1, I_2 = 0] &= (1 - p_1)p_2 - \rho \\
\text{Pr}[I_1 = 0, I_2 = 1] &= p_1(1 - p_2) - \rho \\
\text{Pr}[I_1 = 1, I_2 = 1] &= (1 - p_1)(1 - p_2) + \rho.
\end{align*}
\]

When \( \rho \) increases we face a correlation increasing transformation as defined by **Epstein & Tanny** (1980) and a correlation adverse decision-maker should then dislike an increase in \( \rho \). It can be shown that the distribution of \((I_1a_1, I_2a_2)\) decreases in the \( \preceq_{(1,1) - icv} \) order as \( \rho \) increases, for any positive constants \( a_1 \) and \( a_2 \).

**Denuit, Eeckhoudt & Rey** (2008) consider the bivariate random vector \(((1 - I_1)X_1 + I_1 Y_1, (1 - I_2)X_2 + I_2 Y_2)\) where the random variables \( X_1, X_2, Y_1, \) and \( Y_2 \) are assumed to be mutually independent, independent from \((I_1, I_2)\), and such that \( X_1 \preceq_{s_1 - icv} Y_1 \) and \( X_2 \preceq_{s_2 - icv} Y_2 \). The main result of **Denuit, Eeckhoudt & Rey** (2008) is that \(((1 - I_1)X_1 + I_1 Y_1, (1 - I_2)X_2 + I_2 Y_2)\) decreases in the \( \preceq_{(s_1, s_2) - icv} \)-sense as \( \rho \) increases. This shows that the decision-makers with utilities in \( \mathcal{U}_{(s_1, s_2) - icv} \) dislike any increase in correlation.

### 5.3 Aversion to positive quadrant dependence

Let us briefly discuss another approach to correlation aversion, where independence serves as a reference. Recall that \((X_1, X_2)\) is said to be positive quadrant dependent (PQD, in short) if

\[
(X_1, X_2) \preceq_{(1,1) - icv} (X_1^+, X_2^+)
\]
where \((X_1^+, X_2^+)\) is an independent version of \((X_1, X_2)\), that is, \((X_1^+, X_2^+)\) and \((X_1, X_2)\) have the same univariate marginals and \(X_1^+\) and \(X_2^+\) are mutually independent. This means that the inequality

\[
\Pr[X_1 > t_1, X_2 > t_2] \geq \Pr[X_1^+ > t_1, X_2^+ > t_2] = \Pr[X_1 > t_1] \Pr[X_2 > t_2]
\]

holds true for every \(t_1\) and \(t_2\). This inequality reads “the probability that \(X_1\) and \(X_2\) are simultaneously large is at least as large as if they were independent”. As pointed out by Cardin & Ferretti (2004)

\(u\) submodular \(\iff\) \(\mathbb{E}[u(X_1, X_2)] \leq \mathbb{E}[u(X_1^+, X_2^+)]\) for every PQD \((X_1, X_2)\).

This shows that the decision-makers with a utility function \(u \in \mathcal{U}_{(1,1)-icv}\) prefer independence over PQD.

Now, if we define positive dependence of degree \((s_1, s_2)\) as

\[
(X_1, X_2) \preceq_{(s_1, s_2)-icv} (X_1^+, X_2^+) \iff \text{Cov}[u_1(X_1), u_2(X_2)] \geq 0
\]

for all the non-positive \(u_1 \in \mathcal{U}_{s_1-icv}\) and \(u_2 \in \mathcal{U}_{s_2-icv}\), we then see that the decision-makers with utility functions in \(\mathcal{U}_{(s_1, s_2)-icv}\) are those who prefer independence over positive dependence of degree \((s_1, s_2)\).

Thus, the random couple \((X_1, X_2)\) is positively dependent of degree \((s_1, s_2)\) if the utilities \(u_1(X_1)\) and \(u_2(X_2)\) are positively correlated whatever the \(s_1\)- and \(s_2\)-increasing utility functions \(u_1\) and \(u_2\). In particular, this means that

\[
\text{Cov}[(t_1 - X_1)_{+}^{s_1-1}, (t_2 - X_2)_{+}^{s_2-1}] \geq 0 \quad \text{for all} \; t_1 \text{ and } t_2.
\]

The \((s_1 - 1)\)th power of the shortfall \((t_1 - X_1)_{+}\) of \(X_1\) below the threshold \(t_1\) remains positively correlated with the \((s_2 - 1)\)th power of the shortfall \((t_2 - X_2)_{+}\) of \(X_2\) below the threshold \(t_2\) whatever the \(t_i\)’s. This notion weakens as \(s_1\) and \(s_2\) increase. At the limit, only completely monotone utilities (i.e. mixtures of utilities) are used to define this notion of positive dependence.

### 6 Portfolios and hedging effects

Let us assume that adding the attributes is meaningful (for instance, because both attributes represent financial outcomes expressed in the same currency). We now consider portfolios \(w_1 X_1 + w_2 X_2\) and \(w_1 Y_1 + w_2 Y_2\) made of \((X_1, X_2)\) and \((Y_1, Y_2)\) with non-negative weights \(w_1\) and \(w_2\). Preferences over these portfolios express the attractiveness of the hedging effects existing between the components of the bivariate outcomes of \((X_1, X_2)\) and \((Y_1, Y_2)\).

Bivariate \((s_1, s_2)\)-increasing concave orderings imply interesting univariate ordering results on their transformed components, as shown in the next result.

**Property 6.1.** Given two random couples \((X_1, X_2)\) and \((Y_1, Y_2)\), we have

\[
(X_1, X_2) \preceq_{(s_1, s_2)-icv} (Y_1, Y_2) \Rightarrow w_1 X_1 + w_2 X_2 \preceq_{(s_1 + s_2)-icv} w_1 Y_1 + w_2 Y_2, \text{ for all } w_1 \text{ and } w_2 \geq 0.
\]
Proof. Consider a univariate utility function \( u \in \mathcal{U}_{(s_1+s_2)-icv} \) and define the bivariate function 
\[
 v^{(k_1,k_2)}(x_1, x_2) = w_1^{k_1}w_2^{k_2}u^{(k_1+k_2)}(w_1x_1 + w_2x_2)
\]
we have \( v \in \mathcal{U}_{(s_1,s_2)-icv} \), which achieves the proof. 

This result shows that when all the decision-makers with risk independent utility functions of the form (1.4) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\), any portfolio made of \(Y_1\) and \(Y_2\) is considered as more attractive in the \((s_1 + s_2)\)th stochastic dominance than the corresponding portfolio made of \(X_1\) and \(X_2\). Coming back to the positive dependence of degree \((s_1,s_2)\), we see that in such a case,
\[
w_1X_1 + w_2X_2 \preceq_{(s_1+s_2)-icv} w_1X_1^+ + w_2X_2^+
\]
holds true for all \(w_1 \) and \(w_2 \geq 0\).

Let us now examine the \((1,1)\)-increasing concave, \((1,2)\)-increasing concave, \((2,1)\)-increasing concave, and \((2,2)\)-increasing concave stochastic order relations to illustrate the contents of Property 6.1. To this end, note that Characterization 3.1 reduces to
\[
(X_1, X_2) \preceq_{(1,1)-icv} (Y_1, Y_2) \iff \Pr[X_1 \leq t_1, X_2 \leq t_2] \geq \Pr[Y_1 \leq t_1, Y_2 \leq t_2] \text{ for all } t_1 \text{ and } t_2
\]
\[
(X_1, X_2) \preceq_{(1,2)-icv} (Y_1, Y_2) \iff \begin{cases} 
X_1 \preceq_{2-icv} Y_1 \\
X_2 \preceq_{1-icv} Y_2 \\
\mathbb{E}[t_1 - X_1] \mathbb{1}[X_2 \leq t_2] \geq \mathbb{E}[t_1 - Y_1] \mathbb{1}[Y_2 \leq t_2] 
\end{cases} \text{ for all } t_1 \text{ and } t_2
\]
\[
(X_1, X_2) \preceq_{(2,1)-icv} (Y_1, Y_2) \iff \begin{cases} 
X_1 \preceq_{1-icv} Y_1 \\
X_2 \preceq_{2-icv} Y_2 \\
\mathbb{E}[t_2 - X_2] \mathbb{1}[X_1 \leq t_1] \geq \mathbb{E}[t_2 - Y_2] \mathbb{1}[Y_1 \leq t_1] 
\end{cases} \text{ for all } t_1 \text{ and } t_2
\]
\[
(X_1, X_2) \preceq_{(2,2)-icv} (Y_1, Y_2) \iff \begin{cases} 
(X_1, X_2) \preceq_{(2,1)-icv} (Y_1, Y_2) \\
(X_1, X_2) \preceq_{(1,2)-icv} (Y_1, Y_2) \\
\mathbb{E}[(t_1 - X_1) + (t_2 - X_2)] \geq \mathbb{E}[(t_1 - Y_1) + (t_2 - Y_2)] 
\end{cases} \text{ for all } t_1 \text{ and } t_2
\]

Considering \((X_1, X_2)\) and \((Y_1, Y_2)\) with identical marginals (that is, (1.1) is fulfilled), \((X_1, X_2) \preceq_{(s_1,s_2)-icv} (Y_1, Y_2)\) means that the dependence structure between the \(Y_i\)’s is viewed as more attractive than the dependence structure between the \(X_i\)’s by all the decision-makers with a utility function in \(\mathcal{U}_{(s_1,s_2)-icv}\). In particular, Pearson’s linear correlation coefficient is higher for \((X_1, X_2)\) than for \((Y_1, Y_2)\). If \((X_1, X_2)\) and \((Y_1, Y_2)\) have the same univariate marginals, then it is easily seen that
\[
(X_1, X_2) \preceq_{(1,1)-icv} (Y_1, Y_2) \iff \Pr[X_1 \leq t_1|X_2 \leq t_2] \geq \Pr[Y_1 \leq t_1|Y_2 \leq t_2] \text{ for all } t_1 \text{ and } t_2.
\]
This intuitively means that the knowledge that \(X_2\) is small (that is, \(X_2 \leq t_2\) increases the probability that \(X_1\) is also small (that is, \(X_1 \leq t_1\)) compared to \((Y_1, Y_2)\). The same kind of
interpretation holds in the other cases. Specifically, if \((X_1, X_2)\) and \((Y_1, Y_2)\) have the same univariate marginals, then

\[
(X_1, X_2) \preceq_{(2,1)\text{-}icv} (Y_1, Y_2) \iff \mathbb{E}\left[(t_1 - X_1)_+ \Big| X_2 \leq t_2 \right] \geq \mathbb{E}\left[(t_1 - Y_1)_+ \Big| Y_2 \leq t_2 \right] \quad (6.1)
\]

for all \(t_1\) and \(t_2\) so that the knowledge that \(X_2\) is small (that is, \(X_2 \leq t_2\)) increases the average part of \(X_1\) below any threshold \(t_1\) compared to \((Y_1, Y_2)\). Similarly

\[
(X_1, X_2) \preceq_{(1,2)\text{-}icv} (Y_1, Y_2) \iff \mathbb{E}\left[(t_2 - X_2)_+ \Big| X_1 \leq t_1 \right] \geq \mathbb{E}\left[(t_2 - Y_2)_+ \Big| Y_1 \leq t_1 \right] \quad (6.2)
\]

for all \(t_1\) and \(t_2\). Whenever (6.1) or (6.2) holds true then we have

\[
w_1X_1 + w_2X_2 \preceq_{3\text{-}icv} w_1Y_1 + w_2Y_2
\]

for all \(w_1, w_2 \geq 0\), so that the portfolio \(w_1Y_1 + w_2Y_2\) is preferred over the portfolio \(w_1X_1 + w_2X_2\) by all the profit-seeking, risk averse, prudent, and temperant decision-makers.

Now, when \(s_1 = s_2 = 2\), \((X_1, X_2) \preceq_{(2,2)\text{-}icv} (Y_1, Y_2)\) holds true if, and only if, (6.1)-(6.2) are satisfied as well as

\[
\text{Cov}\left[(t_1 - X_1)_+, (t_2 - X_2)_+\right] \geq \text{Cov}\left[(t_1 - Y_1)_+, (t_2 - Y_2)_+\right] \quad \text{for all } t_1 \text{ and } t_2
\]

which means that the \((t_i - X_i)_+\)'s are more positively correlated than the \((t_i - Y_i)_+\)'s. Whatever the thresholds \(t_1\) and \(t_2\), this means that the shortfalls \((t_1 - X_1)_+\) and \((t_2 - X_2)_+\) are more correlated than the shortfalls \((t_1 - Y_1)_+\) and \((t_2 - Y_2)_+\). In such a case, the stochastic inequality

\[
w_1X_1 + w_2X_2 \preceq_{4\text{-}icv} w_1Y_1 + w_2Y_2
\]

holds true for all \(w_1, w_2 \geq 0\), so that the portfolio \(w_1Y_1 + w_2Y_2\) is preferred over the portfolio \(w_1X_1 + w_2X_2\) by all the profit-seeking, risk averse, prudent, and temperant decision-makers.

7 Conclusion

In this paper, we have established the equivalence between the common preferences of all the decision-makers with independent utility functions and those exhibiting correlation aversion. Specifically, we have extended to the whole class of the \(\preceq_{(s_1, s_2)\text{-}icv}\) orders the seminal ideas contained in the paper by Mosler (1984) for \(s_1 \leq 2\) and \(s_2 \leq 2\) to arbitrary values of \(s_1\) and \(s_2\). The theoretical results derived in the present paper allow for interesting interpretations of empirical studies, where risk independence is often assumed for the preferences of decision-makers.

We have also reviewed different types of risk aversion when bivariate outcomes are considered. The aversion to correlation increasing transformations and to positive dependence of degree \((s_1, s_2)\) allow for a better understanding of the preferences expressed by the \((s_1, s_2)\)-increasing concave multiattribute utility functions.

To end with, portfolios built from two outcomes are compared. The bivariate \((s_1, s_2)\)-increasing concave order relation between bivariate outcomes yields a univariate \((s_1 + s_2)\)-increasing concave order between the resulting portfolios. For \(s_1\) and \(s_2\) less than 2, the dependence structure disliked by risk averse, prudent, and temperant decision-makers are identified. This allows for a better understanding of the actual meaning of these three widely used risk attitudes.
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References


BIVARIATE STOCHASTIC DOMINANCE AND COMMON PREFERENCES OF DECISION-MAKERS WITH RISK INDEPENDENT UTILITIES

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Abstract

The close link between bivariate stochastic dominance relations and the common preferences of the decision-makers with independent multiattribute utility functions is discussed. Specifically, the common preferences of all the decision-makers with a utility function expressing risk independence are shown to coincide with bivariate stochastic dominance expressing correlation aversion. As an application, portfolios are compared to assess the possible hedging effect between two outcomes.

Key words and phrases: Bivariate \((s_1, s_2)\)-increasing concave orders, bivariate stochastic dominance, conditional risk aversion, portfolios.

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1 Introduction and motivation

In this paper, we consider pairs \((X_1, X_2)\) of commodities, often called attributes. We assume that the first attribute is valued in the interval \([a_1, b_1]\) and the second one is valued in the interval \([a_2, b_2]\). For instance, \(X_1\) may represent wealth and \(X_2\) health as in Eeckhoudt, Rey & Schlesinger (2007), or \(X_1\) and \(X_2\) may represent consumption at two different times. Sometimes, bivariate outcomes \((X_1, X_2)\) and \((Y_1, Y_2)\) representing financial consequences are compared after aggregation, that is, on the basis of the respective final wealths \(X_1 + X_2\) and \(Y_1 + Y_2\). Univariate expected utility theory can be used for that purpose. Aggregating the two outcomes is nevertheless not always optimal because preferences may not depend only on final wealths, but on the separate outcomes.

We assume that decision-makers act as if they maximize their respective expected utility. Specifically, a decision-maker with utility function \(u\) prefers \((Y_1, Y_2)\) over \((X_1, X_2)\) if \(\mathbb{E}[u(X_1, X_2)] \leq \mathbb{E}[u(Y_1, Y_2)]\), that is, if the expected utility for \((Y_1, Y_2)\) exceeds that for \((X_1, X_2)\). Faced with the difficulty to deal with general functions \(u\), many specific forms of utility functions have been suggested in the literature.

The easiest approach to evaluate bivariate consequences \((X_1, X_2)\) consists in postulating an additive form for the utility function \(u\), that is, \(u(x_1, x_2) = u_1(x_1) + u_2(x_2)\), where \(u_i\) is the utility function for the \(i\)th attribute, \(i = 1, 2\). Note that instead of adding the outcomes \(X_1\) and \(X_2\), their utilities \(u_1(X_1)\) and \(u_2(X_2)\) are added to compare the bivariate attributes. A decision-maker with such an additive utility function only considers the marginal distribution of the \(X_i\)'s, and not their actual dependence structure. The main advantage of the additive utility function is its simplicity: its assessment reduces to the elicitation of two one-dimensional utility functions \(u_1\) and \(u_2\). A major shortcoming of it is that it ignores the dependence structure of the \(X_i\)'s. Hedging effects cannot be appraised with such a utility function as the dependence structure does not enter the comparison. Specifically, if \((X_1, X_2)\) and \((Y_1, Y_2)\) have identical univariate marginals, that is,

\[
\Pr[X_1 \leq t] = \Pr[Y_1 \leq t] \text{ for all } t \text{ and } \Pr[X_2 \leq t] = \Pr[Y_2 \leq t] \text{ for all } t, \tag{1.1}
\]

then

\[
\mathbb{E}[u(X_1, X_2)] = \mathbb{E}[u_1(X_1)] + \mathbb{E}[u_2(X_2)] = \mathbb{E}[u_1(Y_1)] + \mathbb{E}[u_2(Y_2)] = \mathbb{E}[u(Y_1, Y_2)]
\]

so that all the bivariate outcomes with the same marginals are considered as equivalent when additive utility functions are used.

Risk independence can be postulated to account for possible hedging effects, keeping the mathematical tractability. Let us denote as \(u^{(i,j)}\) the \((i,j)\)th mixed partial derivative of \(u\) with respect to \(x_1\) and \(x_2\), that is, \(u^{(i,j)} = \frac{\partial^{i+j} u}{\partial x_1^i \partial x_2^j}\). The conditional risk aversion for \(X_2\) given \(X_1 = x_1\) is defined as

\[
r_2(x_2 | x_1) = -\frac{u^{(0,2)}(x_1, x_2)}{u^{(0,1)}(x_1, x_2)}.
\]

Similarly, the conditional risk aversion for \(X_1\) given \(X_2 = x_2\) is defined as

\[
r_1(x_1 | x_2) = -\frac{u^{(2,0)}(x_1, x_2)}{u^{(1,0)}(x_1, x_2)}.
\]
Then, according to Keeney (1973), \( X_2 \) is said to be risk independent of \( X_1 \) if \( r_2(x_2|x_1) \) does not depend on \( x_1 \). Similarly, \( X_1 \) is said to be risk independent of \( X_2 \) if \( r_1(x_1|x_2) \) does not depend on \( x_2 \). If all the risk is associated with only one attribute and the other attribute is fixed, risk independence ensures that the decision-maker’s attitude toward risk only depends on that attribute involving the risk. This is also termed as utility independence since the decision-maker’s relative preferences over one attribute, when the other attribute is held fixed at some specified level, are the same regardless of that level.

Given \( X_2 \) is risk independent of \( X_1 \) and \( X_1 \) is risk independent of \( X_2 \) then the utility function \( u \) can be expressed as a sum of univariate utility functions supplemented with their product. Specifically, we know from Theorem 1 in Keeney (1973) that such a utility function \( u \) is either of the form

\[
  u(x_1, x_2) = u_1(x_1) + u_2(x_2) - \gamma(-u_1(x_1))(-u_2(x_2)), \quad \gamma > 0, \tag{1.2}
\]

where both \( u_1 \) and \( u_2 \) are non-decreasing non-positive utility functions, or of the form

\[
  u(x_1, x_2) = u_1(x_1) + u_2(x_2) + \gamma u_1(x_1)u_2(x_2), \quad \gamma > 0, \tag{1.3}
\]

where both \( u_1 \) and \( u_2 \) are non-decreasing non-negative utility functions. If the random couples \((X_1, X_2)\) and \((Y_1, Y_2)\) have the same univariate marginal distributions (that is, (1.1) is fulfilled), the first two terms in (1.2)-(1.3) do not matter and we are essentially interested in utility functions factoring into the product of two univariate utilities. This is why we also consider bivariate utilities of the form

\[
  u(x_1, x_2) = \alpha - \beta(-u_1(x_1))(-u_2(x_2)), \quad \alpha \in \mathbb{R}, \quad \beta > 0, \tag{1.4}
\]

where both \( u_1 \) and \( u_2 \) are non-decreasing non-positive utility functions. Decision-makers with utilities of the form (1.4) consider the attributes as mutually utility independent and substitutional and tend to “diversify” between pairs of attributes. Diversification here means that gaining either a higher level of the first attribute and a lower level of the second one, or vice-versa is more desired than gaining either a lower or a higher level of both attributes. We know from Theorem 2 in Mosler (1984) that all the decision-makers with utilities of the form (1.4) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\) if, and only if, the inequality

\[
  \Pr[X_1 \leq t_1, X_2 \leq t_2] \geq \Pr[Y_1 \leq t_1, Y_2 \leq t_2] \tag{1.5}
\]

holds true for any thresholds \( t_1 \) and \( t_2 \).

It is easily seen from (1.4) that all the decision-makers with utility functions (1.4) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\) if, and only if, the inequality

\[
  \mathbb{E}[h_1(X_1)h_2(X_2)] \geq \mathbb{E}[h_1(Y_1)h_2(Y_2)]
\]

is valid for any non-negative non-increasing functions such that the expectations exist. Taking indicator of left-open intervals \((-\infty, t_i], i = 1, 2\), gives (1.5). In particular, this ensures that \( Y_i \) dominates \( X_i \) in the first-degree stochastic dominance, \( i = 1, 2 \). Provided both outcomes are non-negative, that is \( a_i = 0 \) for \( i = 1, 2 \), it can also be shown that all the decision-makers with utility functions (1.4) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\) if, and only if,
max\{w_1Y_1, w_2Y_2\} is larger than max\{w_1X_1, w_2X_2\} in the first-degree stochastic dominance, whatever the weights \(w_1 > 0\) and \(w_2 > 0\), that is, the inequality
\[
\Pr[\max\{w_1X_1, w_2X_2\} > t] \leq \Pr[\max\{w_1Y_1, w_2Y_2\} > t]
\]
is valid for any threshold \(t\). Decision-makers with utilities (1.4) thus base their choice on the performance of the maximum of the two weighted outcomes.

We also consider utility functions of the form
\[
u(x_1, x_2) = \alpha + \beta u_1(x_1)u_2(x_2), \ \alpha \in \mathbb{R}, \ \beta > 0,
\]
where both \(u_1\) and \(u_2\) are non-decreasing non-negative utility functions. Decision-makers with utility functions of the form (1.6) consider the attributes as mutually utility independent and complementary: such a decision-maker prefers gaining either a lower or a higher level of both attributes rather than gaining either a higher level of the first attribute and a lower level of the second, or vice-versa. We know from Theorem 3 in Mosler (1984) that decision-makers with utilities of the form (1.6) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\) if, and only if, the inequality
\[
\Pr[X_1 > t_1, X_2 > t_2] \leq \Pr[Y_1 > t_1, Y_2 > t_2]
\]
holds true for any thresholds \(t_1\) and \(t_2\).

It is easily seen from (1.6) that all the decision-makers with utility functions (1.6) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\) if, and only if, the inequality
\[
\mathbb{E}[g_1(X_1)g_2(X_2)] \leq \mathbb{E}[g_1(Y_1)g_2(Y_2)]
\]
is valid for any non-negative non-decreasing functions \(g_1\) and \(g_2\) such that the expectations exist. Taking indicator of right-open intervals \((t_i, +\infty), i = 1, 2\), gives (1.7). In particular, this ensures that \(Y_i\) dominates \(X_i\) in the first-degree stochastic dominance, \(i = 1, 2\). Provided both outcomes are non-negative, that is, \(a_i = 0\) for \(i = 1, 2\), it can also be shown that all the decision-makers with utility functions (1.6) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\) if, and only if, \(\min\{w_1Y_1, w_2Y_2\}\) is larger than \(\min\{w_1X_1, w_2X_2\}\) in the first-degree stochastic dominance, whatever the weights \(w_1 > 0\) and \(w_2 > 0\), that is, the inequality
\[
\Pr[\min\{w_1X_1, w_2X_2\} > t] \leq \Pr[\min\{w_1Y_1, w_2Y_2\} > t]
\]
is valid for any threshold \(t\). Decision-makers with utilities (1.6) thus base their choice on the performance of the minimum of the two weighted outcomes.

Now, if \((X_1, X_2)\) and \((Y_1, Y_2)\) have the same univariate marginal distributions (that is, (1.1) is fulfilled) then the identity
\[
\Pr[Y_1 \leq t_1, Y_2 \leq t_2] - \Pr[X_1 \leq t_1, X_2 \leq t_2] = \Pr[Y_1 > t_1, Y_2 > t_2] - \Pr[X_1 > t_1, X_2 > t_2]
\]
is valid for any thresholds \(t_1\) and \(t_2\). We then deduce from (1.5)-(1.7) that all the decision-makers with utility function (1.4) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\) if, and only if, all the decision-makers with utility function (1.6) prefer \((X_1, X_2)\) over \((Y_1, Y_2)\). Once the marginal distributions are fixed, the only difference between \((X_1, X_2)\) and \((Y_1, Y_2)\) lies in the respective dependence structures. We then see that the dependence structure of \((Y_1, Y_2)\) is preferred
over the dependence structure of \((X_1, X_2)\) by all the decision-makers with utility function (1.4) if, and only if, the dependence structure of \((X_1, X_2)\) is preferred over the dependence structure of \((Y_1, Y_2)\) by all the decision-makers with utility function (1.6).

Mosler (1984) also established that if all the decision-makers with utility functions of the form (1.4) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\) then the same conclusion is reached by a much broader class of economic agents with utility dependent preferences, namely those with a non-decreasing utility function such that \(u^{(1,1)} \leq 0\). Similarly, all the decision-makers with utility functions of the form (1.6) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\) if, and only if, all the economic agents with a non-decreasing utility function such that \(u^{(1,1)} \geq 0\) exhibit the same preferences. This means that the results derived under the assumptions of utility independence and given substitu- tional structure carry over to more general situations where utility independence disappears and only the signs of certain partial derivatives remain given (which determine the substitu- tional structure).

In this paper, we concentrate on decision-makers with utility function (1.4). Our aim is to extend the results obtained by Mosler (1984) to products of univariate utilities \(u_i\)'s expressing some desirable properties such as risk aversion, prudence and temperance. We will show that these common preferences agree with the preferences expressed by a much larger class of decision-makers, expressing some correlation aversion, recently studied in Denuit, Eeckhoudt & Rey (2008).

The text is organized as follows. Section 2 briefly recalls the basic results when random variables are considered (univariate stochastic dominance), whereas Section 3 discusses a bivariate extension where pairs of random variables with identical marginals are considered. Section 4 establishes that the bivariate stochastic dominance rules introduced in Section 3 agree with the common preferences of decision-makers with multiplicative utilities. The close link to correlation aversion is discussed in Section 5. As an application, portfolios are compared in Section 6 to assess the hedging effects existing between the two outcomes. The final Section 7 concludes.

### 2 Univariate stochastic dominance: Common preferences of decision-makers with \(s\)-increasing concave utility functions

#### 2.1 Classes of differentiable \(s\)-increasing concave utility functions

For \(s = 1, 2, \ldots\), let us define the class \(\mathcal{U}_s^{-icv}\) of the differentiable \(s\)-increasing concave utility functions as the class containing all the functions \(u\) defined on the real line such that \((-1)^{k+1}u^{(k)} \geq 0\) for \(k = 1, 2, \ldots, s\), where \(u^{(k)}\) stands for the \(k\)th derivative of \(u\). The class \(\mathcal{U}_s^{-icv}\) thus contains the non-decreasing functions with derivatives of degrees 1 to \(s\) with alternating signs. All the commonly used utility functions belong to \(\mathcal{U}_s^{-icv}\) for all \(s\) (as their derivatives alternate in sign, beginning with positive marginal utility). For instance, all the completely monotone utility functions, including the logarithmic, exponential and power utility functions belong to \(\mathcal{U}_s^{-icv}\) for all \(s\).
2.2 Univariate $s$-increasing concave orders

Consider two random variables $X$ and $Y$, many stochastic orderings $\preceq_s$ can be defined by reference to some class $\mathcal{U}$ of measurable functions $u$ by

$$X \preceq_s Y \iff \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \text{ for all } u \in \mathcal{U},$$

(2.1)

provided that the expectations exist. If $X \preceq_s Y$ holds then $Y$ is said to dominate $X$ with respect to $\mathcal{U}$. Note that whether $X \preceq_s Y$ holds or not depends on the marginal distributions of $X$ and $Y$ only, and not on their actual dependence structure. Orderings defined by (2.1) are particularly appealing in expected utility theory, considering $\mathcal{U}_s$ as a set of utility functions $u$. Then, $\preceq_s$ allows to analyze risky decisions for all the agents whose utility function belongs to $\mathcal{U}_s$, without a complete elicitation of the individual utilities.

Recall from Denuit, Lefèvre & Shaked (1998) that the $s$-increasing concave order is defined from (2.1), taking for $\mathcal{U}_s$ the class $\mathcal{U}_{s-icv}$. More precisely, given two random variables $X$ and $Y$, $X$ is said to be smaller than $Y$ in the $s$-increasing concave order, denoted by $X \preceq_{s-icv} Y$ when $\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ for all the functions $u$ in $\mathcal{U}_{s-icv}$, provided the expectations exist. When the first $s-1$ moments are equal, the $s$-increasing concave orders are termed as the $s$-concave orders and coincide with the increasing $s$th degree risk of EKERN (1980). The convex counterparts of the $\preceq_{s-icv}$ orders have been thoroughly studied by Denuit, Lefèvre & Shaked (1998) and Denuit, De Vijlder & Lefèvre (1999).

For $s = 1$ to 4, the order relations $\preceq_{s-icv}$ express the common preferences of well-known classes of decision-makers. For $s = 1$, $\preceq_{1-icv}$ expresses the common preferences of all the profit-seeking decision-makers. For $s = 2$, $\preceq_{2-icv}$ reflects the common preferences of all the profit-seeking risk-averse decision-makers, i.e. those preferring $\mathbb{E}[X]$ over $X$, whatever $X$. For $s = 3$, $\preceq_{3-icv}$ expresses the common preferences of all the profit-seeking, risk-averse and prudent decision-makers. Finally, for $s = 4$, $\preceq_{4-icv}$ expresses the common preferences of all the profit-seeking, risk-averse, prudent and temperant decision-makers.

2.3 Maximal generator of $\preceq_{s-icv}$

Clearly, the properties of the integral orderings $\preceq_{s-icv}$ are inherited from the structures of the generating classes of functions $\mathcal{U}_{s-icv}$. The closure $\overline{\mathcal{U}}_{s-icv}$ of $\mathcal{U}_{s-icv}$ (with respect to the topology of pointwise convergence) contains all the utility functions $u$ such that $u^{(s-2)}$ exists, $(-1)^{k+1}u^{(k)} \geq 0$ for $k = 1, \ldots, s-2$, and $(-1)^s u^{(s-2)}$ is concave (with the convention that $\overline{\mathcal{U}}_{1-icv}$ contains all the non-decreasing functions and $\overline{\mathcal{U}}_{2-icv}$ contains all the non-decreasing concave functions). The class $\overline{\mathcal{U}}_{s-icv}$ can be characterized by sign properties of divided differences. Recall that the $k$th divided difference, $k \geq 1$, of the function $u$ at distinct points $x_0, x_1, \ldots, x_k$, denoted by $[x_0, x_1, \ldots, x_k]u$, is defined recursively by

$$[x_0, x_1, \ldots, x_k]u = \frac{[x_1, x_2, \ldots, x_k]u - [x_0, x_1, \ldots, x_{k-1}]u}{x_k - x_0},$$

(2.2)

starting from $[x_i]u = u(x_i)$, $i = 0, 1, \ldots, k$. These divided differences extend derivatives to less regular functions. Then, $u \in \overline{\mathcal{U}}_{s-icv}$ if, and only if, $(-1)^{k+1}[x_0, x_1, \ldots, x_k]u \geq 0$ for any $x_1, x_1, \ldots, x_k$, $k = 1, 2, \ldots, s$. 

5
Then, the class $\overline{\mathcal{U}}_{s-icu}$ of the $s$-increasing concave functions is the largest class of functions for which the implication $X \preceq_{s-icu} Y \Rightarrow \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ holds true for every ordered pair $(X, Y)$. For this reason, $\overline{\mathcal{U}}_{s-icu}$ is often called the maximal generator of the order $\preceq_{s-icu}$. This means that $\overline{\mathcal{U}}_{s-icu}$ corresponds to the largest class of decision-makers whose preferences are in accordance with $\preceq_{s-icu}$.

2.4 Characterization through integrated left tails

The orders $\preceq_{s-icu}$ can be defined without explicit mention of the utility classes $\overline{\mathcal{U}}_{s-icu}$. Let us denote as $x_+$ the positive part of the real $x$, i.e. $x_+ = \max\{x, 0\}$. In order to establish that $X \preceq_{s-icu} Y$ holds true for some pair $(X, Y)$, it suffices to check for the following conditions.

**Characterization 2.1.** Given two random variables $X$ and $Y$ valued in $[a, b]$, we have $X \preceq_{s-icu} Y$ if, and only if, $\mathbb{E}[(b - X)^k] \geq \mathbb{E}[(b - Y)^k]$ for $k = 1, \ldots, s - 1$ and $\mathbb{E}[(t - X)^{s-1}_+] \geq \mathbb{E}[(t - Y)^{s-1}_+]$ for all $t$.

**Proof.** As pointed out in Denuit, Lefèvre & Mesfioui (1999, equivalence (2.9)), $X \preceq_{s-icu} Y$ if, and only if, the inequality $\mathbb{E}[g(b - X)] \geq \mathbb{E}[g(b - Y)]$ holds true for any function $g$ with positive derivatives of degree 1 to $s$, that is, $g^{(k)}(t) \geq 0$ for $k = 1$ to $s$. Now, whatever the random variable $Z$ valued in the interval $[0, c]$, the following expansion is valid:

$$
\mathbb{E}[g(Z)] = \sum_{k=0}^{s-1} \frac{g^{(k)}(0)}{k!} \mathbb{E}[Z^k] + \int_0^c \mathbb{E}[(Z - t)^{s-1}_+] \frac{g^{(s)}(t)}{(s - 1)!} dt.
$$

(2.3)

Replacing $Z$ with $b - X$ and $c$ with $b - a$ gives the announced result.

Let us denote as $\mathcal{U}_{s-icu}$ the class of functions $x \mapsto -(b - x)^k$ for $k = 1, \ldots, s - 1$, and $x \mapsto -(t - x)^{s-1}_+$ for some real $t$. Then, we see that $X \preceq_{s-icu} Y$ holds true if, and only if, (2.1) is satisfied with $\mathcal{U}_s = \mathcal{U}_{s-icu}$. This means that $\mathcal{U}_{s-icu}$ corresponds to the decision-makers leading the common preferences of those in $\overline{\mathcal{U}}_{s-icu}$. A utility function of the form $x \mapsto -(t - x)^{s-1}_+$ has a relatively simple interpretation. The economic agent has a threshold $t$ and wishes to attain it. Once $t$ is attained, the decision-maker is satisfied and does not value any exceedances over $t$. However, if the wealth $x$ does not attain $t$ then the decision-maker is not fully satisfied and values the situation as $-(t - x)^{s-1}_+$.

Characterization 2.1 allows us to relate $\preceq_{s-icu}$ to the integrated left tails of the underlying distributions. Recall that the integrated left tails $F_X^{[k]}$, $k = 1, 2, \ldots$, of the distribution function $F_X$ for the random variable $X$ are defined by $F_X^{[k]}(t) = \int_t^\infty F_X^{[k-1]}(x) \, dx$, $k = 2, 3, \ldots$, starting from $F_X^{[1]}(t) = F_X(t)$. It can be shown that, for each $k$, either $F_X^{[k]}(t)$ is non-negative and finite for all $t$ or else $F_X^{[k]}(t) = +\infty$ for all $t$. The finiteness of $F_X^{[k]}$ depends only on the left tail of $F_X$. Moreover, $F_X^{[k]}$ is finite if $\mathbb{E}[X^k]$ exists.

It can be shown by induction and Fubini’s Theorem that

$$
F_X^{[s]}(t) = \frac{\mathbb{E}[(t - X)^{s-1}_+]}{(s - 1)!}.
$$

(2.4)

Hence, Characterization 2.1 can be rephrased as follows.
Characterization 2.2. Given two random variables $X$ and $Y$, we have $X \preceq_{s-icv} Y$ if, and only if, $F_X^{[k]}(b) \geq F_Y^{[k]}(b)$ for $k = 1, \ldots, s - 1$ and $F_X^{[s]}(t) \geq F_Y^{[s]}(t)$ for all $t$.

Characterization 2.2 allows us to compare $\preceq_{s-icv}$ to the stochastic dominance relations introduced in Rolski (1976) and Fishburn (1976, 1980). Compared to $\preceq_{s-icv}$, these orderings are defined by pointwise comparison of $F_X^{[s]}$ and $F_Y^{[s]}$, without imposing any constraints on $F_X^{[k]}(b)$ and $F_Y^{[k]}(b)$ for $k = 1, \ldots, s - 1$. This in turn implies additional requirements on the tails of the utility functions agreeing with these orderings, which are avoided with $\preceq_{s-icv}$.

Considering the three sets of utility functions $\mathcal{U}_{s-icv}$, $\mathcal{U}_{s-icv}$ and $\mathcal{U}_{s-icv}$, we clearly have $\mathcal{U}_{s-icv} \subset \mathcal{U}_{s-icv}$ and $\mathcal{U}_{s-icv} \subset \mathcal{U}_{s-icv}$. Decision-makers with utilities in $\mathcal{U}_{s-icv}$ lead the common preferences of those with utilities in $\mathcal{U}_{s-icv}$, in the sense that

$$X \preceq_{s-icv} Y \iff (2.1) \text{ holds with } \mathcal{U}_* = \mathcal{U}_{s-icv}$$

$$\iff (2.1) \text{ holds with } \mathcal{U}_* = \mathcal{U}_{s-icv}$$

$$\iff (2.1) \text{ holds with } \mathcal{U}_* = \mathcal{U}_{s-icv}.$$

3 Bivariate stochastic dominance: Common preferences of decision-makers with $(s_1, s_2)$-increasing concave utility functions

3.1 Classes of differentiable $(s_1, s_2)$-increasing concave functions

Let us now consider utility functions $u$ defined on the real plane. This allows to account for bidimensional consequences. The class $\mathcal{U}_{(s_1, s_2)-icv}$ of the differentiable $(s_1, s_2)$-increasing concave functions is defined as the class of all the functions $u$ such that $(-1)^{k_1+k_2+1}u^{(k_1, k_2)} \geq 0$ for all $k_1 = 0, \ldots, s_1$, $k_2 = 0, \ldots, s_2$, such that $k_1 + k_2 \geq 1$.

Eekhoudt, Rey & Schlesinger (2007) provided equivalence between the signs of the cross-derivatives $u^{(k_1, k_2)}$ and individual preferences within a particular class of simple lotteries. This leads to the concepts of cross-prudence and cross-temperance, exhibited by the elements of $\mathcal{U}_{(s_1, s_2)-icv}$ for $s_1$ and $s_2$ large enough. Specifically, an individual is said to be correlation averse if, and only if, $u^{(1,1)} \leq 0$, cross-prudent in the second outcome if, and only if, $u^{(2,1)} \geq 0$, cross-prudent in the first outcome if, and only if, $u^{(1,2)} \geq 0$, and cross-temperant if, and only if, $u^{(2,2)} \leq 0$. We come back to these issues in Section 5, where the meaning of utilities in $\mathcal{U}_{(s_1, s_2)-icv}$ is discussed.

3.2 Bivariate $(s_1, s_2)$-increasing concave orders

Univariate orderings (2.1) naturally extend to dimension 2. Specifically, let $\mathcal{U}_*$ be a class of measurable functions and let $(X_1, X_2)$ and $(Y_1, Y_2)$ be a pair of bivariate random vectors. Then, $(X_1, X_2)$ is said to be smaller than $(Y_1, Y_2)$ for the integral stochastic ordering $\preceq_*$ generated by $\mathcal{U}_*$ when

$$\mathbb{E}[u(X_1, X_2)] \leq \mathbb{E}[u(Y_1, Y_2)] \text{ for all } u \in \mathcal{U}_*$$

(3.1)
for which the expectations exist. As in the univariate case, these orderings are particularly appealing in expected utility theory, considering \( U_* \) as a set of bivariate utility functions \( u \).

Recall from Denuit, Lefèvre & Mesfioui (1999) that the bivariate \((s_1, s_2)\)-increasing concave order is defined by (3.1) taking \( U_* = \mathcal{U}_{(s_1, s_2)-icv} \). Specifically, \((X_1, X_2)\) is said to be smaller than \((Y_1, Y_2)\) in the \((s_1, s_2)\)-increasing concave ordering, denoted by \((X_1, X_2) \preceq_{(s_1, s_2)-icv} (Y_1, Y_2)\), when \( \mathbb{E}[u(X_1, X_2)] \leq \mathbb{E}[u(Y_1, Y_2)] \) for all the utility functions \( u \) in \( \mathcal{U}_{(s_1, s_2)-icv} \), provided the expectations exist. It is easily seen that

\[
(X_1, X_2) \preceq_{(s_1, s_2)-icv} (Y_1, Y_2) \Rightarrow X_1 \preceq_{s_1-icv} Y_1 \text{ and } X_2 \preceq_{s_2-icv} Y_2
\]

so that each component can be compared through univariate \( s_1 \)– and \( s_2 \)-increasing concave order.

We indicate that some special cases of the orderings \( \preceq_{(s_1, s_2)-icv} \) have been considered before in economics, e.g. in Atkinson & Bourguignon (1982) and Mosler (1984). The \((1,1)\)-increasing concave order corresponds to the \( \preceq_{m1} \) order relation in Mosler (1984), and the \((2,2)\)-increasing concave order corresponds to the \( \preceq_{m3} \) order relation in Mosler (1984). In applied probability, the \((1,1)\)-increasing concave order is known as the lower orthant order, while the \((2,2)\)-increasing concave order is termed as the orthant concave order.

### 3.3 Maximal generator of \( \preceq_{(s_1, s_2)-icv} \)

The maximal generator of \( \preceq_{(s_1, s_2)-icv} \) corresponds to the class \( \overline{\mathcal{U}}_{(s_1, s_2)-icv} \) of bivariate \((s_1, s_2)\)-increasing concave functions, defined by means of bivariate divided differences. Recall that the bivariate divided difference operator [:::] is defined by

\[
\begin{bmatrix}
x_0, \ldots, x_{s_1} \\
y_0, \ldots, y_{s_2}
\end{bmatrix} u = [x_0, \ldots, x_{s_1}] \circ [y_0, \ldots, y_{s_2}] u = [y_0, \ldots, y_{s_2}] \circ [x_0, \ldots, x_{s_1}] u.
\]

(3.2)

Now, the class \( \overline{\mathcal{U}}_{(s_1, s_2)-icv} \) contains all the bivariate \((s_1, s_2)\)-increasing concave utility functions, that is, the utilities \( u \) such that the inequality

\[
(-1)^{k_1+k_2+1} \begin{bmatrix} x_0, \ldots, x_{k_1} \\
y_0, \ldots, y_{k_2}
\end{bmatrix} u \geq 0
\]

holds for all \( x_0, \ldots, x_{k_1}, y_0, \ldots, y_{k_2}, k_1 = 0, \ldots, s_1, k_2 = 0, \ldots, s_2 \) such that \( k_1 + k_2 \geq 1 \).

### 3.4 Characterization through integrated left tails

Let us first characterize the \((s_1, s_2)\)-increasing concave order. Characterizing such an integral stochastic orderings generally consists in substituting for the generating cone of functions \( \mathcal{U}_{(s_1, s_2)-icv} \), either a dense subclass contained in \( \mathcal{U}_{(s_1, s_2)-icv} \), or a larger cone corresponding to the closure of \( \mathcal{U}_{(s_1, s_2)-icv} \) in some suitable topology. In the latter case, we get the maximal generator \( \overline{\mathcal{U}}_{(s_1, s_2)-icv} \) of \( \preceq_{(s_1, s_2)-icv} \). The next result identifies a subclass of the \((s_1, s_2)\)-increasing concave functions which can be used as a test set to establish whether a \( \preceq_{(s_1, s_2)-icv} \) relation holds in a particular situation.

8
Characterization 3.1. Let \((X_1, X_2)\) and \((Y_1, Y_2)\) be two random vectors valued in \([a_1, b_1] \times [a_2, b_2]\). Then, \((X_1, X_2) \succeq_{(s_1, s_2)-icv} (Y_1, Y_2)\) if, and only if,

\[
\begin{align*}
\mathbb{E}[(b_1 - X_1)^{i_1} (b_2 - X_2)^{i_2}] & \geq \ mathbb{E}[(b_1 - Y_1)^{i_1} (b_2 - Y_2)^{i_2}] \\
& \text{for } i_1 = 0, \ldots, s_1 - 1 \text{ and } i_2 = 0, \ldots, s_2 - 1 \\
\mathbb{E}[(b_1 - X_1)^{i_1} (t_2 - X_2)^{s_2-1}] & \geq \ mathbb{E}[(b_1 - Y_1)^{i_1} (t_2 - Y_2)^{s_2-1}] \\
& \text{for } i_1 = 0, \ldots, s_1 - 1 \text{ and every } t_2 \geq 0 \\
\mathbb{E}[(t_1 - X_1)^{s_1-1} (b_2 - X_2)^{i_2}] & \geq \ mathbb{E}[(t_1 - Y_1)^{s_1-1} (b_2 - Y_2)^{i_2}] \\
& \text{for } i_2 = 0, \ldots, s_2 - 1 \text{ and every } t_1 \geq 0 \\
\mathbb{E}[(t_1 - X_1)^{s_1-1} (t_2 - X_2)^{s_2-1}] & \geq \ mathbb{E}[(t_1 - Y_1)^{s_1-1} (t_2 - Y_2)^{s_2-1}] \\
& \text{for every } t_1 \geq 0 \text{ and } t_2 \geq 0.
\end{align*}
\]

Proof. Recall from Property 2.3 in Denuit, Lefèvre & Mesfioui (1999) that \((X_1, X_2) \succeq_{(s_1, s_2)-icv} (Y_1, Y_2)\) holds if, and only if,

\[
\mathbb{E}[g(b_1 - Y_1, b_2 - Y_2)] \leq \mathbb{E}[g(b_1 - X_1, b_2 - X_2)]
\]

for any function \(g\) such that \(g^{(k_1, k_2)} \geq 0\) for all \(k_1 = 0, \ldots, s_1, k_2 = 0, \ldots, s_2\), such that \(k_1 + k_2 \geq 1\). Now, whatever the random variables \(Z_1\) and \(Z_2\) valued in \([0, c_1]\) and \([0, c_2]\), the expectation \(\mathbb{E}[g(Z_1, Z_2)]\) can be represented as

\[
\mathbb{E}[g(Z_1, Z_2)] = \sum_{i_1=0}^{s_1-1} \sum_{i_2=0}^{s_2-1} g^{(i_1,i_2)}(0, 0) \frac{\mathbb{E}[Z_1^{i_1} Z_2^{i_2}]}{i_1! i_2!} \\
+ \sum_{i_1=0}^{s_1-1} \int_{t_2=0}^{c_2} \mathbb{E}\left[ (Z_2 - t_2)^{s_2-1} Z_1^{i_1} \right] \frac{g^{(i_1,i_2)}(0, t_2)}{(s_2-1)! i_1!} dt_2 \\
+ \sum_{i_2=0}^{s_2-1} \int_{t_1=0}^{c_1} \mathbb{E}\left[ (Z_1 - t_1)^{s_1-1} Z_2^{i_2} \right] \frac{g^{(i_1,i_2)}(t_1, 0)}{(s_1-1)! i_2!} dt_1 \\
+ \int_{t_1=0}^{c_1} \int_{t_2=0}^{c_2} \mathbb{E}\left[ (Z_1 - t_1)^{s_1-1} (Z_2 - t_2)^{s_2-1} \right] \frac{g^{(i_1,i_2)}(t_1, t_2)}{(s_1-1)! (s_2-1)!} dt_2 dt_1.
\]

Now, replacing \(Z_i\) with \(b_i - X_i\) and \(c_i\) with \(b_i - a_i\), \(i = 1, 2\), gives the announced result. \(\square\)

If we define \(U_{(s_1, s_2)-icv}\) as the set of utilities \((x_1, x_2) \mapsto -(b_1 - x_1)^{i_1} (b_2 - x_2)^{i_2}\) for \(i_1 = 0, \ldots, s_1 - 1\) and \(i_2 = 0, \ldots, s_2 - 1\), \((x_1, x_2) \mapsto -(b_1 - x_1)^{i_1} (t_2 - x_2)^{s_2-1}\) for \(i_1 = 0, \ldots, s_1 - 1\) and fixed \(t_2\), \((x_1, x_2) \mapsto -(t_1 - x_1)^{s_1-1} (b_2 - x_2)^{i_2}\) for \(i_2 = 0, \ldots, s_2 - 1\) and fixed \(t_1\), and \((x_1, x_2) \mapsto -(t_1 - x_1)^{s_1-1} (t_2 - x_2)^{s_2-1}\) for some fixed \(t_1\) and \(t_2\), we have that

\[
(X_1, X_2) \preceq_{(s_1, s_2)-icv} (Y_1, Y_2) \iff (3.1) \text{ holds with } U_* = U_{(s_1, s_2)-icv}
\]

\[
\iff (3.1) \text{ holds with } U_* = U_{(s_1, s_2)-icv}
\]

\[
\iff (3.1) \text{ holds with } U_* = U_{(s_1, s_2)-icv}.
\]

It can be shown that the expectations of elements of \(U_{(s_1, s_2)-icv}\) are related to the bivariate integrated left tails. Specifically, let \((X_1, X_2)\) be a random vector valued \([a_1, b_1] \times [a_2, b_2],\)
with distribution function $F_{(X_1,X_2)}$. For $(x_1, x_2) \in [a_1, b_1] \times [a_2, b_2]$, we put $F^{(1,1)}_{(X_1,X_2)}(x_1, x_2) \equiv F_{(X_1,X_2)}(x_1, x_2)$, and we define recursively for $s_1$ and $s_2 \geq 1$,

$$F^{(s_1+1,s_2)}_{(X_1,X_2)}(t_1, t_2) = \int_{y_1=a_1}^{t_1} F^{(s_1,s_2)}_{(X_1,X_2)}(y_1, t_2)dy_1. \quad (3.3)$$

$$F^{(s_1,s_2+1)}_{(X_1,X_2)}(t_1, t_2) = \int_{y_2=a_2}^{t_2} F^{(s_1,s_2)}_{(X_1,X_2)}(t_1, y_2)dy_2. \quad (3.4)$$

It is easily shown that

$$F^{(k_1,k_2)}_{(X_1,X_2)}(t_1, t_2) = \mathbb{E} \left[ \left( t_1 - X_1 \right)^{k_1-1} \left( t_2 - X_2 \right)^{k_2-1} \right] \frac{1}{(k_1-1)!(k_2-1)!}, \quad (3.5)$$

which extends Lemma 1 in Scarsini (1985) to the case of unequal $k_1$ and $k_2$. Characterization 3.1 then allows us to state the next result.

**Characterization 3.2.** Let $(X_1, X_2)$ and $(Y_1, Y_2)$ be two random vectors valued in $[a_1, b_1] \times [a_2, b_2]$. Then

$$(X_1, X_2) \preceq_{(s_1,s_2)-icv} (Y_1, Y_2) \iff \begin{cases} F^{(k_1,k_2)}_{(X_1,X_2)}(b_1, b_2) \geq F^{(k_1,k_2)}_{(Y_1,Y_2)}(b_1, b_2), \\
for \ k_1 = 0, \ldots, s_1, \ k_2 = 0, \ldots, s_2,
F^{(k_1,s_2)}_{(X_1,X_2)}(b_1, t_2) \geq F^{(k_1,s_2)}_{(Y_1,Y_2)}(b_1, t_2), \\
for \ all \ t_2 \ and \ k_1 = 1, \ldots, s_1,
F^{(s_1,k_2)}_{(X_1,X_2)}(t_1, b_2) \geq F^{(s_1,k_2)}_{(Y_1,Y_2)}(t_1, b_2), \\
for \ all \ t_1 \ and \ k_2 = 1, \ldots, s_2,
F^{(s_1,s_2)}_{(X_1,X_2)}(t_1, t_2) \geq F^{(s_1,s_2)}_{(Y_1,Y_2)}(t_1, t_2),
for \ all \ t_1 \ and \ t_2. \end{cases}$$

Scarsini (1985) developed a multivariate extension of the higher degree stochastic dominance introduced by Rolski (1976) and Fishburn (1976, 1980). The orderings by Scarsini (1985) however differ from the $\preceq_{(s_1,s_2)-icv}$ orderings examined here in that they take $s_1 = s_2 = s$ and impose only the fourth condition in Characterization 3.2.

## 4 Conditional risk aversion, risk independence, and bivariate stochastic dominance

Characterization 3.1 shows that the $(s_1, s_2)$-increasing concave order is a product stochastic ordering, as it is precisely stated next.

**Proposition 4.1.** Let $(X_1, X_2)$ and $(Y_1, Y_2)$ be two random vectors valued in $[a_1, b_1] \times [a_2, b_2]$. Then,

$$(X_1, X_2) \preceq_{(s_1,s_2)-icv} (Y_1, Y_2) \iff \mathbb{E}[u_1(X_1)u_2(X_2)] \geq \mathbb{E}[u_1(Y_1)u_2(Y_2)]$$

for all the non-positive $u_1 \in \mathcal{U}_{s_1-icv}$ and the non-positive $u_2 \in \mathcal{U}_{s_2-icv}$, or, equivalently, for all the non-positive $u_1 \in \mathcal{U}_{s_1-icv}$ and the non-positive $u_2 \in \mathcal{U}_{s_2-icv}$.
This is a direct consequence of Characterization 3.1, noting that if \( u_1 \in U_{s_1-icv} \) and \( u_2 \in U_{s_2-icv} \) and both \( u_1 \) and \( u_2 \) are non-positive, then \( u = -u_1 u_2 \in U_{(s_1, s_2)-icv} \). Note that we have to control the sign of both \( u_1 \) and \( u_2 \), to fix the sign of \( u^{(k_1,0)} = -u_1^{(k_1)} u_2 \) and of \( u^{(0,k_2)} = -u_1^{(k_2)} u_2 \). We are now ready to establish that the common preferences of all the decision-makers with risk-independent utility function are equivalent with those of all the decision-makers with utility function in \( U_{(s_1, s_2)-icv} \).

**Corollary 4.2.** Let \((X_1, X_2)\) and \((Y_1, Y_2)\) be two random vectors valued in \([a_1,b_1] \times [a_2,b_2] \). Then, \((X_1, X_2) \succeq_{(s_1, s_2)-icv} (Y_1, Y_2)\) holds if, and only if, all the decision-makers whose utility function is of the form (1.4) with \( u_1 \in U_{s_1-icv} \) and \( u_2 \in U_{s_2-icv} \) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\).

The proof of Corollary 4.2 is a direct application of Proposition 4.1.

## 5 Correlation aversion

### 5.1 Risk aversion

There are several concepts of risk aversion in the multivariate case. In this section, we discuss some of them. We refer the reader interested in a detailed study to Dorfleitner & Krapp (2007).

When the decision-maker faces bivariate outcomes, Richard (1975) relates risk aversion to substitutability of goods. More precisely, considering any \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \), and defining the lotteries

\[
\mathcal{L}_1 = \begin{cases} 
(x_1, y_1) \text{ with probability } 0.5 \\
(x_2, y_2) \text{ with probability } 0.5
\end{cases} \quad \text{and} \quad \mathcal{L}_2 = \begin{cases} 
(x_1, y_2) \text{ with probability } 0.5 \\
(x_2, y_1) \text{ with probability } 0.5
\end{cases}
\]

Richard (1975) required preference of \( \mathcal{L}_2 \) over \( \mathcal{L}_1 \) for risk aversion, which means that the bivariate utility function has to be submodular, i.e. the inequality

\[
u(x_1,y_1) + u(x_2,y_2) \leq u(x_1,y_2) + u(x_2,y_1)
\]

has to hold for any \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \). If \( u \) is twice differentiable, it is easily seen that this is equivalent to \( u^{(1,1)} \leq 0 \). The utilities in \( U_{(1,1)-icv} \) are submodular and, thus, exhibit risk aversion in the sense of Richard (1975).

Note that the bivariate risk aversion defined by Richard (1975) does not require any condition on marginal risk aversion. It is not hard to find utility functions \( u \) which simultaneously verify \( u^{(1,1)} \leq 0 \) and \( u^{(2,0)} \geq 0 \) as well as \( u^{(0,2)} \geq 0 \) so that a decision-maker can be bivariate risk averse but still be risk-seeking for one attribute.

There are other ways to define risk aversion for bivariate outcomes. Kihlstrom & Mirman (1974) introduced a second approach, in line with the univariate definition of risk-aversion. Precisely, an economic agent is bivariate risk-averse according to Kihlstrom & Mirman (1974) if given the choice between a random vector \((X_1, X_2)\) and its expected value \((E[X_1], E[X_2])\), he always prefers the second to the first. From Jensen’s inequality, this holds if the Hessian matrix of \( u \) is negative semidefinite, that is, if \( u \) is concave. Note that even if \( u_1 \) and \( u_2 \) involved in (1.4) are concave, the bivariate utility function (1.4) is not necessarily
concave (it is only concave in each argument, or componentwise concave). This is in contrast with the additive case where \( u = u_1 + u_2 \) is concave if, and only if, each \( u_1 \) and \( u_2 \) is concave.

**Finkelstein, Kella & Scarsini (1999)** proved that provided \( \mathbb{E}[X_1|X_2] \) is non-decreasing in \( X_2 \), the inequality

\[
\mathbb{E}[u(X_1, X_2)] \leq \mathbb{E}[u(\mathbb{E}[X_1], X_2)]
\]

holds true provided \( u \) is submodular and concave in its first argument. This relates submodularity to the standard notion of risk aversion (defined as the preference of the expectation of a random variable over the random variable itself) provided the outcomes \( X_1 \) and \( X_2 \) are positively dependent. All the utilities in \( \mathcal{U}_{(s_1,s_2)−icv} \) exhibit risk aversion as defined in Richard (1975). If \( s_1 \geq 2 \) then (5.1) also holds true provided \( X_1 \) and \( X_2 \) are positively dependent.

### 5.2 Aversion to correlation increasing transformation

Another interpretation of the \((s_1, s_2)\)-increasing concavity relates to the concept of increasing correlation transformation due to Epstein & Tanny (1980) and exploited by Denuit, Eeckhoudt & Rey (2008). Specifically, let \( I_1 \) and \( I_2 \) be a couple of binary random variables such that \( \Pr[I_i = 0] = 1 - \Pr[I_i = 1] = p_i \), \( i = 1, 2 \). Without loss of generality, we assume that \( p_1 \leq p_2 \). Let us now consider \( \rho \) such that \( -p_1p_2 \leq \rho \leq p_1(1-p_2) \) and define the joint distribution of \((I_1, I_2)\) as

\[
\begin{align*}
\Pr[I_1 = 0, I_2 = 0] &= p_1p_2 + \rho \\
\Pr[I_1 = 1, I_2 = 0] &= (1-p_1)p_2 - \rho \\
\Pr[I_1 = 0, I_2 = 1] &= p_1(1-p_2) - \rho \\
\Pr[I_1 = 1, I_2 = 1] &= (1-p_1)(1-p_2) + \rho.
\end{align*}
\]

When \( \rho \) increases we face a correlation increasing transformation as defined by Epstein & Tanny (1980) and a correlation averse decision-maker should then dislike an increase in \( \rho \).

It can be shown that the distribution of \((I_1a_1, I_2a_2)\) decreases in the \( \preceq_{(1,1)−icv} \) order as \( \rho \) increases, for any positive constants \( a_1 \) and \( a_2 \).

**Denuit, Eeckhoudt & Rey (2008)** consider the bivariate random vector \((1-I_1)X_1 + I_1Y_1, (1-I_2)X_2 + I_2Y_2)\) where the random variables \( X_1, X_2, Y_1, \) and \( Y_2 \) are assumed to be mutually independent, independent from \((I_1, I_2)\), and such that \( X_1 \preceq_{s_1−icv} Y_1 \) and \( X_2 \preceq_{s_2−icv} Y_2 \). The main result of Denuit, Eeckhoudt & Rey (2008) is that \((1-I_1)X_1 + I_1Y_1, (1-I_2)X_2 + I_2Y_2)\) decreases in the \( \preceq_{(s_1,s_2)−icv} \)-sense as \( \rho \) increases. This shows that the decision-makers with utilities in \( \mathcal{U}_{(s_1,s_2)−icv} \) dislike any increase in correlation.

### 5.3 Aversion to positive quadrant dependence

Let us briefly discuss another approach to correlation aversion, where independence serves as a reference. Recall that \((X_1, X_2)\) is said to be positive quadrant dependent (PQD, in short) if

\[
(X_1, X_2) \preceq_{(1,1)−icv} (X_1^+, X_2^+)
\]

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where \((X_1^+, X_2^+)\) is an independent version of \((X_1, X_2)\), that is, \((X_1^+, X_2^+)\) and \((X_1, X_2)\) have the same univariate marginals and \(X_1^+\) and \(X_2^+\) are mutually independent. This means that the inequality

\[
\Pr[X_1 > t_1, X_2 > t_2] \geq \Pr[X_1^+ > t_1, X_2^+ > t_2] = \Pr[X_1 > t_1] \Pr[X_2 > t_2]
\]

holds true for every \(t_1\) and \(t_2\). This inequality reads “the probability that \(X_1\) and \(X_2\) are simultaneously large is at least as large as if they were independent”. As pointed out by Cardin & Ferretti (2004)

\[
u\text{ submodular} \Leftrightarrow \mathbb{E}[u(X_1, X_2)] \leq \mathbb{E}[u(X_1^+, X_2^+)]\]

for every PQD \((X_1, X_2)\).

This shows that the decision-makers with a utility function \(u \in \overline{U}_{(1,1)} - icv\) prefer independence over PQD.

Now, if we define positive dependence of degree \((s_1, s_2)\) as

\[
(X_1, X_2) \preceq_{(s_1, s_2)} (X_1^+, X_2^+) \Leftrightarrow \text{Cov}[u_1(X_1), u_2(X_2)] \geq 0
\]

for all the non-positive \(u_1 \in \overline{U}_{s_1} - icv\) and \(u_2 \in \overline{U}_{s_2} - icv\), we then see that the decision-makers with utility functions in \(\overline{U}_{(s_1, s_2)} - icv\) are those who prefer independence over positive dependence of degree \((s_1, s_2)\).

Thus, the random couple \((X_1, X_2)\) is positively dependent of degree \((s_1, s_2)\) if the utilities \(u_1(X_1)\) and \(u_2(X_2)\) are positively correlated whatever the \(s_1\)- and \(s_2\)-increasing utility functions \(u_1\) and \(u_2\). In particular, this means that

\[
\text{Cov}[(t_1 - X_1)^{s_1-1}_+, (t_2 - X_2)^{s_2-1}_+] \geq 0 \quad \text{for all } t_1 \text{ and } t_2.
\]

The \((s_1 - 1)\)th power of the shortfall \((t_1 - X_1)_+\) of \(X_1\) below the threshold \(t_1\) remains positively correlated with the \((s_2 - 1)\)th power of the shortfall \((t_2 - X_2)_+\) of \(X_2\) below the threshold \(t_2\) whatever the \(t_i\)'s. This notion weakens as \(s_1\) and \(s_2\) increase. At the limit, only completely monotone utilities (i.e. mixtures of utilities) are used to define this notion of positive dependence.

### 6 Portfolios and hedging effects

Let us assume that adding the attributes is meaningful (for instance, because both attributes represent financial outcomes expressed in the same currency). We now consider portfolios \(w_1X_1 + w_2X_2\) and \(w_1Y_1 + w_2Y_2\) made of \((X_1, X_2)\) and \((Y_1, Y_2)\) with non-negative weights \(w_1\) and \(w_2\). Preferences over these portfolios express the attractiveness of the hedging effects existing between the components of the bivariate outcomes of \((X_1, X_2)\) and \((Y_1, Y_2)\).

Bivariate \((s_1, s_2)\)-increasing concave orderings imply interesting univariate ordering results on their transformed components, as shown in the next result.

**Property 6.1.** Given two random couples \((X_1, X_2)\) and \((Y_1, Y_2)\), we have

\[
(X_1, X_2) \preceq_{(s_1, s_2)} (Y_1, Y_2) \Rightarrow w_1X_1 + w_2X_2 \preceq_{(s_1 + s_2)} w_1Y_1 + w_2Y_2, \text{ for all } w_1 \text{ and } w_2 \geq 0.
\]
Proof. Consider a univariate utility function \( u \in \mathcal{U}_{(s_1+s_2)-icv} \) and define the bivariate function
\[
v^{(k_1,k_2)}(x_1, x_2) = u^{(k_1,k_2)}(x_1) = u^{(k_1,k_2)}(w_1 x_1 + w_2 x_2)
\]
we have \( v \in \mathcal{U}_{(s_1,s_2)-icv} \), which achieves the proof. \( \Box \)

This result shows that when all the decision-makers with risk independent utility functions of the form (1.4) prefer \((Y_1, Y_2)\) over \((X_1, X_2)\), any portfolio made of \(Y_1\) and \(Y_2\) is considered as more attractive in the \((s_1 + s_2)\)th stochastic dominance than the corresponding portfolio made of \(X_1\) and \(X_2\). Coming back to the positive dependence of degree \((s_1, s_2)\), we see that in such a case,
\[
w_1 X_1 + w_2 X_2 \preceq_{(s_1+s_2)-icv} w_1 X_1^+ + w_2 X_2^+
\]
holds true for all \(w_1, w_2 \geq 0\).

Let us now examine the \((1,1)\)-increasing concave, \((1,2)\)-increasing concave, \((2,1)\)-increasing concave, and \((2,2)\)-increasing concave stochastic order relations to illustrate the contents of Property 6.1. To this end, note that Characterization 3.1 reduces to

\[
(X_1, X_2) \preceq_{(1,1)-icv} (Y_1, Y_2) \iff \Pr[X_1 \leq t_1, X_2 \leq t_2] \geq \Pr[Y_1 \leq t_1, Y_2 \leq t_2] \quad \text{for all } t_1 \text{ and } t_2
\]

\[
(X_1, X_2) \preceq_{(1,2)-icv} (Y_1, Y_2) \iff \begin{cases} 
X_1 \preceq_{2-icv} Y_1 \\
X_2 \preceq_{1-icv} Y_2 \\
\mathbb{E}[(t_1 - X_1)_+][X_2 \leq t_2] \geq \mathbb{E}[(t_1 - Y_1)_+][Y_2 \leq t_2] 
\end{cases} \quad \text{for all } t_1 \text{ and } t_2
\]

\[
(X_1, X_2) \preceq_{(2,1)-icv} (Y_1, Y_2) \iff \begin{cases} 
X_1 \preceq_{1-icv} Y_1 \\
X_2 \preceq_{2-icv} Y_2 \\
\mathbb{E}[(t_2 - X_2)_+][X_1 \leq t_1] \geq \mathbb{E}[(t_2 - Y_2)_+][Y_1 \leq t_1] 
\end{cases} \quad \text{for all } t_1 \text{ and } t_2
\]

\[
(X_1, X_2) \preceq_{(2,2)-icv} (Y_1, Y_2) \iff \begin{cases} 
(X_1, X_2) \preceq_{(2,1)-icv} (Y_1, Y_2) \\
(X_1, X_2) \preceq_{(1,2)-icv} (Y_1, Y_2) \\
\mathbb{E}[(t_1 - X_1)_+ + (t_2 - X_2)_+] \geq \mathbb{E}[(t_1 - Y_1)_+ + (t_2 - Y_2)_+] 
\end{cases} \quad \text{for all } t_1 \text{ and } t_2
\]

Considering \((X_1, X_2)\) and \((Y_1, Y_2)\) with identical marginals (that is, (1.1) is fulfilled), \((X_1, X_2) \preceq_{(s_1,s_2)-icv} (Y_1, Y_2)\) means that the dependence structure between the \(Y_i\)'s is viewed as more attractive than the dependence structure between the \(X_i\)'s by all the decision-makers with a utility function in \(\mathcal{U}_{(s_1,s_2)-icv}\). In particular, Pearson’s linear correlation coefficient is higher for \((X_1, X_2)\) than for \((Y_1, Y_2)\). If \((X_1, X_2)\) and \((Y_1, Y_2)\) have the same univariate marginals, then it is easily seen that

\[
(X_1, X_2) \preceq_{(1,1)-icv} (Y_1, Y_2) \iff \Pr[X_1 \leq t_1 | X_2 \leq t_2] \geq \Pr[Y_1 \leq t_1 | Y_2 \leq t_2] \quad \text{for all } t_1 \text{ and } t_2.
\]

This intuitively means that the knowledge that \(X_2\) is small (that is, \(X_2 \leq t_2\)) increases the probability that \(X_1\) is also small (that is, \(X_1 \leq t_1\)) compared to \((Y_1, Y_2)\). The same kind of
interpretation holds in the other cases. Specifically, if \((X_1, X_2)\) and \((Y_1, Y_2)\) have the same univariate marginals, then
\[
(X_1, X_2) \preceq_{(2,1) - icv} (Y_1, Y_2) \iff \mathbb{E}[(t_1 - X_1)_+ | X_2 \leq t_2] \geq \mathbb{E}[(t_1 - Y_1)_+ | Y_2 \leq t_2]
\]
for all \(t_1\) and \(t_2\) so that the knowledge that \(X_2\) is small (that is, \(X_2 \leq t_2\)) increases the average part of \(X_1\) below any threshold \(t_1\) compared to \((Y_1, Y_2)\). Similarly
\[
(X_1, X_2) \preceq_{(1,2) - icv} (Y_1, Y_2) \iff \mathbb{E}[(t_2 - X_2)_+ | X_1 \leq t_1] \geq \mathbb{E}[(t_2 - Y_2)_+ | Y_1 \leq t_1]
\]
for all \(t_1\) and \(t_2\). Whenever (6.1) or (6.2) holds true then we have
\[
w_1X_1 + w_2X_2 \preceq_{3-icv} w_1Y_1 + w_2Y_2
\]
for all \(w_1, w_2 \geq 0\), so that the portfolio \(w_1Y_1 + w_2Y_2\) is preferred over the portfolio \(w_1X_1 + w_2X_2\) by all the profit-seeking, risk averse, prudent, and temperant decision-makers.

Now, when \(s_1 = s_2 = 2\), \((X_1, X_2) \preceq_{(2,2) - icv} (Y_1, Y_2)\) holds true if, and only if, (6.1)-(6.2) are satisfied as well as
\[
\text{Cov}[(t_1 - X_1)_+, (t_2 - X_2)_+] \geq \text{Cov}[(t_1 - Y_1)_+, (t_2 - Y_2)_+]
\]
for all \(t_1\) and \(t_2\), which means that the \((t_i - X_i)_+\)'s are more positively correlated than the \((t_i - Y_i)_+\)'s. Whatever the thresholds \(t_1\) and \(t_2\), this means that the shortfalls \((t_1 - X_1)_+\) and \((t_2 - X_2)_+\) are more correlated than the shortfalls \((t_1 - Y_1)_+\) and \((t_2 - Y_2)_+\). In such a case, the stochastic inequality
\[
w_1X_1 + w_2X_2 \preceq_{4-icv} w_1Y_1 + w_2Y_2
\]
holds true for all \(w_1, w_2 \geq 0\), so that the portfolio \(w_1Y_1 + w_2Y_2\) is preferred over the portfolio \(w_1X_1 + w_2X_2\) by all the profit-seeking, risk averse, prudent, and temperant decision-makers.

## 7 Conclusion

In this paper, we have established the equivalence between the common preferences of all the decision-makers with independent utility functions and those exhibiting correlation aversion. Specifically, we have extended to the whole class of the \(\preceq_{(s_1, s_2) - icv}\) orders the seminal ideas contained in the paper by Mosler (1984) for \(s_1 \leq 2\) and \(s_2 \leq 2\) to arbitrary values of \(s_1\) and \(s_2\). The theoretical results derived in the present paper allow for interesting interpretations of empirical studies, where risk independence is often assumed for the preferences of decision-makers.

We have also reviewed different types of risk aversion when bivariate outcomes are considered. The aversion to correlation increasing transformations and to positive dependence of degree \((s_1, s_2)\) allow for a better understanding of the preferences expressed by the \((s_1, s_2)\)-increasing concave multiattribute utility functions.

To end with, portfolios built from two outcomes are compared. The bivariate \((s_1, s_2)\)-increasing concave order relation between bivariate outcomes yields a univariate \((s_1 + s_2)\)-increasing concave order between the resulting portfolios. For \(s_1\) and \(s_2\) less than 2, the dependence structure disliked by risk averse, prudent, and temperant decision-makers are identified. This allows for a better understanding of the actual meaning of these three widely used risk attitudes.
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