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ON BIVARIATE EXTREMES

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Nonparametric Bayesian Inference on Bivariate Extremes

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Abstract

The tail of a bivariate distribution function in the domain of attraction of a bivariate extreme-value distribution may be approximated by the one of its extreme-value attractor. The extreme-value attractor has margins that belong to a three-parameter family and a dependence structure which is characterised by a spectral measure, that is a probability measure on the unit interval with mean equal to one half. As an alternative to parametric modelling of the spectral measure, we propose an infinite-dimensional model which is at the same time manageable and still dense within the class of spectral measures. Inference is done in a Bayesian framework, using the censored-likelihood approach. In particular, we construct a prior distribution on the class of spectral measures and develop a trans-dimensional Markov chain Monte Carlo algorithm for numerical computations. The method provides a bivariate predictive density which can be used for predicting the extreme outcomes of the bivariate distribution. In a practical perspective, this is useful for computing rare event probabilities and extreme conditional quantiles. The methodology is validated by simulations and applied to a data-set of Danish fire insurance claims.

Keywords. Bayes, Bivariate Extreme Value Distribution, Extreme Conditional Quantiles, MCMC, Prediction, Rare Event Probabilities, Reversible Jumps, Spectral Measure.

1 Introduction

In areas such as engineering or financial risk management, decisions have to be made which depend on the extreme outcomes of two or more variables. Particular examples of interesting questions may be: how high should a dike be in order to withstand exceptionally high levels of a river at several sites simultaneously? How much capital to set aside in order to have sufficient reserve in times of financial crises, affecting the values of multiple financial securities at once?

These kind of problems require inference on the distribution of bivariate (or generally multivariate) extremes. In particular, we are interested in the bivariate density of a random pair \((X_1, X_2)\) on a quadrant \([u_1, \infty) \times [u_2, \infty)\), where \(u_1\) and \(u_2\) are large thresholds, that is \(P(X_1 > u_1)\) and \(P(X_2 > u_2)\) are positive but small. This density

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can not only be used for the computation of rare-event probabilities over joint tail regions, but also of extreme conditional quantiles of one variable given an extreme outcome in the other variable. For instance, it may be of interest to find a level $x_2$ such that for a given probability $p > 0$ (small) and a given value of $x_1$ (large) we have $P(X_2 > x_2 \mid X_1 = x_1) = p$. Up to the best of our knowledge, the latter type of problem has not yet been properly addressed in the extreme-value literature.

More specifically, let $(X_{i1}, X_{i2}), i = 1, \ldots, n,$ be a random sample from a bivariate distribution $F$ in the max-domain of attraction of a bivariate extreme-value distribution $G$. Except for the case of asymptotic independence, the (bivariate) tail of $F$ is well approximated by the one of $G$. The margins of the latter distribution are characterised by three parameters each, for a total of six parameters. We shall refer to these as the tail parameters. In addition, the dependence structure of $G$ is characterised by a spectral measure, which can be any probability measure on the unit interval $[0, 1]$ with mean $1/2$. Thus, an approximation formula for $F$ in a bivariate tail region is obtained by the specification of six marginal tail parameters and a spectral measure. It follows that approximate inference on the tail of $F$ may be done via inference on these parameters. Inference on the spectral measure may be done within parametric families, see for instance [Boldi and Davison (2007), Coles and Tawn (1991, 1994), de Haan et al. (2008), Einmahl et al. (2008), Joe et al. (1992), Ledford and Tawn (1996), Smith (1994)]. Alternatively, one may prefer to proceed nonparametrically, see for instance [de Haan and de Ronde (1998), de Haan and Sinha (1999), Einmahl et al. (2001, 2006), Einmahl and Segers (2009), Schmidt and Stadtmüller (2006)]. Surveys of these methods can be found for instance in the monographs [Coles (2001), Beirlant et al. (2004), de Haan and Ferreira (2006), Kotz and Nadarajah (2000), de Haan and Ferreira (2006)].

Data about extreme events being scarce by nature, the statistical uncertainty in extreme-value analysis is quite substantial. The question is how to deal with this uncertainty for practical purposes. A typical such purpose is prediction, the task being to compute a high return level, that is a level which is exceeded once, on average, during a long future time interval. The uncertainty due to the random nature of future outcomes then has to be combined with the statistical uncertainty on the parameter estimates. In a frequentist setting, it may be unclear how to do so: should one use a high quantile’s point estimate or rather the upper bound of a certain confidence interval? As argued for instance in [Coles and Tawn (1996) and Coles and Tawn (2005)], the Bayesian approach via the predictive density [Aitchison and Dunsmore (1975)] seems more coherent. However, extreme-value dependence structures are essentially infinite-dimensional. Therefore, nonparametric Bayesian methodology for multivariate extremes should be further developed. Our paper aims to take a step in that direction.

Essentially, we extend the censored-likelihood method developed in [Ledford and Tawn (1996)] to the case of arbitrary (i.e. infinite dimensional) spectral measures in a Bayesian setup. To this end, we select a prior on the six marginal tail parameters and on the set of spectral measures. Via the censored likelihood, the joint posterior of the tail parameters and the spectral measure is computed and converted into a posterior distribution for the tail quantities of interest. The actual inference based on the posterior is performed by means of a trans-dimensional Markov chain Monte Carlo algorithm; see for instance [Guillotte and Perron (2008)] for similar work in the context of annual maxima, that is when data can be modelled directly by a bivariate extreme-value distribution. Our methodology enables the evaluation of the predictive density in a bivariate tail region, which can be used, for instance, for prediction of high future levels of one variable given such outcomes of the other one.
The prior selection for the spectral measure is the more delicate part, and this is the main contribution of the paper. We need to put a prior on $H$, where $H$ is the set of all cumulative distribution functions on $[0, 1]$ with mean $1/2$. The actual prior will be concentrated on a certain subset $H' \subset H$ of spectral measures. The latter subset is constructed in such a way that for any desired precision $\varepsilon > 0$ and for any $H \in H$, there exists $H' \in H'$ with
\[
\sup_{w \in [0,1]} |H'(w) - H(w)| \leq \varepsilon.
\]
At the same time, the set $H'$ needs to be sufficiently manageable for actual computations. The spectral measures in our construction are parametrised by a (constrained) vector, the components of which correspond to their atoms at 0 and 1, and to a finite number of points in $(0, 1)$. To each vector in the parameter space corresponds a spectral measure, specified via a cubic spline interpolation, which is absolutely continuous on $(0, 1)$.

The outline of the paper is as follows. Since the model provides an approximation of the bivariate tail of $F$ only, special care has to be taken on how to define the likelihood of the parameters given the data (Section 2). The construction of the subspace of spectral measures $H'$ is done in Section 3. The prior selection, the Bayesian framework, and the MCMC algorithm used for numerical computations are detailed in Section 4. In Section 5 we present the results of a simulation validating our methodology using artificial data and we analyse a data-set of Danish fire insurance claims, which has been previously studied in a univariate context by McNeil (1997) and Resnick (1997). Finally, we conclude with a discussion in Section 6.

2 Modelling bivariate tails

The domain-of-attraction condition on a bivariate cumulative distribution function $F$ yields approximations for $F$ on quadrants of the form $[u_1, \infty) \times [u_2, \infty)$, where $u_1$ and $u_2$ are high thresholds. Here, a high threshold means that $F_j(u)$ is less than but close to 1, where $F_j$ is the marginal distribution, $j \in \{1, 2\}$. From this approximation for $F$, approximations for interesting tail quantities can be derived. Since $F$ is only approximated on a subset of its support, care has to be taken when writing down the likelihood.

The domain-of-attraction condition. We assume the existence of sequences of constants $a_{n,j} > 0$ and $b_{n,j}$, $j \in \{1, 2\}$, and a bivariate cumulative distribution function $G$ with non-degenerate margins such that
\[
\lim_{n \to \infty} F^n(a_{n,1} x_1 + b_{n,1}, a_{n,2} x_2 + b_{n,2}) = G(x_1, x_2)
\]  
for all continuity points $(x_1, x_2)$ of $G$. Here, $G$ is called an extreme-value distribution function and is necessarily of the following form:

- Its marginal distribution functions $G_1$ and $G_2$ are univariate extreme-value distribution functions, so
\[
-\log G_j(x_j) = \left(1 + \xi_j \frac{x_j - \mu_j}{\sigma_j}\right)^{-1/\xi_j}, \quad j \in \{1, 2\},
\]
for $x_j$ such that $\sigma_j + \xi_j(x_j - \mu_j) > 0$, with shape parameter $\xi_j$ (the extreme-value index), location parameter $\mu_j$, and scale parameter $\sigma_j > 0$;
• Its dependence structure is given by

\[ - \log G(x, y) = \ell(- \log G_1(x), - \log G_2(y)), \]

for all \((x, y) \in \mathbb{R}^2\) such that \(G_1(x) > 0\) and \(G_2(y) > 0\), the tail dependence function \(\ell\) admitting the representation

\[ \ell(s, t) = 2 \int_{[0,1]} \max(w/s, (1-w)t) \, H(dw), \quad (s, t) \in [0, \infty)^2, \]

the spectral measure \(H\) being a probability measure on \([0, 1]\) with mean equal to \(\int w \, H(dw) = 1/2\). (Quite often, the name “spectral measure” is reserved for the measure \(2H\).)

Given a pair of large thresholds, \(u_1\) and \(u_2\), it will be convenient to rewrite the marginal distribution functions using a different parametrisation which incorporates the thresholds in the expression of these functions. More precisely, letting \(\eta_j = (\xi_j, \zeta_j, \sigma_j)\), \(j \in \{1, 2\}\), the margins \(G_1\) and \(G_2\) can be written as

\[ - \log G_j(x_j \mid \eta_j) = \zeta_j \left(1 + \frac{x_j - u_j}{\sigma_j}\right)^{-1/\xi_j}, \quad j \in \{1, 2\}, \]

for \(x_j\) such that \(\sigma_j + \xi_j(x_j - u_j) > 0\), where \(\xi_j\) is again the extreme-value index, \(\sigma_j > 0\) is a scale parameter, and \(0 < \zeta_j = -\log G_j(u_j \mid \eta_j) \approx 1 - G_j(u_j \mid \eta_j)\), the marginal probability of exceeding the threshold \(u_j\). Therefore, a bivariate extreme-value distribution function \(G\) is parameterised by its marginal parameter vectors \(\eta_1\) and \(\eta_2\) and its spectral measure \(H\). From now on, we shall make this explicit. Such a cumulative distribution function \((x_1, x_2) \mapsto G(x_1, x_2 \mid H, \eta_1, \eta_2)\) is absolutely continuous if and only if the restriction of the spectral measure \(H\) to the interior \((0, 1)\) of the unit interval is absolutely continuous. Still, the spectral measure is allowed to have atoms at 0 and 1, so as to include, for instance, the case of independence where \(H = \frac{1}{2} (\delta_0 + \delta_1)\), \(\delta_w\) being the Dirac measure at \(w\).

In statistical practice, the spectral measure is often modelled parametrically. In this article, however, we model \(H\) nonparametrically for maximum flexibility.

**The tail approximation.** The tail approximation for \(F\) boils down to stipulating that for large thresholds \(u_1\) and \(u_2\), and for \((x_1, x_2) \in [u_1, \infty) \times [u_2, \infty)\), the form of \(F(x_1, x_2)\) is that of a bivariate extreme-value cumulative distribution function. The justification comes from equation [2.2], and the fact that extreme-value distributions are max-stable: for \(t > 0\), the function \(G^t\) is also a bivariate extreme-value distribution function and it differs from \(G\) by location and scale only. Thus, we assume that for \((x_1, x_2) \in [u_1, \infty) \times [u_2, \infty)\), \(F(x_1, x_2) \approx F(x_1, x_2 \mid H, \eta_1, \eta_2)\), where \(F(x_1, x_2 \mid H, \eta_1, \eta_2)\) has dependence structure given by [2.2], for some spectral measure \(H\) and marginal distributions given by [2.3], for some parameter vectors \(\eta_1\) and \(\eta_2\).

Furthermore, we assume that the approximant \(F(x_1, x_2 \mid H, \eta_1, \eta_2)\) is absolutely continuous on the region \((x_1, x_2) \in [u_1, \infty) \times [u_2, \infty)\). Its density

\[ f(x_1, x_2 \mid H, \eta_1, \eta_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2 \mid H, \eta_1, \eta_2), \quad x_1 > u_1, \ x_2 > u_2, \]

is a (rather complicated) function of the six marginal tail parameters given by \(\eta_1\) and \(\eta_2\) together with the spectral measure \(H\). It is derived in Appendix [A.1].
The censored likelihood. The tail approximation to $F$ is only defined on the region $[u_1, \infty) \times [u_2, \infty)$. Therefore, a delicate task is to define the likelihood contribution of a datum $(x_1, x_2)$ which may or may not fall into this region. We adopt a censoring approach, that is if $x_j \leq u_j$, then we pretend that $x_j$ is censored by $u_j$. Specifically, let $(X_1^*, X_2^*) = (X_1 \vee u_1, X_2 \vee u_2)$ and $d = (1_{[u_1,\infty)}(X_1), 1_{[u_2,\infty)}(X_2))$. If $X_1^* = x_1^*$ and $X_2^* = x_2^*$, define

$$f^*(x_1^*, x_2^* \mid H, \eta_1, \eta_2) = \begin{cases} F(u_1, u_2 \mid H, \eta_1, \eta_2) & \text{if } d = (0, 0), \\ \frac{\partial}{\partial x_1} F(x_1, u_2 \mid H, \eta_1, \eta_2) & \text{if } d = (1, 0), \\ \frac{\partial}{\partial x_2} F(u_1, x_2 \mid H, \eta_1, \eta_2) & \text{if } d = (0, 1), \\ \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2 \mid H, \eta_1, \eta_2) & \text{if } d = (1, 1). \end{cases}$$

The exact expression for $f^*$ is given in Appendix A.1. Let $X = \{(X_{1i}, X_{2i}) : i = 1, \ldots, n\}$, be a random sample from a bivariate cumulative distribution function $F$, with corresponding censored sample $X^* = \{(X_{1i}^*, X_{2i}^*) : i = 1, \ldots, n\}$. The likelihood is defined as

$$L(H, \eta_1, \eta_2 \mid X^*) = \prod_{i=1}^{n} f^*(X_{1i}^*, X_{2i}^* \mid H, \eta_1, \eta_2), \quad (2.4)$$

which depends implicitly on the thresholds $u_1$ and $u_2$.

3 Modelling the spectral measure

Within a fully Bayesian framework, we need not only put a prior on the marginal tail parameter vectors $\eta_1$ and $\eta_2$, but also on the set $\mathcal{H}$ of spectral measures, that is the set of probability measures on $[0, 1]$ with mean $1/2$. Let $\mathcal{H}'$ be the set of spectral measures whose only atoms, if any, are at 0 or 1. Note that the distribution functions of such spectral measures are continuous on $(0, 1)$. For notational convenience, when $H$ is a probability measure on $[0, 1]$, we denote $H(w) := H([0, w])$, for all $w \in [0, 1]$.

In this section, we construct a subset $\mathcal{H}' \subset \mathcal{H}$ of spectral measures that is dense in $\mathcal{H}$ with respect to the topology of uniform convergence of cumulative distribution functions. The set $\mathcal{H}'$ takes the form of a countable union of finite-dimensional parametric families. The selection of a prior distribution on $\mathcal{H}'$ is discussed in the following section.

The construction of $\mathcal{H}'$ is done in two steps: the approximation step (Subsection 3.1), in which a dense class of discrete spectral measures is proposed, and the smoothing step (Subsection 3.2), in which the distribution functions of these discrete measures are smoothed via monotone cubic splines.

3.1 The approximation step

It is well-known that the set of discrete spectral measures is dense in the family of all spectral measures with respect to the topology of weak convergence. The objective of this subsection is to present an explicit construction of such a discrete approximant to a spectral measure in $\mathcal{H}$ (Lemma 3.1 and Proposition 3.2). The approximating spectral measure has the same atoms on 0 and 1 as the original measure and in addition, for a given $m \geq 1$, it has exactly $m$ atoms in $(0, 1)$ that all receive the same mass.
The approximation error, in terms of the largest distance between distribution functions (sup-norm), is bounded by $1/m$. In the next subsection, these approximants are smoothed out, yielding the family $H'$ of spectral measures mentioned above.

**Lemma 3.1.** For any cumulative distribution function $\Phi$ of a random variable $W$ on $[0, 1]$ and for any integer $m \geq 1$ there exists a cumulative distribution function $\Psi$ of a discrete random variable $Y$ on $[0, 1]$ supported on at most $m$ points such that

$$E(W) = E(Y) \quad \text{and} \quad \sup_{w \in [0, 1]} |\Phi(w) - \Psi(w)| \leq 1/m. \quad (3.1)$$

Moreover, if $\Phi$ is continuous on $[0, 1]$, then $Y$ can be uniformly distributed on $m$ points.

**Proof.** Let $\Phi^{-1} : [0, 1] \to [0, 1]$ be the generalised inverse of $\Phi$, that is

$$\Phi^{-1}(t) = \inf \{ w \in [0, 1] : \Phi(w) \geq t \}, \quad \text{for all} \quad t \in [0, 1],$$

and let

$$q_i = \Phi^{-1}(i/m), \quad i = 0, \ldots, m \quad \text{and} \quad q_{m+1} = 1.$$

There exist $m$ points $0 \leq t_1 \leq t_2 \leq \cdots \leq t_m \leq 1$, which solve the following equations

$$\int_{q_{i-1}}^{q_i} \left( \Phi(w) - \frac{i-1}{m} \right) \ dw = \frac{q_i - t_i}{m}, \quad i = 1, \ldots, m. \quad (3.2)$$

Note that $q_{i-1} \leq t_i \leq q_i$ for $i \in \{1, \ldots, m\}$. Let $S = \{y_1, \ldots, y_k\}$, where $y_1 < y_2 < \cdots < y_k$ are the $k \geq 1$ distinct values among $t_1, \ldots, t_m$, and consider the set of multiplicities $M = \{m_1, \ldots, m_k\}$ given by

$$m_i = \# \{ j : t_j = y_i, \quad j = 1, \ldots, m \}, \quad \text{for all} \quad i = 1, \ldots, k.$$

Now, let $Y$ be a random variable on $S$ with $P(Y = y_i) = m_i/m$, for all $i \in \{1, \ldots, k\}$. If $\Psi$ is the cumulative distribution function of $Y$, then we have

$$\int_{q_{i-1}}^{q_i} \Phi(w) \ dw = \int_{q_{i-1}}^{q_i} \Psi(w) \ dw, \quad \text{for all} \quad i = 1, \ldots, m + 1, \quad (3.3)$$

and this implies

$$E(W) = \int_{0}^{1} \Phi(w) \ dw = \sum_{i=1}^{m+1} \int_{q_{i-1}}^{q_i} \Phi(w) \ dw = \sum_{i=1}^{m+1} \int_{q_{i-1}}^{q_i} \Psi(w) \ dw = \int_{0}^{1} \Psi(w) \ dw = E(Y) = \sum_{i=1}^{k} \frac{m_i}{m} y_i.$$

Moreover, for $w \in [0, 1]$, if $w \in [q_m, 1]$, then $\Phi(w) = \Psi(w) = 1$, while if $w \in (0, q_m)$, then there exists $i \in \{1, \ldots, m\}$, for which $w \in [q_{i-1}, q_i)$, so that

$$\frac{i-1}{m} \leq \Phi(w) < \frac{i}{m} \quad \text{and} \quad \frac{i-1}{m} \leq \Psi(w) < \frac{i}{m}.$$ 

It follows that $|\Phi(w) - \Psi(w)| \leq 1/m$, for all $w \in [0, 1]$. Finally, if $\Phi$ is continuous, then $\Phi(q_i) = i/m$, for all $i \in \{0, \ldots, m\}$, $k = m$, and

$$0 = q_0 < y_1 < q_1 < y_2 < \cdots < q_{m-1} < y_{m} < q_m \leq 1.$$

See Figure[1] for an illustration. \hfill \Box
Figure 1: Illustration of equation (3.3) with $m = 4$, $i = 2$, and $\Phi = \text{Beta}(3,3)$. The area of the shaded region is equal to the area under $\Phi$ between $q_1$ and $q_2$.

**Proposition 3.2.** If $H \in \mathcal{H}_c$, $h_0 = H([0])$, $h_1 = H([1])$, and $m \geq 1$, then there exists a discrete spectral measure $H^*$ with $H^*((0)) = h_0$ and $H^*([1]) = h_1$ and, in case $h_0 + h_1 < 1$, with exactly $m$ atoms in $(0, 1)$, such that

$$\sup_{w \in [0,1]} |H(w) - H^*(w)| \leq \frac{1 - h_0 - h_1}{m}. \quad (3.4)$$

**Proof.** First, notice that $0 \leq h_0, h_1 \leq 1/2$. Now, $h_0 = 1/2$ if and only if $h_1 = 1/2$, in which case $H$ is Bernoulli(1/2), and the result follows by setting $H^* \equiv H$. If $0 \leq h_0, h_1 < 1/2$, then there exists a continuous cumulative distribution function $\Phi$ of a random variable $W$ on $[0, 1]$ such that we have the representation

$$H(w) = h_0 \delta_0(w) + (1 - h_0 - h_1) \Phi(w) + h_1 \delta_1(w), \quad \text{for all } w \in [0, 1],$$

where $\delta_\omega(w)$, $w \in [0, 1]$, is the cumulative distribution function of the Dirac measure at $\omega$. By Lemma 3.1, there exists a cumulative distribution function $\Psi$ of a discrete random variable $Y$, uniformly distributed on $\{y_1, \ldots, y_m\}$, with $0 < y_1 < y_2 < \ldots < y_m < 1$, such that $E(W) = E(Y)$ and $\sup_{w \in [0,1]} |\Phi(w) - \Psi(w)| \leq 1/m$. The result follows by considering the discrete spectral measure $H^*$ defined by its distribution function

$$H^*(w) = h_0 \delta_0(w) + (1 - h_0 - h_1) \Psi(w) + h_1 \delta_1(w), \quad \text{for all } w \in [0, 1]. \quad \Box$$

### 3.2 The smoothing step

The second part of the construction of $\mathcal{H}^*$ is motivated by Proposition 3.5 below, which shows that a uniform approximation bound may also be obtained from finite-dimensional families $\mathcal{H}_m$, $m \geq 1$, of spectral measures that are absolutely continuous on (0, 1).

Here we construct $\mathcal{H}_m$, for all $m \geq 1$. The geometrical construction is best understood by occasionally referring to Figure 2. First, note that in Proposition 3.2 the approximating discrete spectral measure $H^*$ depends on $0 \leq h_0, h_1 \leq 1/2$, and, provided $h_0, h_1 < 1/2$, on points $0 < y_1 < \ldots < y_m < 1$ satisfying $(1 - h_0 - h_1)y + h_1 = 1/2$. 


In view of this, for \( m \geq 1 \), let

\[ \Theta_m = \bigcup_{(h_0,h_1) \in [0,1/2]^2} \Theta_{m,h_0,h_1} \]  

where, for \( 0 \leq h_0, h_1 < 1/2 \),

\[ \Theta_{m,h_0,h_1} = (h_0) \times \left\{ (y_1, \ldots, y_m) : 0 < y_1 < \cdots < y_m < 1, \; \bar{y} = \frac{1/2 - h_1}{1 - h_0 - h_1} \right\} \times \{ h_1 \}, \]

and where \( \Theta_{m,1/2,1/2} = \{(1/2, 1/2)\} \). The proof of the following lemma is straightforward.

**Lemma 3.3.** If \( \theta = (h_0, y_1, \ldots, y_m, h_1) \in \Theta_m \) and if \( H^\theta_m \) is the discrete probability measure with atoms \( H^\theta_m(0) = h_0 \), \( H^\theta_m(y_i) = (1 - h_0 - h_1)/m \) for all \( i \in \{1, \ldots, m\} \), and \( H^\theta_m(1) = h_1 \), then \( H^\theta_m \) is a spectral measure.

The next lemma is used to define a mapping of \( \Theta_m \) into the space of spectral measures which are absolutely continuous on \((0, 1)\).

**Lemma 3.4.** If \( \theta = (h_0, y_1, \ldots, y_m, h_1) \in \Theta_m \) and if \( H^\theta_m \) is the discrete spectral measure given in Lemma 3.3, then there exist two probability measures on \([0, 1]\), \( H^-_\theta \) and \( H^+_\theta \), which are absolutely continuous on \((0, 1)\), and such that

\[ H^-_\theta(w) \leq H^\theta_m(w) \leq H^+_\theta(w), \quad \text{for all } w \in [0, 1], \]

\[ \sup_{w \in [0, 1]} |H^\theta_m(w) - H^\theta_\pm(w)| \leq (1 - h_0 - h_1)/m. \]  

(A.3)

**Proof.** Following Fritsch and Butland [1984], we construct two continuously differentiable monotone cubic splines \( \varphi^-_\theta \) and \( \varphi^+_\theta \) for which

\[ \varphi^-_\theta(y_i) = h_0 + (1 - h_0 - h_1) \frac{i - 1}{m}, \quad i \in \{1, \ldots, m\}, \]

with \( \varphi^-_\theta(0) = \varphi^-_\theta(y_1) \) and \( \lim_{\alpha \uparrow 1} \varphi^-_\theta(w) = \varphi^-_\theta(y_m) \), and

\[ \varphi^+_\theta(y_i) = h_0 + (1 - h_0 - h_1) \frac{i}{m}, \quad i \in \{1, \ldots, m\}, \]

with \( \varphi^+_\theta(0) = \varphi^+_\theta(y_1) \) and \( \lim_{\alpha \uparrow 1} \varphi^+_\theta(w) = \varphi^+_\theta(y_m) \). The details of the construction are given in Appendix A.2. Now, let

\[ H^-_\theta(w) = \varphi^-_\theta(w), \quad \text{for all } w \in [0, 1], \quad \text{and} \quad H^+_\theta(1) = 1. \]

We have \( H^-_\theta(w) \leq H^\theta_m(w) \leq H^+_\theta(w) \) for all \( w \in [0, 1] \), as well as (3.6). \( \square \)

The mapping of \( \Theta_m \) into the space of spectral measures which are absolutely continuous on \((0, 1)\) is defined as follows. Let \( \theta \in \Theta_m \). In the trivial case where \( h_0 = 1/2 = h_1 \), let \( H_\theta \) be a Bernoulli \((1/2)\). If \( 0 < h_0, h_1 < 1/2 \), then from Lemma 3.4 we get

\[ a_- = \int_0^1 H^-_\theta(w) \, dw < \int_0^1 H^\theta_m(w) \, dw = 1/2 < \int_0^1 H^+_\theta(w) \, dw = a_+. \]

Now let

\[ a = \frac{a_+ - 1/2}{a_+ - a_-} \quad \text{and} \quad H_\theta = \alpha H^-_\theta + (1 - \alpha) H^+_\theta. \]  

(A.4)

Clearly, in any case, \( H_\theta \) is a spectral measure, and it is absolutely continuous on \((0, 1)\). This, in turn, enables us to define \( \mathcal{M}_m = \{ H_\theta : \theta \in \Theta_m \} \). Figure 2 below illustrates the following proposition.
Figure 2: The illustration shows the distribution functions from Lemma 3.4 and Proposition 3.5. The step function is $H^*_\theta$, the dashed line is $H_{\theta,-}$ and the dash-dotted line is $H_{\theta,+}$. The thick line is the true spectral measure $H$ from Proposition 3.5, in which $H^*_\theta \equiv H^*$, and the dotted line is the smooth $H'$.

**Proposition 3.5.** If $H$ is as in Proposition 3.2, $0 \leq h_0, h_1 \leq 1/2$, and $m \geq 1$, then there exists $H' \in \mathcal{H}_m$ such that

$$\sup_{w \in [0,1]} |H(w) - H'(w)| \leq 2(1 - h_0 - h_1)/m.$$  \hfill (3.8)

**Proof.** By Proposition 3.2 there exists a discrete spectral measure $H^* = H^*_\theta$ defined by $\theta = (h_0, y_1, \ldots, y_m, h_1) \in \Theta_m$ such that (3.4) holds. Now, let $H_{\theta,-}$ and $H_{\theta,+}$ be as in Lemma 3.3 and take $H' = H_\theta$ as in (3.7). In view of (3.6), the bound in (3.8) holds. \hfill □

Finally, in order to allow for maximum flexibility in the model, we take

$$\mathcal{H}' = \bigcup_{m \geq 1} \mathcal{H}_m,$$

as the space of spectral measures on which we shall concentrate a prior. This is done in the following section.

### 4 Bayesian framework

The model approximating the tail of the bivariate distribution $F$ is specified via a spectral measure which belongs to $\mathcal{H}'$ and marginal parameters $(\eta_1, \eta_2) \in \Xi^2$, where

$$\Xi = (-\infty, \infty) \times (0, \infty) \times (0, \infty).$$

The parameter space for $\mathcal{H}'$ is given by $\Theta = \bigcup_{m \geq 1} \Theta_m$, with $\Theta_m$ as in (3.5). Thus, the parameters defining a spectral measure consist of a model index $m$ and, given the model, a parameter vector $\theta = (h_0, y_1, \ldots, y_m, h_1) \in \Theta_m$. We assume a priori that $\eta_1$, $\eta_2$ and $(m, \theta)$ are independent, and in Subsection 4.1 we select priors for each of these quantities.
We develop the Bayesian framework in the same spirit as that of model selection, see for instance [Robert (2007)]. Let \( X = \{(X_{11}, X_{21}), \ldots, (X_{1n}, X_{2n})\} \) be a random sample from \( F \), let \( X' \) be the corresponding censored sample, and assume for the moment that the priors have been selected. The joint posterior for the tail of \( F \) is given by

\[
\pi(H_{m,\theta}, \eta_1, \eta_2 | X') \propto L(H_{m,\theta}, \eta_1, \eta_2 | X') \pi(\theta | m) \pi(m) \pi_1(\eta_1) \pi_2(\eta_2),
\]

where \( L \) is the likelihood given by equation (2.3). In Subsection 4.2 we give an MCMC algorithm which is used for numerical computations in every inference procedure that we propose throughout the rest of the paper.

In particular, for each \( m \geq 1 \), let \( \pi(m | X') \) be the posterior probability of selecting model \( m \). We define the Bayes estimator for the tail of \( F \) as the mixture

\[
(\hat{H}, \hat{\eta}_1, \hat{\eta}_2) = \sum_{m=1}^{\infty} \pi(m | X') (\hat{H}^{(m)}, \hat{\eta}_1^{(m)}, \hat{\eta}_2^{(m)}), \tag{4.1}
\]

where \((\hat{H}^{(m)}, \hat{\eta}_1^{(m)}, \hat{\eta}_2^{(m)})\) is the \( L^2 \)-Bayes estimator inside model \( m \). The estimator (4.1) is evaluated numerically via the sample mean of the trans-dimensional Markov chain constructed in Subsection 4.2.

### 4.1 Selection of the prior

Here, we do not claim any originality in the selection of the prior for the marginal parameters as we simply choose the maximal data information (MDI) prior proposed in [Beirlant et al. (2004)]:

\[
\pi_j(\eta_j) = \exp[E[\log f_j(X | \eta_j)]] \propto \frac{\xi_j}{\sigma_j} \exp[-(1 + \xi_j)(\gamma + \log \xi_j)], \quad j \in \{1, 2\},
\]

where \( f_j \) is derived from (2.3) and \( \gamma = 0.577215 \ldots \) is Euler’s constant. This choice is motivated by the fact that it is an objective prior which is simple to implement. Alternatively, export knowledge on certain marginal return levels may be incorporated in the prior as in [Coles and Tawn (1996)].

The main novelty is the selection of the prior on the spectral measure \( H_{m,\theta} \) via priors on \( m \) and \( \theta \). In particular, we need to specify a prior for the model index \( m \), and for every \( m \) in its (infinite) support, we need to select a density for \( \theta \in \Theta_m \) with respect to an appropriate reference measure. Selecting a density for \( \theta \) is a delicate problem as such a choice can give rise to an intractable normalising constant needed in evaluating the estimator (4.1) using a reversible jumps algorithm as we do.

First, we assume that \( m \) is drawn from a 0-truncated Poisson(\( \lambda \)) distribution, \( \pi(m) = \lambda^m / (e^\lambda - 1)m! \) for \( m \geq 1 \). Next, inside each model \( \Theta_m \), \( m \geq 1 \), we consider an appropriate Hausdorff reference measure. More precisely, for a positive integer \( k \), let \( \mu_k \) be the \( k \)-dimensional Hausdorff measure, see for instance [Billingsley (1986) chapter 19]. The set \( \Theta_m \) is an \((m+1)\)-dimensional surface in \( \mathbb{R}^{m+2} \), with \( 0 < \mu_{m+1}(\Theta_m) < \infty \). We assume that the parameter vector \( \theta = (h_0, y_1, \ldots, y_m, h_1) \) is uniformly distributed on \( \Theta_m \) with respect to \( \mu_{m+1} \):

\[
\pi(\theta | m) = \frac{1}{\mu_{m+1}(\Theta_m)}, \quad \text{for all } \theta \in \Theta_m.
\]

Because of its trans-dimensional nature, implementation of the MCMC algorithm in Subsection 4.2 requires the exact knowledge of \( \pi(\theta | m) \) and thus of the normalising constants \( \mu_{m+1}(\Theta_m) \), for all \( m \geq 1 \). These normalising constants are given by the following result.
Lemma 4.1. For all \( m \geq 1 \), we have

\[
\mu_{m+1}(\Theta_m) = \int_0^{1/2} \int_0^{1/2} \mu_{m-1}(\Theta_{m,h_0,h_1}) \, dh_0 \, dh_1, \quad (4.2)
\]

with

\[
\mu_{m-1}(\Theta_{m,h_0,h_1}) = \frac{\sqrt{m}}{m!(m-1)!} \left( \frac{m(1/2 - h_1)}{1-h_0 - h_1} \right)^{m-1} P\left( Y_{(m)} < \frac{1-h_0 - h_1}{m(1/2 - h_1)} \right),
\]

where \( Y_{(m)} \) is the maximum of \( Y = (Y_1, \ldots, Y_m) \), and \( Y \) is distributed according to a Dirichlet(1, \ldots, 1), that is the uniform distribution over the unit \((m-1)\)-dimensional simplex. Furthermore,

\[
P\left( Y_{(m)} < \frac{1-h_0 - h_1}{m(1/2 - h_1)} \right) = 1 - \sum_{k=1}^{K} (-1)^{k-1} \frac{m!}{k!(m-k)!} \left( 1 - \frac{1-h_0 - h_1}{m(1/2 - h_1)} \right)^{m-k}, \quad (4.3)
\]

where \( K \) is the greatest integer less than \( m(1/2 - h_1)/(1-h_0 - h_1) \).

Proof. For \( m = 1 \), we have \( \mu_2(\Theta_1) = 1/4 \), and the result holds. For \( m > 1 \), let

\[
\Theta'_{m,h_0,h_1} = \{ (h_0) \times \{ (y_1, \ldots, y_m) \in (0,1)^m : \sum_{i=1}^m \frac{1-h_0 - h_1}{y_i} \} \times \{ h_1 \},
\]

so that \( \mu_{m-1}(\Theta'_{m,h_0,h_1}) = m! \mu_{m-1}(\Theta_{m,h_0,h_1}) \). By considering the change of coordinates \( x_i = [(1-h_0 - h_1)/m(1/2 - h_1)] y_i, i = 1, \ldots, m \), let

\[
S = \left\{ 0 < x_1, \ldots, x_m < \frac{1-h_0 - h_1}{m(1/2 - h_1)} ; \sum_{i=1}^m x_i = 1 \right\},
\]

we get

\[
\mu_{m-1}(\Theta'_{m,h_0,h_1}) = \int_{\Theta'_{m,h_0,h_1}} dm_{m-1},
\]

\[
= \int_{\Theta'_{m,h_0,h_1}} \sqrt{m} \, dy_1 \cdots dy_m, \quad (4.4)
\]

\[
= \sqrt{m} \left( \frac{m(1/2 - h_1)}{1-h_0 - h_1} \right)^{m-1} \int_{S_m} dx_1 \cdots dx_m,
\]

\[
= \frac{\sqrt{m}}{(m-1)!} \left( \frac{m(1/2 - h_1)}{1-h_0 - h_1} \right)^{m-1} P\left( Y_{(m)} < \frac{1-h_0 - h_1}{m(1/2 - h_1)} \right),
\]

where \( Y_{(m)} \) is the maximum of \( Y = (Y_1, \ldots, Y_m) \), and \( Y \) is uniformly distributed over the standard \((m-1)\)-dimensional simplex. Note that the term \( \sqrt{m} \) in (4.4) is the value of the (generalised) Jacobian which appears as the integrand when expressing the Hausdorff measure of a smooth surface in terms of the Lebesgue integral, see [Billingsley (1986), p. 253]. Figure 3 illustrates the above quantities. Finally, equation (4.3) is derived from a result which can be found in [Fisher (1929)], namely

\[
P(Y_{(m)} < y) = 1 - \sum_{k=1}^{\lfloor 1/y \rfloor} (-1)^{k-1} \frac{m!}{k!} (1-ky)^{m-1}, \quad \text{for all } y \in [1/m, 1]. \quad \square
\]

Note that for \( m > 1 \), the double integral in (4.2) is to be evaluated using numerical quadrature.
4.2 The MCMC algorithm

We now give the reversible jumps algorithm used for numerical computations. Recall the decomposition \( \Theta = \bigcup_{m \geq 1} \Theta_m \). At each iteration, the spectral measure of the proposed candidate can either belong to the same space \( \Theta_m \) as the spectral measure of the current state, or to \( \Theta_{m-1} \) or to \( \Theta_{m+1} \). Our objective here is to outline the algorithm and to give the geometrical intuition behind each move. The technical details are given in Appendix A.3.

Let \( I = \{0, 1, \ldots, m + 1\} \) be the set of indices of the components of the parameter vector \( \theta = (\theta_0, \ldots, \theta_{m+1}) = (h_0, y_1, \ldots, y_m, h_1) \in \Theta_m \). At every iteration \( 1 \leq t \leq T \), where \( T \) is the length of the chain, choose between Move 1 and Move 2 with equal probability \( p_1 = 1/2 = p_2 \).

**Move 1:** Propose a new spectral measure.

Select between the three following moves with equal probability when \( m > 1 \). In case \( m = 1 \), only the moves 1.1 and 1.2 are considered.

1. **Propose a candidate in \( \Theta_m \).** Select \( (i, j), i < j \in I \), with probability \( 1/\binom{m+2}{2} \), and fix \( \theta_k \) for all \( k \neq i, j \). Propose a random move for the couple \( (\theta_i, \theta_j) \) such that the resulting vector \( \theta' \) remains in \( \Theta_m \).

2. **Propose a candidate in \( \Theta_{m+1} \).** This is done by inserting a new \( y' \) inside \( (0, 1) \), and by moving \( y_k \), a randomly selected point among \( y_1, \ldots, y_m \), inside \( (0, 1) \) so that equilibrium is preserved, that is \( y' = (1/2 - h_1)/(1 - h_0 - h_1) = \tilde{y}, \) where \( y' = \{y'_1, y'_2, \ldots, y'_{m+1}\} \) is the proposal.
1.3 Propose a candidate in $\Theta_{m-1}$. This is the inverse of move 1.2.
First, select two points $y_j \leq \bar{y} < y_k$, randomly among $y_1, y_2, \ldots, y_m$.
Then, remove the one that is closest to the equilibrium point $\bar{y} = (1/2 - h_1)/(1 - h_0 - h_1)$ and move the remaining one inside $(0, 1)$ in such a way that equilibrium is preserved, that is $\bar{y}' = \bar{y}$, where $y' = (y'_1, \ldots, y'_{m-1})$ is the proposal.

Move 2: Propose a new margin.
Here, a new value $\eta_j' = (\xi_j', \zeta_j', \sigma_j') \in \Xi_j$, for either $j = 1$ or 2 is proposed.

The above algorithm generates an ergodic Markov chain on $\Theta$ which can be used, in particular, to approximate the Bayes estimator (4.1), by taking the sample mean of the generated spectral measures. Here, we use it to illustrate the prior on the spectral measure distribution functions $H_{m,\theta}$ induced by $\pi(\theta | m) = \pi(\theta | m)\pi(m)$ specified in the previous subsection. Figure 4 below shows a region $S$ such that for every vertical section $S(w)$, $0 \leq w < 1$, we have $P[H(w) \in S(w)] = 0.95$ a priori, together with the prior pointwise mean $E[H(w)]$.

![Figure 4](image-url)

Figure 4: A region $S$ with $P[H(w) \in S(w)] = 0.95$ a priori for every $w \in [0, 1)$, together with the prior pointwise mean $E[H(w)]$.

5 Examples

In order to provide some validation for our methodology, we first apply it to artificial data in Subsection 5.1. Next, in Subsection 5.2, we analyse a dataset of Danish fire insurance claims, a univariate version of which has been previously studied in McNeil (1997) and Resnick (1997).

5.1 Artificial data

We consider the following bivariate cumulative distribution function:

$$F_r(x_1, x_2) = \left(1 - \frac{1}{x_1}\right)\left(1 - \frac{1}{x_2}\right)\left(1 + \frac{r}{x_1 + x_2}\right), \quad x_1, x_2 \geq 1; \ 0 \leq r \leq 1.$$
For any value of the dependence parameter $r$, both marginal distributions are unit Pareto, which has cumulative distribution function $G(x) = 1 - 1/x$, for $x \geq 1$.

The spectral measure $H_r$ of the extreme-value attractor of $F_r$ has atoms $H_r([0]) = H_r([1]) = (1 - r)/2$ and a constant density $h_r(w) = r, w \in (0, 1)$, that is,

\[
H_r(w) = \begin{cases} 
\frac{1 + rw}{2} & \text{if } 0 \leq w < 1, \\
1 & \text{if } w = 1.
\end{cases}
\]

see e.g. Einmahl and Segers [2009]. We can simulate from $F_r$ using an Accept-Reject algorithm. Here, for every $r \in \{0.2, 0.4, 0.6, 0.8\}$, we simulated 1000 samples each of size $n = 1000$. The thresholds $u_1$ and $u_2$ were set at the 90th percentiles of the marginal samples. Figure 5 shows the Bayes estimates of $H_r$, for every $r$.

![Figure 5](image)

Figure 5: The median of 1000 Bayes estimates (thick line), along with a 95% (pointwise) confidence band (grey region) for the true cumulative distribution function of the spectral measure $H_r$ (dotted line), for $r = 0.2, 0.4, 0.6, 0.8$.

### 5.2 Danish Fire Loss Data

The data set comprises 2167 industrial fire losses and was collected from the Copenhagen Reinsurance Company over the period 1980 to 1990. We are indebted to Alexander McNeil (Heriot-Watt University) for making these data available through his personal homepage on [http://www.ma.hw.ac.uk/~mcneil/data.html](http://www.ma.hw.ac.uk/~mcneil/data.html). The company’s figure for compensatory damage is divided in three categories, namely damage
to building \((X_1)\), damage to furniture and personal property \((X_2)\) and loss of profits due to the incident \((X_3)\), see Figure 6.

![Pairwise scatter plots of loss claims.](image)

(a) Damage to building \((X_1)\) vs Damage to furniture and personal property \((X_2)\)

(b) Damage to building \((X_1)\) vs Loss of profits due to the incident \((X_3)\)

(c) Damage to furniture and personal property \((X_2)\) vs Loss of profits due to the incident \((X_3)\)

Figure 6: Pairwise scatter plots of loss claims.

In [McNeil, 1997] and [Resnick, 1997], the data-set is analysed as if it is univariate, by combining the three categories into one loss figure: \(X_1 + X_2 + X_3\). However, the three types of loss compensations involve different portfolios, and in view of this, it is in the insurance company’s interest to know the dependence among these types of compensations. This is where our methodology becomes useful. Essentially, we are interested in the rare-event probabilities and the extreme conditional quantiles described in the introduction. Here, these tail quantities are derived respectively from the restriction on the region \([u_1, \infty) \times [u_2, \infty)\) of the joint predictive density

\[
f^*(x^*_1, x^*_2 \mid X^*) = \sum_{m \geq 1} \int_{\Theta_m \times \Xi^2} f^*(x^*_1, x^*_2 \mid H_m, \eta_1, \eta_2) \pi(H_m, \eta_1, \eta_2 \mid X^*) \, d\theta \, d\eta_1 \, d\eta_2,
\]

and the conditional predictive density

\[
f^*_2(x^*_2 \mid x^*_1, X^*) = \frac{f^*(x^*_1, x^*_2 \mid X^*)}{f^*(x^*_1 \mid X^*)},
\]

where \(X^* = \{(X^*_{1i}, X^*_{2i}) \mid i = 1, \ldots, 2167\}\) is the censored sample.
A univariate analysis of the data led us to choose the 90th sample quantiles for the thresholds $u_1$ and $u_2$. For descriptive purposes, we now investigate the dependence structures for all pairs of compensatory damage categories. The Bayes estimates and 95% pointwise credibility sets for the cumulative distribution functions of the spectral measures are given in Figure 7.

![Figure 7](image)

(a) Damage to building ($X_1$) vs Damage to furniture and personal property ($X_2$)
(b) Damage to building ($X_1$) vs Loss of profits due to the incident ($X_3$)
(c) Damage to furniture and personal property ($X_2$) vs Loss of profits due to the incident ($X_3$)

Figure 7: Bayes estimate (thick line) and 95% pointwise credibility sets for the cumulative distribution function of the spectral measure for each couple of compensatory damage category.

We focus on the first couple $X_1$ and $X_2$. Figure 7(a) clearly indicates that the two variables are not independent in the region $[u_1, \infty) \times [u_2, \infty)$. The joint predictive density (5.1) is shown in Figure 8. Finally, the mean of the conditional predictive density (5.2), that is the predicted value of the claim $X_2$ given the claim $X_1 = x_1$, and the 95% quantile of the predictive conditional distribution are shown in Figure 9.

6 Discussion

We have provided a nonparametric Bayesian framework for bivariate extreme-value analysis. On the one hand, the nonparametric nature of the dependence structure (spectral measure) is fully respected. On the other hand, for purposes of prediction
Figure 8: Image of the joint tail density (5.1) in the region $[u_1, \infty) \times [u_2, \infty)$.

Figure 9: The data (points), the predicted value of $X_2$ (black line) given the claim $X_1$, along with the 95% pointwise quantiles (grey line) of the conditional predictive density (5.2). The data are also plotted.
of future high levels (even conditionally), the predictive distribution incorporates both process and estimation uncertainty. Actual computations are performed using a trans-dimensional Markov chain constructed via a Metropolis-Hastings algorithm. Software written in (parallel) C++ wrapped in a Python environment may be obtained from the authors.

Conceptually, it is not hard to see how to generalise the approach to arbitrary dimensions. Practically, however, there are some serious obstacles to be overcome. First, in dimension $d$, the censored likelihood branches out in $2^d$ parts. Moreover, the spectral measure sits in the exponential of the bivariate extreme-value distribution, and computation of this distribution’s density becomes tedious. Second, in dimension $d$ the spectral measure may be an arbitrary probability measure on the unit simplex in $\mathbb{R}^d$ satisfying a certain number of moment constraints. It may have a density on each of the $2^d - 1$ faces of the unit simplex (Coles and Tawn, 1991). These densities are to be specified via splines or polynomials or some other set of basis functions. A prior needs to be specified on a manageable but still dense submodel of spectral measures. Then, efficient MCMC methodology should be proposed for numerical computations.

Finally, the bivariate tail approximation via extreme-value distributions is not designed to deal with asymptotic independence, in which case the tail approximation degenerates to exact independence, an approximation which may be unsatisfactory for instance in case of the bivariate Gaussian distribution with a high correlation. A distributional model encompassing both asymptotic independence and dependence has been proposed in Ramos and Ledford (2009), based on Ledford and Tawn (1996) and Ledford and Tawn (1997); see also Resnick (2003) and Beirlant et al. (2004). In this framework, dependence is characterised by a parameter $\eta \in (0, 1]$ and an angular measure $H$ on $(0, 1)$ whose total mass may be infinite and which satisfies a certain moment constraint. Yet another framework to deal with conditional extremes and asymptotic independence is proposed in Heffernan and Tawn (2004); see also Heffernan and Resnick (2007) and Resnick (2008). Again, extremal dependence is described by a possibly infinite spectral measure.

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References


A Appendix

A.1 Censored likelihood

We give explicit formulas for the censored density $f^\star$. Let

$$F_j(x_j) = \exp \{ -\xi_j \left( 1 + \frac{x_j - u_j}{\sigma_j} \right)^{-1/\xi_j} \}, \quad j \in \{1, 2\},$$

so that

$$f_j(x_j) = \frac{d}{dx_j} F_j(x_j) = \xi_j \left( 1 + \frac{x_j - u_j}{\sigma_j} \right)^{-1/\xi_j - 1} F_j(x_j), \quad j \in \{1, 2\},$$

for $x_j \geq u_j$ such that $\sigma_j + \xi_j (x_j - u_j) > 0$. Let

$$F(x_1, x_2) = \exp \{ -\ell(-\log F_1(x_1), -\log F_2(x_2)) \},$$

where

$$\ell(s, t) = 2 \int_0^1 \max(ws, (1 - w)t) H(dw), \quad (s, t) \in [0, \infty)^2.$$

Let $h(w) = \frac{d}{dw} H(w)$, $w \in (0, 1)$, be the Radon-Nikodym derivative of $H$ with respect to the Lebesgue measure on the open unit interval. We have

$$\int_0^1 w h(w) \, dw = 1/2 - H([1]) \quad \text{and} \quad \int_0^1 (1 - w) h(w) \, dw = 1/2 - H([0]).$$

Furthermore,

$$\frac{\partial}{\partial s} \ell(s, t) = 2 \left( H([1]) + \int_0^1 w h(w) \, dw \right),$$

$$\frac{\partial}{\partial t} \ell(s, t) = 2 \left( H([0]) + \int_0^1 (1 - w) h(w) \, dw \right),$$

$$\frac{\partial^2}{\partial s \partial t} \ell(s, t) = -2 \frac{st}{(s + t)^3} h \left( \frac{t}{s + t} \right),$$

and we have the representation

$$\ell(s, t) = s + t + \int_0^\infty \int_0^\infty \frac{\partial^2}{\partial \sigma \partial \tau} \ell(\sigma, \tau) \, d\sigma \, d\tau, \quad (s, t) \in [0, \infty)^2.$$

Finally, $f^\star(x_1^\star, x_2^\star)$ is equal to:

$$F(u_1, u_2) \quad \text{if} \quad x_1 < u_1, x_2 < u_2,$$

$$\frac{\xi_1}{\sigma_1} \left( 1 + \frac{x_1 - u_1}{\sigma_1} \right)^{-1/\xi_1 - 1} \frac{\partial}{\partial s} \ell(-\log F_1(x_1), -\log F_2(u_2)) F(x_1, u_2) \quad \text{if} \quad x_1 > u_1, x_2 < u_2,$$

$$\frac{\xi_2}{\sigma_2} \left( 1 + \frac{x_2 - u_2}{\sigma_2} \right)^{-1/\xi_2 - 1} \frac{\partial}{\partial t} \ell(-\log F_1(u_1), -\log F_2(x_2)) F(u_1, x_2) \quad \text{if} \quad x_1 < u_1, x_2 > u_2,$$

$$\prod_{j=1}^2 \left[ \frac{\xi_j}{\sigma_j} \left( 1 + \frac{x_j - u_j}{\sigma_j} \right)^{-1/\xi_j - 1} \right] \Delta \ell(-\log F_1(x_1), -\log F_2(x_2)) F(x_1, x_2) \quad \text{if} \quad x_1 \geq u_1, x_2 \geq u_2,$$

where

$$\Delta \ell(s, t) = \frac{\partial}{\partial s} \ell(s, t) \frac{\partial}{\partial t} \ell(s, t) - \frac{\partial^2}{\partial s \partial t} \ell(s, t).$$
A.2 Construction of the interpolating monotone cubic spline.

Let \( 0 = y_0 < y_1 < y_2 < \cdots < y_m < y_{m+1} = 1 \) be the set of abscissas, and let \( \varphi_0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq \varphi_m \leq \varphi_{m+1} \) be the set of ordinates. We construct a piecewise cubic polynomial function \( \varphi(\cdot) \) such that

\[
\varphi(y_i) = \varphi_i, \quad \text{for all } i = 0, \ldots, m + 1,
\]

and such that \( \varphi(\cdot) \) is nondecreasing and continuously differentiable on \((0, 1)\).

On every interval \([y_i, y_{i+1}]\), \( i = 0, \ldots, m \), we expand \( \varphi \) around \( y_i \) and we get

\[
\varphi(y) = \frac{d_i + d_{i+1} - 2\Delta_i}{h_i^2} (y - y_i)^2 + \frac{-2d_i - d_{i+1} + 3\Delta_i}{h_i} (y - y_i) + d_i(y - y_i) + \varphi_i,
\]

for all \( y \in [y_i, y_{i+1}] \), where \( \Delta_i = (\varphi_{i+1} - \varphi_i)/h_i \), \( h_i = y_{i+1} - y_i \), and where \( d_i \) and \( d_{i+1} \) are the endpoint derivatives. Thus, the construction of the spline depends only on the specification of the set of endpoint derivatives \( 0 \leq d_0, d_1, d_2, \ldots, d_m, d_{m+1} \). [Fritsch and Carlson (1980)] give necessary and sufficient conditions on these values in order to guarantee monotonicity throughout \([0, 1]\). For instance, setting \( d_i = 0 \), for all \( i = 0, \ldots, m+1 \), produces a continuously differentiable nondecreasing interpolant, although the resulting curve is not very smooth. On the other hand, [Fritsch and Butland (1984)] propose a method for determining the endpoint derivatives which produces smoother curves. In fact, it suffices to set \( d_0 = 0 = d_{m+1} \) and

\[
d_i = \begin{cases} 
\frac{\Delta_{i-1}\Delta_i}{\alpha\Delta_i + (1-\alpha)\Delta_{i-1}} & \text{if } \Delta_{i-1}\Delta_i > 0, \\
0 & \text{otherwise},
\end{cases}
\]

where \( \alpha = \frac{1}{2}(1 + \frac{h}{h_{i+1} + h_i}) \), for \( i = 1, \ldots, m \).

A.3 Implementation of the MCMC algorithm.

Move 1: Assume that the current state is \( \theta = (\theta_0, \ldots, \theta_{m+1}) = (h_0, y_1, \ldots, y_m, h_1) \in \Theta_m \), and let \( \bar{\gamma} = (1/2 - h_1)/(1 - h_0 - h_1) \). Let \( p_1(m) = p_2(m) = p_3(m) = 1/3 \), for all \( m > 1 \), and \( p_1(m) = 1/2 = p_2(m) \), \( p_3(m) = 0 \), if \( m = 1 \). Select move 1.2 or 3, with probability \( p_1 \), \( p_2 \) and \( p_3 \) respectively. For a real interval \( I \), let \( \mathcal{U}(I) \) denote the uniform distribution on \( I \), and let \( \mathcal{A}(I) \) denote its length.

1.1 Propose a new \( \theta' \in \Theta_m \).

1.1.1 Select \((i,j), i < j \in \mathcal{I} = \{0, 1, \ldots, m + 1\}\), with probability \( 1/(m+2)^2 \).

(a) Case \( i = 0 \), \( j = m + 1 \). Draw \( u \sim \mathcal{U}(I_{0;m+1}) \), where

\[
I_{0;m+1} = \left( \max\left\{ \frac{-h_0, -1/2 + (1 - h_0)\bar{\gamma}}{\bar{\gamma}}, 1/2 - h_0 \right\} , 1/2 - h_0 \right) ,
\]

set \( h_0' = h_0 + u \) and \( h_1' = 1/2 - (1+h_0-u)\bar{\gamma} \).

(b) Case \( i = 0, 1 \leq j \leq m \). Draw \( u \sim \mathcal{U}(I_{0,j}) \), where

\[
I_{0,j} = \left( \max\left\{ \frac{-h_0, -(1-h_0-h_1)y_j}{m\bar{\gamma} - y_j} \right\} , \min\left\{ 1/2 - h_0, \frac{(1-h_0-h_1)(1-y_j)}{m\bar{\gamma} + 1 - y_j} \right\} \right) ,
\]

set \( h_0' = h_0 + u \) and \( y_j' = \frac{m(1/2 - h_0') - (1-h_0-h_1-u)(m\bar{\gamma} - y_j)}{1-h_0-h_1-u} \).
1.3 Propose a new $\theta' \in \Theta_{m+1}$.\\
1.3.1 Select $j$ such that $0 < y_j < \bar{y} < y_j < 1$, with probability $1/(m_+ m_-)$, where $m_+ = \#\{k : y_k > \bar{y}, k = 2, \ldots, m\}$ and $m_- = m - m_+$.\\
1.3.2 Let $u = (\bar{y} - y_j) \wedge (y_j - \bar{y})$.
   (a) if $\bar{y} - y_j \leq y_k - \bar{y}$, then set $y^*_k = y_k - u$ and remove $y_j$,
   (b) otherwise, set $y^*_j = y_j + u$ and remove $y_k$.\\
1.3.3 Sort $(y^*_1, \ldots, y^*_m)$ into ascending order and denote the resulting vector as $(y'_1, \ldots, y'_{m+1})$, where $0 < y'_1 < \cdots < y'_{m+1} < 1$. Denote the proposed parameter vector by $\theta'$, so that $\theta' = (h_0, y'_1, \ldots, y'_{m+1}, h_1)$.
1.3.4 Accept $\theta'$ with probability:

$$
\alpha(\theta, \theta') = \frac{L(H_{m-1, \theta}, \eta_1, \eta_2 \mid X^*) p_2(m - 1) [1/(m - 1)] [1/A(I_{jk})]}{L(H_{m, \theta}, \eta_1, \eta_2 \mid X^*) p_3(m) [1/(m - m_s)]},
$$

where

$$
I_{jk} = \begin{cases} 
I_j & \text{if } \bar{y}_j - y_j > y_k - \bar{y}, \\
I_k & \text{otherwise},
\end{cases}
$$

and where $I_j$ and $I_k$ are given by (A.1).

**Move 2:** Assume that the current state is $\eta_1 = (\xi_1, \zeta_1, \sigma_1)$ and $\eta_2 = (\xi_2, \zeta_2, \sigma_2)$ in $\Xi$.

2.1 Select $j \in \{1, 2\}$ with equiprobability.

2.2 Draw $z_k \sim N(0, 1)$, independently for $k = 1, 2, 3$, set

$$
\begin{align*}
\xi_j' &= \xi_j + 0.01z_1, \\
\zeta_j' &= \zeta_j \exp(0.01z_2), \\
\sigma_j' &= \sigma_j \exp(0.01z_3),
\end{align*}
$$

and denote the proposed vector $\eta_j' = (\xi_j', \zeta_j', \sigma_j')$. Also denote $\eta_i' = \eta_i$, $i \neq j$, for notational convenience.

2.3 Accept $\eta_j'$ with probability:

$$
\alpha(\eta_j, \eta_j') = \frac{L(H_{m, \theta}, \eta_1', \eta_2' \mid X^*) \pi_j(\eta_j' \mid \eta_j)}{L(H_{m, \theta}, \eta_1, \eta_2 \mid X^*) \pi_j(\eta_j \mid \eta_j')}.
$$

where $q(\cdot \mid \eta_j)$, is the proposal density.