Optimal funding of a defined benefit pension plan.

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Abstract

In this paper, we address the issue of determining the optimal contribution rate of a stochastic defined benefit pension fund. The affiliate’s mortality is modelled by a jump process and the benefits paid in at retirement are function of the evolution of stochastic salaries. Assets of the fund are invested in cash, stocks and a rolling bond. Interest rates are driven by a Vasicek model. The objective is to minimize both the quadratic spread between the contribution rate and the normal cost, and the quadratic spread between the terminal asset and the mathematical reserve required to cover benefits. The optimization is done under a budget constraint that guarantees the actuarial equilibrium between the current asset and future contributions and benefits. The method of resolution is based on the Cox and Huang’s approach and on dynamic programming.

Keywords: defined benefit, pension fund, asset allocation, optimal rate of contribution.

1 Introduction.

There mainly exist two categories of pension fund: the defined contribution pension plan and the defined benefit pension plan. In the first one, the financial risk is beared by the affiliate: in case of poor performance of assets, his savings may be insufficient to maintain his standard of living at retirement. Whereas in a defined benefit pension plan, the risk is beared by the pension fund: whatsoever the return of assets, benefits paid to pensioners are proportional to his salary. In this context, the choice of the investment policy and of the contribution pattern is hence crucial for the agent financing the fund.

Defined benefit pension plans have been extensively studied in the literature. In the papers of Haberman and Sung (1994, 2005), Boulier et al. (1995), Josa Fombellida and Rincon-Zapareto (2004, 2006), the fund manager keeps assets as close as possible to liabilities by controlling the level of contributions. Cairns (1995, 2000) has discussed the role of objectives in selecting an asset allocation strategy and has analysed some current problems faced by defined benefit pension funds. Huang and Cairns (2006) have studied the optimal contribution rate for defined benefit pension plan when interest rates are stochastic. The most novel features of our work are the modelling of the affiliates’ mortality by a jump process, the use of stochastic interest rates and salaries. Furthermore, we minimize both contribution adjustments and a terminal surplus. By contribution adjustment, we mean the spread between the sponsor’s contribution and the normal cost. Whereas the surplus is here defined as the difference between terminal wealth and fair value of liabilities at retirement. The optimization is done under a budget constraint that ensures the actuarial equilibrium between the current asset and future liabilities.

The presence of random salary and mortality entails that the market is incomplete. The set of equivalent martingale measures counts therefore more than one elements and we need to fix the deflator used by the insurer to value liabilities in order to apply the Cox & Huang (1989) martingale method. This approach was used in a similar setting by Brennan and Xia (2002). The
optimal target wealth process found by the martingale method is not fully replicable but it is possible to determine the investment strategy replicating at best this solution using the dynamic programming principle.

The outline of this paper is as follows. Sections 2 and 3 respectively present the financial market and the defined benefit pension plans. In section 4, the form of the deflator is discussed. Section 5 introduces the optimization problem and in section 6, we propose a solution. Section 7 contains a numerical illustration and the last section concludes.

2 The financial market.

In this section, we introduce the market structure of our model and define the dynamics of interest rates and asset values. The uncertainty involved by the financial market is described by a 2-dimensional standard Brownian motion \( W^P_t = \left( W^{r,P}_t, W^{S,P}_t \right) \) defined on a complete probability space \( (\Omega^f, \mathcal{F}^f, P^f) \). \( \mathcal{F}^f \) is the filtration generated by \( W^P_t \):

\[
\mathcal{F}^f = \left( \mathcal{F}^f_t \right) = \sigma \left\{ \left( W^{r,P}_t, W^{S,P}_t \right) : u \leq t \right\}
\]

\( P^f \) represents the historical financial probability measure. The two Wiener processes \( W^{r,P}_t \) and \( W^{S,P}_t \) are independent. The financial market is complete and there exists therefore a unique equivalent measure under which the discounted prices of assets are martingale. This risk neutral measure is noted \( Q^f \). The assets of the defined benefit pension fund are invested in cash, stocks and rolling bonds. The return of cash is the risk free rate \( r_t \) and is modelled as an Ornstein-Uhlenbeck process (Vasicek model):

\[
dr_t = a \cdot (b - r_t) . dt + \sigma_r . dW^{r,P}_t \tag{2.1}
\]

The constant parameters \( a, b, \sigma_r \) are respectively the speed of mean reversion, the level of mean reversion and the volatility of \( r_t \). Let \( \lambda_r \) be a negative constant defining the dynamic of \( r_t \) under the risk neutral measure \( Q^f \). Under \( Q^f \), the risk free rate is hence solution of the following SDE:

\[
dr_t = a \cdot (b - \sigma_r \cdot \frac{\lambda_r}{a} - r_t) . dt + \sigma_r . \left( dW^{r,P}_t + \lambda_r . dt \right) \tag{2.2}
\]

Where \( W^{r,Q}_t \) is a Wiener process under \( Q^f \). The second category of assets is a rolling bond of maturity \( K \) whose price is noted \( R^K_t \). This bond is a zero coupon bond continuously rebalanced in order to keep a constant maturity and its price obeys to the dynamic:

\[
\frac{dR^K_t}{R^K_t} = r_t . dt - \sigma_r . n(K) . \left( dW^{r,P}_t + \lambda_r . dt \right)
\]

Where \( n(K) \) is a function of the maturity \( K \):

\[
n(K) = \frac{1}{a} \cdot (1 - e^{-a . K})
\]

The risk premium of the rolling bond is written \( \nu_R = -\sigma_r . n(K) . \lambda_r \). The last asset available on the financial market is a stock. Its price process \( S_t \) is modelled by a geometric Brownian motion and is correlated to interest rates fluctuations:

\[
\frac{dS_t}{S_t} = r_t . dt + \sigma_S \cdot \left( dW^{r,P}_t + \lambda_r . dt \right) + \sigma_S \cdot \left( dW^{S,P}_t + \lambda_S . dt \right)
\]

\[
= r_t . dt + \sigma_S . dW^{r,Q}_t + \sigma_S . dW^{S,Q}_t
\]
The constant parameters $\sigma_{Sr}$, $\sigma_S$ and $\lambda_S$ denote respectively the correlation between stocks and risk free rate, the embedded volatility of stocks and the cost of risk. The stocks risk premium is defined by $\nu_S = \sigma_{Sr} \cdot \lambda_r + \sigma_S \cdot \lambda_s$.

### 3 The pension fund.

The pension plan considered in this work provides benefits to affiliates which are defined in terms of a member’s final salary. For the sake of simplicity, one assumes that the pension fund counts initially $n_x$ members of the same age, $x$ and earning the same salary, noted $(A_t)_t$. All members retire at the age $x+T$ and in case of death, no benefits are paid in. The evolution of the individual salary is stochastic and correlated to the financial market. More precisely, one supposes that the dynamic of an affiliate’s salary is defined by the following SDE:

$$
\frac{dA_t}{A_t} = \mu_A(t).dt + \sigma_{Ar}dW_{t}^{r,P} + \sigma_{AS}dW_{t}^{S,P} + \sigma_A dW_{t}^{A,P}
$$

Where $\mu_A(t)$ is the average growth of the salary and $W_{t}^{A,P}$ is a Wiener process that represents the intrinsic randomness of the salary and is independent from $W_{t}^{r,P}$ and $W_{t}^{S,P}$. As this salary risk is not traded, $W_{t}^{A,P}$ is a source of incompleteness. We will come back on this point in the next section. The constants $\sigma_{Ar}$, $\sigma_{AS}$ and $\sigma_A$ denote the correlation of the salary with interest rates, with stocks and the embedded wage volatility. $W_{t}^{A,P}$ is defined on a probability space $(\Omega^a, \mathcal{F}^a, P^a)$ where $\mathcal{F}^a$ is the filtration generated by $W_{t}^{A,P}$.

Benefits are defined in terms of the salary at retirement date. Each pensioner will receive a continuous annuity whose rate $B$ is a fraction, $\alpha$ of the last wage:

$$
B = A_{T} \cdot \alpha
$$

Those benefits are financed during the accumulation phase. $c_t$ is the contribution rate made by the sponsor to the funding process at time $t$.

The value of liabilities will be discussed in the next section. We detail now the jump process modelling the mortality of covered workers. The mortality process is defined on a probability space $(\Omega^m, \mathcal{F}^m, P^m)$ and is independent from the filtration generated by $W_{t}^{r,P}$, $W_{t}^{S,P}$, $W_{t}^{A,P}$. The remaining lifetimes of affiliates are exponential random variables, noted $T_1, T_2, \ldots, T_{n_x}$ and their hazard rate (namely the mortality rate), at time $t$, is written $\mu(x + t)$. $N_t$ points out the total number of deaths observed till time $t$:

$$
N_t = \sum_{i=1}^{n_x} I(T_i \leq t)
$$

Where $I(.)$ is an indicator variable. The filtration $\mathcal{F}^m$ is generated by $N_t$ and the expectation of the infinitesimal variation of $N_t$ verifies:

$$
\mathbb{E}(dN_t|\mathcal{F}^m_t) = (n_x - N_{t-}) \cdot \mu(x+t).dt
$$

As the mortality is not traded in our model, this is a second source of incompleteness. The compensated process $M_t$ of the mortality process is defined as follows:

$$
M_t = N_t - \int_0^t (n_x - N_{u-}) \cdot \mu(x+u).du
$$
And $M_t$ is a martingale under the historical measure $P^m$. The expected number of survivors under $P^m$ is equal to the current numbers of survivors times a survival probability:

$$
\mathbb{E}((n_x - N_t) | \mathcal{F}_t^m) = \mathbb{E}\left(\sum_{i=1}^{n_x} I(T_i > s) | \mathcal{F}_t^m\right)
$$

$$
= \sum_{T_i > s} \mathbb{E}(I(T_i > s) | \mathcal{F}_t^m)
$$

$$
= (n_x - N_t) \cdot \exp\left(-\int_s^t \mu(x + u) \cdot du\right)
$$

$s - t p_{x+t}$ is the actuarial notation for the probability that an individual of age $x + t$ survives till age $x + s$.

4 The deflator and the fair value of liabilities.

Let $(\Omega, \mathcal{F}, P)$ be the probability space resulting from the product of the financial, wage and mortality probability spaces:

$$
\Omega = \Omega^f \times \Omega^a \times \Omega^m \quad \mathcal{F} = \mathcal{F}^f \otimes \mathcal{F}^a \otimes \mathcal{F}^m \lor \mathcal{N} \quad P = P^f \times P^a \times P^m
$$

Where the sigma algebra $\mathcal{N}$ is generated by all subsets of null sets from $\mathcal{F}^f \otimes \mathcal{F}^a \otimes \mathcal{F}^m$. The prices of pension fund liabilities are defined on $(\Omega, \mathcal{F}, P)$. In this setting, the market of pension fund liabilities is incomplete owing to the presence of two unhedgeable risks: the salary risk and the mortality risk. It entails that prices may differ from one insurance company to another. The next subsections describe the insurer’s deflator that is here composed of three elements called abusively the financial, wage and actuarial deflectors.

4.1 Financial deflator.

The completeness of the financial market entails that there exists one unique equivalent measure, namely the risk neutral measure, under which the discounted prices of assets are martingales. This measure is denoted $Q^f$ and is defined by the following change of measure:

$$
\left(\frac{dQ^f}{dP^f}\right)_t = \exp\left(-\frac{1}{2} \int_0^t ||\Lambda_f||^2.du - \int_0^t \Lambda_f.dW_u^{P^f}\right)
$$

where $\Lambda_f = (\lambda_r, \lambda_s)'$. The dynamic of assets under $Q^f$ has been discussed in section 2. The financial deflator $H^f(t, s)$ at time $t$ for a cash flow paid at time $t \leq s$ is equal to the product of the discount factor and of the change of measure:

$$
H^f(t, s) = \exp\left(-\int_0^s r_u.du\right) \cdot \left(\frac{dQ^f}{dP^f}\right)_s
$$

$$
= \exp\left(-\int_t^s r_u.du - \frac{1}{2} \int_t^s ||\Lambda_f||^2.du - \int_t^s \Lambda_f.dW_u^{P^f}\right)
$$

4.2 The wage deflator.

As the intrinsic salary risk is not traded, the market of pension fund liabilities is incomplete and for any $\mathcal{F}^a$ adapted process, $\lambda_{a,t}$, an equivalent probability measure $Q^{a, \lambda_a}$ can be defined by the
following random variable:

\[
\left( \frac{dQ^{a,\lambda_a}}{dP^{a}} \right)_t = \exp \left( \frac{1}{2} \int_0^t |\lambda_{a,u}|^2 . du - \int_0^t \lambda_{a,u}. dW^A_{u}.P^a \right)
\]

And under \(Q^{a,\lambda_a} , dW^A_{u}.Q^{a,\lambda_a} = dW^A_{u}.P^a + \lambda_{a,u}. dt\) is a Brownian motion. For the sake of simplicity, \(\lambda_{a,u}\) is assumed to be constant and denoted \(\lambda_a\) in the sequel of this paper. The dynamic of the salary under \(Q^f \times Q^{a,\lambda_a}\) is:

\[
\frac{dA_t}{A_t} = (\mu_A(t) - \sigma_A. \lambda_r - \sigma_A.S. \lambda_S) . dt
\]

\[
+ \sigma_A . dW^{r,Q^f}_t + \sigma_A.S.dW^{S,Q^f}_t + \sigma_A . dW^A_{t}.Q^{a,\lambda_a}
\]  \hspace{1cm} (4.1)

And \(H^a(t,s)\) denotes the wage deflator at instant \(t\), for a payment occurring at time \(s \geq t\):

\[
H^a(t,s) = \left( \frac{dQ^{a,\lambda_a}}{dP^{a}} \right)_t . \exp \left( -\frac{1}{2} \int_t^s |\lambda_{a,u}|^2 . du - \int_t^s \lambda_{a,u}. dW^A_{u}.P^a \right)
\]

### 4.3 The actuarial deflator.

The second source of incompleteness is the mortality risk. For any \(\mathcal{F}^m\)-predictable process \(h_s\), such that \(h_s > -1\), an equivalent actuarial measure \(Q^{m,h}\) is defined by the random variable solution of the SDE:

\[
d \left( \frac{dQ^{m,h}}{dP^m} \right)_t = \left( \frac{dQ^{m,h}}{dP^m} \right) . h_t . d \left( N_t - \int_0^t (n_x - N_{u-}) . \mu(x + u) . du \right)
\]

\[
= \left( \frac{dQ^{m,h}}{dP^m} \right) . h_t . dM_t
\]  \hspace{1cm} (4.2)

And we have the property that the process \(M^{m,h}_t\) defined by

\[
M^{m,h}_t = N_t - \int_0^t (n_x - N_{u-}) . \mu(x + u) . (1 + h_u) . du
\]

is a martingale under \(Q^{m,h}\). We adopt the notation \(\lambda_{N,u} = (n_x - N_{u-}) . \mu(x + u)\) for the intensity of jumps. The solution of the SDE (4.2) is (for details, see Duffie 2001, appendix I on counting processes):

\[
\left( \frac{dQ^{m,h}}{dP^m} \right)_t = \prod_{T_i \leq t} (1 + h_{T_i}) . \exp \left( -\int_0^t h_u . \lambda_{N,u} . du \right)
\]

\[
= \exp \left( \int_0^t \ln (1 + h_u) . dN_u - \int_0^t h_u . \lambda_{N,u} . du \right)
\]

And \(H^m(t,s)\) denotes the actuarial deflator at instant \(t\), for a payment occurring at time \(s \geq t\), defined by:

\[
H^m(t,s) = \left( \frac{dQ^{m,h}}{dP^m} \right)_t . \exp \left( \int_t^s \ln (1 + h_u) . dN_u - \int_t^s h_u . \lambda_{N,u}. du \right)
\]  \hspace{1cm} (4.3)

Under \(Q^{m,h}\), the expected number of survivors at time \(s\) is equal to the number of survivors at time \(t\) multiplied by a modified probability of survival \(s-(\begin{array}{c} \frac{s-t}{t} \\ \frac{s-t}{t} \end{array})^{b+1}\):

\[
\mathbb{E}^{Q^{a,h}}((n_x - N_s)|\mathcal{F}_t^a) = (n_x - N_t). \exp \left( -\frac{1}{2} \int_t^s \mu(x + u). (1 + h_u). du \right)
\]

\[
\frac{s-t}{t} \left( \begin{array}{c} \frac{s-t}{t} \\ \frac{s-t}{t} \end{array} \right)^{-t}\]

Under \(Q^{m,h}\), the expected number of survivors at time \(s\) is equal to the number of survivors at time \(t\) multiplied by a modified probability of survival \(s-(\begin{array}{c} \frac{s-t}{t} \\ \frac{s-t}{t} \end{array})^{b+1}\):

\[
\mathbb{E}^{Q^{h}}((n_x - N_s)|\mathcal{F}_t^a) = (n_x - N_t). \exp \left( -\frac{1}{2} \int_t^s \mu(x + u). (1 + h_u). du \right)
\]
In the sequel of this work, we restrict our field of research to a constant process $h_u = h$. The reason motivating this choice is that, in this particular case, some interesting analytic results can be presented. Remark that if $h > 0$, $h$ can be seen as a security margin against an adverse evolution of the mortality.

4.4 The deflator and the price of liabilities.

The deflator used to price liabilities, written $H(t, s)$ is in our setting the product of the financial, wage and actuarial deflators:

\[ H(t, s) = \exp \left( -\int_0^s r_u \, du \right) \cdot \left( \frac{dQ^f}{dP^f} \right)_s \cdot \left( \frac{dQ^n_{\lambda_a}}{dP^m} \right)_s \cdot \left( \frac{dQ^m_{\lambda_a}}{dP^m} \right)_s. \]  

The pricing of pension fund liabilities is hence done under a probability measure $Q$ which is equal to the product of $Q^f$, $Q^a$ and $Q^m$. Remark that the expectation of the deflator $H(t, s)$ is equal to the price of a zero coupon bond, denoted $B(t, s)$.

\[ B(t, s) = \mathbb{E}(H(t, s) | \mathcal{F}_t) = \mathbb{E}_Q \left( e^{-\int_t^s r_u \, du | \mathcal{F}_t} \right) \]

And the analytic expression of $B(t, s)$ is reminded in appendix A. The fair value of liabilities at the date of retirement, denoted $L_t$, is defined as the expectation of the deflated value of future contributions and benefits. $L_t$ will be used in the sequel to state the optimization problem. In particular, if $T^m$ is the maximum time horizon of the insurer’s commitment, $L_t$ is equal to:

\[ L_t = \mathbb{E} \left( \int_t^{T^m} H(t, s) \cdot c_s \, ds + \int_T^{T^m} H(t, s) \cdot (n_s - N_s) \cdot B \, ds | \mathcal{F}_t \right) \]

Generally, the minimum asset that the fund must hold to ensure his solvency is set round to $L_t$ (this minimum depends evidently on the local regulation).

5 The optimization problem.

The insurer’s objective is both to minimize the quadratic spread between the contribution rate and a constant target one (namely the normal cost) and to minimize the deviation of the terminal target asset from the mathematical reserve required to cover benefits at the date of retirement. The normal cost, denoted $NC$, is the contribution rate allowing equality between expected assets and liabilities:

\[ NC = \frac{\mathbb{E}(H(0, T), L_T | \mathcal{F}_0)}{\mathbb{E}(\int_0^T H(0, s), ds | \mathcal{F}_0)} \]

The target total asset is denoted $\bar{X}_T$ and the value function is defined as follows:

\[ V(t, x, n, a) = \min_{c_t, \bar{X}_T \in \mathcal{A}_t(x)} \mathbb{E} \left[ \int_t^T u_1 \cdot (c_s - NC)^2 \, ds + u_2 \cdot (\bar{X}_T - L_T)^2 | \bar{X}_t = x, N_t = n, A_t = a \right]. \] (5.1)

Where $u_1$ and $u_2$ are constant weights. The contribution rate and the target wealth are chosen in a set $\mathcal{A}_t(x)$ which is delimited by a constraint ensuring the actuarial equilibrium between future
deflated cash flows and the current asset $x$.

$$
A_t(x) = \left\{ (c_s)_{s \in [t,T]}, \tilde{X}_T \right\} \text{ such that }
\mathbb{E} \left( - \int_t^T H(t,s) c_s ds + H(t,T) \tilde{X}_T | F_t \right) \leq x
$$

(5.2)

In the sequel, this constraint is called the budget constraint. As the market is incomplete, the fact that $\tilde{X}_T$ belongs to $A_t(x)$ doesn’t guarantee that this process is replicable by an adapted investment policy. This point is detailed in section 6.2.

6 The martingale solution.

6.1 Optimal contribution rate and wealth.

Let $y_t \in \mathbb{R}^+$ be the Lagrange multiplier associated to the budget constraint at instant $t$. The Lagrangian is defined by:

$$
\mathcal{L} \left( t, x, n, a, (c_s)_s, \tilde{X}_T, y_t \right) = \mathbb{E} \left( \int_t^T u_1 (c_s - NC)^2 ds + u_2 (\tilde{X}_T - L_T)^2 | F_t \right) - y_t \left( x - \mathbb{E} \left( - \int_t^T H(t,s) c_s ds + H(t,T) \tilde{X}_T | F_t \right) \right)
$$

(6.1)

A sufficient condition to obtain an optimal contribution rate $(c_s^*_s)_{s \in [t,T]}$ and an optimal target wealth $\tilde{X}_T^*$ is the existence of an optimal Lagrange multiplier $y^*_t > 0$ such that the couple $\left( (c_s^*_s)_{s \in [t,T]}, \tilde{X}_T^* \right)$ is a saddle point of the Lagrangian. The value function may therefore be reformulated as:

$$
V(t,x,n,a) = \sup_{y_t \in \mathbb{R}^+} \left( \inf_{(c_s)_s, \tilde{X}_T} \mathcal{L} \left( t, x, n, a, (c_s)_s, \tilde{X}_T, y_t \right) \right)
= \sup_{y_t \in \mathbb{R}^+} V(t,x,n,a,y_t)
$$

(6.2)

And

$$
V(t,x,n,a) = \tilde{V}(t,x,n,a,y_t^*)
$$

It can be proved under technical conditions (see Karatzas and Shreve 1998, for details) that the optimal contribution rate and target wealth are:

$$
c_s^* = y_t^* H(t,s) \frac{1}{2} u_1 + NC
$$

(6.3)

$$
\tilde{X}_T^* = -y_t^* H(t,T) \frac{1}{2} u_2 + L_T
$$

(6.4)

Formally, $c_s^*$ and $\tilde{X}_T^*$ are obtained by setting to zero derivatives of equation (6.1) with respect to $c_s$ and $X_T$. The optimal Lagrange multiplier, $y_t^*$, is such that the budget constraint (5.2) is binding:

$$
y^*_t = \frac{\mathbb{E} \left( H(t,T) L_T | F_t \right) - x - NC \int_t^T \mathbb{E} \left( H(t,s) | F_t \right) ds}{\frac{1}{2} u_1 \int_t^T \mathbb{E} \left( H(t,s)^2 | F_t \right) ds + \frac{1}{2} u_2 \mathbb{E} \left( H(t,T)^2 | F_t \right)}
$$

(6.5)
The numerator of (6.5) is precisely the part of the benefits that are not yet financed: the expected fair value of reserves less the current asset and less the normal cost times a financial annuity. Those unfunded liabilities are denoted in the sequel by $UL_t$:

$$UL_t = \mathbb{E}(H(t,T)L_T|\mathcal{F}_t) - x - NC. \int_t^T \mathbb{E}(H(t,s)|\mathcal{F}_t) \, ds$$

Where $\bar{a}_{t,T}$ is a financial annuity of maturity $T-t$. If we insert (6.3) and (6.4) in the objective (5.1), the value function is rewritten in terms of unfunded liabilities:

$$V(t,x,n,a) = \frac{UL_t^2}{\frac{1}{u_2} \int_t^T \mathbb{E}(H(t,s)^2|\mathcal{F}_t) \, ds + \frac{1}{u_2} \mathbb{E}(H(t,T)^2|\mathcal{F}_t)}$$

The following propositions detail the expectations intervening in the calculation of the Lagrange multiplier (6.5) and of the value function (6.6).

**Proposition 6.1.** Under the assumptions that interest rates follow (2.1), that the deflator is defined by (4.4), and that the process defining the actuarial measure $Q^{a,h}$ is constant, $h_t = h$ with $h > -\frac{1}{2}$, the expectation of the square of the deflator is equal to:

$$\mathbb{E}(H(t,s)^2|\mathcal{F}_t) = \exp\left(\int_t^s (\lambda_r^2 + \lambda_S^2 + \lambda_a^2) \, du\right).$$

$$\exp\left(-\beta^\rho (s-t) + n(s-t)(\beta^\rho - 2.r_t) - \frac{\sigma_r^2}{a} n(s-t)^2\right).$$

$$\sum_{n=1}^{n_x-N_t} \frac{(n_x-N_t)!}{(n_x-N_t-n)! n!} \left(k^n \cdot (s-t)^{2h} \sum_{n=1}^{n_x-N_t-n} (1-s-t)^{2h}ight)^n$$

Where $k$ and $\beta^\rho$ are constant and defined by:

$$\beta^\rho = 2.b - 4.\frac{\sigma_r}{a} \frac{\lambda_r}{\sigma_r^2} - 2.\frac{\sigma_r^2}{a^2}$$

$$k = \frac{(1+h)^2}{(1+2.h)}$$

$s-t\bar{p}^{2h}_{x+t}$ is a probability of survival under a modified measure of probability:

$$s-t\bar{p}^{2h}_{x+t} = \exp\left(-\int_t^s \mu(x+u)(1+2.h) \, du\right)$$

And $n(s-t)$ is a positive decreasing function, null when $s=t$,

$$n(s-t) = \frac{1-e^{-a(s-t)}}{a}$$

The proof is provided in appendix B.

**Proposition 6.2.** The expectation of the deflated value of liabilities, at time $t \leq T$, is:

$$\mathbb{E}(H(t,T)L_T|\mathcal{F}_t) = (n_x-N_t).\alpha. \int_t^T s-t\bar{p}^{h}_{x+t} \mathbb{E}^Q\left(e^{-\int_t^s r_u \, du}.A_T(B(T,s)|\mathcal{F}_t)\right) \, ds$$
Given that \( \tilde{A} \Delta \int \) is such that:

By definition, the set follows:

The optimal target asset \( X_T^* \) is not hedgeable. Indeed, \( X_T^* \) depends on \( L_T \) which has the following expression:

And as \( L_T \) is function both of the mortality and of the salary which are not replicable, it’s obvious that \( X_T^* \) is not hedgeable. However, it is possible to find the investment strategy replicating at best this process. Our reasoning is based on dynamic programming (the interest reader may refer to Fleming and Rishel 1975, for details) and was applied in two papers of Hainaut and Devolder (2006a , 2006b ) which both studies the asset allocation of deterministic insurance liabilities with a stochastic mortality risk. Let \( \tilde{A}_t^* (x) \) be the set of replicable wealth processes. If \( (\pi_t^S, \pi_t^R) \) denote respectively the fraction of the wealth invested in stocks and rolling bonds, \( \tilde{A}_t^* (x) \) is defined as follows:

By definition, the set \( \tilde{A}_t^* (x) \) is included in \( \tilde{A}_t (x) \) and the dynamic of the replicable wealth process is such that:

For a small step of time \( \Delta t \), the dynamic programming principle states that:

Given that \( \tilde{X}_t^* \) is the process minimizing the value function, any other process \( (X_t)_t \in A_t^r (x) \subset A_t (x) \) verifies the inequality:

\[ V(t, x, n, a) \leq \mathbb{E} \left[ \int_t^{t+\Delta t} u_1. (c_s^* - NC)^2 . ds + V(t+\Delta t, X_t^*, N_{t+\Delta t}, A_{t+\Delta t}) \Big| \mathcal{F}_t \right] + \mathbb{E} \left[ \int_t^{t+\Delta t} u_1. (c_s^* - NC)^2 . ds + V(t+\Delta t, X_t^*, N_{t+\Delta t}, A_{t+\Delta t}) \Big| \mathcal{F}_t \right] (6.9) \]
The closest replicable process to \(X_t^t\) is the one minimizing the right hand term of the inequality (6.9). Indeed, the value function \(V(t, x, n, a)\) is quadratic and then local Lipschitz:

\[
\forall O \subset \mathbb{R}, \exists C^O \in \mathbb{R}^+ \mid \|x_1 - x_2\| \leq C^O \mid x_1 - x_2\|
\]

And if \(t + \Delta t\) is the first exit time of \(X_{t+\Delta t}\) or \(X_{t+\Delta t}\) from an interval \(O\) round \(x\), the inequality (6.9) is bounded as follows:

\[
0 \leq \mathbb{E} \left[ V(t + \Delta t, X_{t+\Delta t}, N_{t+\Delta t}, A_{t+\Delta t}) - V(t + \Delta t, \tilde{X}_{t+\Delta t}, N_{t+\Delta t}, A_{t+\Delta t}) \mid \mathcal{F}_t \right]
\]

\[
\leq \mathbb{E} \left[ C^O \mid X_{t+\Delta t} - \tilde{X}_{t+\Delta t} \mid \mathcal{F}_t \right]
\]

Where \(C^O\) is constant. Minimizing \(\mathbb{E} \mid X_{t+\Delta t} - \tilde{X}_{t+\Delta t} \mid \mathcal{F}_t\) is therefore equivalent to minimizing the right hand term of (6.9). The Ito’s lemma for jump processes (see for e.g. Øksendal and Sulem 2004, chapter one), leads to the following expression for the expectation of the value function at time \(t + \Delta t\):

\[
\mathbb{E} \left( V(t + \Delta t, X_{t+\Delta t}, N_{t+\Delta t}, A_{t+\Delta t}) \mid \mathcal{F}_t \right) = V(t, x, n, a) + \mathbb{E} \left( \int_{t}^{t+\Delta t} G^\pi(s, X_s, N_s, A_s).ds \mid \mathcal{F}_t \right) + \mathbb{E} \left( \int_{t}^{t+\Delta t} (V(s, X_s, N_s, A_s) - V(s, X_s, N_s, A_s)) .dN_s \mid \mathcal{F}_t \right)
\]

Where \(G^\pi(s, X_s, N_s, A_s)\) is the generator of the value function:

\[
G^\pi(s, X_s, N_s, A_s) = \\
V_s + a(b - r_s).V_s + \mu_A(s).A_s.V_A + \frac{1}{2} A_s^2(\sigma^2_A + \sigma^2_AR + \sigma^2_{ASS}).V_{AA} + \sigma_{AR}.A_s.V_r + \sigma_{AS}.A_s.(\sigma_{SA}.\pi_s^R + \sigma_{SR}.\pi_s^R + \sigma_{SR}n(K)) .V_{AX} + \left( (\pi_s^R + \pi_s^R \cdot \pi_s^R) .X_s + c_s^R \right) .V_X + X_s .\sigma_r \cdot (\sigma_{SR}^S - \pi_s^R \cdot \sigma_r .n(K)) .V_{Xr} + \frac{1}{2} X_s^2 \cdot \left( \pi_s^S \cdot \sigma_S^2 + \pi_s^R \cdot (\sigma_{SR}^S - \pi_s^R \cdot \sigma_r .n(K))^2 \right) .V_{XX}
\]

\(V_s, V_X, V_r, V_A, V_{XX}, V_{Xr}, V_{XA}, V_{rr}, V_{AA}\) are partial derivatives of first and second orders with respect to time, fund, wage and interest rate. When \(\Delta t\) tends to zero, minimizing the right hand term of the inequality (6.9) is equivalent to minimizing the generator \(G^\pi(s, X_s, N_s, A_s)\). The investment strategy replicating at best the process \(X_t^t\) is then obtained by deriving \(G^\pi(t, X_t, N_t, A_t)\) with respect to \(\pi_t^S\) and \(\pi_t^R\):

\[
\pi_t^{S^*} = \left( \frac{- \mu_r \cdot \sigma_{SR}(K) - \mu_S}{\sigma_{SR}(K)^2} \right) \cdot \frac{V_X}{V_{XX}} \cdot \frac{1}{X_t} - \frac{\sigma_{AS}}{\sigma_S} \cdot \frac{V_{XA}}{V_{XX}} \cdot \frac{A_t}{X_t}\) (6.10)
\]

\[
\pi_t^{R^*} = \left( \frac{- \mu_S \cdot \sigma_S}{\sigma_S^2}(K)^2 - \frac{\mu_r}{\sigma_{SR}^2}(K)^2 \right) \cdot \frac{V_X}{V_{XX}} \cdot \frac{1}{X_t} + \left( \frac{\sigma_{AR}}{\sigma_r(K)} - \frac{\sigma_{AS} \cdot \sigma_{SR}}{\sigma_S \cdot \sigma_{SR}(K)} \right) \cdot \frac{V_{XA}}{V_{XX}} \cdot \frac{A_t}{X_t} + \frac{1}{n(K)} \cdot \frac{V_{Xr}}{V_{XX}} \cdot \frac{1}{X_t}\) (6.11)
\]

As the value function is known (see expression (6.6) ), it suffices to derive it with respect to \(X_t, r_t\) and \(A_t\) to obtain the optimal part of the funds invested in stocks and bonds:

\[
\pi_t^{S^*} = \left( \frac{- \mu_r \cdot \sigma_{SR}}{\sigma_{SR}^2(K)} + \frac{\mu_S}{\sigma_S^2} \right) \cdot \frac{UL_t}{X_t} + \frac{\sigma_{AS}}{\sigma_S} \cdot \mathbb{E} \left( H(t, T).L_T \mid \mathcal{F}_t \right) \) (6.12)
\[ n_t^{R^*} = \left( \frac{\nu S \cdot \sigma_S}{\sigma_S^2 \cdot \sigma_r \cdot n(K)} + \frac{\nu R}{\sigma_r^2 \cdot n(K)^2} \right) \left( 1 + \frac{\sigma_S^2}{\sigma_S^2} \right) \cdot \frac{UL_t}{X_t} \]

\[ - \left( \frac{\sigma_{Ar}}{\sigma_r \cdot n(K)} - \frac{\sigma_{AS} \cdot \sigma_S}{\sigma_S \cdot \sigma_r \cdot n(K)} \right) \cdot \mathbb{E}(H(t, T) \cdot L_T | F_t) \]

\[ + \frac{1}{n(K) \cdot V_{XX}^{t}} \frac{1}{X_t} \]

The correction term has no simple analytic expression:

\[ \frac{1}{n(K)} \cdot \frac{V_{XX}^{t}}{X_t} = \frac{1}{n(K) \cdot X_t} \left( NC \cdot \int_t^T \frac{\partial B(t, s)}{\partial r_t} ds - \frac{\partial \mathbb{E}(H(t, T) \cdot L_T | F_t)}{\partial r_t} \right) + \]

\[ \frac{UL_t}{n(K) \cdot X_t} \left( \frac{1}{u_1} \cdot \int_t^T \mathbb{E}(H(t, s)^2 | F_t) ds + \frac{1}{u_2} \cdot \mathbb{E}(H(t, T)^2 | F_t) \right)^2 \]

where

\[ \frac{\partial B(t, s)}{\partial r_t} = -n(s - t) \cdot B(t, s) \quad \frac{\partial \mathbb{E}(H(t, s)^2 | F_t)}{\partial r_t} = -2 \cdot n(s - t) \cdot \mathbb{E}(H(t, s)^2 | F_t) \]

\[ \frac{\partial \mathbb{E}(H(t, T) \cdot L_T | F_t)}{\partial r_t} = -(n_x - N_t) \cdot \alpha \cdot \int_t^T s - t \cdot p_{x+s} \cdot n(T - s) \cdot \mathbb{E}_Q \left( e^{-\int_t^T r_s \cdot du} \cdot A_T \cdot B(T, s) | F_t \right).

An interesting characteristic of this correction term is that it tends to zero when \( t \to T \). Indeed, all terms intervening in the numerator of the correction term are integrals or function of \( n(T - t) \) which tend to zero when \( t \to T \).

7 Example.

We consider a male population of \( n_x = 10000 \) affiliates, age \( x = 50 \), and who earns a wage \( A_{t=0} = 2500 \) Eur. We assume that all individuals go on retirement at 65 years and receive till their death, a continuous annuity equal to \( \alpha = 20\% \) of the last salary \( A_T \).

Table 7.1: Parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>12.72%</td>
</tr>
<tr>
<td>( b )</td>
<td>3.88%</td>
</tr>
<tr>
<td>( \sigma_r )</td>
<td>1.75%</td>
</tr>
<tr>
<td>( \lambda_r )</td>
<td>-2.36%</td>
</tr>
<tr>
<td>( r_{t=0} )</td>
<td>2.00%</td>
</tr>
<tr>
<td>( K )</td>
<td>8 years</td>
</tr>
<tr>
<td>( \nu R )</td>
<td>2.77%</td>
</tr>
<tr>
<td>( \lambda_S )</td>
<td>34.94%</td>
</tr>
<tr>
<td>( \sigma_S )</td>
<td>15.24%</td>
</tr>
</tbody>
</table>

The normal cost is set to

\[ NC = \mathbb{E}(H(0, T) \cdot L_T | F_0) \]

\[ = 26.763 \text{ Eur} \]
According to equation (6.6), this is the normal cost minimizing the value function at time $t = 0$. Three choices of weights $u_1, u_2$ are tested. In the first test, the asset manager seeks mainly to limit the volatility of the contribution rate: $u_1 = 1$ $u_2 = 0.1$. In the second case studied, $u_1$ and $u_2$ are set to one. In the last test, the aim is mainly to limit the volatility of the terminal surplus: $u_1 = 1$ $u_2 = 10$. We have opted for a Monte Carlo simulations. 6000 sample paths are generated for each test and the discretization step of time $\Delta t$ is set to one year (Contributions and asset allocation are both changed once a year). In the following figures, we compare the average contribution rates, the average surplus.

Figure 7.1: Contribution rates.

![Figure 7.1: Contribution rates.](image)

Figure 7.2: Surplus (% of the fund).

![Figure 7.2: Surplus (% of the fund.](image)

For each set of weights, the contribution rate decreases on average. The lower is the weight granted to minimize the surplus variation, the higher is the decrease of the contribution rate and the lower is the average terminal surplus. The next figure depicts the evolution of the average
The asset allocation for \( u_1 = 1 \) and \( u_2 = 0.1 \). Over the first nine years, huge amounts of cash are borrowed and invested in stocks and bonds. This short position in cash is reduced with time. One year before \( T \), the asset allocation is as follows: 68.3% in bonds, 21.3% in cash and 10.4% in stocks. We also observe that weights mainly influences the contribution rate and the terminal surplus: the average asset allocation for the two other sets of weights are nearly identical to the one displayed in figure (7.3).

Figure 7.3: Asset mix for \( u_1 = 1 \) and \( u_2 = 10 \).

8 Conclusions.

In this paper, we have investigated a model for defined benefits pension plans which incorporates stochastic interest rates, mortality and salary. In particular, we have studied the problem of pension funding from the perspective of an asset manager who wishes to minimize the deviation of contributions and terminal surplus from target ones, under a budget constraint and using a quadratic criterium.

The presence of stochastic mortality and salary entails that the market of pension fund liabilities is incomplete and the set of deflators used to valuate liabilities counts more than one element. In order to apply the Cox & Huang martingale method, it is then necessary to choose a deflator that reflects the pricing preferences of the fund manager. This assumption is not really impeding and correspond to the actuarial practice. Another drawback of the market incompleteness is that the optimal wealth process found by the martingale approach is not perfectly replicable. However, we can find the optimal investment hedging at best this process by a reasoning based on the dynamic programming principle.

We have seen that the optimal contribution rate is the sum of the normal cost and of the unfunded liabilities amortized by a factor, function of the market conditions. The optimal investment strategy also depends on the unfunded liabilities. An illustrative example has been given which shows the dependence between the contribution rate and the weights respectively given to the minimization of the contribution risk and of the surplus risk.
Appendix A.

As mentioned early in section 4.4, the expected value of the deflator, \( \mathbb{E}(H(t, s) \mid \mathcal{F}_t) \), is the price of a zero coupon bond \( B(t, s) \), because of independency of \( W_{r,t}^{P,t}, W_{s,t}^{S,P} \) and \( W_{a,t}^{A,P} \). If interest rates are driven by a Vasicek model (for details on this model, we refer to Cairns 2004), the price of a zero coupon bond is given by

\[
B(t, s) = \exp \left( -\beta(s - t) + n(s - t)(\beta - r_t) - \frac{\sigma_r^2}{2a^2}n(s - t)^2 \right) \tag{8.1}
\]

Where

\[
\beta = bQ - \frac{\sigma_r^2}{2a^2} = b - \sigma_r \frac{\lambda_r}{a} - \frac{\sigma_r^2}{2a^2} \tag{8.2}
\]

and \( n(s - t) \) is a positive decreasing function, null when \( s = t \):

\[
n(s - t) = \frac{1}{a} \left( 1 - e^{-a(s-t)} \right)
\]

The derivative of the bond price with respect to \( r_t \), used in paragraph to calculate the correction term of the optimal bonds strategy (6.13), is:

\[
\frac{\partial B(t, s)}{\partial r_t} = -n(s - t)B(t, s)
\]

Appendix B.

This appendix presents the proof of the proposition 6.1. The deflator (4.4) can be rewritten as follows:

\[
H(t, s) = \exp \left( -\int_t^s r_u \, du - \frac{1}{2} \int_t^s ||\Lambda||^2 \, du - \int_t^s \Lambda \, dW_u^{P} \right) \exp \left( \int_t^s \ln (1 + h) \, dN_u - \int_t^s h \, \lambda_{N,u} \, du \right)
\]

Where \( \Lambda = (\lambda_r, \lambda_s, \lambda_a)' \) and \( W_u^{P} = (W_{r,u}^{P,t}, W_{s,u}^{S,P}, W_{a,u}^{A,P})' \). \( \mathbb{E}^P (H(t, s)^2 \mid \mathcal{F}_t) \) can therefore be decomposed in two independent terms abusively called in the sequel financial and actuarial components which are next calculated separately:

\[
\mathbb{E}^P (H(t, s)^2 \mid \mathcal{F}_t) = \mathbb{E}^P \left( \exp \left( -2 \int_t^s r_u \, du - \int_t^s ||\Lambda||^2 \, du - 2 \int_t^s \Lambda \, dW_u^{P} \right) \mid \mathcal{F}_t \right) \tag{8.3}
\]

\[
\begin{align*}
&\text{Financial component} \\
&\mathbb{E}^P \left( \exp \left( \int_t^s \ln \left( (1 + h)^2 \right) \, dN_u - \int_t^s 2h \, \lambda_{N,u} \, du \right) \mid \mathcal{F}_t \right) \\
&\text{Actuarial component}
\end{align*}
\]

Calculation of the financial component.

The following random variable defines a change of measure from \( P \) to \( \tilde{P} \):

\[
\left( \frac{d\tilde{P}}{dP} \right)_t = \exp \left( -\int_0^t 2\Lambda \, dW_u^{P} - \frac{1}{2} \int_0^t ||\Lambda||^2 \, du \right)
\]

And under \( \tilde{P} \), the following elements are Brownian motions:

\[
dW_{r,u}^{\tilde{P}} = dW_{r,u}^{r,P} + 2\lambda_r \, du
\]
\[ d\tilde{W}_u^{S,\hat{P}} = d\tilde{W}_u^{S,P} + 2.\lambda_S.\text{du} \]
\[ d\tilde{W}_u^{A,\hat{P}} = d\tilde{W}_u^{A,P_u} + 2.\lambda_u.\text{du} \]

The financial component of (8.3) becomes:

\[
\mathbb{E}^\hat{P}\left( \exp\left( -2.\int_t^s r_u.\text{du} - \int_t^s ||\Lambda||^2.\text{du} - 2.\int_t^s \Lambda.d\tilde{W}_u^{P} \right) |\mathcal{F}_t \right) = \exp\left( \int_t^s ||\Lambda||^2.\text{du} \right).\mathbb{E}^\hat{P}\left( e^{-\int_t^s 2.r_u.\text{du}} |\mathcal{F}_t \right)
\]

And as \(2.r_u\) has a mean reverting dynamic under \(\hat{P}\),

\[
d(2.r_u) = a.\left( 2.b - 4.\frac{\sigma_r.\lambda_r}{a} - 2.r_u \right).\text{dt} + 2.\sigma_r.d\tilde{W}_u^{\hat{P}}
\]

it suffices to apply the Vasicek’s formula to obtain that:

\[
\mathbb{E}^\hat{P}\left( e^{-\int_t^s 2.r_u.\text{du}} |\mathcal{F}_t \right) = \exp\left( -\beta^\hat{P}.(s-t) + n(s-t).(\beta^\hat{P} - 2.r_t) - \frac{\sigma_r^2}{a} n(s-t)^2 \right)
\]

Where

\[
\beta^\hat{P} = 2.b - 4.\frac{\sigma_r.\lambda_r}{a} - 2.\frac{\sigma_r^2}{a^2}
\]

And

\[
n(s-t) = \frac{1}{a}.\left( 1 - e^{-a.(s-t)} \right)
\]

**Calculation of the actuarial component.**

Assume that \(h > -\frac{1}{2}\), it is therefore possible to define a positive constant \(k\):

\[
k = \frac{(1 + h)^2}{(1 + 2.h)}
\]

Such that the actuarial component of equation (8.3) can be rewritten as:

\[
\mathbb{E}^P\left( \exp\left( \int_t^s \ln((1 + h)^2).dN_u - \int_t^s 2.h.\lambda_{N,u}.\text{du} \right) |\mathcal{F}_t \right) = \\
\mathbb{E}^{P_a}\left( \exp\left( \int_t^s \ln(k).dN_u \right) \cdot \exp\left( \int_t^s \ln(1 + 2.h).dN_u - \int_t^s 2.h.\lambda_{N,u}.\text{du} \right) |\mathcal{F}_t \right)
\]

(8.4)

The term \( \frac{dQ_{a,2,h}}{dP_a} \) defines a new actuarial measure \(Q_{a,2,h}\), under which the following centered process

\[
M_{t}^{a,2,h} = N_t - \int_0^t (n_x - N_u-).\mu(x + u).(1 + 2.h).\text{du}
\]

is a martingale. And the expected number of survivors at time \(s\), conditionally to instant \(t\) is:

\[
\mathbb{E}^{Q_{a,2,h}}\left( (n_x - N_s) |\mathcal{F}_t \right) = (n_x - N_t).\exp\left( -\int_t^s \mu(x + u).(1 + 2.h).\text{du} \right)
\]

\[
_{s - \hat{P}_{a,2,h}}^{\hat{P}_{a,2,h}}
\]
Equation (8.4) is finally rewritten as the expectation under $Q^{a,2,h}$ of a constant $k$, exponent the number of deaths.

$$E^P \left( \exp \left( \int_t^s \ln \left( (1 + h)^2 \right) dN_u - \int_t^s 2h \lambda(t,N_u) dW_u \right) | F_t \right) = E^{Q^{a,2,h}} \left( k^{N_s - N_t} | F_t \right)$$

Under $Q^{a,2,h}$, the probability of observing $n$ deceases in the interval of time $(t, s)$ is a binomial variable of parameters $(n_x - N_t, 1 - s - t_p^{2,h})$. The expected value of $k^{N_s - N_t}$ is then computable by the following formula:

$$E^P \left( \exp \left( \int_t^s \ln \left( (1 + h)^2 \right) dN_u - \int_t^s 2h \lambda(t,N_u) dW_u \right) | F_t \right)$$

$$= E^{Q^{a,2,h}} \left( k^{N_s - N_t} | F_t \right)$$

$$= \sum_{n=1}^{n_x - N_t} \frac{(n_x - N_t)!}{(n_x - N_t - n)! n!} \left( k^n \cdot (s - t_p^{2,h})^{n_x - N_t - n} \cdot (1 - s - t_p^{2,h})^n \right)$$

**Appendix C.**

The independence between mortality and the other random variables of our model entails that the fair value of the pension fund liabilities is:

$$L_T = E \left( \int_T^{T^m} H(T, s). (n_x - N_s) \cdot B(T, s) ds | F_T \right)$$

$$= (n_x - N_t) \cdot A_T \cdot \int_T^{T^m} s - T_p^{h} B(T, s) ds$$

And that the expectation at time $t \leq T$ of $L_T$ is:

$$E(H(t, T), L_T | F_t)$$

$$= \alpha \cdot (n_x - N_t) \cdot \int_T^{T^m} s - T_p^{h} \cdot E^Q \left( e^{-\int_t^T r_u du} \cdot A_T \cdot B(T, s) | F_t \right) ds$$

The sequel of this paragraph focus then on the calculation of $E^Q \left( e^{-\int_t^T r_u du} \cdot A_T \cdot B(T, s) | F_t \right)$. This step is based on the following four observations. Firstly, $A_T$ is the Doolean-Dade exponential, solution of the SDE (4.1):

$$A_T = A_t \cdot \exp \left( \int_t^T \left( \mu_A(u) - \frac{\sigma_A^2}{2} - \frac{\sigma_S^2}{2} - \frac{\sigma_A^2}{2} \right) du \right)$$

$$\cdot \exp \left( \int_t^T \sigma_A dW_u^{A, Q^{a,\lambda \alpha}} + \int_t^T \sigma_A dW_u^{r \cdot Q^f} + \int_t^T \sigma_A dW_u^{S, Q^f} \right)$$

(8.5)

Secondly, as detailed in appendix A, the price of a zero coupon bond is given by:

$$B(T, s) = \exp \left( -\beta \cdot (s - T) + n(s - T) \cdot (\beta - r_T) - \frac{\sigma^2}{4} \cdot n(s - T)^2 \right)$$

(8.6)

Where $\beta$ is defined by equation (8.2). The last useful elements are related to the fact that interest rates are Gaussian in the Vasicek model.

$$r_T = \left( 1 - e^{-a \cdot (T - t)} \right) b^Q + e^{-a \cdot (T - t)} \cdot r_t + \int_t^T \sigma_r e^{-a \cdot (T - u)} dW_u^{r \cdot Q^f}$$

(8.7)
\[
\int_t^T r_u \, du = b^Q(T - t) + (r_t - b^Q) \cdot n(T - t) + \sigma_r \int_t^T n(T - u) \, dW^r_{u,Q} \tag{8.8}
\]

The proof of such results can be found in Cairns (2004), appendix B. Combining expressions (8.5), (8.6), (8.7) and (8.8) allows us to rewrite \( e^{-\int_t^T r_u \, du} \cdot A_T \cdot B(T, s) \) as an exponential of independent normal random variables and the calculation of \( E^Q \left( e^{-\int_t^T r_u \, du} \cdot A_T \cdot B(T, s) \mid \mathcal{F}_t \right) \) directly results from the expectation of lognormal variables.

**Appendix D.**

In the example presented in this paper, mortality rates obey to a Gompertz-Makeham distribution. The parameters are those defined by the Belgian regulator for the pricing of a life insurance purchased by a man. For an individual of age \( x \), the mortality rate is:

\[
\mu(x) = a_\mu + b_\mu \cdot c^x
\]

Where the parameters \( s_\mu, g_\mu, c_\mu \) take the values showed in the table 8.1.

| \( s_\mu \) | 0.999441703848 |
| \( g_\mu \) | 0.99973441115 |
| \( c_\mu \) | 1.116792453830 |

**References**


