MANAGEMENT OF A PENSION FUND UNDER A VAR CONSTRAINT.

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Abstract. This paper studies the optimal asset allocation problem for a continuous time pension fund under a value at risk constraint. The affiliate’s payments stream is deterministic and the mortality is modelled by a geometric Brownian. The financial market is composed of risky assets driven by geometric Brownian motions. We address the case of a fund manager who wishes to maximize quadratic utility of the solvency ratio, defined as the market value of assets divided by the mathematical reserve under a VAR constraint. The method of Lagrange multipliers is combined with the Hamilton Jacobi Bellman equation to insert the risk management constraint in the framework of resolution. A numerical method is then developed to yield an approximation of the solution of the HJB equation. Finally, we show on examples how the introduction of a VAR constraint modifies the optimal investment strategy.

1. Introduction

During the last decade, the economic role of pension funds has considerably grown: pension fund assets represent actually a large percentage of stock and bond market capitalisations. Amongst the multiple reasons explaining this trend, a low and controlled inflation plays in favor of the capitalization of pension contributions, whereas the announced reforms of social security incite the active population to privilege pension funding and savings insurance.

However, pension funds are exposed to important financial and longevity risks, due to the long time horizon of their commitments. The management of such funds entails therefore a constant monitoring of the risks exposure and a regular rebalancing of assets. This paper is directly related to this topic and proposes a method to determine the optimal investment policy of a portfolio of actuarial liabilities, under a value at risk (VAR) constraint.

Even if risk measures as VAR, are widely adopted by the financial sector, there are relatively few articles combining stochastic control with such risk management criterion. As reference, we quote Yiu (2004) who has proposed an algorithm to solve the optimal investment consumption problem with a VAR constraint. Two others ways were explored to integrate this risk measure in the active management of pension funds. The first one is a Monte Carlo approach : Blake et al. (2001) (2003) have set out a simulation methodology to design a pension fund under a VAR constraint. The second way is a martingale approach, exploited by Gabih and Wunderlich (2005) to determine the optimal investment strategy maximizing the terminal utility of the wealth with a limited VAR. The literature

In this work, we consider a continuous pension plan. The affiliate’s payments stream is deterministic whereas the mortality is stochastic and modelled by a geometric Brownian motion. The financial market is composed of risky assets, also ruled by geometric Brownian motions, without any risk free rate asset. In order to reduce the number of state variables, we introduce the solvency ratio, which is defined as the market value of assets divided by the mathematical reserve. This ratio is an efficient synthetic measure of the available surplus and is recognized by regulators as a good indicator of solvency.

We assume that the fund manager seeks to maximize continuously the utility arising from the solvency ratio, via an adapted investment policy, which is the control variable of our model. The preferences of the manager are reflected by a quadratic utility function having two components. A first one, proportional to the solvency ratio, is related to the fact that the fund manager wants to maximize the solvency ratio. The second component, proportional to the square of the difference between the solvency ratio and a benchmark ratio, penalizes the spread between the current ratio and a target one.

In a first time, the Bellman’s equation, coupled to the utility maximization problem, is solved without risk management constraint. We establish the relation between the value function and the optimal investment policy. This link is used in the constrained case. In a second time, we assume that the fund manager wishes to keep the VAR of the solvency ratio below a certain level. This constraint is inserted in the Bellman’s equation by the method of Lagrange multipliers and a numerical algorithm is proposed. Finally, an example of a portfolio of life annuities is analyzed and we observe that the solution of the constrained problem is identical to the one of the unconstrained problem for certain values of the solvency ratio.

The outline of the paper is as follows. In section 2, we present the actuarial liabilities, the financial market and the dynamic of the pension fund asset. Section 3 introduces the solvency ratio and the objective of the asset manager. Section 4 develops the solution of the unconstrained problem and the relation between the value function and investment policy is established. In section 5, a numerical algorithm, using the method of Lagrangian multipliers, is proposed to solve the VAR constrained problem. Section 6 presents an example and section 7 concludes this work.

2. Liabilities and assets.

We assume that the fund and the affiliate have agreed upon a stream of premiums and a stream of benefits. Considering premiums and benefits respectively as negative and positive liabilities cash flows, the payments streams are merged into a single payment process $L_t$ that is the accumulated payment done till instant $t$, to a living affiliate. The calculation of $L_t$ is
not discussed in this paper and is taken to be predetermined by the policy. The density of the payment process is noted \( l_t \) and is such that \( dL_t = l_t \, dt \). For the sake of simplicity, there is no death benefits.

The fund counts an initial cohort of \( n_0 \) affiliates, of age \( x \). Let \( n_t \) denote the number of alive members at time \( t \). As in the paper of Fonbellida Zapareto (2004), we assume for analytical tractability, that this process is driven by a geometric Brownian motion. If \( \mu(.) \) and \( \sigma(.) \) are respectively the real mean mortality rate and the volatility of this rate, the dynamic of \( n_t \) is:

\[
\frac{dn_t}{n_t} = -\mu(x + t) \, dt + \sigma(x + t) \, dW_t^L
\]

Where \((W_t^L)_{0 \leq t}\) is an one dimensional Brownian motion defined on a probability space \((\Omega^L, \mathcal{F}^L, \mathbb{P}^L)\). \((\mathcal{F}_t^L)_{0 \leq t}\) is the continuous filtration generated by \((W_t^L)_{0 \leq t}\) and \(\mathbb{P}\) is a probability measure on \(\Omega^L\). The solution of equation (2.1) is:

\[
n_t = n_0 e^{\int_0^t \mu(x + z) + \frac{\sigma(x + z)^2}{2} \, dz + \int_0^t \sigma(x + z) \, dW_t^L}
\]

\(n_t\) is not strictly decreasing and therefore doesn’t perfectly fit the mortality process. However, in section 3, this model allows us to reduce the number of state variables. Let \(T_m\) and \(r\) be respectively the time horizon of liabilities and a constant accounting discount rate. The global mathematical reserve of the fund is then defined by:

\[
R_t = \int_0^{T_m} l_s e^{-r(s-t)} \mathbb{E}\left(n_s|\mathcal{F}_t^L\right) \, ds
\]

From the log normality property of \(n_t\), its expectation with respect to \(\mathcal{F}_t^L\) is equal to \(n_t\) times a probability of survival:

\[
\mathbb{E}\left(n_s|\mathcal{F}_t^L\right) = n_t e^{-\int_0^s \mu(x + z) \, dz} = n_t s^{-t} p_{x+s}
\]

Where \(s^{-t} p_{x+s}\) is the actuarial notation for the survival probability from age \(x + t\) till age \(x + s\). The next proposition establishes the dynamic of the reserve.

**Proposition 2.1.** \(\forall t \in [0, T_m]\), \(R_t\) verifies the stochastic differential equation:

\[
dR_t = r R_t \, dt - n_t l_t \, dt + \sigma(x + t) R_t \, dW_t^L
\]

This proposition directly results from the Ito’s lemma.

On the asset side, we consider a financial market composed of \(n\) risky investments, \(F_1(t), F_2(t), \ldots, F_n(t)\) driven by geometric Brownian motions:

\[
dF_i(t) = F_i(t) \left( m_i \, dt + \sum_{j=1}^n \sigma_{i,j} \, dW_j^F \right) \quad F_i(0) = F_{i,0} \quad 1 \leq i \leq n
\]

We assume that \(m_i\) and \(\sigma_{i,j}\) are positive constants. The \(n\) vector \((W_t^F)_{0 \leq t}\) is a Brownian motion defined on a probability space \((\Omega^F, \mathcal{F}^F, \mathbb{P}^F)\) where \((\mathcal{F}_t^F)_{0 \leq t}\) is the filtration generated by \((W_t^F)_{0 \leq t}\). The \((n \times n)\) matrix of \(\sigma_{i,j}\) is noted \(\sigma\) and is the Choleski’s decomposition of the
variance-covariance matrix of the assets $\Sigma = \sigma^\top \sigma$.

Let $A_t$ be the total asset in front of liabilities, at time $t$. The proportion of $A_t$, invested in the $i^{th}$ risky asset at time $t$ is noted $\pi^i_t$ $1 \leq i \leq n$, whereas the vector of $\pi^i_t$ is $\Pi_t$. $A_t$ is therefore solution of the following SDE:

\[
(2.3) \quad dA_t = (m^\top \Pi_t - n_t . l_t) . dt + A_t . \Pi_t^\top . \sigma . dW^F_t
\]

There is no risk free rate asset because cash is used only for short term liquidity and not as a strategic asset. This entails the following constraint:

\[
\pi^n_t = 1 - \sum_{i=1}^{n-1} \pi^i_t
\]

3. The solvency ratio and the objective.

Let $(\Omega, \mathcal{F}, P)$ be the probability space resulting from the combination of the insurance and financial markets.

\[
\Omega = \Omega^a \times \Omega^f \quad \mathcal{F} = \mathcal{F}^a \otimes \mathcal{F}^f \vee \mathcal{N} \quad P = P^a \times P^f
\]

Where the sigma algebra $\mathcal{N}$ is generated by all subsets of null sets from $\mathcal{F}^a \otimes \mathcal{F}^f$. In order to reduce the number of Brownian motions, we define the solvency ratio at time $t$, $S_t$, as the market value of investments, $A_t$, divided by the mathematical reserve, $R_t$. This ratio is a synthetic measure of the level of the surplus and is the main indicator of solvency for the regulators. As the dynamics of assets and liabilities are known and geometric Brownian motions, we easily derive the following proposition:

**Proposition 3.1.** $\forall t \in [0, T_m]$, $S_t$ verifies the stochastic differential equation:

\[
(3.1) \quad dS_t = \frac{dA_t}{R_t} = ( (-r + m^\top \Pi_t + G_t + \sigma \mu (x + t)^2) . S_t - G_t) . dt
\]

\[
+ S_t . \sqrt{\Pi_t^\top . \Sigma . \Pi_t + \sigma \mu (x + t)^2} . dW^S_t
\]

Where $G_t$ is a function of time defined by:

\[
G_t = l_t . \frac{1}{\int_t^{T_m} l_s . e^{-r(s-t)} . x . p_{x+t} . ds}
\]

And $(W^S_t)_{0 \leq t \leq T_m}$ is a one dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$ summarizing the stochastic components of assets and liabilities.

This proposition is proved in appendix, by application of the Ito’s lemma to the ratio $\frac{A_t}{R_t}$.

The solvency ratio being a key indicator, we assume that the fund manager continuously monitor it and optimizes the corresponding expected utility by an adapted investment policy. Let $U(S_t)$ be the utility function. At instant $t$, the optimal control problem is therefore:

\[
(3.2) \quad v(S_t, t) = \max_{\Pi_t \in \mathbb{R}^n} \mathbb{E} \left( \int_t^T e^{-\rho(s-t)} . U(S_s) . ds + e^{-\rho(T-t)} . U(S_T) \right)
\]
Under the constraint that

$$\pi_s^n = 1 - \sum_{i=1}^{n-1} \pi_s^i \quad \forall s \in [t, T]$$

Where $T$ is the time horizon of the optimization and $\rho$ is the discount rate of the utility function. The chosen utility is quadratic and privileges with a weight $k$, a large solvency ratio whereas it penalizes with the weight $(1 - k)$ the square of the spread between the current solvency ratio and a target one, noted $TS$.

(3.3) \[ U(S_t) = k.S_t - (1 - k).\left(S_t - TS\right)^2 \]

Quadratic utilities are widely used in the literature, see for e.g. papers of Haberman et al., in reason of their easy interpretation. In a first time, we solve the optimization problem without VAR constraint.

4. The unconstrained problem.

From the theory of stochastic control (see for e.g. Øksendal 2003 or Fleming and Soner 1993), the value function, $v(S_t, t)$, associated to the optimization problem (3.2) is a $C^{2,1}(\mathbb{R} \times [0, T])$ function, solution of the Hamilton Jacobi Bellman equation:

(4.1) \[ \frac{\partial v(S_t, t)}{\partial t} - \rho.v(S_t, t) + \sup_{\Pi_t} \left( \frac{L^H_t.v(S_t, t) + U(S_t)}{\text{Hamiltonian}} \right) = 0 \]

Where $L^H_t.v(S_t, t)$ is the infinitesimal generator of $v(S_t, t)$:

(4.2) \[ L^H_t.v(S_t, t) = \left(-r + m^\top .\Pi_t + G_t + \sigma_\mu(x + t)^2 \right).S_t - G_t \right) \cdot \frac{\partial v(S_t, t)}{\partial S_t} + \frac{1}{2}.S_t^2 \cdot (\Pi_t^\top .\Sigma .\Pi_t + \sigma_\mu(x + t)^2) \cdot \frac{\partial^2 v(S_t, t)}{\partial S_t^2} \]

Under the terminal condition:

(4.3) \[ v(S_T, T) = U(S_T) \]

And subject to the constraint:

(4.4) \[ \pi_t^n = 1 - \sum_{i=1}^{n-1} \pi_t^i \]

Before any further developments, some matrix notations are introduced. In order to integrate the constraint (4.4) in the framework of resolution, the vector $\Pi_t$ is expressed as a matrix product, which is only a function of the first (n-1) elements of $\Pi_t$.

$$\Pi_t = \begin{pmatrix} \pi_t^1 \\ \vdots \\ \pi_t^{n-1} \\ 1 - \sum_{i=1}^{n-1} \pi_t^i \end{pmatrix} = \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \\ -1 & \ldots & -1 \end{pmatrix} \cdot \begin{pmatrix} \pi_t^1 \\ \vdots \\ \pi_t^{n-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Where $M_1$ is a constant (n, n-1) matrix and $M_2$ is a (n) vector. The vector $\pi_t$ is the vector...
of the (n-1) first elements of \( \Pi_t \). We differentiate the Hamiltonian with respect to \( \pi_t \) in order to obtain the optimal strategy of investment, denoted \( \pi_t^{opt} \):

\[
(4.5) \quad \frac{\partial \text{Hamiltonian}}{\partial \pi_t} = 0 \iff M_1^\top . m . S_t . v_s + S_t^2 . v_{ss} . M_1^\top . \Sigma . (M_1 . \pi_t + M_2) = 0
\]

\( v_s \) and \( v_{ss} \) correspond respectively to the first and the second derivative of the value function with respect to \( S \). To simplify the notations, we define \( N_1 \), a (n-1,n-1) matrix, and \( N_2 \), a (n-1) vector:

\[
N_1 = M_1^\top . \Sigma . M_1 \\
N_2 = M_1^\top . \Sigma . M_2
\]

From equation (4.5), we find \( \pi_t^{opt} \):

\[
(4.6) \quad \pi_t^{opt} = N_1^{-1} . \left( -\frac{M_1^\top . m . v_s}{S_t . v_{ss}} - N_2 \right)
\]

The complete optimal strategy of investment, \( \Pi_t^{opt} \), is therefore:

\[
(4.7) \quad \Pi_t^{opt} = M_1 \left( -\frac{N_1^{-1} . M_1^\top . m . v_s}{S_t . v_{ss}} - N_1^{-1} . N_2 \right) + M_2 \\
= \frac{v_s}{S_t . v_{ss}} . M_1 . N_1^{-1} . M_1^\top . m - M_1 . N_1^{-1} . N_2 + M_2 \\
= \frac{v_s}{S_t . v_{ss}} . (P_1 . m - P_2 - P_2)
\]

Where \( P_1 \) and \( P_2 \) are respectively a (n,n) matrix and a (n) vector. If we inject this last expression of \( \Pi_t^{opt} \) in the Bellman’s equation (4.1) and (4.2), we get that the value function \( v(S_t, t) \) is the solution of the following stochastic differential equation:

\[
(4.8) \quad 0 = \left( \left( -r + m^\top . \left( -\frac{v_s}{S_t . v_{ss}} . P_1 . m - P_2 \right) + G_t + \sigma \mu (x + t)^2 \right) . S_t - G_t \right) . v_s \\
+ \frac{1}{2} . S_t^2 . \left( \left( -\frac{v_s}{S_t . v_{ss}} . P_1 . m - P_2 \right)^\top . \Sigma . \left( -\frac{v_s}{S_t . v_{ss}} . P_1 . m - P_2 \right) + \sigma \mu (x + t)^2 \right) . v_{ss} \\
+ v_t - \rho . v + U(S_t)
\]

Where \( v_t \) is the derivative of the value function with respect to \( t \). As the utility function is a quadratic one, we try a solution \( v(S_t, t) \) with a quadratic structure:

\[
(4.9) \quad v(S_t, t) = a(t) . S_t^2 + b(t) . S_t + c(t)
\]

Then:

\[
(4.10) \quad \frac{\partial v(S_t, t)}{\partial S_t} = v_s = 2 . a(t) . S_t + b(t) \\
\frac{\partial^2 v(S_t, t)}{\partial^2 S_t} = v_{ss} = 2 . a(t) \\
\frac{\partial v(S_t, t)}{\partial t} = v_t = a'(t) . S_t^2 + b'(t) . S_t + c'(t)
\]

Introducing these derivatives in (4.8) and regrouping the terms in \( S^2 \), \( S \) and independent of \( S \), gives three differential equations, for \( a(t) \), \( b(t) \) and \( c(t) \).
\[ a'(t) = -2.a(t).(-r - m^\top.P_2 + G_t + \sigma_\mu(x + t)^2 + P_2^\top.\Sigma.P_1.m - m^\top.P_1.m) \]
\[ b'(t) = -b(t).(-r - m^\top.P_2 + G_t + \sigma_\mu(x + t)^2 + P_2^\top.\Sigma.P_1.m + m^\top.P_1^\top.\Sigma.P_1.m) + (1 - k) \]
\[ c'(t) = \rho.c(t) + b(t).G_t + \left(m^\top.P_1.m - \frac{1}{2}m^\top.P_1^\top.\Sigma.P_1.m\right) \cdot \frac{b^2}{2.a} \]
\[ + (1 - k).T S^2 \]

With the terminal conditions:
\[ a(T) = -(1 - k) \]
\[ b(T) = k + 2.(1 - k).T S \]
\[ c(T) = -(1 - k).T S^2 \]

Functions \( a(t), b(t), c(t) \) can be computed numerically with an Euler Cauchy method (see for e.g. Isaacson, Keller 1994). The symbol \( \tilde{\cdot} \) appoints the result of the numerical integration.

\[ \begin{cases} 
\tilde{a}(t - \Delta t) = \tilde{a}(t) - a'(t).\Delta t & \tilde{a}(T) = -(1 - k) \\
\tilde{b}(t - \Delta t) = \tilde{b}(t) - b'(t).\Delta t & \tilde{b}(T) = k + 2.(1 - k).T S \\
\tilde{c}(t - \Delta t) = \tilde{c}(t) - c'(t).\Delta t & \tilde{c}(T) = -(1 - k).T S^2 
\end{cases} \]

Where \( \Delta t \) is a sufficiently small step of time. The precision of the solution is function of the length of the step of integration \( \Delta t \) and is computed via the Bellman’s equation. If we subdivide the domain of computation into a grid of \( N_S \times N_t \) mesh points, the error \( \epsilon_{t,s} \) at time \( t \) and for a solvency ratio of \( s \), with \( s \in [0, \Delta S, \ldots, N_S.\Delta S] \) and \( t \in [0, \Delta t, \ldots, N_t.\Delta t] \), is valued by:

\[ \epsilon_{t,s} = \bar{v}_t - \rho.\bar{v} + \left(-r + m^\top.\tilde{\Pi}_t^{opt} + G_t + \sigma_\mu(x + t)^2\right).s - G_t \cdot \bar{s} + \frac{1}{2} s^2 \cdot \left(\tilde{\Pi}_t^{opt}.\Sigma.\tilde{\Pi}_t^{opt} + \sigma_\mu(x + t)^2\right).\bar{s} + U(s) \]

Where
\[ \bar{v} = \tilde{a}(t).s^2 + \tilde{b}(t).s + \tilde{c}(t) \]
\[ \bar{v}_s = 2.\tilde{a}(t).s + \tilde{b}(t) \]
\[ \bar{v}_ss = 2.\tilde{a}(t) \]

And
\[ \tilde{\Pi}_t^{opt} = -\frac{\bar{v}_s}{s.\bar{v}_ss}.P_1.m - P_2 \]

The next proposition explicits the link between the optimal investment policy and the value function. A similar relation will be used in the resolution of the VAR constrained problem.
Proposition 4.1. Given the optimal investment strategy \( \Pi_t^{opt} = (M_1, \pi^{opt}_t + M_2) \), the value function \( v(S_t, t) \) is:

\[
v(S_t, t) = a(t).S_t^2 + b(t).S_t + c(t)
\]

Where functions \( a, b, c \) are solutions of the differential equations:

\[
a'(t) = -2.a(t).(-r - m^\top .P_2 + G_t + \sigma_\mu(x + t)^2 + P_2^\top .\Sigma .P_1 .m - m^\top .P_1 .m) \\
(4.17)
\]

\[
b'(t) = -b(t).G_t + \sigma_\mu(x + t)^2 - \rho - r - m^\top .P_1 .m - m^\top .P_2 - 2.a(t).\left( g_t^\top .\Sigma .P_1 .m + g_t^\top .\Sigma .P_2 - G_t - g_t^\top .m \right) \\
(4.18)
\]

\[
c'(t) = \rho .c(t) + b(t).G_t + \sigma_\mu(x + t)^2 - \rho - r - m^\top .P_1 .m - m^\top .P_2 - k - 2.(1 - k).TS \\
(4.19)
\]

Where \( g_t \) is a \((n)\) vector, defined as:

\[
g_t = - (P_2 .S_t + \Pi_t^{opt} .S_t + S_t .P_1 .m) \\
(4.20)
\]

With the terminal conditions (4.14).

The proof is in appendix. Notice that, the optimal strategy is:

\[
\Pi_t^{opt} = - \frac{v_s}{S_t v_{ss}} .P_1 .m - P_2 \\
= - \frac{2.a(t).S_t + b(t)}{2.a(t).S_t} .P_1 .m - P_2
\]

\( g_t \) is therefore only a function of time:

\[
g_t = \frac{b(t)}{2.a(t)}.P_1 .m \\
(4.21)
\]

And it can be easily shown that the system of equations (4.17) (4.18) (4.19) is equivalent to the system (4.11) (4.12) (4.13).

5. Introduction of a value at risk constraint.

In order to control the exposure to market and mortality risks, we now consider that the fund manager seeks to maximize the expected discounted value of his utility function, under a continuous value at risk constraint on the solvency ratio. The value at risk is computed under the assumption that the investment strategy isn’t modified during each successive VAR time horizon, noted \( \Delta t_{var} \). The distribution of the solvency ratio at time \( t + \Delta t_{var} \), is then approached by:

\[
S_{t+\Delta t_{var}} = S_t + \left( (-r + m^\top .\Pi_t + G_t + \sigma_\mu(x + t)^2) .S_t - G_t \right) .\Delta t_{var} \\
+ S_t .\sqrt{\Pi_t^\top .\Sigma .\Pi_t + \sigma_\mu(x + t)^2} .\sqrt{\Delta t_{var}} .\Phi
\]
Where $\Phi$ is a standard normal random variable (average null and variance equal to one). Let $S_{t+\Delta t}^{VAR(\xi)}$ be the $\xi$ percentile of the distribution of the solvency ratio at time $t + \Delta t$. $S_{t+\Delta t}^{VAR(\xi)}$ is approximated by:

$$S_{t+\Delta t}^{VAR(\xi)} = S_t + \left( (-r + m^T \Pi_t + G_t + \sigma_{\mu}(x + t)^2) \cdot S_t - G_t \right) \cdot \Delta t_{var} + S_t \sqrt{\Pi_t^T \Sigma \Pi_t + \sigma_{\mu}(x + t)^2} \cdot \sqrt{\Delta t_{var}} \cdot \phi(\xi)$$

Where $\phi(\xi)$ is the $\xi$ percentile of a standard normal random variable, $\Phi$. The value at risk for a confidence level of $\xi$ is the difference between the expected solvency ratio at time $t + \Delta t$ and $S_{t+\Delta t}^{VAR(\xi)}$.

$$VAR = S_t \sqrt{\Pi_t^T \Sigma \Pi_t + \sigma_{\mu}(x + t)^2} \cdot \sqrt{\Delta t_{var}} \cdot \phi(\xi)$$

If the fund manager limits the VAR to $-R$ ($R$ is a positive constant) the optimization problem (3.2) becomes:

$$v(S_t, t) = \max_{\Pi_t \in \mathbb{R}^n} \mathbb{E} \left( \int_t^T e^{-\rho (s-t)} U(S_s) ds + e^{-\rho (T-t)} U(S_T) \right)$$

Under the constraints:

$$\pi_s^n = 1 - \sum_{i=1}^{n-1} \pi_s^i \quad \forall s \in [s, T]$$

$$-S_s \sqrt{\Pi_s^T \Sigma \Pi_s + \sigma_{\mu}(x + s)^2} \cdot \sqrt{\Delta t_{var}} \cdot \phi(\xi) \leq R \quad \forall s \in [s, T]$$

The constraint (5.2) is directly integrated in the framework of resolution, as in section 4. Let $\lambda_{t,s} \leq 0$ be the Lagrange multiplier related to the VAR constraint (5.3). The value function of the optimization problem (5.1) is therefore solution of the Hamilton Jacobi Bellman equation:

$$\frac{\partial v(S_t, t)}{\partial t} - \rho v(S_t, t) + \max_{\Pi_t \in \mathbb{R}^n, \lambda_{t,s} \in \mathbb{R}^n} \left( \frac{L^{\Pi_t,\lambda_{t,s}} v(S_t, t) + U(S_t)}{\text{Hamiltonian}} \right) = 0$$

Where $L^{\Pi_t,\lambda_{t,s}} v(S_t, t)$ is the infinitesimal generator of $v(S_t, t)$:

$$L^{\Pi_t,\lambda_{t,s}} v(S_t, t) = \left( (-r + m^T \Pi_t + G_t + \sigma_{\mu}(x + t)^2) \cdot S_t - G_t \right) \cdot \frac{\partial v(S_t, t)}{\partial S_t} + \frac{1}{2} S_t^2 \cdot (\Pi_t^T \Sigma \Pi_t + \sigma_{\mu}(x + t)^2) \cdot \frac{\partial^2 v(S_t, t)}{\partial S_t^2} - \lambda_{t,s} \left( R + S_t \sqrt{\Pi_t^T \Sigma \Pi_t + \sigma_{\mu}(x + t)^2} \cdot \sqrt{\Delta t_{var}} \cdot \phi(\xi) \right)$$

Under the terminal condition:

$$v(S_T, T) = U(S_T)$$

And subject to the constraint:

$$\pi_t^n = 1 - \sum_{i=1}^{n-1} \pi_t^i$$
When the optimal solution strictly satisfies the VAR constraint (5.3), the Lagrange multiplier, \( \lambda_{t,s} \) is null. If the constraint is active,

\[
\left( R + S_t \sqrt{\Pi_t^\top \Sigma \Pi_t + \sigma_\mu (x + t)^2 \sqrt{\Delta t_{var}} \phi(\xi)} \right) = 0
\]

the multiplier \( \lambda_{t,s} \) is negative. The stochastic differential equation (5.4) has no analytic solution. For this reason, we propose an iterative algorithm similar to the one of Yiu (2004), which yields a \( C^{2,1} \) approximation \( \hat{v} \), of the exact solution \( v \). \( \hat{\Pi}_t \) is the investment strategy related to \( \hat{v} \).

First, we divide the domain of resolution into a grid of \( N_S \times N_t \) mesh points. Iterations are indexed by \( j \).

1. For each point \((t, s)\), with \( t \in [0, \Delta t, \ldots, N_t \Delta t] \) and \( s \in [0, \Delta S, \ldots, N_s \Delta S] \), we compute the value function \( \hat{v}^{j=0} = v(S_t, t) \) and the optimal strategy \( \hat{\Pi}^{j=0}_t = \Pi^{opt}_t \), of the unconstrained problem. All Lagrange multipliers are set to zero, \( \lambda^{j=0}_{t,s} = 0 \). This solution is the starting point of the algorithm.

2. For all points of the grid, the VAR constraint is then checked. If the constraint is not active \((-VAR < R)\) the multiplier is null \( \lambda^{j+1}_{t,s} = 0 \) and \( \hat{\Pi}^{j+1}_t \) is solution of a similar equation to the one of the unconstrained case.

\[
\lambda^{j+1}_{t,s} = 0
\]

(5.8)

\[
\hat{\Pi}^{j+1}_t = -\frac{\hat{v}_{s}^{\hat{j}}}{s \hat{v}_{ss}^{\hat{j}}} P_1 m - P_2
\]

If the VAR constraint is active \((-VAR \geq R)\), we solve a non linear system in \( \lambda^{j+1}_{t,s} \) and \( \hat{\Pi}^{j+1}_t \). This non linear system is composed of the VAR constraint and of the derivative of the Hamiltonian (5.5) with respect to \( \pi_t \), set to zero.

\[
\begin{align*}
0 & = \left( R + S_t \sqrt{\Pi_t^{j+1 \top} \Sigma \Pi_t^{j+1} + \sigma_\mu (x + t)^2 \sqrt{\Delta t_{var}} \phi(\xi)} \right) \\
0 & = M_1^{\top} m.s. \hat{v}_{s}^{\hat{j}} + s^2 \hat{v}_{ss}^{\hat{j}} M_1^{\top} \Sigma \hat{\Pi}_t^{j+1} \\
-\lambda^{j+1}_{t,s}. s. \phi(\xi) \sqrt{\Delta t_{var}} M_1^{\top} \Sigma \hat{\Pi}_t^{j+1} & = \left( \hat{\Pi}_t^{j+1} \Sigma \hat{\Pi}_t^{j+1} + \sigma_\mu (x + t)^2 \right)^{-\frac{1}{2}} \\
\hat{\Pi}_t^{j+1} & = M_1 \hat{\pi}_t^{j+1} + M_2
\end{align*}
\]

(5.9)

This system is numerically solved by a Newton Raphson method.

3. The last stage consists in the calculation of the value function \( \hat{v}^{j+1} \) according to the strategy of investment \( \hat{\Pi}^{j+1}_t \). This is detailed in the sequel of the paragraph (see proposition 5.1).

4. Go back to step 2 with \( j = j + 1 \), until convergence.
We detail now the method to achieve the third step of the algorithm. In the unconstrained case, the proposition 4.1 links the investment policy $\Pi_t$ and the value function. We use a similar relation to build $\hat{v}$ in function of $\hat{\Pi}_t$:

**Proposition 5.1.** Given an investment strategy $\hat{\Pi}_t = (M_1.\hat{\pi}_t + M_2)$, the value function $\hat{v}(S_t, t)$ of the constrained problem is approximated by:

$$
\hat{v}(S_t, t) \simeq \hat{a}(t).S_t^2 + \hat{b}(S_t, t).S_t + \hat{c}(S_t, t)
$$

(5.10)

Where functions $\hat{a}$, $\hat{b}$, $\hat{c}$ are solutions of the differential equations:

$$
\begin{align*}
\hat{a}'(t) &= -2.\hat{a}(t).(-r - m^T.P_2 + G_t + \sigma(x + t)^2 + P_2^T.\Sigma.P_1.m - m^T.P_1.m) \\
(5.11) &- \hat{a}(t). (P_2^T.\Sigma.P_2 + \sigma(x + t)^2 - \rho + m^T.P_1^T.\Sigma.P_1.m) + (1 - k) \\
\hat{b}'(S_t, t) &= -\hat{b}(S_t, t). (G_t + \sigma(x + t)^2 - \rho - r - m^T.P_1.m - m^T.P_2) \\
(5.12) &- 2.\hat{a}(t). (g_{t,S_t}.P_1.m + g_{t,S_t}.P_2 - G_t - g_{t,S_t}.m) \\
&- k - 2.(1 - k).TS \\
\hat{c}'(S_t, t) &= \rho.\hat{c}(S_t, t) + \hat{b}(S_t, t) (G_t + g_{t,S_t}.m) - \hat{a}(t).g_{t,S_t}.\Sigma.g_{t,S_t} \\
(5.13) &+ (1 - k).TS^2
\end{align*}
$$

Where $g_{t,S_t}$ is a (n) vector, function of time $t$ and of $S_t$, defined as:

$$
g_{t,S_t} = -\left(P_2.S_t + \hat{\Pi}_t.S_t + S_t.P_1.m\right)
$$

(5.14)

With terminal conditions similar to (4.14).

The proof of the proposition 5.1 is similar to the one of the proposition 4.1 if we ignore the dependency of $\hat{b}(S_t, t)$ and $\hat{c}(S_t, t)$ on $S_t$:

$$
\begin{align*}
\frac{\partial \hat{v}(S_t, t)}{\partial S_t} &\simeq 2.\hat{a}(t).S_t + \hat{b}(S_t, t) \\
\frac{\partial^2 \hat{v}(S_t, t)}{\partial^2 S_t} &\simeq 2.\hat{a}(t).S_t \\
\frac{\partial \hat{v}(S_t, t)}{\partial t} &\simeq \hat{a}'(t).S_t^2 + \hat{b}'(S_t, t).S_t + \hat{c}'(S_t, t)
\end{align*}
$$

(5.15)

As in paragraph 6, functions $\hat{a}$, $\hat{b}$, $\hat{c}$ are computed numerically with an Euler Cauchy method. If the symbol $\tilde{\cdot}$ appoints the result of the numerical integration, the error of approximation $\varepsilon_{t,s}$ at time $t$, for a solvency ratio of $s$, and at iteration $j$, is then valued by:

$$
\varepsilon_{t,s}^j = \tilde{v}_s^j - \rho.\tilde{v}_s^j + \left(\left(-r - m^T.\hat{\Pi}_t + G_t + \sigma(x + t)^2\right).s - G_t\right).\tilde{v}_s^j
$$

$$
+ \frac{1}{2}.s^2.\left(\hat{\Pi}_t^T.\Sigma.\hat{\Pi}_t + \sigma(x + t)^2\right).\tilde{v}_{ss}^j + U(s)
$$
With
\[ \bar{v} = \bar{a}^j(t).s^2 + \bar{b}^j(s,t).s + \bar{c}^j(s,t) \]
\[ \tilde{v}_s = 2.\bar{a}^j(t).s + \bar{b}^j(s,t) \]
\[ \tilde{v}_s = 2.\tilde{a}^j(t) \]
\[ \tilde{v}_t = \bar{a}^j(t).s^2 + \bar{b}^j(s,t).s + \bar{c}^j(s,t) \]

A relatively small number of iterations (between 2 and 4) is required to converge to a solution. Again, the precision of the solution is mainly influenced by the size of the step of integration \( \Delta t \).

6. Example.

In this paragraph, we apply the algorithm of the previous section to a portfolio of life annuities. All affiliates are 60 years old and receive a continuous annuity of one, \( l_t = 1 \), till their decease. The discount rate used to value the total mathematical reserve is equal to 3.25% whereas the mortality rates \( \mu(x) \) are given by a Gompertz-Makeham distribution:

\[ \mu(x) = a\mu + b\mu.e^x \]
\[ a\mu = -\ln(s\mu) \]
\[ b\mu = \ln(g\mu).\ln(c\mu) \]

Where the parameters \( s\mu \), \( g\mu \), \( c\mu \) take the values presented in the table 1. The volatility of the mortality rate is set to 50% of the mortality rate, \( \sigma_{\mu}(x + t) = 0.5\mu(x + t) \).

| \( s\mu \) | 0.999441703848 |
| \( g\mu \) | 0.99973441115 |
| \( c\mu \) | 1.116792453830 |

The market is composed of three assets: a fund of stocks \( S_1 \), of long term bonds \( S_2 \) and short term bonds \( S_3 \). The volatilities and correlations of those assets are presented in tables 2 and 3.

| Return, \( m_i \) | Volatilities, \( \sigma_i \) |
|-----------------|-----------------|-----------------|
| Stocks, \( S_1 \) | 7.0% | 20% |
| LT Bonds, \( S_2 \) | 4.5% | 6% |
| ST bonds, \( S_3 \) | 2.0% | 2% |

<table>
<thead>
<tr>
<th>Stocks, ( S_1 )</th>
<th>LT Bonds, ( S_2 )</th>
<th>ST Bonds, ( S_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stocks, ( S_1 )</td>
<td>100%</td>
<td>-10%</td>
</tr>
<tr>
<td>LT Bonds, ( S_2 )</td>
<td>-10%</td>
<td>100%</td>
</tr>
<tr>
<td>ST bonds, ( S_3 )</td>
<td>-3%</td>
<td>85%</td>
</tr>
</tbody>
</table>
We assume that the asset manager maximizes his utility over a period of 15 years. The VAR at 95% is limited to \( R = 5\% \), for a time horizon of one month \( \Delta t_{\text{var}} = 1/12 \). The steps \( \Delta t \) and \( \Delta S \) are respectively set to 0.05 and 0.01. Two utility functions, denoted \( U_1 \) and \( U_2 \), are tested (see table 4). \( U_1 \) grants less importance to the maximization of the ratio than \( U_2 \).

**Table 4. Utility functions : parameters.**

<table>
<thead>
<tr>
<th></th>
<th>( U_1 )</th>
<th>( U_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight ( k )</td>
<td>10%</td>
<td>60%</td>
</tr>
<tr>
<td>Discount rate, ( \rho )</td>
<td>3%</td>
<td>3%</td>
</tr>
<tr>
<td>Target Solvency, ( TS )</td>
<td>104%</td>
<td>104%</td>
</tr>
</tbody>
</table>

The figure 6.1 presents the optimal proportion of the fund, invested in stocks with and without VAR restriction, for the utility function \( U_1 \). When the solvency ratio is smaller than the target, the optimal solution consists to increase the position in stocks. This enhances the expected return of the total asset \( A_t \) and accelerates on average the convergence of the solvency ratio towards the target solvency. The exposure to market risk is however limited by the VAR constraint which bounds the stocks percentage around 30\% of the fund. In the opposite situation, the fund manager should take a short position in stocks, limited to -20\% of stocks so as to reduce the solvency ratio. Nevertheless, this strategy is unrealistic but directly results from the quadratic form of the objective.

**Figure 6.1.** Part of the fund, invested in shares, utility \( U_1 \), with/without VAR.

The influence of the VAR on the optimal asset allocation is detailed in graphs, 6.2 and 6.3 at time \( t=10 \). We call “non speculative zone” the interval of solvency values for which the optimal investment policy involves only long positions. We observe that granting more importance to the maximization of the solvency moves this non speculative zone towards higher values of \( S_t \).
The figure 6.4 presents the values of the Lagrange multipliers for the two utility functions $U_1$ and $U_2$. As for the non speculative zone, we observe that giving more weight to the maximization of the solvency moves the zone of null multipliers towards higher values of $S_t$. 
Figure 6.4. Lagrangian multipliers, year 10, utilities $U_1, U_2$.

The next graph compares the value functions of the constrained and unconstrained problems, for the utility $U_1$.

Figure 6.5. Value function, utility $U_1$.

The value function of the constrained problem is identical to the unconstrained one when
the Lagrange multipliers are null whereas it is clearly inferior to the unconstrained value function when the VAR constraint is active (negative Lagrange multipliers).

The precision of the approximative solution is measured by the norm 2 of the $\epsilon^j_{t,s}$, defined by the equation (5.15). For the utility functions $U_1$, $U_2$ we obtain respectively a precision of $6.56e - 4$ and $1.70e - 3$, after only two iterations. The function $\hat{v} \in C^{2,1}$ is hence a good approximation of the optimal solution $v$.

7. Conclusions.

This paper proposes a numerical method to solve the optimal asset allocation problem under a value at risk constraint. As in our model all sources of risk are geometric Brownian motions, the number of state variables is reduced to one, the solvency ratio, which is both a key indicator to manage the fund and a tool of control used by regulators. In a first step, we detail a numerical method to determine the optimal investment policy maximizing the quadratic utility of the solvency ratio. The optimal asset allocation is precisely determined by a system of three ODE which are easily solved by discretization.

In a second step, we add a VAR constraint to our problem in order to limit the exposure to market and mortality risks. This constraint is inserted in the Bellman’s equation with a Lagrange multiplier. Based upon this equation, we propose an algorithm that calculates an approached solution to the constrained problem. The errors of approximation come from the fact that we apply to the VAR problem, the relation linking the optimal investment strategy to the value of the objective, established in the unconstrained problem (propositions 4.1 and 5.1). Examples of paragraph 6 clearly reveal the influence of the VAR constraint on the optimal allocation of assets: investments in risky assets are bounded, even if the fund has difficulties to remain solvent. We also observe that the constrained value function is identical to the unconstrained one in a certain area of the domain of resolution. This area is delimited by the VAR of the unconstrained optimal policy of investment.

The methodology presented in this work can probably be applied to a wider range of problem than the one developed in this paper. There is still interesting issues that motives further researches. For e.g. it can be interesting to insert a dividend policy, to control contributions or to link the target solvency ratio to the investment policy because the level of solvency required by the regulators depends on the exposure of the fund to the market risk.

8. Appendix.

Proof of proposition 3.1. We calculate the differential of $1/R_t$ by application of the Ito’s lemma:

\[
(8.1) \quad d \left( \frac{1}{R_t} \right) = \left( - (r.R_t - n_t.l_t) \cdot \frac{1}{R_t^2} + \frac{1}{2} \cdot \sigma_{\mu}(x + t)^2 \cdot R_t^2 \cdot \frac{2}{R_t^3} \right) dt - \sigma_{\mu}(x + t) \cdot R_t \cdot \frac{1}{R_t^2} dW_t^L
\]
And by definition of $R_t$, we obtain that:

$$
d\left(\frac{1}{R_t}\right) = \left( - \left( r - \int_t^{T_m} l_s e^{-r(s-t)} s - t p_{x+t} ds \right) \frac{1}{R_t} + \sigma_{\mu}(x + t)^2 \frac{1}{R_t} \right) dt
$$

(8.2)

The differential of the solvency ratio, is then:

$$
d\left(\frac{1}{R_t}\right) = A_t d\left(\frac{1}{R_t}\right) + \frac{1}{R_t} dA_t
$$

(8.3)

$$
dS_t = \left( (-r + m^\top \Pi_t + G_t + \sigma_{\mu}(x + t)^2) S_t - G_t \right) dt + S_t \left( \Pi_t^\top \sigma dW_t^F - \sigma_{\mu}(x + t) dW_t^L \right)
$$

(8.4)

The number of Brownian motions is reduced as follows:

$$
\left( \Pi_t^\top \sigma dW_t^F - \sigma_{\mu}(x + t) dW_t^L \right) \sim N(0, \sqrt{(\Pi_t^\top \Sigma \Pi_t + \sigma_{\mu}(x + t)^2)/dt})
$$

(8.5)

$$
\left( \Pi_t^\top \sigma dW_t^F - \sigma_{\mu}(x + t) dW_t^L \right) = \sqrt{(\Pi_t^\top \Sigma \Pi_t + \sigma_{\mu}(x + t)^2)} dW_t^S
$$

(8.6)

Where $dW_t^S$ is a Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ that summarizes the stochastic behaviour of assets and liabilities. We finally get that:

$$
dS_t = \left( (-r + m^\top \Pi_t + G_t + \sigma_{\mu}(x + t)^2) S_t - G_t \right) dt + S_t \sqrt{(\Pi_t^\top \Sigma \Pi_t + \sigma_{\mu}(x + t)^2)} dW_t^S
$$

(8.7)

□

**Proof of proposition 4.1** Given the optimal strategy of investment $\Pi_t^{opt} = (M_1.\pi_t^{opt} + M_2)$, by definition of $g_t$, we have that:

$$
\Pi_t^{opt} = -\frac{1}{S_t} (S_t P_1 m + g_t) - P_2
$$

(8.8)

Notice that $g_t$ is only a function of time (see equation (4.21)). And if we assume that the value function $v(S_t, t)$ is a quadratic one:

$$
v(S_t, t) = a(t) S^2 + b(t) S + c(t)
$$
The Bellman’s equation becomes:

\[ 0 = a'(t).S_t^2 + b'(t).S_t + c'(t) - \rho.a(t).S_t^2 - \rho.b(t).S_t - \rho.c(t) + U(S_t) + \left( \left( -r + m^\top \left( -\frac{1}{S_t} (S_t.P_1.m + g_t) - P_2 \right) + G_t + \sigma_\mu(x + t)^2 \right) .S_t - G_t \right) .(2.a(t).S_t + b(t)) + \frac{1}{2} S_t^2 . \left( \left( -\frac{1}{S_t} (S_t.P_1.m + g_t) - P_2 \right) ^\top .\Sigma . \left( -\frac{1}{S_t} (S_t.P_1.m + g_t) - P_2 \right) + \sigma_\mu(x + t)^2 \right) .2.a(t) \]

Regrouping the terms in \( S^2, S \) and independent of \( S \), gives us the three differential equations (4.17) (4.18) (4.19)□

REFERENCES