MOMENT BOUNDS ON DISCRETE EXPECTED 
SHORTFALLS, WITH APPLICATIONS

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Abstract
This paper shows how to make the best possible use of the information contained in the 
first few moments (mean, variance and skewness, say) of an integer-valued random variable 
when one is interested in expected shortfalls. This allows to bound various quantities in 
applied probability, including the ruin probabilities, for instance.

Keywords : increasing convex order, stop-loss transform, insurance.

1 Introduction

Given a random variable $X$, many papers have been devoted to the derivation of lower 
and upper bounds on quantities of the form $\mathbb{E}[g(X)]$, for some given measurable function $g$, 
when $X$ belongs to a class of random variables satisfying certain moment conditions; 
see, e.g., Denuit, De Vijlder & Lefèvre (1999) for a review. It is demonstrated there 
that the solution to these problems was given, in most cases, by atomic random variables 
with a similar structure.

The majority of the papers in this vein are devoted to random variables with support in a specified interval. Fewer papers deal with structured supports, as the set of the integers for instance. Discrete random variables (especially integer-valued ones) are often encountered in applied probability problems. Taking the particular form of the support into account allows for more accurate bounds, as demonstrated by Hürlimann (2005) in the determination of the gambler’s ruin probability.

Extrema with respect to the discrete version of the $s$-convex orders are useful for deriving bounds on $\mathbb{E}[g(X)]$ when $g$ is $s$-convex and $X$ is valued in $\mathcal{D}_n = \{0, 1, \ldots, n\}$ (or more generally, in an arbitrary grid of points). Recall that $g : \mathcal{D}_n \to \mathbb{R}$ is $s$-convex if its

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Denuit, Lefèvre & Van Bellegem (2006) and the references therein. However, in many applications, extremal distributions with respect to the stronger convex order (that is, with respect to the 2-convex order) are in fact needed. Only this stochastic order relation allows to get bounds on ruin probabilities or expected shortfalls, for instance. It is well-known that two random variables with equal variances cannot be ordered with respect to the convex order. Therefore, we derive in this paper stochastic bounds with respect to the increasing convex order. In that respect, the present study extends the results obtained by Jansen, Haezendonck & Goovaerts (1986) to discrete random variables.

Expected shortfalls are defined for a random variable $X$ as $E[(X - d)_+]$ where $z_+ = \max\{z, 0\}$ is the positive part of $z$. Viewing $X$ as a financial loss, $(X - d)_+$ is the shortfall or excess of loss over the threshold $d$. This quantity is widely used in actuarial science and mathematical finance. Now, having two random variables $X$ and $Y$, if the inequality $E[(X - d)_+] \leq E[(Y - d)_+]$ holds true for any threshold $d$ then $X$ is said to be smaller than $Y$ in the increasing convex order, written as $X \preceq_{icx} Y$.

The paper is organized as follows. Section 2 aims to derive the supremum with respect to $\preceq_{icx}$ when the mean and the variance are known. Specifically, we consider random variables with discrete support $\mathcal{D}_n = \{0, 1, \ldots, n\}$, and the class $\mathcal{M}_3(\mathcal{D}_n; \mu_1, \mu_2)$ of all random variables with support in $\mathcal{D}_n$ and first two moments $\mu_1$ and $\mu_2$. To ensure that $\mathcal{M}_3(\mathcal{D}_n; \mu_1, \mu_2)$ is not void, we assume that

$$0 < \mu_1 < n \text{ and } \mu_1^2 < \mu_2 < \mu_1 n.$$  

Using the knowledge of $(\mu_1, \mu_2, n)$, the aim of Subsections 2.2-2.3 is to find upper and lower bounds on $E[(X - d)_+]$ valid for any $X \in \mathcal{M}_3(\mathcal{D}_n; \mu_1, \mu_2)$. To this end, we need some preliminary technical results summarized in the Subsection 2.1. In Subsection 2.4, we will see that these results also allow to determine the random variables $X_{\min}$ and $X_{\max}$ such that the stochastic inequalities $X_{\min} \preceq_{icx} X \preceq_{icx} X_{\max}$ hold for all $X \in \mathcal{M}_3(\mathcal{D}_n; \mu_1, \mu_2)$. Note that $X_{\min}$ and $X_{\max}$ do not belong to $\mathcal{M}_3(\mathcal{D}_n; \mu_1, \mu_2)$ but have the same mean.

Let $\mathcal{M}_4(\mathcal{D}_n; \mu_1, \mu_2, \mu_3)$ be the class of all random variables with support in $\mathcal{D}_n$ and first three moments $\mu_1$, $\mu_2$ and $\mu_3$. We assume that $\mu_1$, $\mu_2$ and $\mu_3$ are such that $0 < \mu_1 < n$, $\mu_1^2 < \mu_2 < n\mu_1$ and $\frac{\mu_3}{\mu_2} < \frac{\mu_3}{\mu_2} - \frac{(n\mu_1 - \mu_2)^2}{n - \mu_1^2}$. This ensures that $\mathcal{M}_4(\mathcal{D}_n; \mu_1, \mu_2, \mu_3)$ is not void. In Section 3, we want to find $X_{\min}$ and $X_{\max}$ such that the stochastic inequalities $X_{\min} \preceq_{icx} X \preceq_{icx} X_{\max}$ hold true for any $X \in \mathcal{M}_4(\mathcal{D}_n; \mu_1, \mu_2, \mu_3)$. The structure of Section 3 parallels that of Section 2.

Section 4 discusses two applications. First, we consider the random number of survivors in a given cohort when the forces of mortality obey to the Lee-Carter model. This model accounts for the mortality improvements that pose a challenge for the planning of public retirement systems as well as for the private life annuities business. Specifically, let $N$ be the number of survivors at time $t_0 + d$ from an initial group of $n$ policyholders aged $x_0$ at time $t_0$. In the Lee-Carter model, $N$ obeys to the a mixture of Binomial distributions.
Natural candidates for defining the benefits of longevity bonds or reinsurance treaties covering portfolios of life annuities involve the excess of the actual number of survivors to that expected from a reference life table. If the expected number of survivors at time $t_0 + d$ is $\overline{n}_d$ then the payoff could be related to $(N - \overline{n}_d^+)$. Bounds on $\mathbb{E}[(N - \overline{n}_d^+)]$ are then obtained from the theory developed in Sections 2-3.

In Subsection 4.2, bounds for the eventual probability of ruin in the compound Binomial risk process are derived. In this model, time is measured in discrete time units $t = 0, 1, 2, \ldots$. The number of insured claims is governed by a Binomial process (i.e. in any time period, there occurs 1 or 0 claim, and occurrence of claims in different time intervals are independent events). The claim amounts are independent and identically distributed, valued in $\{1, 2, 3, \ldots\}$, and independent of the Binomial process describing claim occurrence. The premium received in each period is 1 and is assumed to be larger than the net premium. The initial risk reserve is a non-negative integer amount. The probability of eventual ruin is defined as the probability that the surplus of the insurance company ever falls below 0. Numericall illustrations are proposed, that demonstrate the usefulness of the bounds derived in this paper.

To end with, having a random variable $X$ valued in $\mathcal{D}_n$, let us point out that we only have to consider integer values of $d$ to get bounds on $\mathbb{E}[(X - t)^+]$ for any $t \in [0, n]$. Indeed, if $d \leq t < d + 1$, it is easily seen that

$$\mathbb{E}[(X - t)^+] = \mathbb{E}[(X - d)^+] - (t - d)(1 - F_X(d)).$$

Now, since $F_X(d) = 1 + \mathbb{E}[(X - d - 1)^+] - \mathbb{E}[(X - d)^+]$, we obtain

$$\mathbb{E}[(X - t)^+] = \mathbb{E}[(X - d)^+] (1 - (t - d)) + (t - d)\mathbb{E}[(X - d - 1)^+].$$

Henceforth, $d$ will always be thought of as being an integer in $\mathcal{D}_n$.

# 2 Mean-variance bounds on the expected shortfalls

## 2.1 Preliminary results

Define the function

$$\phi_1 : [0, n] \setminus \{\mu_1\} \rightarrow [0, n]
\quad x \mapsto \phi_1(x) = \frac{\mu_2 - \mu_1 x}{\mu_1 - x} = \mu_1 + \frac{\mu_2 - \mu_1^2}{\mu_1 - x}.$$ 

This function is strictly increasing in $x$ from $[0, \phi_1(n)]$ to $[\phi_1(0), n]$ and satisfies $\phi_1(\phi_1(x)) = x$.

Triatomic random variables belonging to $\mathcal{M}_3(\mathcal{D}_n; \mu_1, \mu_2)$ will play a central role in the derivation of extrema. The form of the support of such variables is determined with the help of the function $\phi_1$, as shown in the following result (whose direct proof is omitted).
Proposition 2.1. A triatomic random variable \( X \in \mathcal{M}_3(D_n; \mu_1, \mu_2) \) with ordered support \( \{k_1, k_2, k_3\} \) (\( 0 \leq k_1 < k_2 < k_3 \leq n \)) must satisfy the conditions
\[
 k_2 \leq \phi_1(k_1) \leq k_3 \quad \text{and} \quad k_1 \leq \phi_1(k_3) \leq k_2.
\]
Moreover the weights of \( k_1, k_2, \) and \( k_3 \) are respectively \( p_{k_1} = (\frac{\mu_2 - \mu_1 + k_2 + k_3}{(k_2 - k_1)(k_3 - k_2)}) \) and \( p_{k_2} = (\frac{-\mu_2 + \mu_1 + k_2 - k_3}{(k_2 - k_1)(k_3 - k_2)}) \) and \( p_{k_3} = (\frac{-\mu_2 - \mu_1 + k_2 + k_3}{(k_2 - k_1)(k_3 - k_2)}) \).

2.2 Upper bound on the expected shortfall when two moments are known

Let \( z_1, z_2, w_1, \) and \( w_2 \) be the integers such that \( z_1 < \phi_1(0) = \frac{\mu_2}{\mu_1} \leq z_2 = z_1 + 1 \) and \( w_1 < \phi_1(n) = \frac{\mu_2 - \mu_1 n}{\mu_1 - n} \leq w_2 = w_1 + 1 \). Let also \( f, g, h \) be three functions valued in \( \{0, \ldots, w_1\} \cup \{z_2, \ldots, n\} \) defined by \( f(x) = \frac{x^2 - y(y + 1)}{2x - (y + y + 1)} \), \( g(x) = \frac{x^2 + y + 1}{2} \) and \( h(x) = \frac{x^2 + y}{2} \) where \( y < \phi_1(x) \leq y + 1 \).

The following result extends Theorem 2 in Jansen et al. (1986) to random variables valued in \( D_n \).

Proposition 2.2. For any \( X \in \mathcal{M}_3(D_n; \mu_1, \mu_2) \), the following inequalities hold true:

Case 1 For any integer \( d \) such that \( 0 \leq d \leq f(0) \)
\[
\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(X_{1,\max}^{(2)} - d)_+] = p_{z_1}(z_1 - d) + p_{z_1+1}(z_1 + 1 - d)
\]
where \( X_{1,\max}^{(2)} \) has support \( \{0, z_1, z_1 + 1\} \) and weights \( p_0 = 1 - p_{z_1} - p_{z_1+1}, \) \( p_{z_1} = \frac{-\mu_2 + \mu_1(z_1+1)}{z_1} \) and \( p_{z_1+1} = \frac{\mu_2 - \mu_1 z_1}{z_1+1} \).

Case 2 For any integer \( d \) such that \( d \in [f(0), g(0)] \cup \bigcup_{x \in \{1, \ldots, w_1\}} \{h(x), g(x)\} \)
\[
\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(X_{2,\max}^{(2)} - d)_+] = p_{u_2}(u_2 - d) + p_{u_2+1}(u_2 + 1 - d)
\]
where \( X_{2,\max}^{(2)} \) has support \( \{u_1, u_2, u_2 + 1\} \) \( (u_2 < \phi_1(u_1) \leq u_2 + 1 \) and \( u_1 \in \{1, \ldots, w_1\} \) is such that \( d \in [h(u_1), g(u_1)] \) and \( u_1 = 0 \) if \( d \in [f(0), g(0)] \) and weights \( p_{u_1} = 1 - p_{u_2} - p_{u_2+1}, \) \( p_{u_2} = \frac{-\mu_2 + \mu_1(u_1+u_2+1)-u_1(u_2+1)}{u_2-u_1} \) and \( p_{u_2+1} = \frac{\mu_2 - \mu_1(u_1+u_2)+u_1 u_2}{u_2+1-u_1} \).

Case 3 For any integer \( d \) such that \( d \in \bigcup_{x \in \{z_2, \ldots, n-1\}} \{h(x), g(x)\} \cup \{h(n), f(n)\} \)
\[
\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(X_{3,\max}^{(2)} - d)_+] = p_{v_1}(v_1 - d)
\]
where \( X_{3,\max}^{(2)} \) has support \( \{v_2, v_2 + 1, v_1\} \) \( (v_2 < \phi_1(v_1) \leq v_2 + 1 \) and \( v_1 \in \{z_2, \ldots, n-1\} \) is such that \( d \in [h(v_1), g(v_1)] \) and \( v_1 = n \) if \( d \in [h(n), f(n)] \) and weights \( p_{v_1} = 1 - p_{v_2} - p_{v_2+1}, \) \( p_{v_2} = \frac{-\mu_2 + \mu_1(v_2+v_1+1)-v_1(v_2+1)}{v_1-v_2} \) and \( p_{v_2+1} = \frac{\mu_2 - \mu_1(v_2-v_1)-v_1 v_2}{v_1-v_2-1} \).
Case 4 For any integer \( d \) such that \( f(n) \leq d \leq n \)

\[
\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(X^{(2)}_{1,\max} - d)_+] = p_n(n - d)
\]

where \( X^{(2)}_{1,\max} \) has support \( \{w_1, w_1 + 1, n\} \) and weights \( p_{w_1} = \frac{\mu_2 - \mu_1(w_1 + 1) + (w_1 + 1)n}{n-w_1} \), \( p_{w_1 + 1} = \frac{-\mu_2 + \mu_1(w_1 + 1) - w_1n}{n-w_1+1} \) and \( p_n = 1 - p_{w_1} - p_{w_1+1} \).

**Proof.** The proof is based on the existence of a quadratic polynomial \( p_2(x) = c_0 + c_1x + c_2x^2 \)

such that \( p_2(x) \geq (x - d)_+ \) for all \( x \in \mathcal{D}_n \), with an equality holding at the points of the support of a discrete distribution on \( \mathcal{D}_n \), with the given moments \( \mu_1 \) and \( \mu_2 \). Taking expectation then leaves an upper bound for \( \mathbb{E}[(X - d)_+] \) when \( X \in \mathcal{M}_3(\mathcal{D}_n; \mu_1, \mu_2) \). As \( p_2(x) - (x - d)_+ = c_0 + c_1x + c_2x^2 \) on \( (-\infty, d] \) (quadratic polynomial) and \( p_2(x) - (x - d)_+ = (c_0 + d) + (c_1 - 1)x + c_2x^2 \) on \( [d, +\infty) \) (quadratic polynomial), we have two crossing points on \( (-\infty, d] \) (denoted \( k_1 \) and \( k_2 \)) and two on \( [d, +\infty) \) (denoted \( k_3 \) and \( k_4 \)) with at least three of these points being in \( \mathcal{D}_n \) (cf. Proposition 2.1). The admissible quadratic polynomials are thus in one of the four following types:

- **Case 1:** \( k_1 = y \in (-\infty, 0), k_2 = 0, k_3 = z_1 \in \mathcal{D}_n \) and \( k_4 = z_2 = k_3 + 1 \),

- **Case 2:** \( |k_2 - k_1| \leq 1 \) with \( k_1 \) and/or \( k_2 \) \( \in \mathcal{D}_n \), \( k_3 = 0 \in \mathcal{D}_n \) and \( k_4 = k_3 + 1 \),

- **Case 3:** \( k_1 \in \mathcal{D}_n, k_2 = k_1 + 1 \) and \( |k_4 - k_3| \leq 1 \) with \( k_3 \) and/or \( k_4 \in \mathcal{D}_n \),

- **Case 4:** \( k_1 = w_1 \in \mathcal{D}_n, k_2 = w_2 = k_1 + 1, k_3 = n \) and \( k_4 = y \in (n, +\infty) \).

Let us now discuss the different cases listed above.

**Case 1.** Note that taking a random variable \( X_{1,\max}^{(2)} \) with moments \( \mu_1 \) and \( \mu_2 \) and with support \( \{0, z_1, z_2\} \) \( (z_2 = z_1 + 1) \) implies \( z_1 < \phi_1(0) \leq z_2 \) (see Proposition 2.1). The polynomial we are looking for takes values \( p_2(0) = 0, p_2(z_1) = z_1 - d \) and \( p_2(z_2) = z_2 - d \) so that

\[
p_2(x) = \frac{x(x - z_2)}{z_1(z_1 - z_2)}(z_1 - d) + \frac{x(x - z_1)}{z_2(z_2 - z_1)}(z_2 - d) = \frac{x(x - z_1)}{z_1(z_1 + 1)}(z_1 + 1 - d) - \frac{x(x - z_1 - 1)}{z_1}(z_1 - d).
\]

Let \( y \) be the negative root of \( p_2(x) \) (the other being 0) satisfying

\[
d = \frac{z_1 z_2}{z_1 + z_2 - y}.
\]

As \( y \) tends to \( -\infty \) and to 0, \( d \) tends respectively to 0 and \( \frac{z_1 z_2}{z_1 + z_2} \); which gives the interval of variation of \( d \) in Case 1. Taking \( X_{1,\max}^{(2)} \) with support \( \{0, z_1, z_2\} \) and moments \( \mu_1 \) and \( \mu_2 \), we have

\[
\mathbb{E}[p_2(X_{1,\max}^{(2)})] = \mathbb{E}[(X_{1,\max}^{(2)} - d)_+] = p_0 \cdot 0 + p_{z_1} \cdot (z_1 - d) + p_{z_2} \cdot (z_2 - d)
\]

where \( p_{z_1} = \frac{-\mu_2 + \mu_1(z_1 + 1)}{z_1} \) and \( p_{z_2} = \frac{\mu_2 - \mu_1 z_1}{z_2} \) (see Proposition 2.1).
Case 2. Let $k_1$ be in $\{0, 1, \ldots, w_1\}$ where $w_1 < \phi_1(n) \leq w_1 + 1$ (i.e. $k_1$ is an integer less or equal to $\phi_1(n)$). Let also $k_1 - 1 \leq y \leq k_1 + 1$ if $k_1 \in \{1, \ldots, w_1\}$ and $k_1 \leq y \leq k_1 + 1$ if $k_1 = 0$. Taking a random variable $X_{2,\text{max}}^{(2)}$ with moments $\mu_1$ and $\mu_2$ and support $\{k_1, k_3, k_4\}$ ($k_4 = k_3 + 1$) and considering Proposition 2.1 implies $k_3 < \phi_1(k_1) \leq k_3$.

Now, the desired polynomial takes values $p_2(k_1) = 0$, $p_2(k_3) = k_3 - d$ and $p_2(k_4) = k_4 - d$ so that

$$p_2(x) = \frac{(x - k_1)(x - k_3)}{k_4 - k_1}(k_4 - d) - \frac{(x - k_1)(x - k_4)}{k_3 - k_1}(k_3 - d).$$

Expressing the fact that $y$ (possibly non integer) is one of the two roots of $p_2(x)$ (the other one being $k_1$), we get

$$d = \frac{k_1 y - k_3 k_4}{k_1 + y - (k_3 + k_4)}$$

which is a strictly increasing function of $y$. Consequently, as $\lim_{y \to k_1} d(y) = \frac{k_1 + k_4}{2} = h(k_1)$ ($k_1 \in \{1, \ldots, w_1\}$), $\lim_{y \to 0} d(y) = f(0)$ and $\lim_{y \to k_1 + 1} d(y) = \frac{k_1 + k_4 + 1}{2} = g(k_1)$ ($k_1 \in \{0, 1, \ldots, w_1\}$), we have

$$d \in [f(0), g(0)] \cup \bigcup_{x \in \{1, \ldots, w_1\}} [h(x), g(x)].$$

Case 3. This case is strictly parallel to Case 2. Let $k_3$ be in $\{z_2, z_2 + 1, \ldots, n\}$ where $z_1 < \phi_1(0) \leq z_2 = z_1 + 1$ (i.e. $k_3$ is an integer greater or equal to $\phi_1(0)$). Let also $k_3 - 1 \leq y \leq k_3 + 1$ if $k_3 \in \{z_2, z_2 + 1, \ldots, n - 1\}$ and $k_3 - 1 \leq y \leq k_3$ if $k_3 = n$. Taking a random variable $X_{3,\text{max}}^{(2)}$ with moments $\mu_1$ and $\mu_2$ and support $\{k_1, k_2, k_3\}$ ($k_2 = k_1 + 1$) and considering Proposition 2.1 implies $k_1 < \phi_1(k_3) \leq k_2$.

Now, the desired polynomial takes values $p_2(k_1) = 0$, $p_2(k_2) = 0$ and $p_2(k_3) = k_3 - d$ so that

$$p_2(x) = \frac{(x - k_1)(x - k_2)}{k_3 - k_1}(k_3 - d).$$

Expressing the fact that $p_2(y) = y - d$ (possibly non integer), we get

$$d = \frac{k_3 y - k_1 k_2}{k_3 + y - (k_1 + k_2)}$$

which is a strictly increasing function of $y$. Consequently, as $\lim_{y \to k_3 - 1} d(y) = \frac{k_1 + k_2}{2} = h(k_3)$ ($k_3 \in \{z_2, \ldots, n\}$), $\lim_{y \to k_3 + 1} d(y) = \frac{k_1 + k_2 + 1}{2} = g(k_3)$ ($k_3 \in \{z_2, \ldots, n - 1\}$) and $\lim_{y \to n} d(y) = f(n)$, we have

$$d \in \bigcup_{x \in \{z_2, \ldots, n - 1\}} [h(x), g(x)] \cup [h(n), f(n)].$$

Case 4. Let $y \in (n, +\infty)$. We proceed as in Case 1. Taking a random variable $X_{4,\text{max}}^{(2)}$ with moments $\mu_1$ and $\mu_2$ and support $\{w_1, w_2, n\}$ ($w_2 = w_1 + 1$) and considering always Proposition 2.1 implies $w_1 < \phi_1(n) \leq w_2$. 

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This time, the polynomial have to contain values \( p_2(w_1) = 0 \), \( p_2(w_2) = 0 \) and \( p_2(n) = n - d \) in \( w_1 \), \( x_2 \) and \( n \). We thus get
\[
p_2(x) = \frac{(x - w_1)(x - w_2)}{(n - w_1)(n - w_2)}(n - d).
\]
Expressing the fact that \( p_2(y) \) is equal to \( y - d \), it comes
\[
d = \frac{ny - w_1w_2}{n + y - (w_1 + w_2)}
\]
which is a strictly increasing function of \( y \). As \( y \) tends to \(+\infty\) and to \( n \), \( d \) tends respectively to \( n \) and \( \frac{n^2 - w_1w_2}{2n - (w_1 + w_2)} \); which gives the interval of variation of \( d \) in Case 4.

Now to obtain the maximum stop-loss premium, we take a random variable \( X^{(2)}_{4,\text{max}} \) with support equal to \( \{w_1, w_2, n\} \) and with moments \( \mu_1, \mu_2 \). The maximum is equal to
\[
\mathbb{E}[p_2(X^{(2)}_{4,\text{max}})] = p_n(n - d).
\]

\[\square\]

2.3 Lower bound on the expected shortfall when two moments are known

Following the same lines as for the proof of Proposition 2.2, we get the following result that allows to find lower bounds on the expected shortfall for any \( X \in \mathcal{M}_3(\mathcal{D}_n; \mu_1, \mu_2) \).

**Proposition 2.3.** Let \( d_f \) be the integer such that \( d_f < \phi_1(d) \leq d_f + 1 \). For any \( X \in \mathcal{M}_3(\mathcal{D}_n; \mu_1, \mu_2) \), the following inequalities hold true:

**Case 1** For any integer \( d \) such that \( 0 \leq d \leq \phi_1(n) \)
\[
\mathbb{E}[(X - d)_+] \geq \mathbb{E}[(X_{1,\text{min}}^{(2)} - d)_+] = \mu_1 - d
\]
where \( X_{1,\text{min}}^{(2)} \) has support \( \{d, d_f, d_f+1\} \) and weights \( p_d, p_{d_f} \) and \( p_{d_f+1} \) as in Proposition 2.1.

**Case 2** For any integer \( d \) such that \( \phi_1(n) \leq d \leq \phi_1(0) \)
\[
\mathbb{E}[(X - d)_+] \geq \mathbb{E}[(X_{2,\text{min}}^{(2)} - d)_+] = \frac{\mu_2 - \mu_1 d}{n}
\]
where \( X_{2,\text{min}}^{(2)} \) has support \( \{0, d, n\} \) and weights \( p_0, p_d \) and \( p_n \) as in Proposition 2.1.

**Case 3** For any integer \( d \) such that \( \phi_1(0) \leq d \leq n \)
\[
\mathbb{E}[(X - d)_+] \geq \mathbb{E}[(X_{3,\text{min}}^{(2)} - d)_+] = 0
\]
where \( X_{3,\text{min}}^{(2)} \) has support \( \{d_f, d_f+1, d\} \) and weights \( p_{d_f}, p_{d_f+1} \) and \( p_d \) as in Proposition 2.1.
2.4 \( \preceq_{icx} \) extremal distributions when two moments are known

The bounds on the expected shortfalls are obtained from distributions depending on the threshold \( d \). It is nevertheless possible to obtain extremal distributions in the \( \preceq_{icx} \)-sense.

The maximum distribution \( F_{max}^{(2)} \) is easily obtained from the first order forward difference operator of the function \( \Pi_{max}^{(2)} \), given in Proposition 2.2. Precisely, \( \Pi_{max}^{(2)}(d) \) is equal to \( E[(X_{1,\max}^{(2)} - d)_+] \), \( E[(X_{2,\max}^{(2)} - d)_+] \), \( E[(X_{3,\max}^{(2)} - d)_+] \) or \( E[(X_{4,\max}^{(2)} - d)_+] \) according to the value of \( d \) and

\[
F_{max}^{(2)}(d) = 1 + \Pi_{max}^{(2)}(d + 1) - \Pi_{max}^{(2)}(d).
\]

In fact, knowing that the floor function \( \lfloor \cdot \rfloor \) is piecewise constant, it is easily seen that \( \Pi_{max}^{(2)}(d) \) is piecewise linear and non-increasing in \( d \). Moreover \( \lim_{d \to 0}(\Pi_{max}^{(2)}(d) + d) = \mu_1 \) and \( \lim_{d \to 0}(\Pi_{max}^{(2)}(d)) = 0 \). Consequently, the maximal distribution with respect to \( \preceq_{icx} \) when two moments are known is given in the following result.

**Proposition 2.4.** Let us take the same notations as in Proposition 2.2. Let us furthermore define \( \Pi_{k,\max}^{(2)}(x) = E[(X_{k,\max}^{(2)} - x)_+] \) \( (k = 1, 2, 3, 4) \) and denote \( I_1 = \{f(0), g(0)\} \), \( I_2 = \bigcup_{x \in \{1, \ldots, n\}} [h(x), g(x)) \), \( I_3 = \bigcup_{x \in \{z_2, \ldots, n-1\}} [h(x), g(x)) \) and \( I_4 = \{f(n), g(n)\} \). The maximal distribution \( F_{max}^{(2)} \) with respect to \( \preceq_{icx} \) is

\[
F_{max}^{(2)}(x) = \begin{cases} 
p_0 & 	ext{for } x = 0, 1, \ldots, [f(0)]; \\
1 + \Pi_{2,\max}^{(2)}(x + 1) - \Pi_{2,\max}^{(2)}(x) & \text{for } x \in I_1 \cup I_2 \text{ such that } x + 1 \in I_2; \\
1 + \Pi_{3,\max}^{(2)}(x + 1) - \Pi_{3,\max}^{(2)}(x) & \text{for } x \in I_1 \cup I_2 \text{ such that } x + 1 \in I_3 \cup I_4; \\
1 + \Pi_{3,\max}^{(2)}(x + 1) - \Pi_{3,\max}^{(2)}(x) & \text{for } x \in I_3 \text{ such that } x + 1 \in I_2; \\
1 + \Pi_{3,\max}^{(2)}(x + 1) - \Pi_{3,\max}^{(2)}(x) & \text{for } x \in I_3 \text{ such that } x + 1 \in I_3 \cup I_4; \\
1 - p_n & \text{for } x = [f(n)], \ldots, n-1; \\
1 & \text{for } x = n. 
\end{cases}
\]

The same argument can be used with regard to Proposition 2.3 to derive the minimum distribution \( F_{min}^{(2)} \) and we get

**Proposition 2.5.** Let us take the same notations as in Propositions 2.2 and 2.3. The minimal distribution \( F_{min}^{(2)} \) with respect to \( \preceq_{icx} \) is

\[
F_{min}^{(2)}(x) = \begin{cases} 
0 & \text{for } x = 0, 1, \ldots, w_1 - 1; \\
1 + \frac{\mu_2 - \mu_1 (n+1) + w_1 (n-\mu_1)}{n} & \text{for } x = w_1; \\
1 - \frac{\mu_1}{n} & \text{for } x = w_2, w_2 + 1, \ldots, z_1 - 1; \\
1 - \frac{\mu_2 - \mu_1 z_1}{n} & \text{for } x = z_1; \\
1 & \text{for } x = z_2, z_2 + 1, \ldots, n. 
\end{cases}
\]

Note that the extremal random variables with respective distribution functions \( F_{min}^{(2)} \) and \( F_{max}^{(2)} \) have the same mean \( \mu_1 \).
3 Mean-variance-skewness bounds on the expected shortfalls

3.1 Preliminary results

Define the function

\[ \phi_2(x, y) = \frac{\mu_3 - \mu_2(x + y) + xy\mu_1}{\mu_2 - \mu_1(x + y) + xy}. \]

Note in particular that, for every \( x, y \) such that \( x < c \) and \( y > \phi_1(c) \), the following inequalities hold

\[ x < c < \phi_1(y) < \phi_2(x, y) < \phi_1(x) < \phi_1(c) < y. \]

The solution to the problem under investigation will be given by random variables with support consisting of 4 points in \( D_n \). The following result describes the structure of such elements in \( \mathcal{M}_4(D_n; \mu_1, \mu_2, \mu_3) \).

**Proposition 3.1.** A four-atomic random variable \( X \in \mathcal{M}_4(D_n; \mu_1, \mu_2, \mu_3) \) with ordered support \( \{k_1, k_2, k_3, k_4\} \) (\( 0 \leq k_1 < k_2 < k_3 < k_4 \leq n \)) must satisfy the conditions

\[
\begin{align*}
\mu_3 - \mu_2(k_2 + k_3 + k_4) + \mu_1(k_2k_3 + k_2k_4 + k_3k_4) - k_2k_3k_4 & \leq 0, \\
\mu_3 - \mu_2(k_1 + k_3 + k_4) + \mu_1(k_1k_3 + k_1k_4 + k_3k_4) - k_1k_3k_4 & \geq 0, \\
\mu_3 - \mu_2(k_1 + k_2 + k_4) + \mu_1(k_1k_2 + k_1k_4 + k_2k_4) - k_1k_2k_4 & \leq 0, \\
\mu_3 - \mu_2(k_1 + k_2 + k_3) + \mu_1(k_1k_2 + k_1k_3 + k_2k_3) - k_1k_2k_3 & \geq 0.
\end{align*}
\]

Moreover the weights of \( k_1, k_2, k_3 \) and \( k_4 \) are respectively

\[ p_{k_1} = \frac{\mu_3 - \mu_2(k_2 + k_3 + k_4) + \mu_1(k_2k_3 + k_2k_4 + k_3k_4) - k_2k_3k_4}{(k_1 - k_2)(k_2 - k_3)(k_3 - k_4)}, \]

\[ p_{k_2} = \frac{\mu_3 - \mu_2(k_1 + k_3 + k_4) + \mu_1(k_1k_3 + k_1k_4 + k_3k_4) - k_1k_3k_4}{(k_1 - k_2)(k_2 - k_3)(k_3 - k_4)}, \]

and

\[ p_{k_3} = \frac{\mu_3 - \mu_2(k_1 + k_2 + k_4) + \mu_1(k_1k_2 + k_1k_4 + k_2k_4) - k_1k_2k_4}{(k_4 - k_1)(k_4 - k_2)(k_4 - k_3)}, \]

\[ p_{k_4} = \frac{\mu_3 - \mu_2(k_1 + k_2 + k_3) + \mu_1(k_1k_2 + k_1k_3 + k_2k_3) - k_1k_2k_3}{(k_4 - k_1)(k_4 - k_2)(k_4 - k_3)}. \]

3.2 Upper bound on the expected shortfalls when three moments are known

Let \( y_1, y_2, c_1, c_2, d_1 \) and \( d_2 \) be the integers such that \( c_1 < c \leq c_2 = c_1 + 1, d_1 < \phi_1(c) \leq d_2 = d_1 + 1 \) and \( y_1 < \phi_2(0, n) \leq y_2 = y_1 + 1 \). Let also \( f_1, g_1, h_1, g_2, h_2, g_3, h_3, f_4, g_4 \) and \( h_4 \) be ten functions defined by

\[ f_1(x) = \frac{(y - x)(y + 1 - x)^2(n - x)^2(n - y - 1) - (y - x)^2(y + 1 - x)(n - x)^2(n - y)}{(y + 1 - x)^2(n - x)^2(n - y - 1) - (y - x)^2(n - x)^2(n - y)} \]

\[ + \frac{(n - x)(y - x)^2(y + 1 - x)^2}{(y + 1 - x)^2(n - x)^2(n - y - 1) - (y - x)^2(n - x)^2(n - y)} \]
where \( y < \phi \) and \( z < \phi \).

The following result extends Theorem 3 in Jansen et al. (1986) to the discrete case.
Proposition 3.2. Let $y_1, y_2, c_1, c_2, d_1$ and $d_2$ be the integers such that $c_1 < c \leq c_2 = c_1 + 1$, $d_1 < \phi_1(c) \leq d_2 = d_1 + 1$ and $y_1 < \phi_2(0, n) \leq y_2 = y_1 + 1$. For any $X \in \mathcal{M}_4(\mathcal{D}_n; \mu_1, \mu_2, \mu_3)$, the following inequalities hold true:

Case 1 For any integer $d$ such that $0 \leq d \leq f_1(0)$
\[
\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(X_{1,\text{max}}^{(3)} - d)_+] = p_{y_1}(y_1 - d) + p_{y_2}(y_2 - d) + p_n(n - d)
\]
where $X_{1,\text{max}}^{(3)}$ has support $\{0, y_1, y_2, n\}$ ($y_1 < \phi_2(0, n) \leq y_2$) and $p_0, p_{y_1}, p_{y_2}$ and $p_n$ are as in Proposition 3.1.

Case 2 For any integer $d$ such that $d \in [f_1(0), g_1(0)] \cup \bigcup_{x \in \{1, \ldots, c_1\}} [h_1(x), g_1(x)]$
\[
\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(X_{2,\text{max}}^{(3)} - d)_+] = p_{k_1}(k_3 - d) + p_{k_4}(k_4 - d) + p_n(n - d)
\]
where $X_{2,\text{max}}^{(3)}$ has support $\{k_1, k_3, k_4, n\}$ ($k_3 < \phi_2(k_1, n) \leq k_4$ and $k_1 \in \{1, \ldots, c_1\}$ is such that $d \in [h_1(k_1), g_1(k_1)]$ and $k_1 = 0$ if $d \in [f_1(0), g_1(0)]$) and $p_{k_1}, p_{k_3}, p_{k_4}$ and $p_n$ are as in Proposition 3.1.

Case 3 For any integer $d$ such that $d \in \bigcup_{x \in \{y_2, \ldots, d_1\}} [h_2(x), g_2(x)]$
\[
\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(X_{3,\text{max}}^{(3)} - d)_+] = p_{k_3}(k_3 - d) + p_n(n - d)
\]
where $X_{3,\text{max}}^{(3)}$ has support $\{k_1, k_2, k_3, n\}$ ($k_1 < \phi_2(k_3, n) \leq k_2$ and $k_3 \in \{y_2, \ldots, d_1\}$ is such that $d \in [h_2(k_3), g_2(k_3)]$) and $p_{k_1}, p_{k_2}, p_{k_3}$ and $p_n$ are as in Proposition 3.1.

Case 4 For any integer $d$ such that $g_1(c_1) = h_2(d_2)$
\[
\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(X_{4,\text{max}}^{(3)} - d)_+] = p_d(d_1 - d) + p_{d_2}(d_2 - d)
\]
where $X_{4,\text{max}}^{(3)}$ has support $\{c_1, c_2, d_1, d_2\}$ ($c_1 < c \leq c_2$ and $d_1 < \phi_1(c) \leq d_2$) and $p_{c_1}, p_{c_2}, p_{d_1}$ and $p_{d_2}$ are as in Proposition 3.1.

Case 5 For any integer $d$ such that $\frac{c_1(c_1 + 1) - d_1(d_1 + 1)}{2c_1 + 1 - (2d_1 + 1)} \leq d \leq g_3(c_1) = h_4(d_2)$
\[
\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(X_{5,\text{max}}^{(3)} - d)_+] = p_d(d_1 - d) + p_{d_2}(d_2 - d)
\]
where $X_{5,\text{max}}^{(3)}$ has support $\{c_1, c_2, d_1, d_2\}$ ($c_1 < c \leq c_2$ and $d_1 < \phi_1(c) \leq d_2$) and $p_{c_1}, p_{c_2}, p_{d_1}$ and $p_{d_2}$ are as in Proposition 3.1.

Case 6 For any integer $d$ such that $d \in \bigcup_{x \in \{c_2, \ldots, y_1\}} [h_3(x), g_3(x)]$
\[
\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(X_{6,\text{max}}^{(3)} - d)_+] = p_{k_3}(k_3 - d) + p_{k_4}(k_4 - d)
\]
where $X_{6,\text{max}}^{(3)}$ has support $\{0, k_1, k_3, k_4\}$ ($k_3 < \phi_2(0, k_1) \leq k_4$ and $k_1 \in \{c_2, \ldots, y_1\}$ is such that $d \in [h_3(k_1), g_3(k_1)]$) and $p_0, p_{k_1}, p_{k_3}$ and $p_{k_4}$ are as in Proposition 3.1.
Case 7 For any integer \( d \) such that \( d \in \bigcup_{x \in \{d_x, \ldots, n-1\}} [h_4(x), g_4(x)] \cup [h_4(n), f_4(n)] \)

\[
\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(X_{\alpha_{\text{max}}}^{(3)} - d)_+] = p_{k_3}(k_3 - d)
\]

where \( X_{\alpha_{\text{max}}}^{(3)} \) has support \( \{0, k_1, k_2, k_3\} \) (\( k_1 < \phi_2(0, k_3) \leq k_2 \) and \( k_3 \in \{d_2, \ldots, n-1\} \)) is such that \( d \in [h_4(k_3), g_4(k_3)] \) and \( k_3 = n \) if \( d \in [h_4(n), f_4(n)] \) and \( p_0, p_{k_1}, p_{k_2} \) and \( p_{k_3} \) are as in Proposition 3.1.

Case 8 For any integer \( d \) such that \( f_4(n) \leq d \leq n \)

\[
\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(X_{\alpha_{\text{max}}}^{(3)} - d)_+] = p_n(n - d)
\]

where \( X_{\alpha_{\text{max}}}^{(3)} \) has support \( \{0, y_1, y_2, n\} \) (\( y_1 < \phi_2(0, n) \leq y_2 \)) and \( p_0, p_{y_1}, p_{y_2} \) and \( p_n \) are as in Proposition 3.1.

**Proof.** The proof is based on the existence of a polynomial \( p_3(x) = c_0 + c_1x + c_2x^2 + c_3x^3 \) such that \( p_3(x) \geq (x - d)_+ \) for all \( x \in \mathcal{D}_n \) with an equality holding for the points of the support of a discrete distribution on \( \mathcal{D}_n \), with the given moments \( \mu_1, \mu_2 \) and \( \mu_3 \).

As \( p_3(x) - (x - d)_+ = c_0 + c_1x + c_2x^2 + c_3x^3 \) on \((\infty, d], p_3(x) - (x - d)_+ = (c_0 + d) + (c_1 - 1)x + c_2x^2 + c_3x^3 \) on \([d, +\infty)\) and \( p_3 \) is of degree 3, we have two crossing points on \((-\infty, d] \) and three on \([d, +\infty)\) with at least four of these points being in \( \mathcal{D}_n \) (cf. Proposition 3.1). The admissible polynomials of degree 3 are thus in one of the eight following types:

- **Case 1:** \( k_1 = y \in (-\infty, 0), k_2 = 0, k_3 = y_1 \in \mathcal{D}_n, k_4 = y_2 = y_1 + 1 \) and \( k_5 = n \),
- **Case 2:** \( |k_2 - k_1| \leq 1 \) with \( k_1 \) and/or \( k_2 \in \mathcal{D}_n, k_3 = k_4 = k_5 + 1 \) and \( k_5 = n \),
- **Case 3:** \( k_1 \in \mathcal{D}_n, k_2 = k_1 + 1, k_3 = k_4 - k_3 \) \( \leq 1 \) with \( k_3 \) and/or \( k_4 \in \mathcal{D}_n \) and \( k_5 = n \),
- **Case 4:** \( k_1 = c_1 \in \mathcal{D}_n, k_2 = c_2 = k_1 + 1, k_3 = d_1, k_4 = d_2 = k_3 + 1 \) and \( k_4 = y \in (n, +\infty) \),
- **Case 5:** \( k_1 = y \in (-\infty, 0), k_2 = c_1 \in \mathcal{D}_n, k_3 = c_2 = k_2 + 1, k_4 = d_1 \) and \( k_5 = d_2 = k_4 + 1 \),
- **Case 6:** \( k_1 = 0, |k_3 - k_2| \leq 1 \) with \( k_2 \) and/or \( k_3 \in \mathcal{D}_n, k_4 \in \mathcal{D}_n \) and \( k_5 = k_4 + 1 \),
- **Case 7:** \( k_1 = 0, k_2 \in \mathcal{D}_n, k_3 = k_2 + 1 \) and \( |k_5 - k_4| \leq 1 \) with \( k_4 \) and/or \( k_5 \in \mathcal{D}_n \),
- **Case 8:** \( k_1 = 0, k_2 = y_1 \in \mathcal{D}_n, k_3 = y_2 = y_1 + 1, k_4 = n \) and \( k_5 = y \in (n, +\infty) \).

Cases 1, 2, 3 and 4 correspond to 2 crossing points on \((-\infty, d] \) (denoted \( k_1 \) and \( k_2 \)) and 3 on \([d, +\infty) \) (denoted \( k_3, k_4 \) and \( k_5 \)). Cases 5, 6, 7 and 8 correspond to 3 crossing points on \((-\infty, d] \) (denoted \( k_1, k_2 \) and \( k_3 \)) and 2 on \([d, +\infty) \) (denoted \( k_3 \) and \( k_4 \)).

**Case 1.** Let \( y \in (-\infty, 0) \). Before going on, note that taking a random variable with moments \( \mu_1, \mu_2 \) and \( \mu_3 \) and support \( \{0, y_1, y_2, n\} \) \( y_2 = y_1 + 1 \) implies \( y_1 < \phi_2(0, n) \leq y_2 \) (see Proposition 3.1).
The polynomial we are looking for takes values \( p_3(0) = 0, \ p_3(y_1) = y_1 - d, \ p_3(y_2) = y_2 - d \) and \( p_3(n) = n - d \) so that

\[
p_3(x) = \frac{x(x - y_2)(x - n)}{y_1(n - y_1)}(y_1 - d) - \frac{x(x - y_1)(x - n)}{y_2(n - y_2)}(y_2 - d) + \frac{x(x - y_1)(x - y_2)}{n(n - y_1)(n - y_2)}(n - d).
\]

To find the expression of \( d \), we must express that \( y \) is one of the roots of \( p_3(x) \). That leads to

\[
d = \frac{ny_1y_2(n - y_2)(y - y_2)(y - n) - ny_1y_2(n - y_1)(y - y_1)(y - n) + ny_1y_2(y - y_1)(y - y_2)}{ny_2(n - y_2)(y - y_2)(y - n) - ny_1(n - y_1)(y - y_1)(y - n) + y_1y_2(y - y_1)(y - y_2)}
\]

which is a strictly increasing function of \( y \). As \( y \) tends to \(-\infty\) and to \( 0 \), \( d \) tends respectively to \( 0 \) and \( \frac{y_1^2y_2^2(n - y_2) - y_1y_2n^2(n - y_1) + y_1^2y_2n}{y_1^2y_2^2(n - y_2) - y_1y_2^2n^2(n - y_1) + y_1y_2^2n} \). This gives the interval of variation of \( d \) in Case 1.

Now, to obtain the maximum stop-loss premium, we take a random variable \( X_{1,\text{max}}^{(3)} \) with support \( \{0, y_1, y_2, n\} \) and with moments \( \mu_1, \mu_2 \) and \( \mu_3 \). The maximal expected shortfall is equal to

\[
E[p_3(X_{1,\text{max}}^{(3)})] = p_{y_1}(y_1 - d) + p_{y_2}(y_2 - d) + p_n(n - d)
\]

where \( p_{y_1}, p_{y_2} \) and \( p_n \) are the weights associated to the atoms \( y_1, y_2 \) and \( n \) and are given by Proposition 3.1.

**Case 2.** Let \( k_1 \) be in \( \{0, 1, \ldots, c_1\} \) where \( c_1 < c \leq c_1 + 1 \) (i.e. \( k_1 \) is an integer in \( [0, c]\)). Let also \( k_1 - 1 \leq y \leq k_1 + 1 \) if \( k_1 \in \{1, \ldots, c_1\} \) and \( k_1 \leq y \leq k_1 + 1 \) if \( k_1 = 0 \). Taking a random variable with moments \( \mu_1, \mu_2 \) and \( \mu_3 \) and support \( \{k_1, k_3, k_4, n\} \) \( (k_4 = k_3 + 1) \) and considering Proposition 3.1 implies \( k_3 < \phi_2(k_1, n) \leq k_4 \).

Now, the desired polynomial takes values \( p_3(k_1) = 0, \ p_3(k_3) = k_3 - d, \ p_3(k_4) = k_4 - d \) and \( p_3(n) = n - d \) in \( k_1, k_3, k_4 \) and \( n \), so that

\[
p_3(x) = \frac{(x - k_1)(x - k_4)(x - n)}{(k_3 - k_1)(n - k_3)}(k_3 - d) - \frac{(x - k_1)(x - k_3)(x - n)}{(k_4 - k_1)(n - k_4)}(k_4 - d)
\]

\[
+ \frac{(x - k_1)(x - k_3)(x - k_4)}{(n - k_1)(n - k_3)(n - k_4)}(n - d).
\]

Expressing the fact that \( y \) (possibly non integer) is one of the roots of \( p(x) \), we get

\[
d = \frac{k_3(k_4 - k_1)(n - k_1)(n - k_4)(y - k_4)(y - n)}{n(k_3 - k_1)(k_4 - k_1)(y - k_3)(y - k_4)}
\]

\[
\frac{(k_4 - k_1)(n - k_1)(n - k_4)(y - k_4)(y - n)}{(k_3 - k_1)(n - k_1)(n - k_3)(y - k_3)(y - n)}
\]

\[
+ \frac{(k_3 - k_1)(k_4 - k_1)(y - k_3)(y - k_4)}{(k_3 - k_1)(k_4 - k_1)(y - k_3)(y - k_4)}
\]

which is a strictly increasing function of \( y \). Consequently, as \( \lim_{y \to k_1-1} d(y) = h_{1}(k_1) \) \( (k_1 \in \{1, \ldots, c_1\}) \), \( \lim_{y \to k_1} d(y) = f_1(0) \) (and \( \lim_{y \to k_1+1} d(y) = g_1(k_1) \) \( (k_1 \in \{0, \ldots, c_1\}) \), we have

\[
d \in [f_1(0), g_1(0)] \cup \bigcup_{x \in \{1, \ldots, c_1\}} [h_1(x), g_1(x)].
\]
Now, to obtain the maximum stop-loss premium, we take a random variable $X^{(3)}_{\text{max}}$ with support $\{k_1, k_3, k_4, n\}$ and with moments $\mu_1, \mu_2$ and $\mu_3$, so that the maximal expected shortfall is equal to

$$\mathbb{E}[p(X^{(3)}_{\text{max}})] = p_{k_3}(k_3 - d) + p_{k_4}(k_4 - d) + p_n(n - d)$$

where $p_{k_3}$, $p_{k_4}$ and $p_n$ are the weights associated to the atoms $k_3$, $k_4$ and $n$ and are given by Proposition 3.1.

**Case 3.** Let $k_3$ be in $\{y_2, y_2 + 1, \ldots, d_1\}$ where $y_1 < \phi_2(0, n) \leq y_2 = y_1 + 1$ and $d_1 < \phi_1(c) \leq d_1 + 1$ (i.e. $k_3$ is an integer in $[\phi_2(0, n), \phi_1(c)]$). Let also $k_3 - 1 \leq y \leq k_3 + 1$. Taking a random variable with moments $\mu_1, \mu_2$ and $\mu_3$ and support $\{k_1, k_2, k_3, n\}$ ($k_2 = k_1 + 1$) and considering Proposition 3.1 implies $k_1 < \phi_2(k_3, n) \leq k_2$.

Now, the desired polynomial takes values $p_3(k_1) = 0$, $p_3(k_2) = 0$, $p_3(k_3) = k_3 - d$ and $p_3(n) = n - d$ so that

$$p_3(x) = \frac{(x - k_1)(x - k_2)(x - k_3)}{(n - k_1)(n - k_2)(n - k_3)}(n - d) - \frac{(x - k_1)(x - k_2)(x - n)}{(k_3 - k_1)(k_3 - k_2)(k_3 - n)}(k_3 - d).$$

Expressing the fact that $y$ (possibly non integer) is such that $p(y) = y - d$, we get

$$d = \frac{n(k_3 - k_1)(k_3 - k_2)(y - k_1)(y - k_2)(y - k_3) - k_3(n - k_1)(n - k_2)(y - k_1)(y - k_2)(y - n) - y(k_3 - k_1)(k_3 - k_2)(n - k_1)(n - k_2)(n - k_3)}{(k_3 - k_1)(k_3 - k_2)(y - k_1)(y - k_2)(y - k_3) - (n - k_1)(n - k_2)(y - k_1)(y - k_2)(y - n) - (k_3 - k_1)(k_3 - k_2)(n - k_1)(n - k_2)(n - k_3)}$$

which is a strictly increasing function of $y$. Consequently, as $\lim_{y \to k_3 - 1} d(y) = h_2(k_3)$ ($k_3 \in \{y_2, \ldots, d_1\}$), and $\lim_{y \to k_3 + 1} d(y) = g_2(k_3)$ ($k_3 \in \{y_2, \ldots, d_1\}$), we have

$$d \in \bigcup_{x \in \{y_2, \ldots, d_1\}} [h_2(x), g_2(x)].$$

Now, to obtain the maximum stop-loss premium, we take a random variable $X^{(3)}_{\text{max}}$ with support $\{k_1, k_2, k_3, n\}$ and with moments $\mu_1, \mu_2$ and $\mu_3$, so that the maximal expected shortfall is equal to

$$\mathbb{E}[p_3(X^{(3)}_{\text{max}})] = p_{k_3}(k_3 - d) + p_n(n - d)$$

where $p_{k_3}$ and $p_n$ are the weights associated to the atoms $k_3$ and $n$ and are given by Proposition 3.1.

**Case 4.** Let $y \in (n, +\infty)$. Before going on, note that taking a random variable with moments $\mu_1, \mu_2$ and $\mu_3$ and support $\{c_1, c_2, d_1, d_2\}$ ($c_2 = c_1 + 1$ and $d_2 = d_1 + 1$) implies $c_1 < c \leq c_2$ and $d_1 < \phi_1(c) \leq d_2$ (see Proposition 3.1).

The polynomial we are looking for takes values $p_3(c_1) = 0$, $p_3(c_2) = 0$, $p_3(d_1) = d_1 - d$ and $p_3(d_2) = d_2 - d$ in $c_1$, $c_2$, $d_1$ and $d_2$, so that

$$p_3(x) = \frac{(x - c_1)(x - c_2)(x - d_1)}{(d_2 - c_1)(d_2 - c_2)}(d_2 - d) - \frac{(x - c_1)(x - c_2)(x - d_2)}{(d_1 - c_1)(d_1 - c_2)}(d_1 - d).$$

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Now, expressing the fact that \( p_3(y) \) is equal to \( y - d \), it comes

\[
d = \frac{d_2(d_1 - c_1)(d_1 - c_2)(y - c_1)(y - c_2)(y - d_1) - d_1(d_2 - c_1)(d_2 - c_2)(y - c_1)(y - c_2)(y - d_2)}{(d_1 - c_1)(d_1 - c_2)(y - c_1)(y - c_2)(y - d_1) - (d_2 - c_1)(d_2 - c_2)(y - c_1)(y - c_2)(y - d_2) - (d_1 - c_1)(d_1 - c_2)(d_2 - c_1)(d_2 - c_2)}
\]

which is a strictly increasing function of \( y \). As \( y \) tends to \( n \) and to \( +\infty \), \( d \) tends respectively to \( \frac{d_2(n-1)}{(n-1)(n-c_1)(n-c_2)} \) and \( \frac{d_2}{(d_1-c_1)(d_1-c_2)} \). This gives the interval of variation of \( d \) in Case 4.

Now, to obtain the maximum stop-loss premium, we take a random variable \( X_{4,\text{max}}^{(3)} \) with support \( \{c_1, c_2, d_1, d_2\} \) and with moments \( \mu_1, \mu_2 \) and \( \mu_3 \), so that the maximal expected shortfall is equal to

\[
\mathbb{E}[p(X_{4,\text{max}}^{(3)})] = p_{d_1}(d_1 - d) + p_{d_2}(d_2 - d)
\]

where \( p_{d_1} \) and \( p_{d_2} \) are the weights associated to the atoms \( d_1 \) and \( d_2 \) and are given by Proposition 3.1.

**Case 5.** Let \( y \in (-\infty, 0) \). The polynomial we are looking for contains values \( p(c_1) = 0 \), \( p(c_2) = 0 \), \( p(d_1) = d_1 - d \) and \( p(d_2) = d_2 - d \) in \( c_1, c_2, d_1 \) and \( d_2 \) and thus takes the same form as in Case 4. To find the expression of \( d \), we must express that \( y \) is one of the roots of \( p(x) \). That leads to

\[
d = \frac{d_2(d_1 - c_1)(d_1 - c_2)(y - d_1) - d_1(d_2 - c_1)(d_2 - c_2)(y - d_2)}{(d_1 - c_1)(d_1 - c_2)(y - d_1) - (d_2 - c_1)(d_2 - c_2)(y - d_2) - (d_1 - c_1)(d_1 - c_2)(d_2 - c_1)(d_2 - c_2)}
\]

which is a strictly increasing function of \( y \). As \( y \) tends to \( -\infty \) and to \( 0 \), \( d \) tends respectively to \( \frac{d_2}{(d_1-c_1)(d_1-c_2)-d_1(d_2-c_1)(d_2-c_2)} \) and \( \frac{d_2}{(d_1-c_1)(d_1-c_2)-d_1(d_2-c_1)(d_2-c_2)} \). This gives the desired results for Case 5.

**Case 6.** Let \( k_1 \) be in \( \{c_2, c_2 + 1, \ldots, y_1\} \) where \( c_1 < c < c_2 = c_1 + 1 \) and \( y_1 < \phi_2(0, n) \leq y_1 + 1 \) (i.e. \( k_1 \) is an integer in \( [c, \phi_2(0, n)] \)). Let also \( k_1 - 1 \leq y \leq k_1 + 1 \). Taking a random variable with moments \( \mu_1, \mu_2 \) and \( \mu_3 \) and support \( \{0, k_1, k_3, k_4\} \) \( (k_4 = k_3 + 1) \) and considering Proposition 3.1 implies \( k_3 < \phi_2(0, k_1) \leq k_4 \).

Now, the desired polynomial takes values \( p_3(0) = 0, p_3(k_1) = 0, p_3(k_3) = k_3 - d \) and \( p_3(k_4) = k_4 - d \) in \( 0, k_1, k_3 \) and \( k_4 \), so that

\[
p(x) = \frac{x(x - k_1)(x - k_3)}{k_4(k_4 - k_1)}(k_4 - d) - \frac{x(x - k_1)(x - k_4)}{k_3(k_3 - k_1)}(k_3 - d).
\]

Expressing the fact that \( y \) (possibly non integer) is one of the roots of \( p_3(x) \), we get

\[
d = \frac{k_3k_4(k_3 - k_1)(y - k_3) - k_3k_4(k_4 - k_1)(y - k_4)}{k_3(k_3 - k_1)(y - k_3) - k_4(k_4 - k_1)(y - k_4)}
\]
which is a strictly increasing function of \( y \). Consequently, as \( \lim_{y \to k_1-1} d(y) = h_3(k_1) \) (\( k_1 \in \{c_2, \ldots, y_1\} \)) and \( \lim_{y \to k_1+1} d(y) = g_3(k_1) \) (\( k_1 \in \{c_2, \ldots, y_1\} \)), we have

\[
d \in \bigcup_{x \in \{c_2, \ldots, y_1\}} [h_3(x), g_3(x)].
\]

Now, to obtain the maximum stop-loss premium, we take a random variable \( X^{(3)}_{6, \text{max}} \) with support \( \{0, k_1, k_3, k_4\} \) and with moments \( \mu_1, \mu_2 \) and \( \mu_3 \), so that the maximal expected shortfall is equal to

\[
\mathbb{E}[p_3(X^{(3)}_{6, \text{max}})] = p_{k_3}(k_3 - d) + p_{k_4}(k_4 - d)
\]

where \( p_{k_3} \) and \( p_{k_4} \) are the weights associated to the atoms \( k_3 \) and \( k_4 \) and are given by Proposition 3.1.

**Case 7.** Let \( k_3 \) be in \( \{d_2, d_2 + 1, \ldots, n\} \) where \( d_1 < \phi_1(c) \leq d_2 = d_1 + 1 \) (i.e. \( k_3 \) is an integer \( \{\phi_1(c), n\} \)). Let also \( k_3 - 1 \leq y \leq k_3 + 1 \) if \( k_3 \in \{d_2, \ldots, n-1\} \) and \( k_3 - 1 \leq y \leq k_3 \) if \( k_3 = n \). Taking a random variable with moments \( \mu_1, \mu_2 \) and \( \mu_3 \) and support \( \{0, k_1, k_2, k_3\} \) \((k_2 = k_1 + 1)\) and considering Proposition 3.1 implies \( k_1 < \phi_2(0, k_3) \leq k_2 \).

Now, the desired polynomial takes values \( p_3(0) = 0, p_3(k_1) = 0, p_3(k_2) = 0 \) and \( p_3(k_3) = k_3 - d \) so that

\[
p_3(x) = \frac{x(x - k_1)(x - k_2)}{k_3(k_3 - k_1)(k_3 - k_2)}(k_3 - d).
\]

Expressing the fact that \( y \) (possibly non integer) is such that \( p_3(y) = y - d \), we get

\[
d = \frac{k_3y(y - k_1)(y - k_2) - k_3y(k_3 - k_1)(k_3 - k_2)}{y(y - k_1)(y - k_2) - k_3(k_3 - k_1)(k_3 - k_2)}
\]

which is a strictly increasing function of \( y \). Consequently, as \( \lim_{y \to k_3-1} d(y) = h_4(k_3) \) (\( k_3 \in \{d_2, \ldots, n\} \)), \( \lim_{y \to k_3+1} d(y) = g_4(k_3) \) (\( k_3 \in \{d_2, \ldots, n-1\} \)) and \( \lim_{y \to n} d(y) = f_4(n) \), we have

\[
d \in \bigcup_{x \in \{d_2, \ldots, n-1\}} [h_4(x), g_4(x)] \cup [h_4(n), f_4(n)].
\]

Now, to obtain the maximum stop-loss premium, we take a random variable \( X^{(3)}_{7, \text{max}} \) with support \( \{0, k_1, k_2, k_3\} \) and with moments \( \mu_1, \mu_2 \) and \( \mu_3 \), so that the maximal expected shortfall is equal to

\[
\mathbb{E}[p_3(X^{(3)}_{7, \text{max}})] = p_{k_3}(k_3 - d)
\]

where \( p_{k_3} \) is the weight associated to the atom \( k_3 \) and is given by Proposition 3.1.

**Case 8.** Let \( y \in (n, +\infty) \). The polynomial we are looking for takes values \( p_3(0) = 0, p_3(y_1) = 0, p_3(y_2) = 0 \) and \( p_3(n) = n - d \), so that

\[
p_3(x) = \frac{x(x - y_1)(x - y_2)}{n(n - y_1)(n - y_2)}(n - d).
\]
Now, expressing the fact that \( p_3(y) \) is equal to \( y - d \), it comes

\[
d = \frac{ny(y - y_1)(y - y_2) - ny(n - y_1)(n - y_2)}{y(y - y_1)(y - y_2) - n(n - y_1)(n - y_2)}
\]

which is a strictly increasing function of \( y \) (strictly positive derivative with respect to \( y \)). As \( y \) tends to \( n \) and to \( +\infty \), \( d \) tends respectively to \( \frac{n^2(2n - y_1 - y_2)}{(n-y_1)(n-y_2)+n(2n-y_1-y_2)} \) and \( n \). This gives the interval of variation of \( d \) in Case 8.

Now, to obtain the maximum stop-loss premium, we take a random variable \( X_{8,\text{max}}^{(3)} \) with support \( \{0, y_1, y_2, n\} \) and with moments \( \mu_1, \mu_2 \) and \( \mu_3 \), so that the maximal expected shortfall is equal to

\[
\mathbb{E}[p_3(X_{8,\text{max}}^{(3)})] = p_n(n - d)
\]

where \( p_n \) is the weight associated to the atom \( n \) and is given by Proposition 3.1.

### 3.3 Lower bounds on the expected shortfalls

Following the reasoning held in the proof of Proposition 3.2, we get the next result providing lower bounds on expected shortfalls.

**Proposition 3.3.** For a fixed integer \( d \), let \( d_0 \) and \( d_n \) be the integers such that \( d_0 < \phi_2(0,d) \leq d_0 + 1 \) and \( d_n < \phi_2(d,n) \leq d_n + 1 \). For any \( X \in \mathcal{M}_4(D_n;\mu_1,\mu_2,\mu_3) \), the following inequalities hold true:

**Case 1** For any integer \( d \) such that \( 0 \leq d \leq c \)

\[
\mathbb{E}[(X - d)_+] \geq \mathbb{E}[(X_{1,\text{min}}^{(3)} - d)_+] = \mu_1 - d
\]

where \( X_{1,\text{min}}^{(3)} \) has support \( \{d, d_n, d_n + 1, n\} \) and weights \( p_d, p_{d_n}, p_{d_n+1} \) and \( p_n \) as in Proposition 3.1.

**Case 2** For any integer \( d \) such that \( c \leq d \leq \phi_2(0,n) \)

\[
\mathbb{E}[(X - d)_+] \geq \mathbb{E}[(X_{2,\text{min}}^{(3)} - d)_+] = \frac{-\mu_3 + \mu_2(d + 2d_0 + 1) - \mu_1d(2d_0 + 1)}{d_0(d_0 + 1)}
\]

where \( X_{2,\text{min}}^{(3)} \) has support \( \{0, d, d_0, d_0 + 1\} \) and weights \( p_0, p_d, p_{d_0} \) and \( p_{d_0+1} \) as in Proposition 3.1.

**Case 3** For any integer \( d \) such that \( \phi_2(0,n) \leq d \leq \phi_1(c) \)

\[
\mathbb{E}[(X - d)_+] \geq \mathbb{E}[(X_{3,\text{min}}^{(3)} - d)_+] = p_n(n - d)
\]

where \( X_{3,\text{min}}^{(3)} \) has support \( \{d_n, d_n + 1, d, n\} \) and weights \( p_{d_n}, p_{d_n+1}, p_d \) and \( p_n \) as in Proposition 3.1.
Case 4 For any integer \(d \leq n\)
\[
\mathbb{E}[(X - d)_+] \geq \mathbb{E}[(X_{4, \min}^{(3)} - d)_+] = 0
\]
where \(X_{4, \min}^{(3)}\) has support \(\{0, d_0, d_0 + 1, d\}\) and weights \(p_0, p_{d_0}, p_{d_0+1}\) and \(p_d\) as in Proposition 3.1.

3.4 \(\preceq_{\text{icx}}\) extremal distributions when three moments are known

Following the same lines as in Subsection 3.4, but considering functions \(f_1, h_1, g_2, h_2, f_2, g_3, h_3, f_4, g_4\) and \(h_4\) instead of \(f, g\) and \(h\), we can get extremal distributions in the \(\preceq_{\text{icx}}\)-sense that do not depend on the threshold \(d\).

**Proposition 3.4.** Let us take the same notations as in Proposition 3.2. Let us furthermore define \(\Pi_{k, \max}^{(3)}(x) = \mathbb{E}[(X_{k, \max}^{(3)} - x)_+] \ (k = 1, \ldots, 8)\) and denote \(J_1 = [f_1(0), g_1(0)], \ J_2 = \bigcup_{x \in [1, \ldots, c_1]} [h_1(x), g_1(x)], \ J_3 = \bigcup_{x \in [g_2, \ldots, d_1]} [h_2(x), g_2(x)], \ J_4 = [h_2(d_2), \frac{c_1(c_1+1) - d_2(d_2+1)}{2c_1+1-(2d_1+1)}], \ J_5 = \frac{c_1(c_1+1)-d_1(d_1+1)}{2c_1+1-(2d_1+1)} g_1(c_1), \ J_6 = \bigcup_{x \in [g_2, \ldots, g_3]} [h_3(x), g_3(x)] \ J_7 = \bigcup_{x \in [d_2, \ldots, n-1]} [h_4(x), g_4(x)], \ J_8 = [h_4(n), f_4(n)]\). The maximal distribution \(F_{\max}^{(3)}\) with respect to \(\preceq_{\text{icx}}\) is

\[
F_{\max}^{(3)}(x) = \begin{cases} 
  p_0 & \text{for } x = 0, 1, \ldots, [f_1(0)]; \\
  1 + \Pi_{2, \max}^{(3)}(x + 1) - \Pi_{2, \max}^{(3)}(x) & \text{for } x \in J_1 \cup J_2 \text{ such that } x + 1 \in J_2; \\
  1 + \Pi_{3, \max}^{(3)}(x + 1) - \Pi_{3, \max}^{(3)}(x) & \text{for } x \in J_1 \cup J_2 \text{ such that } x + 1 \in J_3; \\
  1 + \Pi_{4, \max}^{(3)}(x + 1) - \Pi_{4, \max}^{(3)}(x) & \text{for } x \in J_3 \text{ such that } x + 1 \in J_3; \\
  1 + \Pi_{5, \max}^{(3)}(x + 1) - \Pi_{5, \max}^{(3)}(x) & \text{for } x \in J_1 \cup J_2 \text{ such that } x + 1 \in J_1; \\
  1 + \Pi_{6, \max}^{(3)}(x + 1) - \Pi_{6, \max}^{(3)}(x) & \text{for } x \in J_3 \text{ such that } x + 1 \in J_4; \\
  1 + \Pi_{7, \max}^{(3)}(x + 1) - \Pi_{7, \max}^{(3)}(x) & \text{for } x = [h_2(d_2)], \ldots, [g_1(c_1)]; \\
  1 + \Pi_{8, \max}^{(3)}(x + 1) - \Pi_{8, \max}^{(3)}(x) & \text{for } x \in J_5 \text{ such that } x + 1 \in J_6; \\
  1 + \Pi_{9, \max}^{(3)}(x + 1) - \Pi_{9, \max}^{(3)}(x) & \text{for } x \in J_6 \text{ such that } x + 1 \in J_7 \cup J_8; \\
  1 + \Pi_{10, \max}^{(3)}(x + 1) - \Pi_{10, \max}^{(3)}(x) & \text{for } x \in J_7 \text{ such that } x + 1 \in J_7 \cup J_8; \\
  1 + \Pi_{11, \max}^{(3)}(x + 1) - \Pi_{11, \max}^{(3)}(x) & \text{for } x \in J_8 \text{ such that } x + 1 \in J_7 \cup J_8; \\
  1 + \Pi_{12, \max}^{(3)}(x + 1) - \Pi_{12, \max}^{(3)}(x) & \text{for } x = [f_4(n)], \ldots, n - 1; \\
  1 - p_n & \text{for } x = n.
\end{cases}
\]

**Proposition 3.5.** In the notation of Propositions 3.2 and 3.3 and defining \(\Pi_{k, \min}^{(3)}(x) = \ldots\)}
\( E[(X_{k,\min}^{(3)} - x)_+] \quad (k = 1, 2, 3, 4) \), the minimal distribution \( F_{\min}^{(3)} \) with respect to \( \preceq_{icx} \) is

\[
F_{\min}^{(3)}(x) = \begin{cases} 
0 & \text{for } x = 0, 1, \ldots, c_1 - 1; \\
1 + \Pi_{2,\min}^{(3)}(c_2) - \Pi_{1,\min}^{(3)}(c_1) & \text{for } x = c_1; \\
1 + \Pi_{2,\min}^{(3)}(x + 1) - \Pi_{2,\min}^{(3)}(x) & \text{for } x = c_2, c_2 + 1, \ldots, y_1 - 1; \\
1 + \Pi_{3,\min}^{(3)}(y_2) - \Pi_{2,\min}^{(3)}(y_1) & \text{for } x = y_1; \\
1 + \Pi_{3,\min}^{(3)}(x + 1) - \Pi_{3,\min}^{(3)}(x) & \text{for } x = y_2, y_2 + 1, \ldots, d_1 - 1; \\
1 - \Pi_{3,\min}^{(3)}(d_1) & \text{for } x = d_1; \\
1 & \text{for } x = d_2, d_2 + 1, \ldots, n.
\end{cases}
\]

Remark that we even have that the extremal random variables with respective distribution functions \( F_{\min}^{(3)} \) and \( F_{\max}^{(3)} \) have the same mean \( \mu_1 \).

4 Applications

4.1 Lee-Carter stochastic modelling for dynamic mortality

Let us denote as \( T_x(t) \) the remaining lifetime of an individual aged \( x \) on January the first of year \( t \). Here and below, \( x \) and \( t \) are always assumed to be integers. This individual will die at age \( x + T_x(t) \) in year \( t + T_x(t) \). The mortality force at age \( x \) during calendar year \( t \), denoted as \( \mu_x(t) \), is defined as

\[
\mu_x(t) = \lim_{\Delta \to 0} \Pr\left[ x < T_0(t - x) \leq x + \Delta | T_0(t - x) > x \right].
\]

Furthermore, we assume that the age-specific mortality rates are constant within bands of age and time, but allowed to vary from one band to the next. Specifically, given any integer age \( x \) and calendar year \( t \), it is supposed that

\[
\mu_{x+\xi}(t + \tau) = \mu_x(t) \text{ for } \xi \geq 0, \tau < 1.
\]

The classical Lee-Carter relational model assumes that

\[
\ln \mu_x(t) = \alpha_x + \beta_x \kappa_t.
\]

Interpretation of the parameters involved in this model is quite simple. The value of \( \alpha_x \) is an average of \( \ln \mu_x(t) \) over time \( t \) so that \( \exp \alpha_x \) is the general shape of the mortality schedule. The actual forces of mortality change according to an overall mortality index \( \kappa_t \) modulated by an age response \( \beta_x \). The shape of the \( \beta_x \) profile tells which rates decline rapidly and which slowly over time in response of change in \( \kappa_t \). The time factor \( \kappa_t \) is intrinsically viewed as a stochastic process and Box-Jenkins techniques are then used to model and forecast \( \kappa_t \).
Here, as in the majority of papers devoted to empirical applications of the Lee-Carter methodology, a simple random walk with drift is selected to model the dynamics of the \( \kappa_t \)'s. According to this model, the \( \kappa \)'s obey to

\[
\kappa_t = \kappa_{t-1} + \theta + \xi_t \quad \text{with iid } \xi_t \sim \mathcal{N}(0, \sigma^2),
\]

(1)

where \( \theta \) is known as the drift parameter and “\( \sim \mathcal{N}(0, \sigma^2) \)” means “is Normally distributed with mean 0 and variance \( \sigma^2 \)”.

We will assume in the remainder of this paper that the values \( \kappa_1, \ldots, \kappa_{t_0} \) are known but that the \( \kappa_{t_0+k} \)'s \( (k = 1, 2, \ldots) \) are unknown and have to be projected from (1). To forecast the time index at time \( t_0 + k \) with all data available up to \( t_0 \), we use the representation

\[
\kappa_{t_0+k} = \kappa_{t_0} + k\theta + \sum_{j=1}^{k} \xi_{t_0+j}.
\]

For any non-negative integer \( d \), let \( dP_{x_0} \) be the \( d \)-year (random) survival probability for an individual aged \( x_0 \) in year \( t_0 \) given the trajectory of the time index \( \kappa = (\kappa_{t_0}, \ldots, \kappa_{t_0+\omega-x_0}) \), where \( \omega \) is the ultimate age of the life table. More specifically, 

\[
dP_{x_0} = \Pr[T_{x_0}(t_0) > d|\kappa].
\]

In the Lee-Carter framework and for integer \( d \), this probability writes

\[
dP_{x_0} = \prod_{j=0}^{d-1} p_{x_0+j}(t_0+j) = \exp \left( -\sum_{j=0}^{d-1} \exp(\alpha_{x_0+j} + \beta_{x_0+j}\kappa_{t_0+j}) \right) = \exp(-S_d)
\]

with 

\[
S_d = \sum_{j=0}^{d-1} \exp(\alpha_{x_0+j} + \beta_{x_0+j}\kappa_{t_0+j}) = \sum_{j=0}^{d-1} \delta_j \exp(X_j),
\]

where \( \delta_j = \exp(\alpha_{x_0+j}) \) and \( X_j = \beta_{x_0+j}\kappa_{t_0+j} \). Conditional upon \( \kappa_{t_0} \), we have that \( X_j \sim \mathcal{N}\text{or}(\mu_j, \sigma_j^2) \) with

\[
\mu_j = \beta_{x_0+j}(\kappa_{t_0} + j\theta) \quad \text{and} \quad \sigma_j^2 = (\beta_{x_0+j})^2 j\sigma^2,
\]

with the convention that a normally distributed random variable with zero variance is constantly equal to the mean. Note that \( dP_{x_0} \) is a random variable since it involves the \( \kappa_{t_0+j} \)'s obeying to (1). Remark also that \( S_d \) is a linear combination of correlated LogNormal random variables.

Let \( N \) be the number of survivors at time \( t_0 + d \) from an initial group of \( n \) policyholders aged \( x_0 \) at time \( t_0 \). Given \( \kappa \), \( N \) obeys to the Binomial distribution with exponent \( n \) and parameter \( dP_{x_0} \). Let \( \nu_1, \nu_2, \nu_3, \ldots \) denote the moments of \( dP_{x_0} \) (i.e. \( \nu_k = \mathbb{E}[dP_{x_0}^k], \quad k \in \mathbb{N} \)).

As suggested in Denuit & Dhaene (2006), natural candidates for defining the benefits of longevity bonds or reinsurance treaties covering portfolios of life annuities involve the excess of the actual number of survivors to that expected from a reference life table (for instance, the Belgian regulatory life tables MR for males and FR for females). If the expected number of survivors at time \( t_0 + d \) is denoted \( \bar{n}_d \) then the payoff could be related...
to \((N - \bar{n}_d)_+\) and the aim is thus to obtain accurate and easy-to-compute bounds on \(\mathbb{E}[(N - \bar{n}_d)_+]\). This will be done using the method introduced previously. The actual number of survivors \(N\) is a random variable with discrete support \(\{0, 1, \ldots, n\}\). Consequently, to use the desired method, we have to know the first moments of \(N\). The moments of \(N\) are functions of the moments of \(d\).

\[\mu_1 = \mathbb{E}[N] = \mathbb{E}[\mathbb{E}[N|\kappa]] = n\nu_1,\]

\[\mu_2 = \mathbb{E}[N^2] = \mathbb{E}[\mathbb{E}[N^2|\kappa]] = n\nu_1 + n(n - 1)\nu_2,\]

and

\[\mu_3 = \mathbb{E}[N^3] = \mathbb{E}[\mathbb{E}[N^3|\kappa]] = n\nu_1 + 3n(n - 1)\nu_2 + n(n - 1)(n - 2)\nu_3.\]

### 4.1.1 Numerical results

As an illustration, we consider three portfolios of 100 policyholders aged 65 at times 1999, 2002 and 2006. The moments of \(N\) (see Tables 2 and 3) are computed for \(d = 1, 5, 10, 15, 20, 25\) and 30 using the estimated parameters from the Poisson log-bilinear regression method of Brouhs, Denuit & Vermunt (2002) summarized in Table 1.

Having the moments of \(N\), it is then easy to bound \(\mathbb{E}[(N - \bar{n}_d)_+]\) using the results of Section 2 (using 2 moments, \(\mu_1\) and \(\mu_2\)) or of Section 3 (using 3 moments, \(\mu_1\), \(\mu_2\) and \(\mu_3\)). See Tables 4 and 5. Comparing with upper and lower bounds given in Jansen et al. (1986) viewing \(N\) as valued in the interval \([0, n]\) (thus, not exploiting the particular form of the support \(D_n\)), we see that considering the particular discrete form of the support of the random variables in presence implies sharper results. The zero lower bounds come from Case 3 of Proposition 2.3.

Tables 6 and 7 illustrate the remarkable accuracy of the bounds on \(\mathbb{E}[(N - k)_+]\) for various \(k\)’s. Figures 1 and 2 display the graph of the upper and lower bounds on the expected shortfalls for \(N\), together with the corresponding distribution functions deduced from Propositions 2.4-2.5 in case of two moments, and 3.4-3.5 in case of three moments.

### 4.2 Bounds for the ruin probability

Following Gerber (1988), Shiu (1989) and De Vijlder (1996) for example, we examine the ruin problem for the compound Binomial risk process. In this model, time is measured in discrete time units \(t = 0, 1, 2, \ldots\). The number of insured claims is governed by a Binomial process \(\{N(t), \ t \geq 0\}\) with parameter \(q\), \(0 < q < 1\) (i.e. in any time period, there occurs 1 or 0 claim with probabilities \(q\) and \(1 - q\) respectively, and occurrence of claims in different time intervals are independent events). The claim amounts \(X_k, k \geq 1\), are independent and distributed as a random variable \(X\) valued in \(\{1, 2, 3, \ldots\}\), with mean \(\mu_X\); they are independent of the Binomial process \(\{N(t), \ t \geq 0\}\). The premium received in each period is 1 and is assumed to be larger than the net premium, which means that \(1 > q\mu_X\). The initial risk reserve is a non-negative integer amount \(z\).
| Age | $\alpha_x$ Men | $\beta_x$ Men | Year | $\kappa_t$ Men | | Age | $\alpha_x$ Women | $\beta_x$ Women | Year | $\kappa_t$ Women |
|-----|---------------|---------------|------|----------------|-----|-----|---------------|---------------|------|----------------|-----|-----|---------------|---------------|------|----------------|-----|-----|---------------|---------------|
| 73  | -2.80         | 0.0371        | 2007 | -10.89         | -15.91       | 74  | -2.70         | 0.0351        | 2008 | -11.20         | -16.46       | 75  | -2.61         | 0.0325        | 2009 | -11.51         | -17.00       | 76  | -2.52         | 0.0328        | 2010 | -11.83         | -17.55       |
| 77  | -2.43         | 0.0313        | 2011 | -12.14         | -18.09       | 78  | -2.34         | 0.0300        | 2012 | -12.45         | -18.64       | 79  | -2.24         | 0.0299        | 2013 | -12.76         | -19.19       | 80  | -2.15         | 0.0288        | 2014 | -13.08         | -19.73       |
| 81  | -2.07         | 0.0243        | 2015 | -13.39         | -20.28       | 82  | -1.98         | 0.0236        | 2016 | -13.70         | -20.82       | 83  | -1.89         | 0.0236        | 2017 | -14.02         | -21.37       | 84  | -1.81         | 0.0217        | 2018 | -14.33         | -21.91       |
| 85  | -1.70         | 0.0248        | 2019 | -14.64         | -22.46       | 86  | -1.63         | 0.0232        | 2020 | -14.96         | -23.01       | 87  | -1.53         | 0.0213        | 2021 | -15.27         | -23.55       | 88  | -1.47         | 0.0182        | 2022 | -15.58         | -24.10       |
| 89  | -1.39         | 0.0183        | 2023 | -15.90         | -24.64       | 90  | -1.32         | 0.0146        | 2024 | -16.21         | -25.19       | 91  | -1.25         | 0.0132        | 2025 | -16.52         | -25.73       | 92  | -1.21         | 0.0038        | 2026 | -16.84         | -26.28       |
| 93  | -1.15         | 0.0056        | 2027 | -17.15         | -26.83       | 94  | -1.11         | -0.0004       | 2028 | -17.46         | -27.37       | 95  | -1.09         | -0.0137       | 2029 | -17.78         | -27.92       | 96  | -1.07         | -0.0236       | 2030 | -18.09         | -28.46       |
| 97  | -1.07         | -0.0355       | 2031 | -18.40         | -29.01       | 98  | -1.13         | -0.0598       | 2032 | -18.72         | -29.56       | 99  | -1.14         | -0.0185       | 2033 | -18.72         | -29.56       |

Table 1: Estimated parameters from the Poisson log-bilinear regression method of [1], with $\theta_M = -0.31324$, $\sigma^2_M = 1.077735$, $\theta_W = -0.54574$ and $\sigma^2_W = 1.239885$
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Table 2: Moments of $N$ for Belgian males in function of the year the age 65 is reached.
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Table 3: Moments of $N$ for Belgian females in function of the year the age 65 is reached.
Table 4: Bounds on $E[(N - \bar{n}_d)_+]$ for Belgian males according to the year the age 65 is reached.
Discrete bounds using two moments

<table>
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<th>2006</th>
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Discrete bounds using three moments

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Continuous bounds using two moments

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Continuous bounds using three moments

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Table 5: Bounds on $\mathbb{E}[(N - \bar{n}_d)_+]$ for Belgian females according to the year the age 65 is reached.
Table 6: Bounds on $\mathbb{E}[(N - k)_+]$ for various $k$'s for Belgian males reaching the age 65 during year 1999 with $\bar{d} = 15$ and knowing 2 or 3 moments.
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<th>( \Pi_{\text{max}}^{(2)} )</th>
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Table 7: Bounds on \( \mathbb{E}[(N - k)_+] \) for various \( k \)'s for Belgian females reaching the age 65 during year 1999 with \( d = 15 \) and knowing 2 or 3 moments.

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Figure 1: Bounds on the expected shortfalls $\mathbb{E}[(N - k)_+]$ for $k = 0, \ldots, 100$ (left panel) together with the corresponding distribution functions (right panel), obtained with 2 moments of $N$. Males are on the top.
Figure 2: Bounds on the expected shortfalls $E[(N - k)_+]$ for $k = 0, \ldots, 100$ (left panel) together with the corresponding distribution functions (right panel), obtained with 3 moments of $N$. Males are on the top.
Denoting by $U_X(t)$ the risk reserve at time $t$, i.e.

$$U_X(t) = z + t - \sum_{k=1}^{N(t)} X_k, \quad t = 0, 1, 2, \ldots,$$

the probability of eventual ruin $\psi_X(z)$ is defined as $\psi_X(z) = \Pr[U_X(t) < 0 \mid U_X(0) = z]$. Note that this definition is that of SHIU (1989) and differs from that of GERBER (1988) where ruin occurs when $U_X(t) \leq 0$. Let $\phi_X(z) = 1 - \psi_X(z)$ be the probability of non-ruin.

Let us recall the formula below that can be found, for example, in SHIU (1989) or in DE VIJOLDER (1996, Theorem 11 p. 251). In the notation above,

$$\phi_X(z) = \phi_X(0) \sum_{k=0}^{\infty} \left( \psi_X(0) \right)^k H_X^{*}(k)(z),$$

where

$$\psi_X(0) = q(\mu_X - 1)/(1 - q),$$

and

$$H_X^{*}(m) = \sum_{j=1}^{m} h_X^{*}(j),$$

$h_X^{*}(k)$ being the $k$-fold convolution of the discrete probability density function $h_X$ defined by

$$h_X(j) = \frac{1 - F_X(j)}{\mu_X - 1}, \quad j = 1, 2, \ldots.$$ 

Based on this formula, DENUIT & LEFÈVRE (1999) proved that

$$X \preceq_{\text{icx}} Y \Rightarrow \psi_X(z) \leq \psi_Y(z) \text{ for all } z = 0, 1, 2, \ldots.$$ 

The extrema for $\preceq_{\text{icx}}$ derived in Sections 2-3 allow us to derive bounds for the eventual probability of ruin in the compound Binomial ruin process. Some numerical illustrations of these bounds are presented in Table 9. They are computed using the same parameters values as in DENUIT & LEFÈVRE (1997). In Table 8, we recall the bounds obtained in DENUIT & LEFÈVRE (1997). Table 9 displays the results obtained with the $\preceq_{\text{icx}}$ extrema derived in this paper (in Propositions 2.4-2.5 in case of two moments, and in Propositions 3.4-3.5 in case of three moments). We clearly see that the upper bounds are much improved compared to the previous works. The improvements for the lower bounds is moderate.

5 Discussion

The results derived in this paper can be generalized in different ways. First, they can be extended to variables valued in an arbitrary ordered grid $\{e_0, e_1, \ldots, e_n\}$ of points.
Table 8: Bounds on the eventual probability of ruin \((q = 0.1, \mu_X = 5.5, n = 50, \mu_2 = 30.5, 32.5, 36.5)\) obtained in Denuit & Lefèvre (1997).

<table>
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<tr>
<th>Initial surplus level</th>
<th>(\psi_{\min})</th>
<th>(\psi_{\max})</th>
<th>(\psi_{\max}) (\mu_2 = 30.5)</th>
<th>(\psi_{\max}) (\mu_2 = 32.5)</th>
<th>(\psi_{\max}) (\mu_2 = 36.5)</th>
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<tr>
<td>z = 0</td>
<td>0.500000</td>
<td>0.500000</td>
<td>0.595256</td>
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<td>z = 9</td>
<td>0.073517</td>
<td>0.452162</td>
<td>0.219333</td>
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<td>z = 12</td>
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<td>0.009112</td>
<td>0.399747</td>
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<td>z = 24</td>
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<td>0.362047</td>
<td>0.079057</td>
<td>0.141021</td>
<td>0.180833</td>
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</table>

Table 9: Bounds on the eventual probability of ruin \((q = 0.1, \mu_X = 5.5, n = 50, \mu_2 = 30.5, 32.5, 36.5)\).
It can also be extended to any number of moments superior to three. However, analytical computations become more challenging. For example, if four moments $\mu_1, \mu_2, \mu_3$ and $\mu_4$ are known, satisfying $0 < \mu_1 < \mu_2 < \mu_3 < \mu_4$, we need to show the existence of a polynomial $p_4(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$ such that $p_4(x) \geq (x - d)_+$ for all $x \in D_n$ with an equality holding for the points of the support of a discrete distribution on $D_n$, with the given moments $\mu_1, \mu_2, \mu_3$ and $\mu_4$. As previously, taking expectation in the previous inequality gives an upper bound on the stop-loss premium of distributions on $D_n$ with given four moments, and this upper bound is attained for the mentioned discrete distribution.

First, it is easy to prove that a five-atomic random variable valued in $D_n$ with first four moments $\mu_1, \mu_2, \mu_3$ and $\mu_4$ and with support $\{k_1, k_2, k_3, k_4, k_5\}$ ($0 \leq k_1 < k_2 < k_3 < k_4 < k_5 \leq n$) must satisfy the following conditions

1. $\mu_4 - \mu_3(k_2 + k_3 + k_4 + k_5) + \mu_2(k_2 k_3 + k_2 k_4 + k_2 k_5 + k_3 k_4 + k_3 k_5 + k_4 k_5) - \mu_1(k_2 k_3 k_4 + k_2 k_3 k_5 + k_2 k_4 k_5 + k_3 k_4 k_5) + k_2 k_3 k_4 k_5 \geq 0,$

2. $\mu_4 - \mu_3(k_1 + k_3 + k_4 + k_5) + \mu_2(k_1 k_3 + k_1 k_4 + k_1 k_5 + k_3 k_4 + k_3 k_5 + k_4 k_5) - \mu_1(k_1 k_3 k_4 + k_1 k_3 k_5 + k_1 k_4 k_5 + k_1 k_4 k_5) + k_1 k_3 k_4 k_5 \leq 0,$

3. $\mu_4 - \mu_3(k_1 + k_2 + k_4 + k_5) + \mu_2(k_1 k_2 + k_1 k_4 + k_1 k_5 + k_2 k_4 + k_2 k_5 + k_4 k_5) - \mu_1(k_1 k_2 k_4 + k_1 k_2 k_5 + k_1 k_4 k_5 + k_1 k_4 k_5) + k_1 k_2 k_4 k_5 \geq 0,$

4. $\mu_4 - \mu_3(k_1 + k_2 + k_3 + k_5) + \mu_2(k_1 k_2 + k_1 k_3 + k_1 k_5 + k_2 k_3 + k_2 k_5 + k_3 k_5) - \mu_1(k_1 k_2 k_3 + k_1 k_2 k_5 + k_1 k_3 k_5 + k_2 k_3 k_5) + k_1 k_2 k_3 k_5 \leq 0,$

5. $\mu_4 - \mu_3(k_1 + k_2 + k_3 + k_4) + \mu_2(k_1 k_2 + k_1 k_3 + k_1 k_4 + k_2 k_3 + k_2 k_4 + k_3 k_4) - \mu_1(k_1 k_2 k_3 + k_1 k_2 k_4 + k_1 k_3 k_4 + k_2 k_3 k_4) + k_1 k_2 k_3 k_4 \geq 0.$

Then, remark that we shall have 2 (resp. 3, 4) crossing points on $(-\infty, d]$ and 4 (resp. 3, 2) on $[d, +\infty)$ with at least five of these points being in $D_n$. The admissible polynomials of degree four are thus in one of the next 14 types:

- **Case 1:** $k_1 = y \in (-\infty, 0), k_2 = 0, k_3 \in D_n, k_4 = k_3 + 1, k_5 \in D_n$ and $k_6 = k_5 + 1,$
- **Case 2:** $k_1 \in D_n, k_2 = k_1 + 1, k_3 \in D_n, k_4 = k_3 + 1, k_5 \in D_n$ and $|y - k_5| \leq 1 (k_6 = y),$
- **Case 3:** $k_1 \in D_n, k_2 = k_1 + 1, k_3 \in D_n, |y - k_3| \leq 1 (k_4 = y), k_5 \in D_n$ and $k_6 = k_5 + 1,$
- **Case 4:** $k_1 \in D_n, |y - k_1| \leq 1 (k_2 = y), k_3 \in D_n, k_4 = k_3 + 1, k_5 \in D_n$ and $k_6 = k_5 + 1,$
- **Case 5:** $k_1 \in D_n, k_2 = k_1 + 1, k_3 \in D_n, k_4 = k_3 + 1, k_5 = n$ and $y \in (n, +\infty)$ ($k_6 = y),$
- **Case 6:** $y \in (-\infty, 0) (k_1 = y), k_2 \in D_n, k_3 = k_2 + 1, k_4 \in D_n$ and $k_5 = k_4 + 1$ and $k_6 = n.$

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• Case 7: $k_1 = 0, k_2 \in D_n, k_3 = k_2 + 1, k_4 \in D_n, |y - k_4| \leq 1 (k_5 = y)$ and $k_6 = n$.

• Case 8: $k_1 = 0, k_2 \in D_n, |y - k_2| \leq 1 (k_3 = y), k_4 \in D_n, k_5 = k_4 + 1$ and $k_6 = n$.

• Case 9: $k_1 = 0, k_2 \in D_n, k_3 = k_2 + 1, k_4 \in D_n$ and $k_5 = k_4 + 1$ and $y \in (n, +\infty)$ ($k_6 = y$).

• Case 10: $y \in (-\infty, 0)$ ($k_1 = y$), $k_2 = 0, k_3 \in D_n, k_4 = k_3 + 1, k_5 \in D_n$ and $k_6 = k_5 + 1$.

• Case 11: $k_1 \in D_n, k_2 = k_1 + 1, k_3 \in D_n, k_4 = k_3 + 1, k_5 \in D_n$ and $|y - k_5| \leq 1$ ($k_6 = y$).

• Case 12: $k_1 \in D_n, k_2 = k_1 + 1, k_3 \in D_n, |y - k_3| \leq 1 (k_4 = y), k_5 \in D_n$ and $k_6 = k_5 + 1$.

• Case 13: $k_1 \in D_n, |y - k_2| \leq 1 (k_2 = y), k_3 \in D_n, k_4 = k_3 + 1, k_5 \in D_n$ and $k_6 = k_5 + 1$.

• Case 14: $k_1 \in D_n, k_2 = k_1 + 1, k_3 \in D_n, k_4 = k_3 + 1, k_5 = n$ and $y \in (n, +\infty)$ ($k_6 = y$).

Cases 1, 2, 3, 4 and 5 correspond to 2 crossing points on $(-\infty, d]$ (denoted $k_1$ and $k_2$) and to 4 on $[d, +\infty)$ (denoted $k_3, k_4, k_5$ and $k_6$). Cases 6, 7, 8 and 9 correspond to 3 crossing points on $(-\infty, d]$ (denoted $k_1, k_2$ and $k_3$) and to 3 on $[d, +\infty)$ (denoted $k_4, k_5$ and $k_6$). Cases 10, 11, 12, 13 and 14 correspond to 4 crossing points on $(-\infty, d]$ (denoted $k_1, k_2, k_3$ and $k_4$) and to 2 on $[d, +\infty)$ (denoted $k_5$ and $k_6$). Of course, the computations become heavier.

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References


