Abstract

Extremal distributions have been extensively used in the actuarial literature in order to derive bounds on functionals of the underlying risks, such as stop-loss premiums or ruin probabilities, for instance. In this paper, the idea is extended to a dynamic setting. Specifically, convex bounds on multiplicative processes are derived. Despite their relative simplicity, the extremal processes produce reasonably accurate bounds on option prices in the classical trinomial model for incomplete markets.

Keywords: Convex order, extremal distributions, incomplete market, risk-neutral martingales, option pricing, trinomial model.

1 Introduction

Stochastic orderings are probabilistic tools to compare random variables or random vectors. Mathematically speaking, they are partial order relations defined on sets of probability distributions. Many papers have been devoted to the derivation of bounds in some stochastic order on a given random variable $S$. These bounds use some information about the random variable $S$, like moments, support, unimodality, etc. Relying on extrema with respect to some order relation, the actuary acts in a conservative way by basing his decisions on the least attractive risk that is consistent with the incomplete available information. The extrema correspond to the “worst” and the “best” risk. See, e.g., Denuit, De Vylder & Lefèvre (1999) and the references therein.

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In this paper, we will use the convex order, defined as follows: given two random variables $S$ and $T$, $S$ is said to be smaller than $T$ in the convex order, denoted as $S \preceq_{cx} T$, if the inequality $E[\phi(S)] \leq E[\phi(T)]$ holds for all the convex functions $\phi : \mathbb{R} \to \mathbb{R}$, provided the expectations exist. The intuitive meaning of $S \preceq_{cx} T$ is that $S$ is less variable than $T$. The multivariate version of $\preceq_{cx}$ is easily obtained by considering convex functions $\phi : \mathbb{R}^n \to \mathbb{R}$. However, this ordering does not allow for interesting applications (so that we will rather consider in this paper $\preceq_{cx}$-inequalities among linear combinations of the components of the random vectors to be compared, as explained below).

In this paper, we consider multiplicative discrete-time processes \( \{X_n, n = 1, 2, \ldots\} \) obtained as follows. Starting from a sequence \( \{Y_n, n = 1, 2, \ldots\} \) of positive independent random variables, we define recursively the \( X_n \)'s as \( X_{n+1} = X_n Y_{n+1}, \ n = 1, 2, \ldots \) with \( X_1 = Y_1 \). Such a process can be seen as a multiplicative random walk with relative increase \( Y_n \) at time \( n \). It is widely used in finance to model the price of financial instruments (where \( X_n \) is the exponential of some process with independent increments). Our aim is to derive lower and upper bounds on the process \( \{X_n, n = 1, 2, \ldots\} \) in the sense that any positive linear combination of the \( X_n \)'s is bounded in the convex order by the corresponding linear combinations of the components of the extremal processes. This is similar to the works by Koshevoy & Mosler (1996,1997,1998) where orderings between random vectors \( \mathbf{X} \) and \( \mathbf{Y} \) defined by $a_1 X_1 + a_2 X_2 + \cdots + a_n X_n \preceq_{cx} a_1 Y_1 + a_2 Y_2 + \cdots + a_n Y_n$ for all constants \( a_1, a_2, \ldots, a_n \) are studied.

The results derived in this paper are applied to discrete-time contingent claims pricing models. The underlying assets are assumed to follow a discrete-time process and trading only takes place at some prespecified dates. In this paper, we consider an incomplete market framework, so that the risk-neutral probability measure is not unique and we are in presence of a class of risk-neutral measures. The aim is thus to find the risk-neutral probability measures that imply the lower and upper bounds on the price of the claim and that are elements of the class of admissible prices. Examples within a trinomial model (i.e. a model where the change in the value of the stock between two trading times can attain three different values) are discussed.

The connection between the papers devoted to extremal distributions that appeared in the actuarial literature and financial pricing in incomplete markets is as follows. The class of risk-neutral probability measures is considered as a class of distributions with fixed support and first moment. Then, extremal elements are identified within the set of risk-neutral distributions, leading to bounds on the prices of contingent claims. This bridge between actuarial risk theory and financial mathematics seems to be promising.

The paper is organized as follows. The extremal processes are built in Section 2. Section 3 describes the application to financial pricing in the trinomial model. Numerical illustrations are provided there. The final Section 4 concludes.
2 Extremal processes

2.1 Definitions

Let us denote as $Y_i^-$ and $Y_i^+$ two positive random variables such that $Y_i^- \preceq_{cx} Y_i \preceq_{cx} Y_i^+$ holds for all $i$. Assume for instance that the support of $Y_i$ is in $[a_i, b_i]$ and that $\mathbb{E}[Y_i] = \mu_i$. Then if we define the random variables $Y_i^-$ and $Y_i^+$ as $Y_i^- = \mu_i$ almost surely, and

$$Y_i^+ = \begin{cases} a_i & \text{with probability } \frac{b_i - \mu_i}{b_i - a_i} \\ b_i & \text{with probability } \frac{\mu_i - a_i}{b_i - a_i}, \end{cases}$$

we have $Y_i^- \preceq_{cx} Y_i \preceq_{cx} Y_i^+$. Other choices for the $\preceq_{cx}$-bounds are possible, according to the amount of information available about the $Y_i$’s (support, moments, unimodality, ageing notions, etc.). See, e.g., Courtois & Denuit (2005) and the references therein.

All the random variables $Y_1, Y_2, \ldots, Y_i^-, Y_i^+, \ldots$ are assumed to be independent. Starting from $X^-_1 = Y_1^-$ and $X^+_1 = Y_1^+$, we define the extremal processes $\{X^-_n, n = 1, 2, \ldots\}$ and $\{X^+_n, n = 1, 2, \ldots\}$ by $X^-_i = X^-_{i-1} Y^-_i$ and $X^+_i = X^+_{i-1} Y^+_i$ for $i = 2, 3, \ldots$.

2.2 Convex ordered marginals

We expect that a convex ordering holds between $X^-_i$, $X^+_i$ and $X_i$. To prove that this is indeed the case, we will need the following useful lemma.

**Lemma 2.1.** Let $T_1, T_2, Z_1, Z_2$ be independent and positive random variables such that $T_1 \preceq_{cx} T_2$ and $Z_1 \preceq_{cx} Z_2$. Then, $T_1 Z_1 \preceq_{cx} T_2 Z_2$ holds.

**Proof.** Let $\phi$ be a convex function, and let us denote as $F_{T_1}, F_{T_2}, F_{Z_1}$ and $F_{Z_2}$ the distribution functions of $T_1, T_2, Z_1$ and $Z_2$, respectively. From

$$\mathbb{E}[\phi(T_1 Z_1)] = \int_0^\infty \mathbb{E}[\phi(t Z_1)] dF_{T_1}(t)$$

$$\leq \int_0^\infty \mathbb{E}[\phi(t Z_2)] dF_{T_1}(t) \text{ since } Z_1 \preceq_{cx} Z_2$$

$$= \int_0^\infty \mathbb{E}[\phi(T_2 z)] dF_{Z_2}(z)$$

$$\leq \int_0^\infty \mathbb{E}[\phi(T_2 z)] dF_{Z_2}(z) \text{ since } T_1 \preceq_{cx} T_2$$

$$= \mathbb{E}[\phi(T_2 Z_2)],$$

we conclude that the announced $\preceq_{cx}$-inequality indeed holds. □

We are now ready to prove the next result that shows that the processes $\{X^-_n, n = 1, 2, \ldots\}$, $\{X_n, n = 1, 2, \ldots\}$ and $\{X^+_n, n = 1, 2, \ldots\}$ have indeed $\preceq_{cx}$-ordered univariate marginals.
 Proposition 2.2. The stochastic inequalities $X_i^- \preceq_{cx} X_i \preceq_{cx} X_i^+$ hold for all $i$.

Proof. Let us prove the announced result using an iterative argument. The result is obviously true for $i = 1$, since it reduces to $Y_1^- \preceq_{cx} Y_1 \preceq_{cx} Y_1^+$. Now, assume that the result holds for $i = 1, 2, \ldots, n$ and let us prove it for $n + 1$. Let us apply Lemma 2.1 in our setting. Taking $T_1 = T_2 = X_n^-$ and $Z_1 = Y_{n+1}^-$, $Z_2 = Y_{n+1}$, we get

$$X_n Y_{n+1} = X_{n+1} \preceq_{cx} X_n Y_{n+1}.$$  

Now, taking $T_1 = T_2 = Y_{n+1}$ and $Z_1 = X_n^-$, $Z_2 = X_n$, we have

$$X_n Y_{n+1} \preceq_{cx} X_n Y_{n+1} = X_{n+1}.$$  

We then conclude that $X_{n+1}^- \preceq_{cx} X_{n+1}$ by transitivity. The proof of $X_{n+1} \preceq_{cx} X_{n+1}^+$ follows along the same lines.

\[\square\]

2.3 Convex ordered linear combinations

Let us now prove that any positive linear combination of the $X_i$’s is bounded from below and from above in the $\preceq_{cx}$-sense by the same combination of the $X_i^-$’s and of the $X_i^+$’s.

Proposition 2.3. Whatever the positive constants $\alpha_1, \ldots, \alpha_n$, the stochastic inequalities

$$\sum_{j=1}^{n} \alpha_j X_{i_j}^- \preceq_{cx} \sum_{j=1}^{n} \alpha_j X_{i_j} \preceq_{cx} \sum_{j=1}^{n} \alpha_j X_{i_j}^+$$

hold for any $i_1 < i_2 < \ldots < i_n$ and integer $n$.

Proof. We only prove the stochastic inequality $\sum_{j=1}^{n} \alpha_j X_{i_j}^- \preceq_{cx} \sum_{j=1}^{n} \alpha_j X_{i_j}$; the reasoning to establish $\sum_{j=1}^{n} \alpha_j X_{i_j} \preceq_{cx} \sum_{j=1}^{n} \alpha_j X_{i_j}^+ \preceq_{cx}$ is similar. The result is obviously true for $n = 1$. Let us first establish the result for $n = 2$. To this end, let us write

$$\alpha_1 X_{i_1}^- + \alpha_2 X_{i_2}^- = X_{i_1}^- (\alpha_1 + \alpha_2 Y_{i_1+1} \ldots Y_{i_2})$$

and $\alpha_1 X_{i_1} + \alpha_2 X_{i_2} = X_{i_1} (\alpha_1 + \alpha_2 Y_{i_1+1} \ldots Y_{i_2})$.

Since $Y_{i_1+1} \ldots Y_{i_2} \preceq_{cx} Y_{i_1+1} Y_{i_2} Y_{i_2}$ and since $\preceq_{cx}$ is closed under changes of scale and origin, Lemma 2.1 then gives $\alpha_1 X_{i_1}^- + \alpha_2 X_{i_2}^- \preceq_{cx} \alpha_1 X_{i_1} + \alpha_2 X_{i_2}$, as announced. Now, let us assume that the result holds for $n$ and let us establish it for $n + 1$. First, note that

$$\alpha_1 X_{i_1}^- + \ldots + \alpha_n X_{i_n}^- + X_{i_{n+1}}^- (\alpha_n + \alpha_{n+1} Y_{i_{n+1}} \ldots Y_{i_{n+1}})$$

The recurrence relation ensures that, given $Y_{i_{n+1}} \ldots Y_{i_n} = t$, the stochastic inequality

$$\alpha_1 X_{i_1}^- + \ldots + X_{i_n}^- (\alpha_n + \alpha_{n+1} Y_{i_{n+1}} \ldots Y_{i_n}) \preceq_{cx} \alpha_1 X_{i_1} + \ldots + X_{i_n} (\alpha_n + \alpha_{n+1} Y_{i_{n+1}} \ldots Y_{i_n})$$

holds true. Since $Y_{i_{n+1}} \ldots Y_{i_n}$ is independent from both $X_{i_1}, \ldots, X_{i_n}$ and $X_{i_1}, \ldots, X_{i_n}$, the $\preceq_{cx}$-inequality also holds unconditionally, so that we get

$$\alpha_1 X_{i_1}^- + \ldots + \alpha_n X_{i_n}^- \preceq_{cx} \alpha_1 X_{i_1} + \ldots + X_{i_n} (\alpha_n + \alpha_{n+1} Y_{i_n+1} \ldots Y_{i_{n+1}})$$

since $Y_{i_{n+1}} \ldots Y_{i_n} \preceq_{cx} Y_{i_{n+1}} \ldots Y_{i_{n+1}}$. This ends the proof. \[\square\]
3 Applications to the trinomial model for stock prices

3.1 Description of the model

In the trinomial asset pricing model, we begin with an initial stock price $S_0 = 1$. There are three possible numbers, $d$, $1$ and $u$, with $0 < d < 1 < u$, such that at the next period, the stock price will be either $dS_0$, $S_0$ or $uS_0$. Typically, we take $d$ and $u$ to satisfy $0 < d < 1 < u$, so change of the stock price from $S_0$ to $dS_0$ represents a downward movement, and change of the stock price from $S_0$ to $uS_0$ represents an upward movement. Therefore, at each time step, the stock price either goes up by a factor $u$ or down by a factor $d$ or does not move.

\[ S_{n+1} = S_n \cdot J_{n+1} \]

The physical probabilities associated with the downward ($d$), stationary ($1$) and upward ($u$) movements are respectively equal to 10%, 51.26% and 38.74% as in Hull (2002). The financial pricing of contingent claims is not made under the physical (or historical) probability distribution, but well under the risk-neutral one. Recall that a risk-neutral probability measure is a measure that agrees with the physical probability measure about which price paths have zero probability, and under which the discounted prices of all primary assets are martingales. The condition for the model to be free of arbitrage oppor-
tunities is the existence of a risk-neutral probability measure and the price is then obtained by taking the expectation of the discounted payoff under such a measure.

If there exist claims that are not attainable, then the market is said to be incomplete. In this case there are infinitely many risk-neutral measures. The trinomial model is known to be incomplete. The space $\mathcal{P}$ of all risk-neutral probability measures is taken such that under all risk-neutral probability measures $\tilde{\mathbb{P}}$, the discounted stock price process $\{\tilde{S}_{n+1}/(1+r)^{n+1}, n = 0, 1, \ldots\}$ is a martingale with respect to the natural filtration, i.e.

$$
\mathbb{E}_{\tilde{\mathbb{P}}} \left[ \frac{S_{n+1}}{(1+r)^{n+1}} \right] S_0, S_1, \ldots, S_n = \frac{S_n}{(1+r)^n} \quad \text{for any } \tilde{\mathbb{P}} \in \mathcal{P}.
$$

### 3.2 The set $\mathcal{P}$ of risk-neutral probability measures

Let us denote as $X_n$ the discounted stock price, that is, $X_n = \frac{S_n}{(1+r)^n}$, starting from $X_0 = S_0$. The process $\{X_n, \ n = 1, 2, \ldots\}$ admits the representation $X_n = X_{n-1}Y_n$ with

$$(1 + r)Y_n = \begin{cases} 
\tilde{p}_2 & \text{with probability } \tilde{p}_2, \\
1 & \text{with probability } \tilde{p}_1, \\
\tilde{p}_3 & \text{with probability } \tilde{p}_3.
\end{cases}$$

By convention, $X_1 = Y_1$.

Within the trinomial model, every risk-neutral probability measure $\tilde{\mathbb{P}}$ corresponds to a triplet $(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ of positive real numbers satisfying $\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 = 1$, where the risk-neutral probabilities $\tilde{p}_1, \tilde{p}_2$ and $\tilde{p}_3$ are respectively associated with a stationary $(d)$, downward $(u)$ and upward $(u)$ movement of the stock price process. The class $\mathcal{P}$ of risk-neutral probability measures can then be identified with the set of admissible triplets.

All the risk-neutral probability measures, henceforth denoted as $\tilde{\mathbb{P}} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$, must be equivalent to the historical measure (in the sense that the set of events that have probability 0 under $\tilde{\mathbb{P}}$ is the same as the set of events that have probability 0 under the physical measure $\mathbb{P}$) and such that $\mathbb{E}_{\tilde{\mathbb{P}}}[X_{n+1}|X_n] = X_n$ for all $n$. So $(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ have to verify the following system

$$
\begin{cases}
\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 = 1 \\
\tilde{p}_2 \cdot d + \tilde{p}_1 \cdot 1 + \tilde{p}_3 \cdot u = 1 + r
\end{cases}
$$

with $0 < \tilde{p}_i < 1$ ($i = 1, 2, 3$). This is equivalent to say that all risk-neutral probability measures $(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ must be such that $\tilde{p}_1 = 1 - \frac{r}{u-1} - \tilde{p}_2 \frac{u-d}{u-1}$ and $\tilde{p}_3 = \frac{r}{u-1} + \tilde{p}_2 \frac{1-d}{u-1}$ with $0 < \tilde{p}_2 < \frac{u-1}{u-d}$.

### 3.3 Extremal price processes

Denuit & Lefèvre (1997) and Denuit, Lefèvre & Mesfioui (1999) derived $\preceq_{cx}$-bounds on random variables valued in $\{0, 1, \ldots, n\}$. These extremal distributions can be generalized to the case of random variables valued in an arbitrary set $\mathcal{E}_n = \{e_0, \ldots, e_n\}$,
with \(e_0 < e_1 < \ldots < e_n\), in the spirit of Denut, Lefèvre & Utev (1999). Specifically, consider a random variable \(S\) valued in \(\mathcal{E}_n\) with mean \(\mu\). Defining
\[
S_{\text{disc}}^\text{min} = \left\{ e_k \right\} \text{ with probability } \frac{e_{k+1} - \mu}{e_{k+1} - e_k},
\]
where \(e_k \in \mathcal{E}_{n-1}\) is such that \(e_k < \mu \leq e_{k+1}\), and
\[
S_{\text{disc}}^\text{max} = \left\{ e_0 \right\} \text{ with probability } \frac{\mu - e_0}{e_n - e_0},
\]
we have \(S_{\text{disc}}^\text{min} \preceq \mathcal{S} \preceq \mathcal{S}_{\text{disc}}^\text{max}\). Knowing that \(\mathbb{E}_\mathcal{P}[Y_n] = 1\) for all \(n\), we see easily that the random variables \(Y_n^-\) and \(Y_n^+\) such that the stochastic inequalities \(Y_n^- \preceq Y_n \preceq Y_n^+\) hold true are defined by
\[
(1 + r)Y_n^- = \left\{ \begin{array}{ll}
1 & \text{with probability } \frac{u-(1+r)}{u-1}, \\
u & \text{with probability } \frac{u-1}{u-1},
\end{array} \right.
\]
and
\[
(1 + r)Y_n^+ = \left\{ \begin{array}{ll}
d & \text{with probability } \frac{u-(1+r)}{u-d}, \\
u & \text{with probability } \frac{u-1}{u-d}.
\end{array} \right.
\]
The processes \(\{X_n^-, n = 1, 2, \ldots\}\) and \(\{X_n^+, n = 1, 2, \ldots\}\) are then defined by \(X_n^- = X_{n-1}^- Y_n^-\) and \(X_n^+ = X_{n-1}^+ Y_n^+\), starting from \(X_1^- = Y_1^-\) and \(X_1^+ = Y_1^+\).

The stochastic processes \(\{X_n^-, n = 1, 2, \ldots\}\) and \(\{X_n^+, n = 1, 2, \ldots\}\) are trinomial models with probabilities associated to \((1, d, u)\) being respectively \(\left(\frac{u-(1+r)}{u-1}, 0, \frac{r}{u-1}\right)\) and \(\left(0, \frac{u-(1+r)}{u-d}, \frac{(1+r)-d}{u-d}\right)\). These two sets of probabilities do not correspond to risk neutral measures (since the support is not the physical one). The two extremal processes are obtained by letting the probability associated to \(d\) (i.e. \(\tilde{P}_n\)) converging to its minimal and maximal possible values (i.e. \(0\) and \(\frac{u-(1+r)}{u-d}\)). Figure 2 describes 100 trajectories of the minimal and the maximal processes with \(u = 1.1, d = 0.9\).

### 3.4 Numerical results

#### 3.4.1 European call option

The owner of a European call option has the right to buy a stock for \(K\) (strike price) at a certain future time \(N\). We denote by \(S_0\) the current value of the stock price and we make the assumption that the considered stock price follows a trinomial model with \(N\) periods of time. Considering \(\tilde{\mathcal{P}}\) in the set \(\mathcal{P}\) of risk-neutral probability measures, a possible price of this European call is given by \(\frac{1}{(1+r)^N}\mathbb{E}_{\tilde{\mathcal{P}}}[S_N - K]_+\). Every possible price satisfies
\[
\frac{1}{(1+r)^N}\mathbb{E}_{\tilde{\mathcal{P}}}[S_N - K]_+ \leq \frac{1}{(1+r)^N}\mathbb{E}_{\tilde{\mathcal{P}}}[S_N - K]_+ \leq \frac{1}{(1+r)^N}\mathbb{E}_{\tilde{\mathcal{P}}}[S_N^+ - K]_+,
\]
Figure 2: 100 trajectories of the extremal trinomial process.
where

\[ \mathbb{E}_\tilde{P}[(S_N^- - K)_+] = \sum_{i=0}^{N} \binom{N}{i} \left( \frac{r}{u - 1} \right)^i \left( \frac{u - (1 + r)}{u - 1} \right)^{N-i} (S_0 u^i - K)_+ \]

and

\[ \mathbb{E}_\tilde{P}((S_N^+ - K)_+) = \sum_{i=0}^{N} \binom{N}{i} \left( \frac{(1 + r) - d}{u - d} \right)^i \left( \frac{u - (1 + r)}{u - d} \right)^{N-i} (S_0 u^i d^{N-i} - K)_+ . \]

Table 1 displays the bounds obtained on the call price for different maturities and strike prices. As in Hull (2002), we consider \( u = 1.1 \) and \( d = 0.9 \). The annual risk-free rate is 12%. A period of time corresponds to 3 months. The range of possible values for the price of the call option is not too large.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( K )</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
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<td>0.0938558</td>
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<td>0.02793458</td>
<td>0.0625706</td>
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<tr>
<td></td>
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<td>0.01396729</td>
<td>0.0312853</td>
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<tr>
<td>6 months</td>
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</tr>
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<td>0.0626412</td>
</tr>
<tr>
<td>1 year</td>
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<td>0.1714171</td>
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<tr>
<td></td>
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<tr>
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<td>1.05</td>
<td>0.1655976</td>
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Table 1: Bounds on the price of a European call option.

3.4.2 Asian call option

An arithmetic Asian call option with exercise date \( N \), exercise price \( K \) and \( M \) averaging dates generates a pay-off \( \left( \frac{1}{M} \sum_{i=1}^{M-1} S_{N-i} - K \right)_+ \). This contingent claim is traded at time 0 for a price

\[ \frac{1}{(1 + r)^N} \mathbb{E}_{\tilde{P}} \left[ \left( \frac{1}{M} \sum_{i=1}^{M-1} S_{N-i} - K \right)_+ \right] \]

where \( \tilde{P} \) is in the set \( \mathcal{P} \) of all risk-neutral probability measures.

As a numerical illustration, we consider the same parameter values as for the European call. Moreover, the averaging dates are taken to be all the dates during the life of the option
(including maturity), i.e. $M = N$. Results are displayed in Table 2. The bounds displayed in Table 2 are computed by simulation using 10,000 random gene rations (standard errors attached to these approximations are also given). Again, the intervals of admissible prices is not too large.

<table>
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<th>$K$</th>
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<th>Std Error</th>
<th>Maximum</th>
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<tr>
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<td>0.0684%</td>
<td>0.1022506</td>
<td>0.1161%</td>
</tr>
</tbody>
</table>

Table 2: Bounds on the price of an Asian call option.

4 Discussion

In this paper, extremal elements in the class of risk-neutral probability measures are investigated, leading to bounds on the prices of contingent claims. This promising approach also leaves some open questions. It is well-known that improvements of the $\preceq_{\text{cx}}$-bounds are possible when the underlying distributions are unimodal (and are given by mixtures of uniform distributions). See, e.g., Denuit, De Vijlder & Lefèvre (1999). Unimodality is often satisfied under the physical probability measure. An interesting question could be to investigate the possible transmission of unimodality to the class of risk-neutral distributions. The same problem could be investigated with ageing notions.

Of course, alternative approaches could be investigated. For instance, convex bounds on the conditional distributions could be derived. From the definition of the process $\{X_n, \ n = 1, 2, \ldots\}$ we see that $E[X_{n+1}|X_1, \ldots, X_n] = X_nE[Y_{n+1}]$. If $E[Y_n] = 1$ for all $n$ (as it is usually the case in the financial applications, after a suitable change of measure) then $E[X_{n+1}|X_1, \ldots, X_n] = X_n$. The idea is then to construct the extremal processes from the extremal conditional distributions from the knowledge of the support $(a, b)$ and the conditional mean $X_n$.

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