II.3. POLYNOMIAL ANALYSIS

Didier HENRION
henrion@laas.fr

Belgian Graduate School on
Systems, Control, Optimization and Networks

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Polynomial methods

Based on the algebra of **polynomials** and **polynomial matrices**, typically involve
- linear Diophantine equations
- quadratic spectral factorization

Pioneered in central Europe during the 70s mainly by Vladimír Kučera from the former Czechoslovak Academy of Sciences

Network funded by the European commission

[EUROPOLY](http://www.utia.cas.cz/europoly)

Polynomial matrices also occur in Jan Willems' behavioral approach to systems theory

**Alternative** to state-space methods developed during the 60s most notably by Rudolf Kalman in the USA, rather based on
- linear Lyapunov equations
- quadratic Riccati equations
A scalar transfer function can be viewed as the ratio of two polynomials.

**Example**
Consider the mechanical system:

\[ G(s) = \frac{y(s)}{u(s)} = \frac{1}{ms^2 + k_1 s + k_2} \]

Neglecting static and Coulomb frictions, we obtain the linear transfer function.
Similarly, a MIMO transfer function can be viewed as the ratio of polynomial matrices

\[ G(s) = N_R(s)D_R^{-1}(s) = D_L^{-1}(s)N_L(s) \]

the so-called matrix fraction description (MFD)

Lightly damped structures such as oil derricks, regional power models, earthquakes models, mechanical multi-body systems, damped gyroscopic systems are most naturally represented by second order polynomial MFDs

\[ (D_0 + D_1 s + D_2 s^2)y(s) = N_0 u(s) \]

Example
The (simplified) oscillations of a wing in an air stream is captured by properties of the quadratic polynomial matrix [Lancaster 1966]

\[
D(s) = \begin{bmatrix}
121 & 18.9 & 15.9 \\
0 & 2.7 & 0.64 \\
11.9 & 3.64 & 15.5
\end{bmatrix} + \begin{bmatrix}
7.66 & 2.45 & 2.1 \\
0.23 & 1.04 & 0.223 \\
0.6 & 0.756 & 0.658
\end{bmatrix}s + \begin{bmatrix}
17.6 & 1.28 & 2.89 \\
1.28 & 0.824 & 0.413 \\
2.89 & 0.413 & 0.725
\end{bmatrix}s^2
\]
First-order polynomial MFD

Example
RCL network

Applying Kirchoff’s laws and Laplace transform we get

\[
\begin{bmatrix}
1 & -Ls \\
Cs & 1 + RCs
\end{bmatrix}
\begin{bmatrix}
y_1(s) \\
y_2(s)
\end{bmatrix}
= \begin{bmatrix}
0 \\
Cs
\end{bmatrix} u(s)
\]

and thus the first-order left system MFD

\[
G(s) = \begin{bmatrix}
1 & -Ls \\
Cs & 1 + RCs
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
Cs
\end{bmatrix}.
\]
Second-order polynomial MFD

Example

mass-spring system

\[
\begin{align*}
& M \ddot{x} + C \dot{x} + Kx = 0 \\
& \text{where e.g. } n = 250, m_i = 1, \kappa_i = 5, \tau_i = 10 \text{ except } \kappa_1 = \kappa_n = 10 \text{ and } \tau_1 = \tau_n = 20
\end{align*}
\]

Vibration of system governed by 2nd-order differential equation \( M \ddot{x} + C \dot{x} + Kx = 0 \) where e.g. \( n = 250, m_i = 1, \kappa_i = 5, \tau_i = 10 \) except \( \kappa_1 = \kappa_n = 10 \) and \( \tau_1 = \tau_n = 20 \)

Quadratic matrix polynomial

\[
D(s) = Ms^2 + Cs + K
\]

with

\[
\begin{align*}
M &= I \\
C &= \text{tridiag}(-10, 30, -10) \\
K &= \text{tridiag}(-5, 15, -5).
\end{align*}
\]
More examples of polynomial MFDs

Higher degree polynomial matrices can also be found in *aero-acoustics* (3rd degree) or in the study of the spatial stability of the Orr-Sommerfeld equation for plane Poiseuille flow in *fluid mechanics* (4rd degree)

For more info see Nick Higham’s homepage at [www.ma.man.ac.uk/~higham](http://www.ma.man.ac.uk/~higham)
Stability analysis for polynomials

Well established theory - LMIs are of no use here!

Given a continuous-time polynomial

\[ p(s) = p_0 + p_1s + \cdots + p_{n-1}s^{n-1} + ps^n \]

with \( p_n > 0 \) we define its \( n \times n \) Hurwitz matrix

\[
H(p) = \begin{bmatrix}
p_{n-1} & p_{n-3} & 0 & 0 \\
p_n & p_{n-2} & \vdots & \vdots \\
0 & p_{n-1} & 0 & 0 \\
0 & p_n & p_0 & 0 \\
\vdots & \vdots & p_1 & 0 \\
0 & 0 & p_2 & p_0
\end{bmatrix}
\]

Hurwitz stability criterion: Polynomial \( p(s) \) is stable iff all principal minors of \( H(p) \) are \( > 0 \)

Adolf Hurwitz
(Hanover 1859 - Zürich 1919)
Robust stability analysis for polynomials

Analyzing stability robustness of polynomials is a little bit more interesting..

Here too computational complexity depends on the uncertainty model

In increasing order of complexity, we will distinguish between

• single parameter uncertainty $q \in [q_{\text{min}}, q_{\text{max}}]$
• interval uncertainty $q_i \in [q_{i\text{min}}, q_{i\text{max}}]$
• polytopic uncertainty $\lambda_1 q_1 + \cdots + \lambda_N q_N$
• multilinear uncertainty $q_0 + q_1 \cdot q_2 \cdot q_3$

LMIs will not show up very soon..
..just basic linear algebra
Single parameter uncertainty and eigenvalue criterion

Consider the uncertain polynomial

\[ p(s, q) = p_0(s) + qp_1(s) \]

where

- \( p_0(s) \) nominally stable with positive coefs
- \( p_1(s) \) such that \( \deg p_1(s) < \deg p_0(s) \)

The largest stability interval

\[ q \in ]q_{\text{min}}, q_{\text{max}}[ \]

such that \( p(s, q) \) is robustly stable is given by

\[
q_{\text{max}} = \frac{1}{\lambda_{\text{max}}^+(-H_0^{-1}H_1)} \\
q_{\text{min}} = \frac{1}{\lambda_{\text{min}}^-(-H_0^{-1}H_1)}
\]

where \( \lambda_{\text{max}}^+ \) is the max positive real eigenvalue
\( \lambda_{\text{min}}^- \) is the min negative real eigenvalue
\( H_i \) is the Hurwitz matrix of \( p_i(s) \)
Higher powers of a single parameter

Now consider the continuous-time polynomial

\[ p(s, q) = p_0(s) + qp_1(s) + q^2p_2(s) + \cdots + q^m p_m(s) \]

with \( p_0(s) \) stable and \( \text{deg} \ p_0(s) > \text{deg} \ p_i(s) \)

Using the zeros (roots of determinant) of the polynomial Hurwitz matrix

\[ H(p) = H(p_0) + qH(p_1) + q^2H(p_2) + \cdots + q^m H(p_m) \]

we can show that

\[ q_{\text{min}} = \frac{1}{\lambda_{\text{min}}^{-1}(M)} \]
\[ q_{\text{max}} = \frac{1}{\lambda_{\text{max}}^{-1}(M)} \]

where

\[
M = \begin{bmatrix}
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & I & 0 \\
0 & 0 & \cdots & I \\
-H_0^{-1}H_m & \cdots & -H_0^{-1}H_2 & -H_0^{-1}H_1
\end{bmatrix}
\]

is a block companion matrix
MIMO systems

Uncertain multivariable systems are modeled by uncertain polynomial matrices

\[ P(s, q) = P_0(s) + qP_1(s) + q^2P_2(s) + \cdots + q^mP_m(s) \]

where \( p_0(s) = \det P_0(s) \) is a stable polynomial

We can apply the scalar procedure to the determinant polynomial

\[ \det P(s, q) = p_0(s) + qp_1(s) + q^2p_2(s) + \cdots + q^r p_r(s) \]

Example

MIMO design on the plant with left MFD

\[
A^{-1}(s, q)B(s, q) = \begin{bmatrix}
    s^2 & q \\
    q^2 + 1 & s
\end{bmatrix}^{-1}
\begin{bmatrix}
    s + 1 & 0 \\
    q & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    s^2 + s - q^2 & -q \\
    qs^2 - (q^2 + 1)s - (q^2 + 1) & s^2
\end{bmatrix}
\]

with uncertain parameter \( q \in [0, 1] \)
MIMO systems: example

Using some design method, we obtain a controller with right MFD

\[ Y(s)X^{-1}(s) = \begin{bmatrix} 94 - 51s & -18 + 17s \\ -55 & 100 \end{bmatrix} \begin{bmatrix} 55 + s & -17 \\ -1 & 18 + s \end{bmatrix} \]

Closed-loop system with characteristic denominator polynomial matrix

\[ D(s, q) = A(s, q)X(s) + B(s, q)Y(s) = D_0(s) + qD_1(s) + q^2D_2(s) \]

Nominal system poles: roots of \( \det D_0(s) \)

Applying the eigenvalue criterion on \( \det D(s, q) \) yields the stability interval

\[ q \in ]-0.93, 1.17[ \supset [0, 1] \]

so the closed-loop system is robustly stable
Independent uncertainty

So far we have studied polynomials affected by a single uncertain parameter

\[ p(s, q) = (6 + q) + (4 + q)s + (2 + q)s^2 \]

However in practice several parameters can be uncertain, such as in

\[ p(s, q) = (6 + q_0) + (4 + q_1)s + (2 + q_2)s^2 \]

**Independent** uncertainty structure: each component \( q_i \) enters into only one coefficient

**Interval** uncertainty: independent structure and uncertain parameter vector \( q \) belongs to a given box, i.e. \( q_i \in [q_i^-, q_i^+] \)

**Example**
Uncertain polynomial

\[ (6 + q_0) + (4 + q_1)s + (2 + q_2)s^2, \quad |q_i| \leq 1 \]

has interval uncertainty, also denoted as

\[ [5, 7] + [3, 5]s + [1, 3]s^2 \]

Some coefficients can be fixed, e.g.

\[ 6 + [3, 5]s + 2s^2 \]
Kharitonov’s polynomials

Associated with the interval polynomial
\[ p(s, q) = \sum_{i=0}^{n} [q_i^-, q_i^+] s^i \]
are four Kharitonov’s polynomials

\[
\begin{align*}
  p^{--}(s) &= q_0^- + q_1^- s + q_2^+ s^2 + q_3^+ s^3 + q_4^- s^4 + q_5^- s^5 + \cdots \\
  p^{-+}(s) &= q_0^- + q_1^+ s + q_2^+ s^2 + q_3^- s^3 + q_4^- s^4 + q_5^+ s^5 + \cdots \\
  p^{+-}(s) &= q_0^+ + q_1^- s + q_2^- s^2 + q_3^+ s^3 + q_4^- s^4 + q_5^- s^5 + \cdots \\
  p^{++}(s) &= q_0^+ + q_1^+ s + q_2^- s^2 + q_3^- s^3 + q_4^+ s^4 + q_5^+ s^5 + \cdots 
\end{align*}
\]

where we assume \( q_n^- > 0 \) and \( q_n^+ > 0 \)

**Example**
Interval polynomial
\[ p(s, q) = [1, 2] + [3, 4]s + [5, 6]s^2 + [7, 8]s^3 \]
Kharitonov’s polynomials

\[
\begin{align*}
  p^{--}(s) &= 1 + 3s + 6s^2 + 8s^3 \\
  p^{-+}(s) &= 1 + 4s + 6s^2 + 7s^3 \\
  p^{+-}(s) &= 2 + 3s + 5s^2 + 8s^3 \\
  p^{++}(s) &= 2 + 4s + 5s^2 + 7s^3 
\end{align*}
\]
Kharitonov’s theorem

In 1978 the Russian researcher Vladimír Kharitonov proved the following fundamental result

A continuous-time interval polynomial is robustly stable iff its four Kharitonov polynomials are stable.

Instead of checking stability of an infinite number of polynomials we just have to check stability of four polynomials, which can be done using the classical Hurwitz criterion.

Peter and Paul fortress in St Petersburg
Affine uncertainty

Sadly, Kharitonov’s theorem is valid only
• for continuous-time polynomials
• for independent interval uncertainty
so that we have to use more general tools in practice

When coefficients of an uncertain polynomial \( p(s, q) \) or
a rational function \( n(s, q)/d(s, q) \) depend affinely on parameter \( q \), such as in

\[
a^T q + b
\]

we speak about affine uncertainty

The above feedback interconnection

\[
\frac{n(s, q)x(s)}{d(s, q)x(s) + n(s, q)y(s)}
\]

preserves the affine uncertainty structure of the plant
Polytopes of polynomials

A family of polynomials $p(s, q)$, $q \in \mathbb{Q}$ is said to be a polytope of polynomials if
- $p(s, q)$ has an affine uncertainty structure
- $Q$ is a polytope

There is a natural isomorphism between a polytope of polynomials and its set of coefficients

**Example**

$p(s, q) = (2q_1 - q_2 + 5) + (4q_1 + 3q_2 + 2)s + s^2$, $|q_i| \leq 1$

Uncertainty polytope has 4 generating vertices

- $q^1 = [-1, -1]$  
- $q^2 = [-1, 1]$  
- $q^3 = [1, -1]$  
- $q^4 = [1, 1]$

Uncertain polynomial family has 4 generating vertices

- $p(s, q^1) = 4 - 5s + s^2$  
- $p(s, q^2) = 2 + s + s^2$  
- $p(s, q^3) = 8 + 3s + s^2$  
- $p(s, q^4) = 6 + 9s + s^2$

Any polynomial in the family can be written as

$$p(s, q) = \sum_{i=1}^{4} \lambda_i p(s, q^i), \quad \sum_{i=1}^{4} \lambda_i = 1, \quad \lambda_i \geq 0$$
Interval polynomials

Interval polynomials are a **special case** of polytopic polynomials

\[ p(s, q) = \sum_{i=0}^{n} [q_i^-, q_i^+] s^i \]

with at most \(2^{n+1}\) generating vertices

\[ p(s, q^k) = \sum_{i=0}^{n} q_i^k s^i, \quad q_i^k = \begin{cases} q_i^- & \text{or} \\ q_i^+ \end{cases} \quad 1 \leq k \leq 2^{n+1} \]

**Example**
The interval polynomial

\[ p(s, q) = [5, 6] + [3, 4]s + 5s^2 + [7, 8]s^3 + s^4 \]

can be generated by the \(2^3 = 8\) vertex polynomials

\[
\begin{align*}
p(s, q^1) & = 5 + 3s + 5s^2 + 7s^3 + s^4 \\
p(s, q^2) & = 6 + 3s + 5s^2 + 7s^3 + s^4 \\
p(s, q^3) & = 5 + 4s + 5s^2 + 7s^3 + s^4 \\
p(s, q^4) & = 6 + 4s + 5s^2 + 7s^3 + s^4 \\
p(s, q^5) & = 5 + 3s + 5s^2 + 8s^3 + s^4 \\
p(s, q^6) & = 6 + 3s + 5s^2 + 8s^3 + s^4 \\
p(s, q^7) & = 5 + 4s + 5s^2 + 8s^3 + s^4 \\
p(s, q^8) & = 6 + 4s + 5s^2 + 8s^3 + s^4 
\end{align*}
\]
The edge theorem

Let \( p(s, q), q \in Q \) be a polynomial with invariant degree over polytopic set \( Q \)

Polynomial \( p(s, q) \) is robustly stable over the whole uncertainty polytope \( Q \)
iff \( p(s, q) \) is stable along the edges of \( Q \)

In other words, it is enough to check robust stability of the single parameter polynomial
\[
\lambda p(s, q^{i_1}) + (1 - \lambda)p(s, q^{i_2}), \quad \lambda \in [0, 1]
\]
for each pair of vertices \( q^{i_1} \) and \( q^{i_2} \) of \( Q \)

This can be done with the eigenvalue criterion
Interval feedback system

Example
We consider the interval control system

\[
K(s, q) \rightarrow [6, 8]s^2 + [9.5, 10.5], \quad d(s, q) = s(s^2 + [14, 18])
\]

and characteristic polynomial

\[
K[9.5, 10.5] + [14, 18]s + K[6, 8]s^2 + s^3
\]

For \( K = 1 \) we draw the 12 edges of its root set

The closed-loop system is robustly stable
More about uncertainty structure

In typical applications, uncertainty structure is more complicated than interval or affine.

Usually, uncertainty enters highly non-linearly in the closed-loop characteristic polynomial.

We distinguish between
- multilinear uncertainty, when each uncertain parameter $q_i$ is linear when other parameters $q_j, i \neq j$ are fixed
- polynomial uncertainty, when coefficients are multivariable polynomials in parameters $q_i$

We can define the following hierarchy on the uncertainty structures:

interval $\subset$ affine $\subset$ multilinear $\subset$ polynomial
Examples of uncertainty structures

Examples
The uncertain polynomial

\[(5q_1 - q_2 + 5) + (4q_1 + q_2 + q_3)s + s^2\]

has affine uncertainty structure

The uncertain polynomial

\[(5q_1 - q_2 + 5) + (4q_1q_3 - 6q_1q_3 + q_3)s + s^2\]

has multilinear uncertainty structure

The uncertain polynomial

\[(5q_1 - q_2 + 5) + (4q_1 - 6q_1 - q_3^2)s + s^2\]

has polynomial (here quadratic) uncertainty structure

The uncertain polynomial

\[(5q_1 - q_2 + 5) + (4q_1 - 6q_1q_3^2 + q_3)s + s^2\]

has polynomial uncertainty structure
We will focus on multilinear uncertainty because it arises in a wide variety of system models such as:

- **multiloop systems**

\[
G_1 \quad \quad G_2 \quad \quad G_3
\]

\[
H_1 \quad \quad H_2
\]

Closed-loop transfer function

\[
\frac{y}{u} = \frac{G_1 G_2 G_3}{1 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3}
\]

- **state-space models with rank-one uncertainty**

\[
\dot{x} = A(q)x, \quad A(q) = \sum_{i=1}^{n} q_i A_i, \quad \text{rank } A_i = 1
\]

and characteristic polynomial

\[
p(s, q) = \det(sI - A(q))
\]

- **polynomial MFDs with MIMO interval uncertainty**

\[
G(s) = A^{-1}(s, q) B(s, q), \quad C(s) = Y(s)X^{-1}(s)
\]

and closed-loop characteristic polynomial

\[
p(s, q) = \det(A(s, q)X(s) + B(s, q)Y(s))
\]
Robust stability analysis for multilinear and polynomial uncertainty

Unfortunately, there is no systematic computational tractable necessary and sufficient robust stability condition

On the one hand, sufficient condition through polynomial value sets, the zero exclusion condition and the mapping theorem

On the other hand, brute-force method: intensive parameter gridding, expensive in general

No easy trade-off between computational complexity and conservatism
Checking robust stability can be
• easy (polynomial-time algorithms) or more
• difficult (exponential complexity)
depending namely on the uncertainty model

We focused on polytopic uncertainty:
• Interval scalar polynomials
  Kharitonov’s theorem (ct only)
• Polytope of scalar polynomials
  (affine polynomial families)
  Edge theorem
• Interval matrix polynomials
  (multi-affine polynomial families)
  Mapping theorem
• Polytopes of matrix polynomials
  (polynomic polynomial families)
Lessons from robust analysis: lack of extreme point results

Ensuring robust stability of the parametrized polynomial

\[ p(s, q) = p_0(s) + qp_1(s) \]

\[ q \in [q_{\min}, q_{\max}] \]

amounts to ensuring robust stability of the whole segment of polynomials

\[ \lambda p(s, q_{\min}) + (1 - \lambda)p(s, q_{\max}) \]

\[ \lambda = \frac{q_{\max} - q}{q_{\max} - q_{\min}} \in [0, 1] \]

A natural question arises: does stability of two vertices imply stability of the segment?

Unfortunately, the answer is no

Example
First vertex: \(0.57 + 6s + s^2 + 10s^3\) stable
Second vertex: \(1.57 + 8s + 2s^2 + 10s^3\) stable
But middle of segment:
\(1.07 + 7s + 1.50s^2 + 10s^3\) unstable
Lessons from robust analysis:
lack of edge results

In the same way there is lack of vertex results for affine uncertainty, there is a lack of edge results for multilinear uncertainty

Example
Consider the uncertain polynomial

\[
p(s, q) = (4.032q_1q_2 + 3.773q_1 + 1.985q_2 + 1.853) + (1.06q_1q_2 + 4.841q_1 + 1.561q_2 + 3.164)s + (q_1q_2 + 2.06q_1 + 1.561q_2 + 2.871)s^2 + (q_1 + q_2 + 2.56)s^3 + s^4
\]

with multilinear uncertainty over the polytope \(q_1 \in [0, 1], q_2 \in [0, 3]\), corresponding to the state-space interval matrix

\[
p(s, q) = \det(sI - \begin{bmatrix} [-1.5, -0.5] & -12.06 & -0.06 & 0 \\ -0.25 & -0.03 & 1 & 0.5 \\ 0.25 & -4 & -1.03 & 0 \\ 0 & 0.5 & 0 & [-4, 1] \end{bmatrix})
\]

The four edges of the uncertainty bounding set are stable, however for \(q_1 = 0.5\) and \(q_2 = 1\) polynomial \(p(s, q)\) is unstable..
Non-convexity of stability domain

Main problem: the stability domain in the space of polynomial coefficients $p_i$ is non-convex in general

Discrete-time stability domain in $(q_1, q_2)$ plane for polynomial $p(z, q) = (-0.825 + 0.225q_1 + 0.1q_2) + (0.895 + 0.025q_1 + 0.09q_2)z + (-2.475 + 0.675q_1 + 0.3q_2)z^2 + z^3$

How can we overcome the non-convexity of the stability conditions in the coefficient space?
Handling non-convexity

Basically, we can pursue two approaches:

• we can approximate the non-convex stability domain with a convex domain (segment, polytope, sphere, ellipsoid, LMI)

• we can address the non-convexity with the help of non-convex optimization (global or local optimization)
Stability polytopes

Largest hyper-rectangle around a nominally stable polynomial

\[ p(s) + r \sum_{i=0}^{n} [-\varepsilon_i, \varepsilon_i] s^i \]

obtained with the eigenvalue criterion applied on the 4 Kharitonov polynomials

In general, there is no systematic way to obtain more general stability polytopes, namely because of computational complexity
(no analytic formula for the volume of a polytope)

Well-known candidates:

- ct LHP: outer approximation (necessary stab cond)
  positive cone \( p_i > 0 \)

- dt unit disk: inner approximation (sufficient stab cond)
  diamond \( |p_0| + |p_1| + \cdots + |p_{n-1}| < 1 \)
Stability region (second degree)

**Necessary** stab cond in dt: convex hull of stability domain is a polytope whose \( n + 1 \) vertices are polynomials with roots +1 or -1

**Example**
When \( n = 2 \): triangle with vertices

\[
(z + 1)(z + 1) = 1 + 2z + z^2 \\
(z + 1)(z - 1) = -1 + z^2 \\
(z - 1)(z - 1) = 1 - 2z + z^2
\]
Stability region (third degree)

Example
Third degree dt polynomial: two hyperplanes and a non-convex hyperbolic paraboloid with a saddle point at \( p(z) = p_0 + p_1 z + p_2 z^2 + z^3 = z(1 + z^2) \)

\[
(z + 1)(z + 1)(z + 1) = 1 + 3z + 3z^2 + z^3 \\
(z + 1)(z + 1)(z - 1) = -1 - z + z^2 + z^3 \\
(z + 1)(z - 1)(z - 1) = 1 - z - z^2 + z^3 \\
(z - 1)(z - 1)(z - 1) = -1 + 3z - 3z^2 + z^3
\]
Stability ellipsoids

A weighted and rotated hyper-sphere is an ellipsoid

$\mathbb{E} = \{ p : (p - \bar{p})^* P (p - \bar{p}) \leq 1 \}$

We are interested in inner ellipsoidal approximations of stability domains

where

- $p$ coef vector of polynomial $p(s)$
- $\bar{p}$ center of ellipsoid
- $P$ positive definite matrix
Hermite stability criterion

The polynomial \( p(s) = p_0 + p_1s + \cdots + p_n s^n \) is stable if and only if

\[
H(x) = \sum_i \sum_j p_i p_j H_{ij} > 0
\]

where matrices \( H_{ij} \) are given and depend on the root clustering region only.

Examples for \( n = 3 \):

continuous-time stability

\[
H(p) = \begin{bmatrix}
2p_0 p_1 & 0 & 2p_0 p_3 \\
0 & 2p_1 p_2 - 2p_0 p_3 & 0 \\
2p_0 p_3 & 0 & 2p_2 p_3
\end{bmatrix}
\]

discrete-time stability

\[
H(p) = \begin{bmatrix}
p_3^2 - p_0^2 & p_2 p_3 - p_0 p_1 & p_1 p_3 - p_0 p_2 \\
p_2 p_3 - p_0 p_1 & p_2^2 + p_3^2 - p_0^2 - p_1^2 & p_2 p_3 - p_0 p_1 \\
p_1 p_3 - p_0 p_2 & p_2 p_3 - p_0 p_1 & p_3^2 - p_0^2
\end{bmatrix}
\]
Inner ellipsoidal approximation

Our objective is then to find \( \bar{p} \) and \( P \) such that the ellipsoid

\[
E = \{ p : (p - \bar{p})^* P (p - \bar{p}) \leq 1 \}
\]

is a convex inner approximation of the actual non-convex stability region

\[
S = \{ p : H(p) \succ 0 \}
\]

that is to say

\[
E \subset S
\]

Naturally, we will try to enlarge the volume of the ellipsoid as much as we can

Using the Hermite matrix formulation, we can derive (details omitted) a suboptimal LMI formulation (not given here) with decision variables \( \bar{p} \) and \( P \)
Stability ellipsoids

Example
Discrete-time second degree polynomial

\[ p(z) = p_0 + p_1 z + z^2 \]

We solve the LMI problem and we obtain

\[
\begin{bmatrix}
1.5625 & 0 \\
0 & 1.2501
\end{bmatrix} \quad \bar{p} = \begin{bmatrix}
0.2000 \\
0
\end{bmatrix}
\]

which describes an ellipse \( E \) inscribed in the exact triangular stability domain \( S \).
Stability ellipsoids

**Example**
Discrete-time third degree polynomial

\[ p(z) = p_0 + p_1z + p_2z^2 + z^3 \]

We solve the LMI problem and we obtain

\[
 P = \begin{bmatrix}
 2.3378 & 0 & 0.5397 \\
 0 & 2.1368 & 0 \\
 0.5397 & 0 & 1.7552
\end{bmatrix}
 \quad \bar{x} = \begin{bmatrix}
 0 \\
 0.1235 \\
 0
\end{bmatrix}
\]

which describes a convex ellipse \( E \) inscribed in the exact stability domain \( S \) delimited by the non-convex hyperbolic paraboloid

Very simple scalar convex sufficient stability condition

\[
2.4166p_0^2 + 2.2088p_1^2 + 1.8143p_2^2 - 0.5458p_1 + 1.1158p_0p_2 \leq 1
\]
Volume of stability ellipsoid

In the discrete-time case, the well-known sufficient stability condition defines a diamond

\[ D = \{ p : |p_0| + |p_1| + \cdots + |p_{n-1}| < 1 \} \]

For different values of degree \( n \), we compared volumes of exact stability domain \( S \), ellipsoid \( E \) and diamond \( D \)

<table>
<thead>
<tr>
<th></th>
<th>( n = 2 )</th>
<th>( n = 3 )</th>
<th>( n = 4 )</th>
<th>( n = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stability domain ( S )</td>
<td>4.0000</td>
<td>5.3333</td>
<td>7.1111</td>
<td>7.5852</td>
</tr>
<tr>
<td>Ellipsoid ( E )</td>
<td>2.2479</td>
<td>1.4677</td>
<td>0.7770</td>
<td>0.3171</td>
</tr>
<tr>
<td>Diamond ( D )</td>
<td>2.0000</td>
<td>1.3333</td>
<td>0.6667</td>
<td>0.2667</td>
</tr>
</tbody>
</table>

\( E \) is “larger” than \( D \), yet very small wrt \( S \)

In the last part of this course, we will propose better LMI inner approximations of the stability domain