Orthonormal Basis Functions for Continuous-Time Systems and L_p Convergence

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Abstract

In this paper, model sets for linear-time invariant continuous-time systems that are spanned by fixed pole orthonormal bases are investigated. These bases generalise the well known Laguerre and two–parameter Kautz cases. It is shown that the obtained model sets are everywhere dense in the Hardy space $H_1(\Pi)$ under the same condition as previously derived by the authors for the denseness in the (Π is the open right half plane) Hardy spaces $H_p(\Pi)$, $1 < p < \infty$. As a further extension, the paper shows how orthonormal model sets, that are everywhere dense in $H_p(\Pi)$, $1 \leq p < \infty$ and which have a prescribed asymptotic order may be constructed. Finally, it is established that the Fourier series formed by orthonormal basis functions converge in all spaces $H_p(\Pi)$ and ($\mathbf{D}$ is the open unit disk) $H_p(\mathbf{D})$, $1 < p < \infty$. The results in this paper have application in system identification, model reduction and control system synthesis.

Keywords: Orthonormal basis functions, continuous-time, Fourier series, L_p convergence.

1 Notation

$\mathbf{C}$	the field of complex numbers.
$\mathbf{R}$	the field of real numbers.
Π	the open right half plane $\{s \in \mathbf{C} : \mathrm{Re}\{s\} > 0\}$.
$\overline{\Pi}$	the closed right half plane $\{s \in \mathbf{C} : \mathrm{Re}\{s\} \geq 0\}$.
$\mathbf{D}$	the open unit disk $\{z \in \mathbf{C} : \lvert z\rvert < 1\}$.
$\mathbf{T}$	the unit circle $\{z \in \mathbf{C} : \lvert z\rvert = 1\}$.
$H_p(\Pi)$	the Hardy spaces of functions f analytic on Π and such that $\lVert f\rVert_p^p = (1/2\pi)\sup_{x>0}\int_{-\infty}^{\infty}\lvert f(x+jy)\rvert^p \,\mathrm{d}y < \infty$, $0 < p < \infty$ and $\lVert f\rVert_\infty = \sup_{s\in\Pi}\lvert f(s)\rvert < \infty$.

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$A(\Pi)$	the right half plane algebra $\{f : f \in H_\infty(\Pi)$ and continuous on $\overline{\Pi}\}$.
$A(\mathbf{D})$	the disk algebra $\{f : f$ analytic on $\mathbf{D}$ and continuous on $\overline{\mathbf{D}}\}$.
sp$A$	the linear span of A.
$\overline{a}$	the complex conjugate of a.
$O(s^{-m})$	The notation $f(s) = O(s^{-m})$ as $s \to \infty$ means that $$\limsup_{\lvert s\rvert\to\infty} \lvert s\rvert^m \lvert f(s)\rvert < \infty$$

2 Introduction

A fundamental idea in various areas of applied mathematics, control theory, signal processing and system analysis is that of decomposing (perhaps infinite dimensional) descriptions of linear time invariant dynamics in terms of an orthonormal basis. This approach is of greatest utility when accurate system descriptions are achieved with only a small number of basis functions. In recognition of this, there has been much work over the past several decades [M, B, E, R, S1] and, with renewed interest, more recently [WM, W2, W1, HVB, WP, NHG, BGS, NG] on the construction, analysis and application of rational orthonormal bases suitable for providing linear system characterisations.

In a system theoretic context, the applications of these orthonormal basis ideas have been manifold, but nevertheless have concentrated mainly on the discrete time setting [W1, W2, VHB, O1, O2, WP, NG, BGS]. Motivated largely by problems of estimation from frequency domain data [AIN, MAL, PGRSH, CW], but also with control system analysis and synthesis in mind [H, GGRS] this and the companion paper [AN2] focus attention on the continuous time scenario by considering the set of basis functions defined by a choice of numbers $\{a_k\} \in \Pi$ as

$$\begin{aligned} B_n(s) &\triangleq \frac{\sqrt{2\mathrm{Re}\{a_n\}}}{s+a_n}\varphi_{n-1}(s), \qquad n \geq 1 \\ \varphi_n(s) &\triangleq \prod_{k=1}^{n} \frac{s-\overline{a_k}}{s+a_k}, \qquad n \geq 1 \end{aligned} \tag{1}$$

with $B_0(s) = \varphi_0(s) \equiv 1$. With respect to the usual inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(j\omega)\overline{g(j\omega)}\,\mathrm{d}\omega$$

on $H_2(\Pi)$ these functions are orthonormal. Previous work on continuous time orthonormal bases has concentrated on special cases of the basis (1) wherein all the $\{a_k\}$ are the same real number $a_k = a \in \mathbf{R}$ in which case the ensuing basis is known as the 'Laguerre' basis [M2, M1, CW, P1], or the case of all the $\{a_k, a_{k+1}\}$ being the same complex conjugate pair $a_k = a, a_{k+1} = \overline{a}$ [WM].

For the general basis (1) studied here the only restriction on the pole choice $\{a_k\}$ is via the following result which was recently established in [AN2].

Theorem 2.1 *The linear span of the set of basis functions $\{B_n(s)\}_{n\geq 0}$ is everywhere dense in all of the spaces $H_p(\Pi)$, $1 < p < \infty$ and $A(\Pi)$ if and only if*

$$\sum_{n=1}^{\infty} \frac{\mathrm{Re}\{a_n\}}{1+|a_n|^2} = \infty. \tag{2}$$

The first result of this paper is, via Theorem 3.1, to extend this result to the case (which has important applications in a robust control context) of $p = 1$.

A function in $f(s) \in H_p(\Pi)$ is said to have 'asymptotic order' m if $f(s) = O(s^{-m})$ as $s \to \infty$. Clearly the bases defined by (1) have asymptotic order 1, but as illustrated in other work on continuous time orthonormal bases such as [WM], for the purposes of model error approximation and minimisation, there is great utility in being able to construct bases of asymptotic order greater than 1. Accordingly, Theorem 4.1 in § 4 establishes a method to construct an infinite set of orthonormal bases, each of which have arbitrary asymptotic order, and whose linear span is everywhere dense in $H_p(\Pi)$ for all $1 \leq p < \infty$.

Up until and including § 4 the paper has established that approximants with arbitrarily small $H_p(\Pi)$ norm approximation error exist, but not what they might be. In § 5 a specific (and obvious) approximant is considered which is the generalised Fourier series approximant. There, via Theorem 5.1, it is established that this approximant is, in fact, of arbitrarily small $H_p(\Pi)$ norm distance from the function being approximated for any $1 < p < \infty$. The paper concludes by showing how this continuous time result may be used to establish an equivalent discrete time one.

3 Complete Model Sets in $H_1(\Pi)$

The paper begins by presenting the following result which extends the result in Theorem 2.1 to include the $H_1(\Pi)$ space.

Theorem 3.1 *The linear span of the set of basis functions $\{B_n\}_{n\geq 0}$ is everywhere dense in all of the spaces $H_p(\Pi)$, $1 \leq p < \infty$ (and $A(\Pi)$) if and only if (2) holds.*

Proof. The necessity of (2) is proven in [AN2]. To prove the sufficiency, let $f \in H_1(\Pi)$ be a given function. Let $\epsilon > 0$ be also a given number. It is known that every function f in $H_1(\Pi)$ can be factored as $f = gh$ for some $g, h \in H_2(\Pi)$ (see, for example Garnett [G]). Choose a function ϕ and a set of basis elements $\{B_1, \cdots, B_n\}$ such that $\phi \in \mathrm{sp}\{B_k\}_{k=1}^n$ and

$$\|g - \phi\|_2 < \frac{1}{2\,(1 + \|h\|_2)}\,\epsilon. \tag{3}$$

This is possible since by Theorem 2.1, $\mathrm{sp}\{B_k\}_{k\geq 1}$ is everywhere dense in $H_2(\Pi)$.

Next choose a sufficiently large number m so that the elements in the set $\{a_1, \cdots, a_n\}$ with finite multiplicities are not included in the set $\{a_{m+1}, a_{m+2}, \cdots\}$. Set $m = 0$ when every element in $\{a_1, \cdots, a_n\}$ has infinite multiplicity. Define a new set of basis functions in $H_2(\Pi)$ by

$$\widetilde{B}_k(s) = \frac{\sqrt{2\mathrm{Re}\{a_{k+m}\}}}{s + a_{k+m}}\widetilde{\varphi}_{k-1}(s), \qquad \widetilde{\varphi}_k(s) = \prod_{i=m+1}^{k+m} \frac{s - \overline{a_i}}{s + a_i}, \qquad k \geq 1$$

and $\widetilde{\varphi}_0 \equiv 1$. Notice that the new basis functions coincide with (1) when $m = 0$. Since

$$\sum_{k=m+1}^{\infty} \frac{\mathrm{Re}\{a_k\}}{1+|a_k|^2} = \infty,$$

$\mathrm{sp}\{\widetilde{B}_k\}_{k\geq 1}$ is also everywhere dense in $H_2(\Pi)$. Therefore, there exists a finite set of basis elements $\{\widetilde{B}_k\}_{k=1}^N$ and $\psi \in \mathrm{sp}\{\widetilde{B}_k\}_{k=1}^N$ such that

$$\|h-\psi\|_2 < \frac{1}{2\left(\|g\|_2+\epsilon\right)}\ \epsilon. \tag{4}$$

Let $\Phi = \phi\psi$. The purpose of using new basis functions to approximate h should be clear now. If we had instead chosen ψ from $\mathrm{sp}\{B_k\}_{k\geq 1}$, then the product $\phi\psi$ would not have been necessarily in the linear span $\mathrm{sp}\{B_k\}_{k\geq 1}$.

Applying the Cauchy–Schwarz and triangle inequalities provides, via (3) and (4) that

$$\begin{aligned} \|f-\Phi\|_1 &\leq \|h\|_2\|g-\phi\|_2 + \|\phi\|_2\|h-\psi\|_2 \\ &< \frac{\|h\|_2}{2\left(1+\|h\|_2\right)}\ \epsilon + \left[\|g\|_2 + \frac{\epsilon}{2\left(1+\|h\|_2\right)}\right] \frac{1}{2\left(\|g\|_2+\epsilon\right)}\ \epsilon \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

It remains to show that $\Phi \in \mathrm{sp}\{B_1, B_2, \cdots\}$. To this end it is first established that $\Phi \in \mathrm{sp}\, Q$ where

$$Q = \left\{\frac{1}{s+a_1}, \frac{1}{(s+a_2)^{M(2)}}, \cdots\right\} \tag{5}$$

and $M(k)$ denotes the multiplicity of a_k in the set $\{a_1, \cdots, a_k\}$. Thus Q is precisely the set containing all possible partial fraction expansion terms of basis functions $B_1, B_2, \cdots$. Since

$$\phi \in \mathrm{sp}\{B_1, \cdots, B_n\} \subset \mathrm{sp}\left\{\frac{1}{s+a_1}, \frac{1}{(s+a_2)^{M(2)}}, \cdots, \frac{1}{(s+a_n)^{M(n)}}\right\}$$

and

$$\psi \in \mathrm{sp}\{\widetilde{B}_1, \cdots, \widetilde{B}_N\} \subset \mathrm{sp}\left\{\frac{1}{s+a_{m+1}}, \frac{1}{(s+a_{m+1})^{\widetilde{M}(2)}}, \cdots, \frac{1}{(s+a_{m+N})^{\widetilde{M}(N)}}\right\}$$

where $\widetilde{M}(k)$ is the multiplicity of a_{m+k} in the set $\{a_{m+1}, \cdots, a_{m+k}\}$, Φ can be written for some coefficients $\{c_k\}, \{d_k\}$ as

$$\Phi = \sum_{k=1}^{n}\sum_{\ell=1}^{N} \frac{c_k d_\ell}{(s+a_k)^{M(k)}(s+a_{m+\ell})^{\widetilde{M}(\ell)}}.$$

Suppose $a_k \neq a_{m+\ell}$. Then the summand above admits a partial fraction expansion

$$\sum_{i=1}^{M(k)} \frac{e_{k,l,i}}{(s+a_k)^i} + \sum_{i=1}^{\widetilde{M}(\ell)} \frac{f_{k,l,i}}{(s+a_{m+\ell})^i}$$

for some coefficients $e_{k,l,i}, f_{k,l,i}$ where the partial fraction terms above are in sp$Q$. In the other case $a_k = a_{m+l}$, we have

$$\frac{c_k d_\ell}{(s+a_k)^{M(k)}(s+a_{m+\ell})^{\widetilde{M}(\ell)}} = \frac{c_k d_\ell}{(s+a_k)^{M(k)+\widetilde{M}(\ell)}}$$

which is again in sp$Q$ since for $m = 0$, a_k has infinite multiplicity and for $m \neq 0$, the sets $\{a_1, \cdots, a_m\}$ and $\{a_{m+1}, \cdots, a_{m+N}\}$ are disjoint and the equality $a_k = a_{m+l}$ is attained only if a_k has infinite multiplicity by the construction of the latter set. Hence $\Phi \in \mathrm{sp}\, Q$.

To complete the proof, it is sufficient to show that sp$Q \subset \mathrm{sp}\{B_n\}_{n\geq 1}$. Then this will imply that $\Phi \in \mathrm{sp}\{B_n\}_{n\geq 1}$. Since f and ϵ are arbitrary, this will then show that $\mathrm{sp}\{B_k\}_{k\geq 1}$ is everywhere dense in $H_1(\Pi)$.

Let L be a given positive integer. Write the partial fraction expansions of the basis functions $B_1, B_2, \cdots, B_L$ in the following linear equation form

$$\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_L \end{bmatrix} = \begin{bmatrix} \alpha_{11} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{L1} & \alpha_{L2} & \cdots & \alpha_{LL} \end{bmatrix} \begin{bmatrix} \frac{1}{s+a_1} \\ \frac{1}{(s+a_2)^{M(2)}} \\ \vdots \\ \frac{1}{(s+a_L)^{M(L)}} \end{bmatrix}. \tag{6}$$

The degree of B_k is k which implies that $\alpha_{kk} \neq 0$ for all $k \leq L$ and thus the lower triangular matrix above is invertible. Hence for $i = 1, 2, \cdots, L$

$$\frac{1}{(s+a_i)^{M(i)}} \in \mathrm{sp}\{B_k, k = 1, \cdots, L\} \subset \mathrm{sp}\{B_k\}_{k\geq 1}.$$

Since L is arbitrary, it follows that sp$Q \subset \mathrm{sp}\{B_n\}_{n\geq 1}$. ♣

The proof of Theorem 3.1 is an interesting application of the factorization $f = gh$ where $f \in H_1(\Pi)$ and $g, h \in H_2(\Pi)$. The completeness proof in [AN2] for the spaces $A(\Pi)$ and $H_p(\Pi)$, $1 < p < \infty$ does not apply to $H_1(\Pi)$ since the basis functions (1) themselves are not in $H_1(\Pi)$.

Theorem 3.1 shows in particular that a first-order time-delayed system described by the transfer function

$$G(s) = \frac{e^{-s\tau}}{s+a}, \quad \tau, a > 0$$

and commonly seen in process engineering can be approximated by models spanned by the basis functions (1) with arbitrarily small approximation errors in the $H_1(\Pi)$ norm. Note that G is not in $H_1(\Pi)$, yet $G \in H_p(\Pi)$, for all $1 < p < \infty$ and $A(\Pi)$.

The completeness result (3.1) has also potential applications on robust identification of continuous-time systems. An abstract framework that solves robust identification problems formulated for systems lying in *seperable* Banach spaces is provided in [P2]. This framework has found important applications in the identification of discrete-time systems [AN1].

4 Orthonormal Basis Functions with Prescribed Asymptotic Order

This section presents a derivation of model sets that are everywhere dense in $H_p(\Pi)$ for $1 < p < \infty$ and for which the orthonormal basis functions $B_n(s)$ defining the sets each have a prescribed asymptotic order. That is, $B_n(s) = O(s^{-m})$ as $s \to \infty$. The basis functions studied in the previous section all have asymptotic order $m = 1$. The problem of synthesis of bases of arbitrary asymptotic order has been investigated in the literature for various specific cases of choice a_k of pole position or of asymptotic order m [S2, M, C1, M2].

In contrast to these previous specific cases available in the literature, the following result provides a recipe to construct bases of arbitrary asymptotic order m and with arbitrary pole position a_k (that satisfies (2)). It is easy to see that suitably chosen linear combinations of the basis functions (1) yield a set of functions with a prescribed asymtotic order. The difficult part is to show that this set is everywhere dense.

Theorem 4.1 *Suppose that (2) is satisfied. Let $P(s)$ be an mth order polynomial with roots in the complement of $\overline{\Pi}$ and $m > 1$. Let*

$$\psi_n(s) \triangleq \frac{B_n(s)}{P(s)} - \sum_{k=1}^{n-1} \left\langle \frac{B_n}{P}, \psi_k \right\rangle \frac{\psi_k(s)}{\|\psi_k\|_2^2}, \qquad \psi_1(s) \triangleq \frac{B_0(s)}{P(s)}. \tag{7}$$

Then the basis functions

$$\phi_n(s) \triangleq \frac{\psi_n(s)}{\|\psi_n\|_2}, \qquad n \geq 1 \tag{8}$$

are orthonormal and have the asymptotic order $\phi_n(s) = O(s^{-m})$ as $s \to \infty$. Moreover $sp\{\phi_n\}_{n\geq 1}$ is everywhere dense in $H_p(\Pi)$ for all $1 \leq p < \infty$.

Proof. Let $f \in H_p(\Pi)$ be a given function. Let $\epsilon > 0$ be also a given number. Approximate f by a function $g \in A(\Pi)$ that has the properties $\|f - g\|_p < \epsilon$ and

$$\lim_{|s|\to\infty} |s|^m |g(s)| = 0, \quad s \in \Pi.$$

This is possible since such functions form a dense subset of $H_p(\Pi)$ (see for example, Garnett [G, Cor. 3.3 in Chap. II]). Take $h(s) = P(s)g(s)$. Then $h \in A(\Pi)$ and since by Theorem 2.1, $\mathrm{sp}\,\{1, B_1, B_2, \cdots\}$ is everywhere dense in $A(\Pi)$ there exists a function $F \in \mathrm{sp}\{1, B_1, B_2, \cdots\}$ such that $\|h - F\|_\infty < \epsilon$ which implies that

$$\left| g(s) - \frac{F(s)}{P(s)} \right| < \frac{\epsilon}{|P(s)|}, \qquad s \in \Pi.$$

Therefore

$$\left\| f - \frac{F}{P} \right\|_p < \epsilon + \left\| \frac{1}{P} \right\|_p \epsilon.$$

Since f and ϵ are arbitrary, it follows that the linear span

$$Q = \mathrm{sp}\left\{ \frac{1}{P}, \frac{B_1}{P}, \frac{B_2}{P}, \cdots \right\}$$

is everywhere dense in $H_p(\Pi)$ for all $1 \leq p < \infty$. Finally, observe that (7) is the Gram-Schmidt orthogonalisation procedure applied to Q. ♣

The basis functions (7) are useful in any modelling applications that require certain roll off rates at high frequencies. For example in modal analysis, mechanical structures for vibration studies are represented by models in the form

$$\widehat{G}_{2n}(s) = \sum_{k=1}^{n} \frac{a_k}{s^2 + 2\xi_k \omega_k s + \omega_k^2}$$

where damping coefficients satisfy $0 < \xi_k < 1$ for all k and ω_k's denote natural frequencies. The frequency response of $\widehat{G}_{2n}$ rolls off 40 dB per decade.

5 Convergence of Generalised Fourier Series in $H_p(\Pi)$

Let $\{B_k\}_{k\geq 1}$ be a set of basis functions which satisfy (2). Then $\text{sp}\{B_k\}_{k\geq 1}$ is an everywhere dense set of basis functions for $H_2(\Pi)$ and every $f \in H_2(\Pi)$ has a Fourier series expansion

$$\widehat{f}_n(s) \triangleq \sum_{k=1}^{n} \langle f, B_k \rangle B_k(s) \tag{9}$$

that converges to f in the $L_2(j\mathbf{R})$–norm. When $B_k = z^k$, $k = 1, 2, \cdots$ and the underlying space is $H_p(\mathbf{D})$, it is well known that every $f \in H_p(\mathbf{D})$ has a Fourier series which also converges in the $L_p(\mathbf{T})$–norm for all $1 < p < \infty$. In this section it is shown that the same is true for the basis functions in (1). First it is necessary to establish that the maps $f \mapsto \widehat{f}_n$ are bounded.

Lemma 5.1 *Let $\widehat{f}_n$ be as in (9). Then there exists a constant $C_p < \infty$, which depends only on p, such that for all $1 < p < \infty$*

$$\|f - \widehat{f}_n\|_p \leq C_p \|f\|_p. \tag{10}$$

Proof. Let $\psi(s) = f(-js)$. Then ψ is analytic on the upper half plane $\text{Im}\{s\} > 0$ and $(1/2\pi)\int_{-\infty}^{\infty} |\psi(t)|^p dt = \|f\|_p^p$. Hence $\psi(s)$ can be represented by a Cauchy integral [D, Theorem 11.8] as

$$\psi(s) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{\psi(t)}{t-s}\,\mathrm{d}t, \qquad \text{Im}\{s\} = y > 0$$

and consequently

$$f(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(j\omega)}{s - j\omega}\,\mathrm{d}\omega, \qquad s \in \Pi. \tag{11}$$

For the basis functions $\{B_k\}_{k\geq 1}$ defined by (1), the following well-known Christoffel-Darboux formula (see [AN2] for a sample derivation) holds

$$\sum_{k=1}^{n} \overline{B_k(j\omega)} B_k(s) = \frac{1 - \overline{\varphi_n(j\omega)}\varphi_n(s)}{s - j\omega}, \qquad s \in \Pi. \tag{12}$$

Therefore from (9), and (11)–(12)

$$\begin{aligned} \widehat{f}_n(s) &= f(s) - \frac{\varphi_n(s)}{2\pi}\int_{-\infty}^{\infty} \frac{f(j\omega)\overline{\varphi_n(j\omega)}}{s - j\omega}\,\mathrm{d}\omega, \qquad s \in \Pi \\ &= f(s) - \varphi_n(s)\frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{f(-j\omega)\overline{\varphi_n(-j\omega)}}{\omega - js}\,\mathrm{d}\omega \\ &\triangleq f(s) - \varphi_n(s)\widetilde{H}_n(s). \end{aligned} \tag{13}$$

Put $\widetilde{f}_n(\omega) = f(-j\omega)\overline{\varphi_n(-j\omega)}$ and $\widetilde{F}_n(js) = \widetilde{H}_n(s)$, $s \in \Pi$. Then $\widetilde{F}_n$ is analytic on the upper half plane and from (13) the following representation holds for $\widetilde{F}_n$

$$\widetilde{F}_n(s) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{\widetilde{f}_n(\omega)}{\omega - s}\,\mathrm{d}\omega, \qquad \text{Im}\{s\} = y > 0. \tag{14}$$

The map $\widetilde{f}_n \mapsto \widetilde{F}_n$ defined by (14) is a bounded linear operator from $L_p(\mathbf{R})$ onto $H_p(\{y > 0\})$ (see [RR, Theorem 5.32]). Hence

$$\|f - \widehat{f}_n\|_p = \|\widetilde{H}_n\|_p = \|\widetilde{F}_n\|_p \leq C_p\|\widetilde{f}_n\|_p = C_p\|f\|_p \tag{15}$$

where C_p is a constant which depends only on p. ♣

For the precise value of C_p, the reader is referred to Chapter III in Garnett [G]. Let X_n denote the linear space spanned by the functions $B_k,\ k = 1, 2, \cdots, n$ and define

$$e_n(f;p) = \min_{g \in X_n} \|f - g\|_p. \tag{16}$$

Thus $e_n(f;p)$ is the best $H_p(\Pi)$ norm approximation error of f by functions in X_n. Since $\bigcup_{n=1}^{\infty} X_n$ is everywhere dense in $H_p(\Pi)$ $(1 \leq p < \infty)$ and $A(\Pi)$, the quantity $e_n(f;p)$ defined by (16) monotonically tends to zero as $n \to \infty$. Using Lemma 5.1, the following main result of this section can now be established.

Theorem 5.1 *Consider the partial sums of the Fourier series defined by (9). Let $e_n(f;p)$ be as in (16). Suppose that (2) holds. Then for all $1 < p < \infty$ and $f \in H_p(\Pi)$*

$$\|f - \widehat{f}_n\|_p \leq C_p\, e_n(f;p) \tag{17}$$

and

$$\lim_{n \to \infty} \|f - \widehat{f}_n\|_p = 0. \tag{18}$$

Proof. Let $f \in H_p(\Pi)$ and g be the minimizing solution in (16). Let $\psi = f - g$. Observe that $\widehat{g}_n = g$ since $g \in X_n$. Due to the linearity of Fourier series, notice also that $\widehat{\psi}_n = \widehat{f}_n - \widehat{g}_n$. Hence from Lemma 5.1

$$\|f - \widehat{f}_n\|_p = \|\psi - \widehat{\psi}_n\|_p \leq C_p\, \|\psi\|_p = C_p\, e_n(f;p).$$

♣

The inequality (17) shows that the approximation error of the linear estimate (9) is in the order of the best achievable error for every choice of basis functions when the approximated system lies in $H_p(\Pi)$ $(1 < p < \infty)$. The choice of basis functions on the other hand depends on the class of systems. This subject will not be pursued here.

The remainder of this section will be consumed with the extension of Theorem 5.1 to the discrete–time orthonormal basis functions studied in [NG, AN1, NHG] defined on $\mathbf{D} \cup \mathbf{T}$ by

$$\mathcal{B}_n(z) \triangleq \frac{\sqrt{1 - |\xi_n|^2}}{1 - \overline{\xi_n} z}\phi_{n-1}(z), \qquad \phi_n(z) \triangleq \prod_{k=1}^{n} \frac{z - \xi_k}{1 - \overline{\xi_k} z}, \qquad \phi_0(z) \triangleq 1. \tag{19}$$

These basis functions were considered in [AN1] for the purpose of robust estimation. In particular, it was shown that model sets spanned by (19) are complete in $H_p(\mathbf{D})$ for all $1 \leq p < \infty$ and $A(\mathbf{D})$ if and only if the sequence of complex numbers $\xi_k \in \mathbf{D}$ satisfies

$$\sum_{n=1}^{\infty}(1 - |\xi_n|) = \infty. \tag{20}$$

The following is the discrete-time version of Lemma 5.1. In what follows, the notation $\|\cdot\|_p$ refers to the L_p–norms on the unit circle and the inner product for two functions $f, g \in H_2(\mathbf{T})$ is defined as

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{j\omega})\overline{g(e^{j\omega})}\,\mathrm{d}\omega.$$

Lemma 5.2 *Let $\widehat{f}_n$ denote the partial sums of the Fourier series of $f \in H_p(\mathbf{D})$, that is*

$$\widehat{f}_n(z) = \sum_{k=1}^{n}\langle f, \mathcal{B}_k\rangle \mathcal{B}_k(z). \tag{21}$$

Then there exists a constant $C_p < \infty$, which depends only on p, such that

$$\|f - \widehat{f}_n\|_p \le C_p\|f\|_p, \qquad 1 < p < \infty. \tag{22}$$

Proof. The proof is very similar to that of Lemma 5.1. Accordingly, it is omitted, and instead the required modifications of Lemma 5.1 are specified. The Cauchy formula for $f \in H_1(\mathbf{D})$ is

$$f(z) = \frac{1}{2\pi}\int_{0}^{2\pi} \frac{f(e^{j\theta})}{1 - e^{-j\theta}z}\,\mathrm{d}\theta \tag{23}$$

and the well-known Christoffel-Darboux formula (which can be proven by induction) is

$$\sum_{k=1}^{n}\overline{\mathcal{B}_k(e^{j\theta})}\mathcal{B}_k(z) = \frac{1 - \overline{\phi_n(e^{j\theta})}\phi_n(z)}{1 - e^{-j\theta}z}, \qquad z \in \mathbf{D}.$$

♣

Theorem 5.2 *Consider the partial sums of the Fourier series defined by (21). Let $e_n(f;p)$ denote the best $H_p(\mathbf{D})$ norm approximation error of f by functions in $sp\{\mathcal{B}\}_{k=1}^n$. Then for all $1 < p < \infty$ and $f \in H_p(\mathbf{D})$*

$$\|f - \widehat{f}_n\|_p \le C_p\, e_n(f;p) \tag{24}$$

and if (20) holds

$$\lim_{n\to\infty}\|f - \widehat{f}_n\|_p = 0. \tag{25}$$

In Theorems 5.1–5.2, the cases $p = 1$ and $p = \infty$ can not be included since the Riesz maps $L_p(\mathbf{R}) \to H_p(\{y > 0\})$ and $L_p(\mathbf{T}) \to H_p(\mathbf{D})$ defined respectively by (14) and (23) are not bounded for $p = 1$ and $p = \infty$.

Under additional system and/or basis *smoothness* assumptions, partial sums in (9) and (21) can be computed by fast Fourier transform techniques with guaranteed error convergence behaviour. We omit the details (see [GKL, WM]).

6 Conclusions

This paper has provided an analysis of the approximation properties of certain general classes of rational orthonormal basis functions. The nature of the results was such as to establish that for linear-time invariant continuous-time system modelling, arbitrarily small H_p norm approximation error was possible for any $p \in [1, \infty)$ and furthermore, this may be provided while at the same time using bases with arbitrary asymptotic order. Finally, a specific construction of the system approximant via Fourier decomposition was shown to be one in which the H_p norm error is arbitrarily small for any $p \in (1, \infty)$. The results have application in the analysis and design of robust estimation and control strategies.

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