A SUBSPACE-BASED METHOD FOR SOLVING LAGRANGE-SYLVESTER INTERPOLATION PROBLEMS

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Outline

Interpolation of matrix valued rational functions analytic at infinity from frequency domain data ...

1. Lagrange interpolation studied by Antoulas and Anderson using a tool called *Löwner* matrix also with additional constraints such as bounded real, positive real *etc.*
2. Generating system approach studied by Antoulas, Ball, Kang, Willems, Gohberg, and Rodman.
3. Applications of interpolation theory to control and system theory and estimation (see, for example, the monographs: Ball, Gohberg, and Rodman; Nikolski).

Consider a multi-input/multi-output, linear-time invariant discrete-time system represented by the state-space equations:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

where $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^m$ and $y(t) \in \mathbf{R}^p$ are the input and the output.

Transfer function

$$G(z) = D + C(zI_n - A)^{-1}B$$

is stable and $\{A, B\}$ and $\{A, C\}$ are controllable and observable.

Given: samples of $G(z)$ and its derivatives at L distinct points $z_k \in \mathcal{D}$

$$\frac{d^j G(z_k)}{dz^j} = w_{kj}, \quad j = 0, 1, \cdots, N_k; \quad k = 1, 2, \cdots, L.$$

Find: $(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D})$, a minimal realization of $G(z)$.

Lagrange-Sylvester rational interpolation problem.

• *Obvious solution!* Reduce the problem first to a system of independent scalar problems and obtain a minimal solution by eliminating unobservable or/and uncontrollable modes.

- (Bi)tangential and contour integral versions treated for example, in Ball, Gohberg, and Rodman.

- *Related problems:* Nonhomogeneous interpolation with metric constraints; Nevanlinna-Pick interpolation; Partial realization.

Outline

Take the z-transform of the state-space equations:

$$\begin{aligned} zX(z) &= AX(z) + BU(z) \\ Y(z) &= CX(z) + DU(z) \end{aligned}$$

where $X(z)$ denotes the z-transforms of $x(k)$ defined by

$$U(z) \triangleq \sum_{k=0}^{\infty} u(k) z^{-k}.$$

Let $X_j(z)$ be the resulting state z-transform when $u(k) = e_j$.

Define the compound state z-transform matrix:

$$X_{\mathrm{C}}(z) \triangleq [X_1(z) \;\; X_2(z) \;\; \cdots \;\; X_m(z)] \,.$$

Then, $G(z)$ can implicitly be described as

$$G(z) = CX_{\mathrm{C}}(z) + D$$

with

$$zX_{\mathrm{C}}(z) = AX_{\mathrm{C}}(z) + B.$$

By recursive use, we obtain the relation

$$z^k G(z) \;\; = \;\; CA^k X_{\mathrm{C}}(z) + Dz^k + \sum_{j=0}^{k-1} CA^{k-1-j} B z^j \,, \quad k \geq 1.$$

The impulse response coefficients of $G(z)$:

$$g_k = \begin{cases} D, & k = 0; \\ CA^{k-1}B, & k \geq 1. \end{cases}$$

Thus,

$$z^k G(z) = CA^k X_{\mathrm{C}}(z) + \sum_{j=0}^{k} g_{k-j}\, z^j, \qquad k \geq 0.$$

Hence,

$$\begin{bmatrix} G(z) \\ zG(z) \\ \vdots \\ z^{q-1}G(z) \end{bmatrix} = \mathcal{O}_q X_{\mathrm{C}}(z) + \Gamma_q \begin{bmatrix} I_m \\ zI_m \\ \vdots \\ z^{q-1}I_m \end{bmatrix}$$

where

$$\mathcal{O}_q \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix}, \quad \Gamma_q \triangleq \begin{bmatrix} g_0 & 0 & \cdots & 0 \\ g_1 & g_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_{q-1} & g_{q-2} & \cdots & g_0 \end{bmatrix}.$$

- $\mathcal{O}_q$, *extended observability matrix*, has full rank n if (C, A) is observable and $q \geq n$.

Let

$$\mathcal{Z}_q(z) \triangleq \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{q-1} \end{bmatrix}, \quad \mathcal{J}_{q,2} \triangleq \begin{bmatrix} 0 & & \cdots & & 0 \\ 1 & 0 & & & \\ 0 & 1 & 0 & & \\ \vdots & & & \ddots & \vdots \\ 0 & & \cdots & 1 & 0 \end{bmatrix} \in \mathrm{R}^{q \times q},$$

$$\mathcal{J}_{q,1} = I_q, \qquad \mathcal{J}_{q,2}^0 = I_q.$$

- $\mathcal{J}_{q,2}$ obtained by shifting the elements of $\mathcal{J}_{q,1}$ one row down and filling its first row with zeros.

Let $\mathcal{J}_{q,j}$ denote the matrix obtained by $j-1$ repeated applications of this process to $\mathcal{J}_{q,1}$.

- Note the following relations

$$\mathcal{J}_{q,j} = \begin{cases} \mathcal{J}_{q,2}^{j-1}, & j \leq q \\ 0, & j > q. \end{cases}$$

Thus,

$$\Gamma_q = \sum_{j=0}^{q-1} \mathcal{J}_{q,1+j} \otimes g_j$$

A compact expression:

$$\mathcal{Z}_q(z) \otimes G(z) \quad = \quad \mathcal{O}_q X_{\mathrm{C}}(z) + \sum_{j=0}^{q-1} [\mathcal{J}_{q,2}^{j} \otimes g_j] \; [\mathcal{Z}_q(z) \otimes I_m] \, .$$

• Forms the basis of the frequency domain subspace identification algorithms (McKelvey, Akçay, and Ljung; 1996).

(Subspace ID: evaluate this equation at a set of distinct points on the unit circle and stack into columns of long matrices yielding a matrix equation *affine* in $\mathcal{O}_q$. Then, recover the range space of $\mathcal{O}_q$ by a projection.)

Differentiate $\mathcal{Z}_q(z) \otimes G(z)$ l times with respect to z:

$$\begin{aligned} H_q^{(l)}(z) &= \sum_{j=0}^{l} \binom{l}{j} \left[\mathcal{Z}_q^{(j)}(z) \otimes G^{(l-j)}(z)\right] \\ &= \mathcal{O}_q \frac{\mathrm{d}^l X_{\mathrm{C}}(z)}{\mathrm{d}z^k} + \sum_{j=0}^{q-1} [\mathcal{J}_{q,2}^{j} \otimes g_j] \left[\mathcal{Z}_q^{(l)}(z) \otimes I_m\right], \quad l \geq 0 \end{aligned}$$

where

$$H_q(z) \triangleq \mathcal{Z}_q(z) \otimes G(z).$$

Augment $H_q(z_k)$ and the first N_k derivatives of $H_q(z)$ at z_k in a data matrix:

$$\mathcal{H}_k \triangleq \begin{bmatrix} H_q(z_k) & H_q'(z_k) & \cdots & H_q^{(N_k)}(z_k) \end{bmatrix}, \quad k = 1, \cdots, L.$$

A compact expression for $\mathcal{H}_k$ in terms of the elementary matrices:

$$\mathcal{D}_{N_k+1} \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 2 & & \\ & & 0 & \cdots & \\ \vdots & & & \ddots & N_k \\ 0 & & & \cdots & 0 \end{bmatrix} \in \mathbf{R}^{(N_k+1)\times(N_k+1)}$$

and

$$\mathcal{W}_k \triangleq \left[\mathcal{Z}_q(z_k) \;\; \mathcal{Z}_q'(z_k) \;\; \cdots \;\; \mathcal{Z}_q^{(N_k)}(z_k) \right], \quad k = 1, \cdots, L,$$

is derived as

$$\mathcal{H}_k = \sum_{j=0}^{N_k} \frac{1}{j!} [\mathcal{W}_k \, \mathcal{D}_{N_k+1}^{j}] \otimes w_{kj}, \quad k = 1, \cdots, L.$$

- $\mathcal{D}_{N_k+1}^{j} = 0$ for all $j > N_k$.

An alternative compact expression for $\mathcal{H}_k$:

$$\mathcal{H}_k = \mathcal{O}_q \, \mathcal{X}_k + \sum_{j=0}^{q-1} [\mathcal{J}_{q,2}^{j} \otimes g_j] \, [\mathcal{W}_k \otimes I_m], \quad k = 1, \cdots, L$$

where

$$\mathcal{X}_k \triangleq \left[X_{\mathrm{C}}(z_k) \;\; X'_{\mathrm{C}}(z_k) \;\; \cdots \;\; X_{\mathrm{C}}^{(N_k)}(z_k) \right], \quad k = 1, \cdots, L.$$

The derivatives of $\mathcal{Z}_q(z)$?

Let

$$\mathcal{T}_q \triangleq \begin{bmatrix} 0! & 0 & \cdots & 0 \\ 0 & 1! & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (q-1)! \end{bmatrix} \in \mathbf{R}^{q \times q}.$$

Then, it is easy to verify that

$$\frac{\mathrm{d}^l \mathcal{Z}_q(z)}{\mathrm{d}z^l} = \mathcal{T}_q \mathcal{J}_{q,2}^l \mathcal{T}_q^{-1} \mathcal{Z}_q(z), \qquad l \geq 0.$$

Now, collect $\mathcal{H}_k$, $\mathcal{X}_k$, and $\mathcal{W}_k$ in the compound matrices:

$$\begin{aligned} \mathcal{H} &\triangleq [\mathcal{H}_1\ \mathcal{H}_2\ \cdots\ \mathcal{H}_L], \\ \mathcal{X} &\triangleq [\mathcal{X}_1\ \mathcal{X}_2\ \cdots\ \mathcal{X}_L], \\ \mathcal{W} &\triangleq [\mathcal{W}_1\ \mathcal{W}_2\ \cdots\ \mathcal{W}_L]. \end{aligned}$$

Hence,

$$\mathcal{H} = \mathcal{O}_q\, \mathcal{X} + \sum_{j=0}^{q-1} [\mathcal{J}_{q,2}^j \otimes g_j]\, [\mathcal{W} \otimes I_m].$$

An equation involving only real-valued matrices

$$\widehat{\mathcal{H}} = \mathcal{O}_q \, \widehat{\mathcal{X}} + \sum_{j=0}^{q-1} [\mathcal{J}_{q,2}^{j} \otimes g_j] \, \mathcal{F}$$

where

$$\begin{aligned} \widehat{\mathcal{H}} &\triangleq [\mathrm{Re}\mathcal{H} \;\; \mathrm{Im}\mathcal{H}] \, , \\ \widehat{\mathcal{X}} &\triangleq [\mathrm{Re}\mathcal{X} \;\; \mathrm{Im}\mathcal{X}] \, , \\ \mathcal{F} &\triangleq [\mathrm{Re}\mathcal{W} \;\; \mathrm{Im}\mathcal{W}] \otimes I_m \, . \end{aligned}$$

Total number of interpolation conditions:

$$N \triangleq \sum_{k:z_k \in \mathbf{R}} (N_k + 1) + \sum_{k:z_k \in \mathbf{C}-\mathbf{R}} 2(N_k + 1).$$

- $\widehat{\mathcal{H}} \in \mathbf{R}^{pq \times mN}$, $\mathcal{F} \in \mathbf{R}^{mq \times mN}$, and $\widehat{\mathcal{X}} \in \mathbf{R}^{n \times mN}$.

- The first stage is complete: $\widehat{\mathcal{H}}$ is affine in $\mathcal{O}_q$!

Outline

The projection matrix onto the null space of $\mathcal{F}$:

$$\mathcal{F}^{\perp} \triangleq I_{mN} - \mathcal{F}^{T}(\mathcal{F}\mathcal{F}^{T})^{-1}\mathcal{F}$$

Then,

$$\widehat{\mathcal{H}}\mathcal{F}^{\perp} = \mathcal{O}_q \, \widehat{\mathcal{X}}\mathcal{F}^{\perp}.$$

- $Range(\widehat{\mathcal{H}}\mathcal{F}^{\perp}) = Range(\mathcal{O}_q)$ if no rank cancelations occur!

- Sufficient condition: " $Range(\mathcal{F}^T) \bigcap Range(\widehat{\mathcal{X}}^T) = Empty$. "

Lemma 1 *Suppose that $N \geq q + n$ and the eigenvalues of A do not coincide with the distinct complex numbers z_k. Then,*

$$\text{rank} \begin{bmatrix} \mathcal{F} \\ \widehat{\mathcal{X}} \end{bmatrix} = qm + n \iff (A, B) \text{ controllable.}$$

- Since A is stable, $Range(\widehat{\mathcal{H}}\mathcal{F}^{\perp}) = Range(\mathcal{O}_q)$.

QR-factorization

$$\begin{bmatrix} \mathcal{F} \\ \widehat{\mathcal{H}} \end{bmatrix} = \begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}.$$

$$\widehat{\mathcal{H}}\mathcal{F}^{\perp} = R_{22}Q_2^T,$$

- Use $R_{22} \in \mathbf{R}^{pq \times m(N-q)}$ in the extraction of the observability range space since Q_2^T is a matrix of full rank.

Outline

- Use the singular value factorization of $\widehat{\mathcal{H}}\mathcal{F}^{\perp}$ to get A and C:

$$\begin{aligned} \widehat{\mathcal{H}}\mathcal{F}^{\perp} &= \widehat{U}\widehat{\Sigma}\widehat{V}^T \\ &= \begin{bmatrix} \widehat{U}_s & \widehat{U}_o \end{bmatrix} \begin{bmatrix} \widehat{\Sigma}_s & 0 \\ 0 & \widehat{\Sigma}_o \end{bmatrix} \begin{bmatrix} \widehat{V}_s^T \\ \widehat{V}_o^T \end{bmatrix} \end{aligned}$$

where $\widehat{\Sigma}_s \in \mathbf{R}^{n\times n}$. Let

$$\widehat{A} = (J_1\widehat{U}_s)^{\dagger} J_2 \widehat{U}_s, \qquad \widehat{C} = J_3 \widehat{U}_s$$

where $X^{\dagger} = (X^T X)^{-1} X^T$ and

$$\begin{aligned} J_1 &= \begin{bmatrix} I_{(q-1)p} & 0_{(q-1)p\times p} \end{bmatrix}, \\ J_2 &= \begin{bmatrix} 0_{(q-1)p\times p} & I_{(q-1)p} \end{bmatrix}, \end{aligned}$$

$$J_3 = \begin{bmatrix} I_p & 0_{p\times(q-1)p} \end{bmatrix}.$$

- If (C, A) is observable, $(J_1\widehat{U}_s)^\dagger$ exists if and only if $q > n$.

Then, from Lemma 1 for some $T \in \mathbf{R}^{n\times n}$,

$$\widehat{A} = T^{-1}AT, \qquad \widehat{C} = CT.$$

Outline

Repeated application of the differentiation formula

$$\frac{d}{dz}X^{-1} = -X^{-1}\frac{dX}{dz}X^{-1}$$

to $X_{\mathrm{C}}(z) = (zI_n - A)^{-1}B$ yields the derivatives of $G(z)$:

$$G^{(j)}(z) = \delta_{0j}\,D + (-1)^j j!\,C(zI_n - A)^{-j-1}B,\ j \geq 0$$

where δ_{ks} is the Kronecker delta.

- The derivatives are linear in B and D for given A and C.

Let

$$\mathcal{G}_k \triangleq \begin{bmatrix} w_{k0} \\ w_{k1} \\ \vdots \\ w_{kN_k} \end{bmatrix}, \qquad \mathcal{G} \triangleq \begin{bmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \\ \vdots \\ \mathcal{G}_L \end{bmatrix}.$$

$$\mathcal{Y}_k \triangleq \begin{bmatrix} C(z_kI_n - A)^{-1} & I_p \\ -C(z_kI_n - A)^{-2} & 0 \\ \vdots & \\ (-1)^{N_k}N_k!\,C(z_kI_n - A)^{-N_k-1} & 0 \end{bmatrix}, \qquad \mathcal{Y} \triangleq \begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \\ \vdots \\ \mathcal{Y}_L \end{bmatrix}$$

Determine B and D by solving the linear LS problem:

$$\widehat{B}, \widehat{D} = \arg\min_{B,D} \left\| \widehat{\mathcal{G}} - \widehat{\mathcal{Y}} \begin{bmatrix} B \\ D \end{bmatrix} \right\|_F^2$$

provided that $\mathcal{Y}$ is not rank deficient where

$$\widehat{\mathcal{G}} \triangleq \begin{bmatrix} \mathrm{Re}\mathcal{G} \\ \mathrm{Im}\mathcal{G} \end{bmatrix} \in \mathbf{R}^{pN \times m},$$

$$\widehat{\mathcal{Y}} \triangleq \begin{bmatrix} \mathrm{Re}\mathcal{Y} \\ \mathrm{Im}\mathcal{Y} \end{bmatrix} \in \mathbf{R}^{pN \times (n+p)},$$

A sufficient condition

Lemma 2 *Suppose that $N > n$ and the eigenvalues of A do not coincide with the distinct complex numbers z_k. Then,*

$$\text{rank}\mathcal{Y} = p + n \qquad \Longleftrightarrow \qquad (C, A) \text{ observable.}$$

- If $N \geq q + n$ and $q > n$, then

$$\widehat{B} = T^{-1}B, \qquad \widehat{D} = D$$

and

$$\widehat{G}(z) \triangleq \widehat{C}(zI_n - \widehat{A})^{-1}\widehat{B} + \widehat{D} = G(z).$$

Outline

Algorithm

1. Given the data, compute the matrices $\widehat{\mathcal{H}}$ and $\mathcal{F}$.
2. Perform the QR-factorization.
3. Calculate the singular value decomposition with $\widehat{\mathcal{H}}\mathcal{F}^{\perp}$ replaced by R_{22}.
4. Determine the system order by inspecting the singular values, and partition the singular value decomposition such that $\widehat{\Sigma}_s$ contains the n largest singular values.
5. With J_1, J_2, and J_3, calculate $\widehat{A}$ and $\widehat{C}$.
6. Solve the least-squares problem for $\widehat{B}$ and $\widehat{D}$.

Theorem *Consider the above algorithm with the noise-free frequency domain data of a discrete-time stable system of order n. If* $N \geq q + n$, $q > n$, *then the quadruplet* $(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D})$ *is a minimal realization of* $G(z)$.

- Extends an interpolation result in McKelvey, Akçay, and Ljung (1996) for uniformly spaced points on the unit circle to arbitrary interpolation points in the complement of the open unit disk (including derivatives).

Outline

Differences between the algorithm and the Löwner matrix based approach (Anderson and Antoulas; 1990):

- Formation of the data matrices
- Determination of the minimal order.

Similarities between the algorithm and the Löwner matrix based approach (Anderson and Antoulas; 1990):

- Both rely on the factorization of the data matrices as a product of two matrices related to the observability and controllability concepts.
- The solvability conditions are the same.

Numerical example

System in the state-space representation:

$$A = \begin{bmatrix} -0.5 & 0.5 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & -0.25 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

i.e, $n = 4, p = 2, m = 3$. This system has the transfer function:

$$G(z) = \begin{bmatrix} \frac{z^2+3z+1.5}{z^2+z+0.5} & G_{12}(z) & 0 \\ G_{21}(z) & G_{22}(z) & \frac{z+1.25}{z+0.25} \end{bmatrix}$$

where

$$\begin{aligned} G_{12}(z) &= -\frac{z^3+0.5z^2+0.5z+0.75}{z^3+0.5z^2-0.25}, \\ G_{21}(z) &= \frac{2z^2+1.25z+0.5}{z^3+1.25z^2+0.75z+0.125}, \\ G_{22}(z) &= \frac{z^3+3.25z^2+2.5z+0.75}{z^3+1.25z^2+0.75z+0.125}. \end{aligned}$$

Interpolation data:

$$z_1 = 1 + i,\ z_2 = 1 - i,\ N_1 = N_2 = 0,\ z_3 = 2,\ N_3 = 4$$

$$\begin{aligned} w_{10} &= \begin{bmatrix} 1.9333 & -0.8667 & 0 \\ 0.8878 & 1.9545 & 1.4878 \end{bmatrix} \\ &- \begin{bmatrix} 0.5333 & -0.4000 & 0 \\ 0.5236 & 0.6569 & 0.3902 \end{bmatrix} i, \end{aligned}$$

$$\begin{aligned} w_{20} &= \begin{bmatrix} 1.9333 & -0.8667 & 0 \\ 0.8878 & 1.9545 & 1.4878 \end{bmatrix} \\ &+ \begin{bmatrix} 0.5333 & -0.4000 & 0 \\ 0.5236 & 0.6569i & 0.3902 \end{bmatrix} i, \end{aligned}$$

$$
\begin{aligned}
W_{30} &= \begin{bmatrix} 1.7692 & -1.2051 & 0 \\ 0.7521 & 1.8291 & 1.4444 \end{bmatrix}, \\
W_{31} &= \begin{bmatrix} -0.2840 & 0.2433 & 0 \\ -0.2804 & -0.3395 & -0.1975 \end{bmatrix}, \\
W_{32} &= \begin{bmatrix} 0.2003 & -0.4251 & 0 \\ 0.2084 & 0.2757 & 0.1756 \end{bmatrix}, \\
W_{33} &= \begin{bmatrix} -0.2000 & 0.9844 & 0 \\ -0.2333 & -0.3341 & -0.2341 \end{bmatrix}, \\
W_{34} &= \begin{bmatrix} 0.2456 & -2.8518 & 0 \\ 0.3531 & 0.5390 & 0.4162 \end{bmatrix}.
\end{aligned}
$$

$q = 5 \Longrightarrow N = 9; \quad N \geq q + n, \ q > n.$

Results:

$$\widehat{A} = \begin{bmatrix} 0.5204 & -0.1361 & 0.3199 & 0.5352 \\ 0.0882 & -0.4983 & 0.4848 & -0.1035 \\ 0.0052 & 0.0820 & -0.4810 & 0.7195 \\ -0.0295 & 0.1919 & -0.3546 & -0.2911 \end{bmatrix},$$

$$\widehat{C} = \begin{bmatrix} 0.8460 & 0.2123 & -0.2149 & -0.3233 \\ -0.0721 & 0.8069 & 0.5289 & 0.1046 \end{bmatrix},$$

$$\widehat{B} = \begin{bmatrix} 1.0502 & -0.5390 & -0.0816 \\ 2.8626 & 1.8321 & 0.9041 \\ -0.1545 & 1.0984 & 0.4896 \\ -1.4555 & -0.9375 & 0.0547 \end{bmatrix},$$

$$\widehat{D} = \begin{bmatrix} 1.0000 & -1.0000 & -0.0000 \\ -0.0000 & 1.0000 & 1.0000 \end{bmatrix}.$$

- $(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}) \sim (A, B, C, D)$. (Max. error: 5.9746×10^{-14}).

Finding Q-Parameter

Example: Active suspension design for a quarter-car model

Closed-loop transfer function:

$$T_{zw} = G_{11} + G_{12}(Y - MQ)\widetilde{M}G_{21}, \qquad Q \in \mathcal{R}H_\infty, \qquad (1)$$

where $T_{zw} \in \mathbf{R}^{3\times 1}$; $Y, M, \widetilde{M} \in \mathcal{R}H_\infty$ are some matrices in a double coprime factorization of G_{22} over $\mathcal{R}H_\infty$; and G_{11}, G_{12}, G_{21}, G_{22} are some (open loop) block matrices.

Problem: find a $Q \in \mathcal{R}H_\infty$ satisfying (1) given $T_{z_k w}(s)$.

- $T_{z_l w}$, $l \neq k$ are uniquely determined by $T_{z_k w}$ (trade-offs).

- $T_{z_k w}$ and/or its derivatives are subject to certain interpolation conditions at $s = 0$, $s = \infty$, and some finite and nonzero invariant frequencies.

- Quite often a $T_{z_k w}$ with desirable features and satisfying (1) and the interpolation conditions can be constructed.

Solution: evaluate (1) and/or its derivatives at a set of sufficiently many and arbitrarily selected frequencies to formulate a bitangential interpolation problem. Next, use the subspace-based algorithm to obtain a minimal realization of Q. (Türkay and Akçay; 2008).

- A new algorithm for the Lagrange-Sylvester interpolation of rational matrix functions analytic at ∞ was introduced.
- A necessary and sufficient condition in terms of the total multiplicity of the interpolation nodes for the existence and uniqueness of a minimal interpolant was formulated.
- The algorithm is insentitive to inaccuracies in the interpolation data.