On the uniform approximation of discrete–time systems by generalized Fourier Series

Hüseyin Akçay

***Abstract*—In this paper, model sets for linear time-invariant discrete-time systems spanned by fixed orthonormal bases are studied. It is shown that the Fourier series of the system transfer function with respect to these bases converges uniformly on the unit circle if the frequency response of the system is Dini-Lipschitz continuous.**

***Keywords*: Orthogonal functions, generalized Fourier series, uniform convergence, discrete–time system, Dini-Lipschitz continuous.**

I. INTRODUCTION

The use of rational orthogonal basis functions for the decomposition of linear time–invariant dynamics has a long history in modelling and identification of dynamical systems [1], [2], [3], [4], [5], [6]. The main advantage of this approach is that *a priori* information about the system can easily be incorporated in the basis construction by a choice of basis poles, which leads to accurate system descriptions with a small number of basis functions and the estimation and analysis problems become simple due to the linear–in–the parameters model structure.

In this paper, we will consider a particular class of basis functions introduced in [7] and defined by a choice of complex numbers z_n in the open unit disk: $\mathbf{D} = \{z : |z| < 1\}$ as $B_0(z) = \sqrt{1-|z_0|^2}/(1-\overline{z_0}\,z)$ and for $n = 1, 2, \cdots$,

$$\begin{aligned} B_n(z) &\triangleq \frac{\sqrt{1-|z_n|^2}}{1-\overline{z_n}\,z}\,\phi_n(z), \\ \phi_n(z) &\triangleq \prod_{k=0}^{n-1} \frac{z-z_k}{1-\overline{z_k}\,z}, \quad \phi_0(z) \triangleq 1 \end{aligned} \tag{1}$$

which are orthonormal with respect to the inner product:

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,\overline{g(e^{i\theta})}\,\mathrm{d}\theta. \tag{2}$$

These basis functions generalize the well-known finite–pulse response, the *Laguerre*, and the *two–parameter Kautz* functions [8] and the more recently introduced *generalized orthonormal basis* functions [9] and the *rational wavelet* basis [10]. In contrast to the Laguerre and the two–parameter Kautz bases, where all the poles are fixed at the same value, the basis defined by (1) enjoys increased flexibility of pole location. An application example that illustrates the utility of the continuous-time versions of the basis functions (1) for flexible structure modelling was presented in [11].

With regard to the basis defined by (1), the approximation issues have been addressed in [12], [13]. In particular, it was established in [12] (see also Chap. 7 in [14]) that the linear span of the basis functions in (1) is everywhere dense in $H_p(\mathbf{D})$ $(1 \le p < \infty)$, the Hardy spaces of functions analytic on $\mathbf{D}$, and the disk algebra $A(\mathbf{D}) = H_\infty(\mathbf{D}) \cap C(\mathbf{T})$, where $C(\mathbf{T})$ is the space of complex functions continuous on the unit circle $\mathbf{T}$ provided that

$$\sum_{k=0}^{\infty}(1-|z_k|) = \infty. \tag{3}$$

It was also shown that, by using a min–max criterion, these bases lead to robust estimators for which error bounds in different norms can be explicitly quantified [12]. Moreover in [13], it was established that the Fourier series formed by the orthonormal basis functions in (1), whose partial sums are defined by

$$\mathcal{S}_n f(z) = \sum_{k=0}^{n}\langle f, B_k\rangle B_k(z), \tag{4}$$

converges in all spaces $H_p(\mathbf{D})$ $(1 < p < \infty)$ and for the estimate (4), tight approximation error bounds in the L_p norms were computed.

The completeness and approximation results for the stable discrete–time systems were extended in [15] to include also unstable systems by complementing the basis functions in (1) with the orthonormal functions defined by a choice of numbers $x_n \in \mathbf{D}$ for $n = 1, 2, \cdots$ as

$$\begin{aligned} B_{-n} &\triangleq \frac{\sqrt{1-|x_n|^2}}{z-x_n}\,\psi_{n-1}, \\ \psi_n &\triangleq \prod_{k=1}^{n}\frac{1-\overline{x_k}\,z}{z-x_k}, \quad \psi_0 \triangleq 1 \end{aligned} \tag{5}$$

which are orthogonal to the functions in (1).

In this paper, we will assume that the basis defined by (1) and (5) is uniformly bounded, i.e. it satisfies

$$\sup_n \{|z_n|, |x_n|\} = r < 1. \tag{6}$$

Since the completeness conditions: (3) and

$$\sum_{k=1}^{\infty}(1-|x_k|) = \infty$$

are obviously satisfied by the basis functions in (1) and (5) subject to (6), they are complete in $L_p(\mathbf{T})$ $(1 \le p < \infty)$, the Lebesque spaces on $\mathbf{T}$, and $C(\mathbf{T})$ [15].

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Department of Electrical and Electronics Engineering, Anadolu University, 26470 Eskişehir, Turkey. Tel. +90 222 335 0580 -X 6459. Fax: +90 222 323 9501. E-mail: huakcay@anadolu.edu.tr.

The partial sums of the generalized Fourier series of an integrable complex function f on $\mathbf{T}$ are defined by

$$\mathcal{S}_{n,m}f(z) = \sum_{k=-m}^{n} \langle f, B_k \rangle B_k(z). \tag{7}$$

We will study approximation of functions from a particular subset of $C(\mathbf{T})$ by the sums in (7) in the supremum norm

$$\|f\|_\infty = \sup_\theta \left| f(e^{i\theta}) \right|. \tag{8}$$

When $f(e^{i\theta})$ is a continuous function on $\mathbf{T}$, we write

$$\omega_f(\delta) = \sup_{|x-y|\le\delta} |f(e^{ix}) - f(e^{iy})| \tag{9}$$

for the *modulus of continuity* of $f(e^{i\theta})$. A function $f(e^{i\theta})$ is said *Dini-Lipschitz continuous* if

$$\omega_f(\delta)\, \ln(1/\delta) \to 0 \qquad (\delta \to 0). \tag{10}$$

For the trigonometric basis $\{z^{\pm k}\}$, it is well known that $\mathcal{S}_{n,m}f \to f$ uniformly on the unit circle as $n, m \to \infty$ if $f(e^{i\theta})$ is Dini-Lipschitz continuous. The main result of this paper is to establish an analogous result for the basis functions defined by (1) and (5) as follows.

Theorem 1.1: Suppose $m = O(n^\nu)$ $(\nu > 0)$. Let $\mathcal{S}_{n,m}f$ be as in (7). Assume that the orthonormal functions in (1) and (5) are uniformly bounded. If f has a Dini-Lipschitz continuous frequency response $f(e^{i\theta})$, then

$$\|\mathcal{S}_{n,m}f - f\|_\infty \to 0 \qquad (n \to \infty).$$

In the course of proving this theorem, we will show that $\|\mathcal{S}_{n,m}\| = O(\ln(n+m))$. This implies that the orthonormal functions defined by (1) and (5) can not form a basis for the space $C(\mathbf{T})$ if they are uniformly bounded.

Note that the Fourier coefficients in (7) can be estimated from noisy measurements of the frequency response $f(e^{i\omega})$ by a least–squares method. In [16], strong consistency of a fairly general class of least–squares algorithms has been established under mild stochastic noise assumptions.

II. Proof of Theorem 1.1

Note that the partial sums in (7) evaluated at $z = e^{i\theta}$ can be written as

$$\mathcal{S}_{n,m}f(e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{iy})\, L_{n,m}(y;\theta)\,\mathrm{d}y$$

where $L_{n,m}(y;\theta)$ is the *so-called* Dirichlet kernel defined by

$$L_{n,m}(y;\theta) = \sum_{k=-m}^{n} \overline{B_k(e^{iy})}\, B_k(e^{i\theta}).$$

Hence

$$\begin{aligned} \|\mathcal{S}_{n,m}\| &= \sup_{f\in C(\mathbf{T}), \|f\|_\infty = 1} \|\mathcal{S}_{n,m}f\|_\infty \\ &= \sup_\theta \|L_{n,m}(\cdot\,;\theta)\|_1. \end{aligned} \tag{11}$$

We shall first give a lemma.

Lemma 2.1 (Christoffel-Darboux formulae): For $z\overline{\zeta} \neq 1$,

$$\sum_{k=0}^{n} \overline{B_k(\zeta)}\, B_k(z) = \frac{1 - \overline{\phi_{n+1}(\zeta)}\,\phi_{n+1}(z)}{1 - \overline{\zeta}\, z}, \tag{12}$$

$$\sum_{k=-m}^{-1} \overline{B_k(\zeta)}\, B_k(z) = \frac{1 - \overline{\psi_m(\zeta)}\,\psi_m(z)}{\overline{\zeta}\, z - 1}. \tag{13}$$

Proof. The proof of (12) can be found, for instance in [17]. A concise proof is by induction. For $n = 0$,

$$\frac{1 - \overline{\phi_1(\zeta)}\phi_1(z)}{1 - \overline{\zeta}\, z} = \frac{1 - |z_0|^2}{(1 - z_0\,\overline{\zeta})(1 - \overline{z_0}\, z)} = \overline{B_0(\zeta)}\, B_0(z)$$

while for $n > 0$

$$\begin{aligned} \frac{1 - \overline{\phi_{n+1}(\zeta)}\phi_{n+1}(z)}{1 - \overline{\zeta}\, z} &= \sum_{k=0}^{n-1} \overline{B_k(\zeta)}\, B_k(z) \\ &\quad + (1 - |z_n|^2)\, \frac{\overline{\phi_n(\zeta)}\phi_n(z)}{(1 - z_n\,\overline{\zeta})(1 - \overline{z_n}\, z)}. \end{aligned}$$

To obtain (13), make the substitutions $z \to 1/z$, $\zeta \to 1/\zeta$, and $x_k \to \overline{x_k}$ in (12). ∎

The following lemma will be instrumental in proving Theorem 1.1.

Lemma 2.2: Let $e^{i\theta} - z_k = r_k(\theta)\, e^{ia_k(\theta)}$ be the polar decomposition of $z - z_k$ where $[0, 2\pi)$ branch is used both for θ and $a_k(\theta)$. Then

$$a_k(\theta) \equiv a_k(s) + \frac{\theta - s}{2} + \frac{1}{2}\int_s^\theta \left| B_k(e^{iy}) \right|^2\, \mathrm{d}y \;(\mathrm{mod} 2\pi). \tag{14}$$

Proof. For the notation, we refer to Figure 1. Observe that if $\overset{\wedge}{\mathrm{BOA}}$ angle increases as A rotates in the counterclockwise direction with respect to B, then $\overset{\wedge}{\mathrm{BCA}}$ angle also increases.

Fig. 1. The points A, B, C, O are respectively $e^{i\theta}$, e^{is}, z_k, 0. The points p, y, and x are the intersection of the chords OC and AC with the unit circle.

Let us first show that for almost all θ,

$$2\lim_{s\to\theta} \frac{a_k(\theta) - a_k(s)}{\theta - s} = 1 + |B_k(e^{i\theta})|^2. \tag{15}$$

From elementary geometry notice the similarity of the two triangles $\overset{\triangle}{\mathrm{pCA}}$ and $\overset{\triangle}{\mathrm{xCy}}$, which is due to the pair-wise equality of the six angles:

$$\overset{\wedge}{\mathrm{pAx}} = \overset{\wedge}{\mathrm{pyx}}, \qquad \overset{\wedge}{\mathrm{ypA}} = \overset{\wedge}{\mathrm{Axy}}, \qquad \overset{\wedge}{\mathrm{pCA}} = \overset{\wedge}{\mathrm{xCy}}\ .$$

Thus

$$\mathrm{Cx} \;=\; \frac{\mathrm{Cy}\cdot\mathrm{pC}}{\mathrm{AC}} \;=\; \frac{(1-|z_k|)\,(1+|z_k|)}{r_k(\theta)}. \tag{16}$$

A second fact from elementary geometry provides

$$\overset{\wedge}{\mathrm{AxB}} = \frac{\overset{\wedge}{\mathrm{AOB}}}{2} \;=\; \frac{\theta-s}{2}. \tag{17}$$

A third fact from trigonometry yields from (16) and (17),

$$\sin\,\overset{\wedge}{\mathrm{CBx}} = \frac{\mathrm{Cx}}{\mathrm{CB}}\,\sin\,\overset{\wedge}{\mathrm{CxB}} = \frac{\left(1-|z_k|^2\right)}{r_k(\theta)\,r_k(s)}\,\sin\left(\frac{\theta-s}{2}\right). \tag{18}$$

Finally

$$a_k(\theta)-a_k(s) \;= \overset{\wedge}{\mathrm{ACB}} = \overset{\wedge}{\mathrm{AxB}} \;+\; \overset{\wedge}{\mathrm{CBx}}\,. \tag{19}$$

Hence from (17)–(19)

$$\begin{aligned} 2\,\lim_{s\to\theta}\frac{a_k(\theta)-a_k(s)}{\theta-s} \quad &= \quad \lim_{s\to\theta}\frac{\sin\left(a_k(\theta)-a_k(s)\right)}{\sin\left(\frac{\theta-s}{2}\right)} \\ &= \quad \lim_{s\to\theta}\frac{\sin\,\overset{\wedge}{\mathrm{AxB}}+\sin\,\overset{\wedge}{\mathrm{CBx}}}{\sin\left(\frac{\theta-s}{2}\right)} \\ &= \quad 1+\frac{1-|z_k|^2}{r_k^2(\theta)} \\ &= \quad 1+|B_k(e^{i\theta})|^2. \end{aligned}$$

Hence $a_k(\theta)$ is differentiable on $(0,2\pi)$ except a possible point where it is not continuous.

Integrating (15), we obtain (14). A complication arises when either $\theta=0$ or $a_k(\theta)=0$ since a specific branch, say $[0,2\pi)$, had to be chosen for θ and $a_k(\theta)$. However, this is not a problem at all since by an application of the residue theorem the following formula

$$\int_s^{s+2\pi}\left|B_k(e^{iy})\right|^2\,\mathrm{d}y = 2\pi$$

holds for all s. ∎

Corollary 2.3:

$$\begin{aligned} \phi_n(e^{i\theta})\,\overline{\phi_n(e^{is})} \quad &= \quad \exp\left(i\int_s^\theta\sum_{k=0}^{n-1}\left|B_k(e^{iy})\right|^2\,\mathrm{d}y.\right), \\ \psi_m(e^{i\theta})\,\overline{\psi_m(e^{is})} \quad &= \quad \exp\left(-i\int_s^\theta\sum_{k=-m}^{-1}\left|B_k(e^{iy})\right|^2\,\mathrm{d}y.\right). \end{aligned}$$

Proof. Write the numerator and denumerator factors of $\phi_{n+1}(e^{i\theta})$ and $\psi_m(e^{i\theta})$ in polar forms as

$$e^{i\theta}-z_k = r_k(\theta)\,e^{ia_k(\theta)}, \qquad e^{i\theta}-x_k = r_k'(\theta)\,e^{ia_{-k}(\theta)}.$$

Since

$$1-\overline{z_k}\,e^{i\theta} = r_k(\theta)\,\exp\left(i\left[\theta-a_k(\theta)\right]\right),$$

we have

$$\phi_n(e^{i\theta}) = \exp\left(i\left[-n\,\theta+2\sum_{k=0}^{n-1}a_k(\theta)\right]\right)$$

and

$$\psi_m(e^{i\theta}) = \exp\left(i\left[m\,\theta-2\sum_{k=-m}^{-1}a_k(\theta)\right]\right).$$

Now the previous lemma completes the proof. ∎

A key consequence of this result is that it facilitates a simple formulation of the Dirichlet kernel as follows.

Lemma 2.4:

$$L_{n,m}(s;\theta) = e^{i\Phi}\,\frac{\sin\lambda(s;\theta)}{\sin\left(\frac{\theta-s}{2}\right)}. \tag{20}$$

where

$$\Phi = \frac{1}{2}\int_s^\theta\left(\sum_{k=0}^{n}\left|B_k(e^{iy})\right|^2-1-\sum_{k=-m}^{-1}\left|B_k(e^{iy})\right|^2\right)\,\mathrm{d}y,$$

$$\lambda\left(s;\theta\right) = \frac{1}{2}\int_s^\theta\sum_{k=-m}^{n}\left|B_k(e^{iy})\right|^2\,\mathrm{d}y. \tag{21}$$

Proof. This follows from the corollary, (12) and (13), and

$$e^{i\theta}-e^{-i\theta} = 2i\sin\theta.$$

∎

Now we are ready to prove the main result of this paper. First, from Lemma 2.4 we derive upper and lower bounds on $\|\mathcal{S}_{n,m}\|$.

Lemma 2.5: Let $\|\mathcal{S}_{n,m}\|$ be as in (11). Suppose that the basis defined by (1) and (5) satisfies (6). Then

$$\frac{\gamma^2+1}{2\pi\gamma^4}\,\ln\left(\frac{P}{\gamma}-1\right)\le\|\mathcal{S}_{n,m}\|\le\frac{\pi\gamma^2}{2}+\ln\left(\frac{P}{\gamma}\right) \tag{22}$$

where $P=n+m+1$ and

$$\gamma = \frac{1+r}{1-r}. \tag{23}$$

Proof. We start with the derivation of the lower bound. For a given θ, $\lambda\left(\theta-x;\theta\right)$ defined by (21) is strictly increasing on $[-\pi,\pi]$ as a function of x. Thus for each integer number k, the equation:

$$\lambda\left(\theta-x;\theta\right) = \frac{\pi}{2}\,k \tag{24}$$

has a unique solution denoted by x_k. We claim that if x_k is restricted to the interval $[-\pi,\pi]$, then k must be restricted to a finite interval $[-M,N]$, *that is*, the following inequalities:

$$x_{-(M+1)}<-\pi\le x_{-M}, \qquad x_N\le\pi<x_{N+1} \tag{25}$$

are satisfied for some finite numbers M and N. We prove this claim by estimating upper and lower bounds for M and N in the above inequalities.

From (24) and (21),

$$\begin{aligned} \frac{\pi}{2} \quad &= \quad \lambda\left(\theta-x_{k+1};\theta\right)-\lambda\left(\theta-x_k;\theta\right) \\ &= \quad \frac{1}{2}\int_{\theta-x_{k+1}}^{\theta-x_k}\sum_{k=-m}^{n}\left|B_k(e^{iy})\right|^2\,\mathrm{d}y. \end{aligned} \tag{26}$$

Since for all y,

$$\begin{aligned} \frac{1}{\gamma} \le \left|B_k(e^{iy})\right|^2 &= \frac{1-|z_k|^2}{|1-\overline{z_k}\,e^{iy}|^2} \le \gamma, \qquad k \ge 0 \\ \frac{1}{\gamma} \le \left|B_k(e^{iy})\right|^2 &= \frac{1-|x_k|^2}{|1-\overline{x_k}\,e^{iy}|^2} \le \gamma \qquad k < 0, \end{aligned} \tag{27}$$

we have from (26),

$$\frac{\pi}{\gamma P} \le x_{k+1} - x_k \le \frac{\pi\gamma}{P}. \tag{28}$$

Summing the left and right hand sides of the inequalities in (28) from $k = 0$ to J or from $k = -J$ to 0 and noting that $x_0 = 0$, we obtain

$$\frac{\pi}{\gamma P}\, J \le |x_{\pm J}| \le \frac{\pi\gamma}{P}\, J, \qquad J \ge 0. \tag{29}$$

The above inequalities evaluated for $J = N$ and $J = N + 1$ or $J = -M$ and $J = -(M + 1)$, and the inequalities in (25) yield

$$\frac{P}{\gamma} - 1 < N \le \gamma P, \qquad \frac{P}{\gamma} - 1 < M \le \gamma P \tag{30}$$

which proves our claim.

From the fact that

$$|\sin(x)| \le |x|, \qquad \text{for all } x, \tag{31}$$

$$\begin{aligned} \|L_{n,m}(\cdot\,;\theta)\|_1 &\ge \frac{1}{2\pi} \sum_{k=-M}^{N-1} \int_{x_k}^{x_{k+1}} \left| \frac{\sin\lambda\,(\theta - x;\theta)}{\sin\left(\frac{x}{2}\right)} \right| \mathrm{d}x \\ &\ge \frac{1}{\pi} \sum_{k=-M}^{N-1} \int_0^{x_{k+1}-x_k} \frac{|\sin\lambda\,(\theta - x_k - s;\theta)|}{\max\{|x_k|, |x_{k+1}|\}}\, \mathrm{d}s. \end{aligned} \tag{32}$$

An application of the sin expansion formula:

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

to the identity:

$$\lambda\,(\beta_k - s;\theta) = \lambda\,(\beta_k - s;\beta_k) + \lambda\,(\beta_k;\theta)$$

where $\beta_k = \theta - x_k$ results in

$$\begin{aligned} \sin\lambda\,(\beta_k - s;\theta) &= \sin\lambda\,(\beta_k - s;\beta_k)\,\cos\lambda\,(\beta_k;\theta) \\ &\quad + \cos\lambda\,(\beta_k - s;\beta_k)\,\sin\lambda\,(\beta_k;\theta) \\ &= \begin{cases} (-1)^{k/2} \sin\lambda\,(\beta_k - s;\beta_k)\,, & k \;\text{ even} \\ (-1)^{(k-1)/2} \cos\lambda\,(\beta_k - s;\beta_k)\,, & k \;\text{ odd.} \end{cases} \end{aligned}$$

Since $\lambda(\beta_k - s;\beta_k)$ is an increasing function,

$$\lambda\,(\beta_k - s;\beta_k) \le \frac{\pi}{2}, \qquad 0 \le s \le x_{k+1} - x_k.$$

Thus from,

$$|\sin(x)| \ge \frac{2}{\pi}\,|x|, \qquad |x| \le \frac{\pi}{2}, \tag{33}$$

we get

$$\begin{aligned} \sin\lambda\,(\beta_k - s,\beta_k) &\ge \frac{2}{\pi}\,\lambda\,(\beta_k - s,\beta_k) \\ &= \frac{1}{\pi} \int_{\beta_k - s}^{\beta_k} \sum_{k=-m}^{n} \left|B_k(e^{iy})\right|^2 \mathrm{d}y \\ &\ge \frac{P\,s}{\pi\gamma} \end{aligned}$$

where the last inequality has followed from (27). Hence if k is an even integer, from (28)

$$\begin{aligned} \int_0^{x_{k+1}-x_k} |\sin\lambda\,(\beta_k - s;\theta)|\,\mathrm{d}s &\ge \frac{P}{2\pi\gamma}\,(x_{k+1} - x_k)^2 \\ &\ge \frac{\pi}{2\gamma^3 P}. \end{aligned} \tag{34}$$

Considering the terms in (32) for even indices, thus we have from (29) and (34)

$$\begin{aligned} &\frac{1}{\pi} \sum_{\substack{-M \le k < N \\ k:\,\text{even}}} \int_0^{x_{k+1}-x_k} \frac{|\sin\lambda\,(\beta_k - s;\theta)|}{\max\{|x_k|, |x_{k+1}|\}}\,\mathrm{d}s \\ &\ge \frac{1}{2\pi\gamma^4} \left(\sum_{\substack{-M \le k \le -2 \\ k:\,\text{even}}} \frac{1}{|k|} + \sum_{\substack{0 \le k < N \\ k:\,\text{even}}} \frac{1}{k+1} \right) \\ &\ge \frac{1}{2\pi\gamma^4} \sum_{k=1}^{\min\{M,N\}-1} \frac{1}{k}. \end{aligned} \tag{35}$$

Now we consider the terms in (32) for odd indices. The graph of $\cos(x)$ has the property:

$$\cos(x) \ge 1 - \frac{2x}{\pi}, \qquad 0 \le x \le \frac{\pi}{2}.$$

Thus for all $0 \le s \le x_{k+1} - x_k$ from (27),

$$\begin{aligned} \cos\lambda\,(\beta_k - s;\beta_k) &\ge 1 - \frac{2}{\pi}\,\lambda\,(\beta_k - s;\beta_k) \\ &= 1 - \frac{1}{\pi} \int_{\beta_k - s}^{\beta_k} \sum_{k=-m}^{n} \left|B_k(e^{iy})\right|^2 \mathrm{d}y \\ &\ge 1 - \frac{\gamma P\,s}{\pi}. \end{aligned}$$

Hence if k is an odd integer,

$$\begin{aligned} \int_0^{x_{k+1}-x_k} |\sin\lambda\,(\beta_k - s;\theta)|\,\mathrm{d}s &\ge \int_0^{\frac{\pi}{\gamma P}} \left(1 - \frac{\gamma P}{\pi}\,s\right) \mathrm{d}s \\ &= \frac{\pi}{2\gamma P}. \end{aligned} \tag{36}$$

Thus from (29) and (36),

$$\begin{aligned} &\frac{1}{\pi} \sum_{\substack{-M \le k < N \\ k:\,\text{odd}}} \int_0^{x_{k+1}-x_k} \frac{|\sin\lambda\,(\beta_k - s;\theta)|}{\max\{|x_k|, |x_{k+1}|\}}\,\mathrm{d}s \\ &\ge \frac{1}{2\pi\gamma^2} \left(\sum_{\substack{-M \le k \le -1 \\ k:\,\text{odd}}} \frac{1}{|k|} + \sum_{\substack{1 \le k < N \\ k:\,\text{odd}}} \frac{1}{k+1} \right) \\ &\ge \frac{1}{2\pi\gamma^2} \sum_{k=1}^{\min\{M,N\}-1} \frac{1}{k}. \end{aligned} \tag{37}$$

It follows from (32), (35), and (37)

$$\|L_{n,m}(\cdot\,;\theta)\|_1 \ge \frac{\gamma^2 + 1}{2\pi\gamma^4} \sum_{k=1}^{\min\{M,N\}-1} \frac{1}{k}. \tag{38}$$

Hence from (38), (30), and the following inequality

$$\sum_{k=1}^{N-1} \frac{1}{k} \geq \int_1^N \frac{\mathrm{d}x}{x} = \ln N,$$

we obtain a lower bound on $\|L_{n,m}(\cdot\,;\theta)\|_1$ as follows

$$\|L_{n,m}(\cdot\,;\theta)\|_1 \geq \frac{\gamma^2+1}{2\pi\gamma^4} \ln\left(\frac{P}{\gamma}-1\right).$$

An upper bound on $\|L_{n,m}(\cdot\,;\theta)\|_1$ is derived as follows

$$\begin{aligned}\|L_{n,m}(\cdot\,;\theta)\|_1 &= \frac{1}{2\pi}\int_{x_{-1}}^{x_1} \left|\frac{\sin\lambda(\theta-x;\theta)}{\sin\left(\frac{x}{2}\right)}\right| \mathrm{d}x \\ &\quad + \frac{1}{2\pi}\int_{x\notin[x_{-1},x_1]} \left|\frac{\sin\lambda(\theta-x;\theta)}{\sin\left(\frac{x}{2}\right)}\right| \mathrm{d}x \\ &\leq \frac{1}{2\pi}\int_{x_{-1}}^{x_1} \frac{\pi\gamma P}{2}\,\mathrm{d}x + \frac{1}{2}\int_{x\notin[x_{-1},x_1]} \frac{\mathrm{d}x}{|x|} \\ &= \frac{\gamma P}{4}\,(x_1 - x_{-1}) + \frac{1}{2}\,[2\ln\pi - \ln(-x_{-1}) - \ln x_1] \\ &\leq \frac{\pi\gamma^2}{2} + \ln\left(\frac{P}{\gamma}\right)\end{aligned}$$

where in deriving the above inequalities, (31), (33), and (29) have been used. Since the lower and upper bounds on $\|L_{n,m}(\cdot\,;\theta)\|_1$ derived above are independent of θ, they bound $\|\mathcal{S}_{n,m}\|$ from below and above. ∎

From the lemma, it follows that the orthonormal functions defined by (1) and (5) can not also form a basis for $L_(\mathbf{T})$ if they are uniformly bounded.

We need the following technical lemma whose proof can be found in [15].

Lemma 2.6: Let $\mathbf{A}(r_1,r_2) = \{z : r_1 \leq |z| \leq r_2\}$, where $r_1 < 1$ and $r_2 > 1$ are two given positive numbers. Suppose that $f(z)$ is analytic and bounded by M_f in a region that contains $\mathbf{A}(r_1,r_2)$. Let $\mathcal{S}_{n,m}f$ be as in (7). Then

$$\|f - \mathcal{S}_{n,m}f\|_\infty \leq \frac{M_f\,r_2}{r_2-1}\exp\left(-\frac{r_2-1}{2r_2}\sum_{k=0}^{n}(1-|z_k|)\right) + \frac{M_f\,r_1}{1-r_1}\exp\left(-\frac{1-r_1}{2}\sum_{k=1}^{m}(1-|x_k|)\right).$$

Now we complete the proof of the main result. Let $X_{n,m}$ denote the linear space spanned by the basis functions $B_k(z)$, $k = -m, \cdots, n$ and define

$$\delta_{n,m}(f) = \min_{g\in X_{n,m}} \|f-g\|_\infty. \tag{39}$$

Thus $\delta_{n,m}(f)$ is the best approximation error of f in the $L_\infty(\mathbf{T})$ norm by functions in $X_{n,m}$. Let $\tau_{n,m}$ be the unique minimizing solution in (39). Let $\widehat{\delta}_K(f)$ denote the best approximation error of f in the $L_\infty(\mathbf{T})$ norm by trigonometric polynomials $h_K(z) = \sum_{k=-K}^{K} c_k z^k$ and let $\widehat{\tau}_K$ be the unique minimizer. Note that

$$\|\widehat{\tau}_K\|_\infty \leq \widehat{\delta}_K(f) + \|f\|_\infty \leq 2\,\|f\|_\infty.$$

Hence $\|c\|_\infty \leq 2\,\|f\|_\infty$. Since $\widehat{\tau}_K$ is analytic on $\mathbf{A}(0,\infty)$, we can use Lemma 2.6 with $\mathbf{A}(1/2,2)$. Then

$$\begin{aligned}M_{\widehat{\tau}_K} &= \sup_{z\in\mathbf{A}(1/2,2)} |\widehat{\tau}_K(z)| \\ &\leq \|c\|_\infty \sup_{z\in\mathbf{A}(1/2,2)} \sum_{k=-K}^{K} |z|^k \\ &\leq 2^{K+2}\,\|f\|_\infty.\end{aligned} \tag{40}$$

Thus from Lemma 2.6,

$$\begin{aligned}\|\widehat{\tau}_K - \mathcal{S}_{n,m}\widehat{\tau}_K\|_\infty &\leq 2M_{\widehat{\tau}_K}\exp\left[-(n+1)\,\frac{1-r}{4}\right] \\ &\quad + M_{\widehat{\tau}_K}\exp\left(-m\,\frac{1-r}{4}\right) \\ &\leq 3M_{\widehat{\tau}_K}\exp\left(-\min\{m,n\}\,\frac{1-r}{4}\right) \\ &\leq 12\,\|f\|_\infty \exp\left(K - \min\{m,n\}\,\frac{1-r}{4}\right)\end{aligned} \tag{41}$$

where the last inequality has followed from (40). For each pair n and m, choose K such that

$$\frac{1}{3} \leq \frac{4K}{(1-r)\min\{m,n\}} \leq \frac{1}{2}. \tag{42}$$

Since $\mathcal{S}_{n,m}\widehat{\tau}_K \in X_{n,m}$, an application of the triangle inequality yields

$$\begin{aligned}\delta_{n,m}(f) &\leq \|f - \mathcal{S}_{n,m}\widehat{\tau}_K\|_\infty \\ &\leq \|f - \widehat{\tau}_K\|_\infty + \|\widehat{\tau}_K - \mathcal{S}_{n,m}\widehat{\tau}_K\|_\infty \\ &= \widehat{\delta}_K(f) + \|\widehat{\tau}_K - \mathcal{S}_{n,m}\widehat{\tau}_K\|_\infty.\end{aligned} \tag{43}$$

The first term on the right hand side of the above inequality is bounded from a theorem of Jackson [18, p. 144]:

$$\widehat{\delta}_K(f) \leq \omega_f\left(\frac{\pi}{K+1}\right). \tag{44}$$

Hence if $m = O(n^\nu)$ $(\nu > 0)$, then for some absolute constant $C > 0$,

$$\ln P \leq C\,\ln K,$$

and thus from (44),

$$\widehat{\delta}_K(f)\,\ln P \to 0 \qquad (n\to\infty). \tag{45}$$

From (41) and (42), under the same condition $m = O(n^\nu)$ $(\nu > 0)$, we also have

$$\|\widehat{\tau}_K - \mathcal{S}_{n,m}\widehat{\tau}_K\|_\infty\,\ln P \to 0 \qquad (n\to\infty). \tag{46}$$

It follows from (43), (45), and (46)

$$\delta_{n,m}(f)\,\ln P \to 0 \qquad (n\to\infty).$$

The linearity of the operators $\mathcal{S}_{n,m}$ and the fact that $\mathcal{S}_{n,m}\tau_{n,m} = \tau_{n,m}$ (since $\tau_{n,m} \in X_{n,m}$) complete the proof as follows

$$\begin{aligned}\|f - \mathcal{S}_{n,m}f\|_\infty &= \|f - \tau_{n,m} + \mathcal{S}_{n,m}(\tau_{n,m} - f)\|_\infty \\ &\leq (1 + \|\mathcal{S}_{n,m}\|)\,\delta_{n,m}(f) \\ &\leq C_1\,\delta_{n,m}(f)\,\ln P \to 0 \qquad (n\to\infty)\end{aligned}$$

where $C_1 > 0$ is an absolute constant. ∎

A totally different proof of this theorem appeared very recently in [19].

Corollary 2.7: Let $\mathcal{S}_n f$ be as in (4). Assume that the orthonormal functions defined by (1) are uniformly bounded. If $f \in A(\mathbf{D})$ has a Dini-Lipschitz continuous frequency response $f(e^{i\theta})$, then

$$\|\mathcal{S}_n f - f\|_\infty \to 0 \qquad (n \to \infty).$$

Recall that a discrete–time ℓ_2 bounded–input/bounded–output (BIBO) stable system has a transfer function $f(z)$ in $H_\infty(\mathbf{D})$. If in addition, the system is ℓ_∞ BIBO stable, then its transfer function is in $A(\mathbf{D})$. Corollary 2.7 then tells us that any $f \in A(\mathbf{D})$ in the Dini-Lipschitz class can be recovered asymptotically by its Fourier series (4). This is the largest uniform converge set of the Fourier series (4) since from Lemma 2.5, we have

$$\|\mathcal{S}_n\| = \sup_{f\in A(\mathbf{D}), \|f\|_\infty = 1} \|\mathcal{S}_n f\|_\infty = O(\ln n).$$

An important consequence of this result is that the orthonormal functions defined by (1) can not form a basis for the disk algebra if they are uniformly bounded. This result applies not only to rational orthonormal systems defined by (1), but also to arbitrary uniformly bounded orthonormal bases in $H_2(\mathbf{D})$ [20].

Whether there exists an orthonormal basis for $H_2(\mathbf{D})$ defined by (1) such that every function in $A(\mathbf{D})$ has a convergent Fourier series with respect to this basis is unknown. A necessary but insufficient condition is that the closure of $\{z_n\}$ covers entire unit circle [21].

The situation is quite different if one considers orthonormal systems other than the rational system defined by (1). There are certainly orthonormal bases for $H_2(\mathbf{D})$ which consists of rational functions (even polynomials) and also form bases in the disk algebra. See for example, the construction in [22].

III. Conclusions

This paper has provided a preliminary study of the uniform convergence properties of a certain general class of rational orthonormal basis functions. The main result was to establish that the Fourier series with respect to uniformly bounded orthonormal bases converged uniformly in the space of Dini-Lipschitz continuous functions.

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Hüseyin Akçay was born in Antalya, Turkey in 1958. He received the Engineer degree from the İstanbul Technical University in 1981, the M.Sc degree from the Massachusetts Institute of Technology in 1988, and the Ph.D degree from the University of Michigan, Ann Arbor in 1992, all in mechanical engineering, and the M.A. degree in mathematics from the University of Michigan, Ann Arbor in 1991. He held visiting positions with Linköping, Newcastle, and Bremen universities. He received a research fellowship award in 1998 from the Alexander Von Humbold Foundation. He worked at the Tübitak, Marmara Research Center, Gebze, Turkey as Research Scientist. He is currently professor of Electrical and Electronics Engineering in Anadolu University. His research interests include system identification, signal processing, condition monitoring and fault detection.