Synchronization and control

Henk Nijmeijer, Henri Huijberts, Sasha Pogromsky
Eindhoven University of Technology, University of London

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Cooperation with: Ilya Blekhman, Sasha Fradkov
Alejandro Rodriguez-Angeles, Torsten Lilge,
Iven Mareels, Giovanni Santoboni
Rob Willems

Introduction

History of synchronization

- Christiaan Huijgens (1670): pendulum clocks synchronize...
- Rayleigh (1877): organ tubes sound in unison..., tuning forks,...
- 1900: electrical and electromechanical systems (Van der Pol)
- Rotating bodies, e.g. moon vs earth
- Chaos synchronization
- (Secure) communication: Pecora and Carroll (1990)
- Controlled synchronization: communication, ship mooring, robot coordination, sound and light show,...

Contents

This presentation

- Control view on synchronization
- Dynamics
- Observer theory vs synchronization
- Controlled synchronization
- Examples of simple/chaotic systems
- Theory: literature
- No complete review

Christiaan Huijgens,

born April 14, 1629 , The Hague, died July 8, 1695 , The Hague

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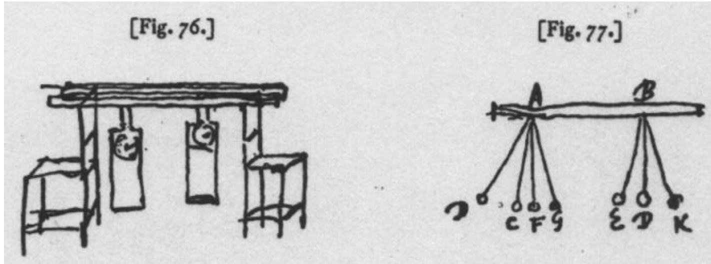

Huijgens' notebook, "Horloges Marines (Et Sympathie Des Horloges)",

1 March, 1665.

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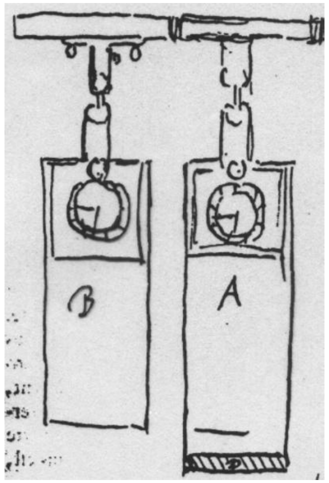

A figure from Huijgens' notebook, 22 Febr. 1665

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Neuzen zetten menstruaties precies gelijk

Vrouwen die nauw samenleven in een groep, worden op den duur allemaal rond dezelfde dagen ongesteld. Amerikaanse onderzoekers hebben het oude vermoeden bevestigd dat geurstoffen daarvoor verantwoordelijk zijn.

De negen 'geurdonors' dro
toe op bepaalde dagen in hu
een douche acht uur lang w
tjes onder hun oksels, die n
vieren geknipt, met alcohol
en ingevroren werden. D
proefpersonen roken tijdens
struatiecycli dagelijks aan e
ontdooide watjes.
'Opmerkelijk was nu dat al
was gedragen in de dage
eisprong van de geurdonor,
bij de vrouwen die de geur
bijna twee dagen eerder da
op gang kwam. Omgekeerd
als het watje de okselgeur b
de dagen na de ovulatie, de c

Women living together have

synchronous menstrual cycles.

An application of master slave synchronization

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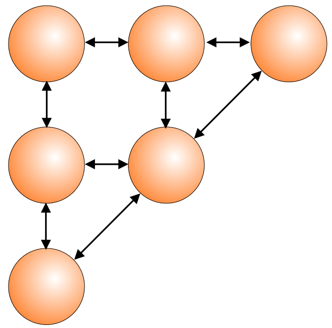

Synchronization in networks: different synchronization modes, partial synchronization.

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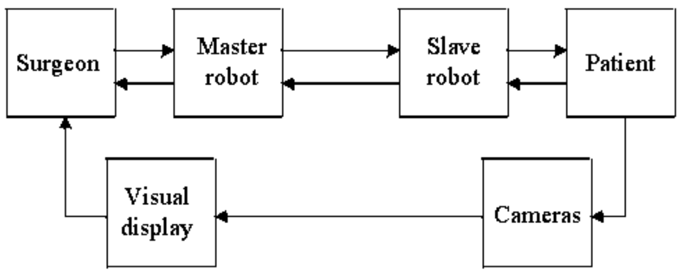

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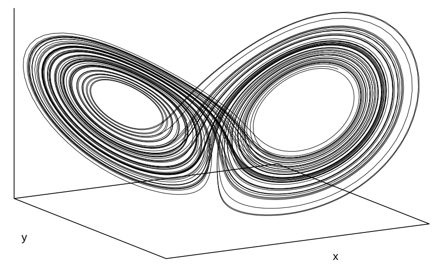
This workshop:

- Control theory: observers, controlled synchronization
- Dynamics: synchronization in networks
- Applications: communication and coordination of mechanical systems

Contents

- Introduction
- An observer view on synchronization
- Communication and synchronization
- Synchronization in diffusive networks
- Controlled synchronization
- Coordination of mechanical systems



The Lorenz attractor.

Synchronization and observers

Two points of view on synchronization

Peccora and Carroll (1990), Lorenz system

Transmitter (master) system :

$$\dot{x}_1 = \sigma(x_2 - x_1)$$
$$\dot{x}_2 = rx_1 - x_2 - x_1x_3$$
$$\dot{x}_3 = -bx_3 + x_1x_2$$

Two points of view on synchronization

Peccora and Carroll (1990), Lorenz system

Transmitter (master) system :	Receiver (slave) system ("copy" of master)
$\dot{x}_1 = \sigma(x_2 - x_1)$	
$\dot{x}_2 = rx_1 - x_2 - x_1x_3$	$\dot{\widehat{x}}_2 = rx_1 - \widehat{x}_2 - x_1\widehat{x}_3$
$\dot{x}_3 = -bx_3 + x_1x_2$	$\dot{\widehat{x}}_3 = -b\widehat{x}_3 + x_1\widehat{x}_2$

No $\widehat{x}_1$-dynamics, because x_1 already known.

$$e_2 = x_2 - \hat{x}_2, \quad e_3 = x_3 - \hat{x}_3$$

$$\dot{e}_2 = -e_2 - x_1 e_3$$
$$\dot{e}_3 = -b e_3 + x_1 e_2$$

Lyapunov function: $V(e_2, e_3) = e_2^2 + e_3^2$

$$\dot{V} = -2e_2^2 - 2be_3^2$$

$$(e_2, e_3) \to (0,0), \quad \text{as} \quad t \to \infty$$

Control viewpoint: Slave is partial observer for master.

$$e_1 = x_1 - \hat{x}_1, \quad e_2 = x_2 - \hat{x}_2, \quad e_3 = x_3 - \hat{x}_3$$

$$\dot{e}_1 = \sigma(e_2 - e_1)$$
$$\dot{e}_2 = -e_2 - x_1 e_3$$
$$\dot{e}_3 = -b e_3 + x_1 e_2$$

Lyapunov function: $V(e_1, e_2, e_3) = 1/\sigma e_1^2 + e_2^2 + e_3^2$

$$\dot{V} = -2(e_1 - 1/2 e_2)^2 - 3/2 e_2^2 - 2be_3^2$$

$$(e_1, e_2, e_3) \to (0,0,0), \quad \text{as} \quad t \to \infty$$

Control viewpoint: slave is full observer for master.

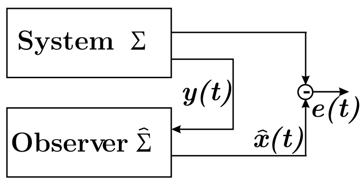
Two points of view on synchronization

Peccora and Carroll (1990), Lorenz system

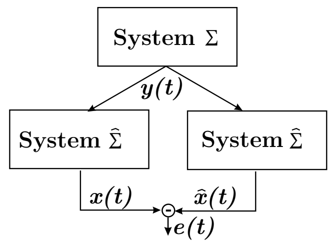
Transmitter (master) system :	Receiver (slave) system ("copy" of master)
$\dot{x}_1 = \sigma(x_2 - x_1)$	$\dot{\hat{x}}_1 = \sigma(\hat{x}_2 - \hat{x}_1)$
$\dot{x}_2 = r x_1 - x_2 - x_1 x_3$	$\dot{\hat{x}}_2 = r x_1 - \hat{x}_2 - x_1 \hat{x}_3$
$\dot{x}_3 = -b x_3 + x_1 x_2$	$\dot{\hat{x}}_3 = -b \hat{x}_3 + x_1 \hat{x}_2$

$\dot{x}_1 = \sigma(x_2 - x_1)$	$\dot{\hat{x}}_1 = \sigma(\hat{x}_2 - \hat{x}_1)$
$\dot{x}_2 = r x_1 - x_2 - x_1 x_3$	$\dot{\hat{x}}_2 = r x_1 - \hat{x}_2 - x_1 \hat{x}_3$
$\dot{x}_3 = -b x_3 + x_1 x_2$	$\dot{\hat{x}}_3 = -b \hat{x}_3 + x_1 \hat{x}_2$

Two related problems:



Synchronization/observer problem



Convergent systems, Demidovich

Synchronization/Observer problem statement

$$\dot{x} = f(x)$$
$$y = h(x)$$

$x(t) \in \mathbb{R}^n$, state; $y(t) \in \mathbb{R}$, output, or measurement.

Observer: given $y(t)$, $t \geq 0$, reconstruct asymptotically $x(t), t \geq 0$.

Reduced observer: given $y(t)$, $t \geq 0$, reconstruct asymptotically $x(t)$ modulo $y(t)$.

So if slave can be chosen freely, synchronization problem is equivalent to observer problem.

Example

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = ax_1 + bx_2, \quad y = x_1$$

observer: estimate for (x_1, x_2)

How to find observer? Try "Pecora & Carroll copy":

$$\dot{\widehat{x}}_1 = \widehat{x}_2$$
$$\dot{\widehat{x}}_2 = ay + b\widehat{x}_2 = ax_1 + b\widehat{x}_2$$

$$e_1 = x_1 - \widehat{x}_1, \;\; e_2 = x_2 - \widehat{x}_2$$

$$\dot{e}_1 = e_2, \;\; \dot{e}_2 = be_2$$

Does give reconstruction of x_2 iff $b < 0$, but not reconstruction of x_1!

Example

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = ax_1 + bx_2, \quad y = x_1$$

reduced observer: estimate for x_2.

How to find reduced observer? Try "Pecora & Carroll copy":

$$\dot{\widehat{x}}_2 = ay + b\widehat{x}_2 = ax_1 + b\widehat{x}_2$$

$$e_2 = x_2 - \widehat{x}_2$$

$$\dot{e}_2 = be_2$$

Asymptotic reconstruction if and only if $b < 0$.

Alternative:

$$\dot{\widehat{x}}_1 = \widehat{x}_2 + k_1 e_1$$
$$\dot{\widehat{x}}_2 = a\widehat{x}_1 + b\widehat{x}_2 + k_2 e_1$$

Suitable k_1 and k_2 yield

$$(e_1, e_2) \to (0, 0), \quad \text{as} \;\; t \to \infty$$

(Reduced observer: similar)

Linear systems (no complex dynamics):

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n$$
$$y = Cx$$

$$\dot{\widehat{x}} = A\widehat{x} + K(y - \widehat{y})$$
$$\widehat{y} = C\widehat{x}$$

$$\dot{e} = (A - KC)e$$

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Nonlinear systems?

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n$$
$$y = h(x)$$

Try:

$$\dot{\widehat{x}} = f(\widehat{x}) + k(\widehat{x}, y)$$

with $k(\widehat{x}, y) = 0$ if $h(\widehat{x}) = y$

Required for synchronization: $x \to \widehat{x}$, as $t \to \infty$ for any initial $x(0), \widehat{x}(0)$.

Find $k(\cdot,\cdot)$!

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$A - KC$ has arbitrary pole location iff system is observable, i.e.

$$\text{rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n$$

Thus synchronization $\widehat{x} \to x$ for suitably chosen K.

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When does an observer exist?

1. Linear error dynamics

Example: Chua circuit

$$\dot{x}_1 = \alpha(-x_1 + x_2 - \varphi(x_1))$$
$$\dot{x}_2 = x_1 - x_2 + x_3$$
$$\dot{x}_3 = -\lambda x_2$$

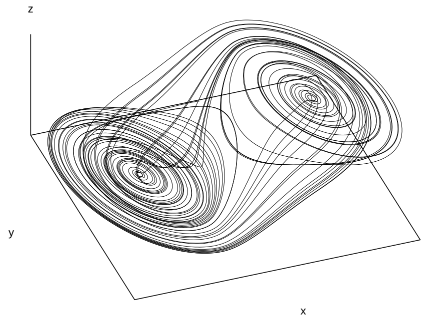
$$\varphi(x_1) = m_1 x_1 + m_2(|x_1 + 1| - |x_1 - 1|)$$

with $m_1 = -5/7$, $m_2 = -3/7$, $23 < \lambda < 31$, $\alpha = 15.6$.

Double scroll chaotic attractor

Output: $y = x_1$

Thus nonlinearity $\varphi(x_1)$ is measurable!



The double scroll attractor.

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2. Linearizable Error Dynamics

Example: Rössler system, $a, b, c > 0$

$$
\begin{aligned}
\dot{x}_1 &= -x_2 - x_3 \\
\dot{x}_2 &= x_1 + a x_2 \\
\dot{x}_3 &= c + x_3(x_1 - b) \\
y &= x_3
\end{aligned}
$$

NB $\quad x_3(0) > 0 \Longrightarrow x_3(t) > 0,\ \forall t > 0.$

New coordinates:

$$(\xi_1, \xi_2, \xi_3) = (x_1, x_2, \log x_3)$$

New output: $\quad y = \log y = \xi_3$

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Observer:

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \alpha(-\hat{x}_1 + \hat{x}_2 - \varphi(x_1)) + k_1 e_1 \\
\dot{\hat{x}}_2 &= \hat{x}_1 - \hat{x}_2 + \hat{x}_3 + k_2 e_1 \\
\dot{\hat{x}}_3 &= -\lambda \hat{x}_2 + k_3 e_1
\end{aligned}
$$

$$
\begin{aligned}
\dot{e}_1 &= (k_1 - \alpha) e_1 + e_2 \\
\dot{e}_2 &= (k_2 + 1) e_1 - e_2 + e_3 \\
\dot{e}_3 &= k_3 e_1 - \lambda e_2
\end{aligned}
$$

Note: Linear Observable System, $(e_1, e_2, e_3) \to (0, 0, 0)$ for suitable choice of k_1, k_2, k_3.

Arbitrarily fast!

Similar design for Lur'e systems.

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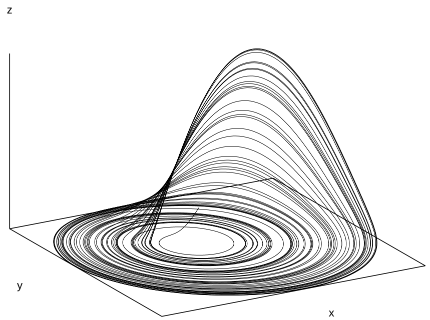

Trajectories of the Rössler system.

In new coordinates:

$$
\begin{aligned}
\dot{\xi}_1 &= -\xi_2 - e^{\ _3} \\
\dot{\xi}_2 &= \xi_1 + a\xi_2 \\
\dot{\xi}_3 &= \xi_1 + (-b + ce^{-\ _3}) \\
y &= \xi_3
\end{aligned}
$$

- Linear part is observable
- Nonlinear part is measurable

Observer:

$$
\begin{aligned}
\dot{\widehat{\xi}}_1 &= -\widehat{\xi}_2 - e^{\ _3} + k_1(\xi_3 - \widehat{\xi}_3) \\
\dot{\widehat{\xi}}_2 &= \widehat{\xi}_1 + a\widehat{\xi}_2 + k_2(\xi_3 - \widehat{\xi}_3) \\
\dot{\widehat{\xi}}_3 &= \widehat{\xi}_1 + (-b + ce^{-\ _3}) + k_3(\xi_3 - \widehat{\xi}_3)
\end{aligned}
$$

3. High-gain Observer

$$\dot{x} = f(x), \quad y = h(x)$$

Assume:

- $f(x)$ satisfies Lipschitz condition on positively invariant compact domain Ω.
- The n functions $h(x), L_f h(x), L_f^2 h(x), \ldots$ (Iterated Lie derivatives of h in the direction of f) define new coordinates on domain Ω.

There exists an observer of the form

$$\dot{\widehat{x}} = f(\widehat{x}) + K(y - h(\widehat{x}))$$

with K suitable $(n, 1)$-vector.

Example: Lorenz system on compact domain.

Error dynamics:

$$
\begin{aligned}
\dot{e}_1 &= -e_2 + k_1 e_3 \\
\dot{e}_2 &= e_1 + a e_2 + k_2 e_3 \\
\dot{e}_3 &= e_1 + k_3 e_3
\end{aligned}
$$

Suitable $k_1, k_2, k_3 \Longrightarrow$ synchronization

4. Time rescaling

Suppose that for the system

$$\dot{x} = f(x), \quad y = h(x)$$

there exist new coordinates ξ such that

$$\dot{\xi} = s(y)(A\xi + \varphi(y)), \quad y = C\xi$$

with some $s(y) > 0$.

New time: $d\tau = s(y)dt$.

$$\frac{d\xi}{d\tau} = (A\xi + \varphi(y))$$

In new time – linear error dynamics provided (A, C) is observable (detectable).

5. Partial observers and partial synchronization

$$\dot{x} = f(x), \quad y = h(x), \quad z = g(x)$$

Problem: Reconstruct $z(t)$ instead of $x(t)$.

An idea of possible solution: to find new coordinates (ξ_1, ξ_2) s.t. the system is in a cascade form:

$$\dot{\xi}_1 = p(\xi_1)$$
$$\dot{\xi}_2 = q(\xi_1, \xi_2)$$
$$y = w(\xi_1), \quad z = v(\xi_1)$$

where the ξ_1-subsystem admits an observer.

Full order observer:

$$\widehat{x}(k+1) = \widehat{f}(\widehat{x}(k), y(k)), \quad \widehat{x}(0) = \widehat{x}_0 \in \mathbb{R}^n$$

where $\widehat{x} \in \mathbb{R}^n$, and $\widehat{f}$ is a smooth mapping on $\mathbb{R}^n$ parametrized by y, such that the error $e(k) := x(k) - \widehat{x}(k)$ asymptotically converges to zero as $k \to \infty$ for all initial conditions x_0 and $\widehat{x}_0$.

6. Discrete-time observers

$$x(k+1) = f(x(k)), \quad x(0) = x_0 \in \mathbb{R}^n$$

$$y(k) = h(x(k))$$

where $y \in \mathbb{R}^1$ and $h : \mathbb{R}^n \to \mathbb{R}^1$ is the smooth output map.

Problem: *how to reconstruct the state trajectory $x(k, x_0)$ on the basis of the measurements $y(k)$?*

Systems in Lur'e form

$$x(k+1) = Ax(k) + \varphi(y(k)), \quad y(k) = Cx(k),$$

where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^1$ is the scalar output, $\varphi : \mathbb{R}^1 \to \mathbb{R}^n$, (C, A) detectable.

Observer:

$$\begin{cases} \widehat{x}(k+1) = A\widehat{x}(k) + \varphi(y(k)) + L(y(k) - \widehat{y}(k)) \\ \widehat{y}(k) = C\widehat{x}(k) \end{cases}$$

Error dynamics:

$$e(k+1) = (A - LC)e(k).$$

Observation: The representation of a system in Lur'e form is *coordinate dependent.*

Question: Is it possible to transform a system into Lur'e form by means of a nonlinear coordinate change?

Local results due to Lin and Byrnes:

A discrete–time system with single output is locally equivalent to a system in Lur'e form with observable pair (C, A) via a coordinate change $z = T(x)$ if and only if

(i) the pair $(\partial h(0)/\partial x, \partial f(0)/\partial x)$ is observable,

(ii) the Hessian matrix of the function $h \circ f^n \circ \mathcal{O}^{-1}(s)$ is diagonal, where $x = \mathcal{O}^{-1}(s)$ is the inverse map of

$$\mathcal{O}(x) = \left[h(x), h \circ f(x), \dots, h \circ f^{n-1}(x)\right]^T,$$

with $h \circ f(x) := h(f(x)), f^1 := f, f^j := f \circ f^{j-1}$.

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Observable form:

$$\begin{aligned} s_1(k+1) &= s_2(k) \\ &\vdots \\ s_{n-1}(k+1) &= s_n(k) \\ s_n(k+1) &= f_s(s) \\ y(k) &= s_1(k) \end{aligned}$$

(Alternative result) A discrete–time system with single output is locally equivalent to a system in Lur'e form with observable pair (C, A) via a coordinate change $z = T(x)$ if and only if for the observable form there exist functions $\varphi_1, \cdots, \varphi_n : \mathbb{R} \to \mathbb{R}$ such that

$$f_s(s) = \varphi_1(s_1) + \cdots + \varphi_n(s_n)$$

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Alternative formulation. If the pair $(\partial h(0)/\partial x, \partial f(0)/\partial x)$ is observable, there exist new coordinates $s_i = h \circ f^{i-1}(x)$ $(i = 1, \cdots, n)$ such that in these new coordinates the system takes a so-called *obser able form*

$$\begin{aligned} s_1(k+1) &= s_2(k) \\ &\vdots \\ s_{n-1}(k+1) &= s_n(k) \\ s_n(k+1) &= f_s(s) \\ y(k) &= s_1(k) \end{aligned}$$

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Example. Bouncing ball.

$$\begin{cases} x_1(k+1) = x_1(k) + x_2(k) \\ x_2(k+1) = \alpha x_2(k) - \beta \cos(x_1(k) + x_2(k)) \\ y(k) = h(x(k)) = x_1(k) \end{cases} \tag{1}$$

with $x_1(k)$ the phase of the table at the k-th impact, $x_2(k)$ proportional to the velocity of the ball at the k-th impact, α the coefficient of restitution, ω the angular frequency of table oscillation, A its amplitude, and $\beta = 2\omega^2(1+\alpha)A/g$.

Condition i):

$$\frac{\partial f(0)}{\partial x} = \begin{bmatrix} 1 & 1 \\ 0 & \alpha \end{bmatrix}, \quad \frac{\partial h(0)}{\partial x} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Condition ii):

$$\mathcal{O}(x) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x \qquad (2)$$

$s = \mathrm{col}(s_1, s_2) := \mathcal{O}(x)$

$$f_s(s) := h \circ f^2 \circ \mathcal{O}^{-1}(s) = -\alpha s_1 + (1+\alpha)s_2 - \beta \cos s_2$$

Hessian is diagonal

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Transformation into extended Lur'e form

Extended Lur'e form:

$$\begin{cases} x(k+1) &= Ax(k) + \varphi(y(k), y(k-1), \cdots, y(k-N)) \\ y(k) &= Cx(k) \end{cases}$$

Observer for the extended Lur'e form:

$$\begin{cases} \widehat{x}(k+1) &= A\widehat{x}(k) + \varphi(y(k), \cdots, y(k-N)) \\ & \quad +L(y(k) - \widehat{y}(k)) \\ \widehat{y}(k) &= C\widehat{x}(k) \end{cases}$$

When can a system be transformed into an extended Lur'e form?

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New coordinates:

$$\begin{cases} z_1 = -\alpha x_1 + x_2 + \beta \cos x_1 \\ z_2 = x_1 \end{cases}$$

In new coordinates:

$$\begin{cases} z_1(k+1) = -\alpha z_2(k) \\ z_2(k+1) = z_1(k) + (1+\alpha) z_2(k) - \beta \cos z_2(k). \end{cases}$$

Observer:

$$\begin{cases} \widehat{z}_1(k+1) &= -\alpha \widehat{z}_2(k) + {}_1(z_2(k) - \widehat{z}_2(k)) \\ \widehat{z}_2(k+1) &= \widehat{z}_1(k) + (1+\alpha)\widehat{z}_2(k) - \beta \cos z_2(k) + {}_2(z_2(k) - \widehat{z}_2(k)) \end{cases}$$

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Assume that the mapping $\mathcal{O}$ is a local di eomorphism. Let $N \in \{0, \cdots, n-1\}$ be given. Then there is a local transformation into an e tended Lur'e form with bu er N if and only if there locally e ist functions $\varphi_{N+1}, \cdots, \varphi_n : \mathbb{R}^{N+1} \to \mathbb{R}$ such that the function f_s in the observable form

$$\begin{cases} s_1(k+1) &= s_2(k) \\ & \vdots \\ s_n(k+1) &= f_s(s(k)) \\ y(k) &= s_1(k) \end{cases}$$

where $f_s(s) := h \circ f^n \circ \mathcal{O}^{-1}(s)$, satisfies

$$f_s(s_1, \cdots, s_n) = \sum_{i=N+1}^{n} \varphi_i(s_i, \cdots, s_{i-N})$$

7. Extended Kalman Filters

Nonlinear discrete time dynamics with a linear output map:

$$x(k+1) = f(x(k)) + w(k), \quad x \in \mathbb{R}$$
$$y(k) = Hx(k) + v(k)$$

where:

- $w(k)$ is noise in dynamics of transmitter, assumed to satisfy $E(w(k)) = 0$, $E(w(k)w(l)^T) = Q\delta_{kl}$ $(Q > 0)$,
- $v(k)$ is measurement noise, assumed to satisfy $E(v(k)) = 0$, $E(v(k)v(l)^T) = R\delta_{kl}$.

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Under certain conditions (differentiability, stochastic observability, boundedness of state trajectories and noise signals), "practical" convergence of the observer can be guaranteed, i.e., there exists a $\rho > 0$ and a $\tau \in \mathbb{Z}_+$ such that

$$\|x(k) - \hat{x}(k)\| \le \rho, \quad \forall k \ge \tau$$

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Extended Kalman Filter:

$$\hat{x}(k) = \hat{x}(k \mid k-1) + K(k)[y(k) - H\hat{x}(k \mid k-1)]$$
$$\hat{x}(k+1 \mid k) = f(\hat{x}(k))$$

$$P(k) = [I - K(k)H]P(k \mid k-1)$$
$$P(k+1 \mid k) = F(k)P(k)F(k)^T + Q$$

$$K(k) = P(k \mid k-1)H^T[HP(k \mid k-1)H^T + R]^{-1}$$

$$F(k) = \frac{\partial f}{\partial x}(x)\Big|_{x=\hat{x}(k)}$$

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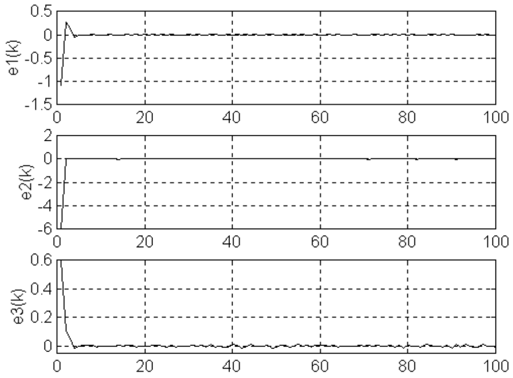
Example: coupled logistic maps

$$x_1(k+1) = (1-\epsilon)\mu x_1(k)(1-x_1(k)) + \epsilon x_2(k) + w_1(k)$$
$$x_2(k+1) = (1-\epsilon)\mu x_2(k)(1-x_2(k)) + \epsilon x_3(k) + w_2(k)$$
$$x_3(k+1) = (1-\epsilon)\mu x_3(k)(1-x_3(k)) + \epsilon x_1(k) + w_3(k)$$
$$y = x_2(k) + v(k)$$

with $Q = \mathrm{diag}(10^{-6}, 10^{-6}, 10^{-6})$, $R = 10^{-5}$, $\mu = 3.7$, $\epsilon = 0.35$.



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8. Alternative methods:

- Bilinear systems
- Physics-based Observers (see "Coordination of Mechanical Systems")

BUT no fully general method exists that works for all systems!

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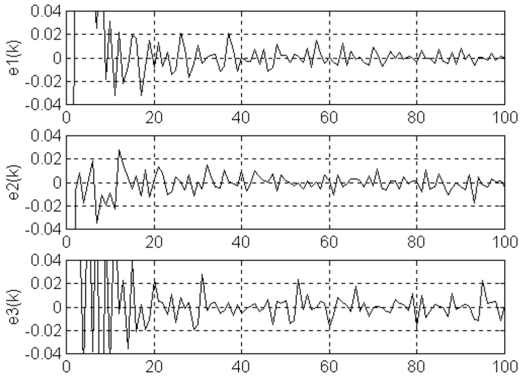

Practical synchronization with $\rho = 0.04$.

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Contents

Communication and synchronization

1. Introduction

Pecora and Carroll, 1990:

> "The ability to design synchronizing systems in nonlinear and, especially, chaotic systems may open interesting opportunities for application of chaos to communications, exploiting the unique features of chaotic signals."

Why and how?

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If Σ_T is a chaotic for constant $\lambda_{\min} \le \lambda(t) \le \lambda_{\max}$:

- It produces chaotic (seemingly random) signals, even when $\lambda(t)$ is (mainly) slowly time-varying.
- This means that the original message $\lambda(t)$ is "hidden" inside the seemingly random encoded message y.
- This gives the possibility to use a chaotic communication scheme for private communication.
- Note: originally it was thought that chaotic communication schemes would even provide *secure* communication. The work of numerous code breakers, however, has shown that is too much to ask for

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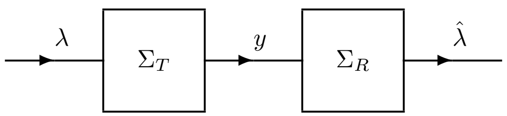

- λ : message, "coded" by modulating a parameter
- $\lambda_{\min} \le \lambda(t) \le \lambda_{\max}$, $\lambda(t)$ (mainly) slowly varying
- Σ_T : transmitter
- y : encoded message
- $\widehat{\lambda}$: decoded message

Problem: design Σ_R such that $|\lambda(t) - \widehat{\lambda}(t)|$ is small.

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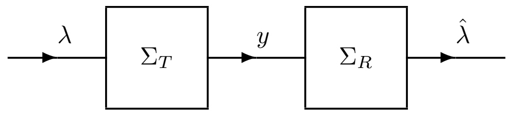
2. Control viewpoint



Design Σ_R such that $|\hat{\lambda}(t) - \lambda(t)|$ is small.

From a control point of view:

- System inversion
- Parameter estimation
- Adaptive observers

Will pay attention to last two points

Parameter estimation

Parameter estimation methods are well established for linear systems

However, are dealing with chaotic systems, which are nonlinear.

Still, if appropriate decomposition/transformation of system as well as synchronizing subsystem exists, linear parameter estimation methods can still be used.

Chaos helps in the convergence of estimates, because chaotic signals are persistently exciting

Parameter estimation: Example (Corron & Hahs, 1997)

Transmitter is a Lorenz system:

$$\Sigma_T \left\{ \begin{array}{rcl} \dot{x}_1 & = & 10(x_2 - x_1) \\ \dot{x}_2 & = & \lambda x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 & = & x_1 x_2 - \frac{8}{3} x_3 \\ y & = & x_2 \end{array} \right.$$

Then the system

$$\hat{\Sigma} \left\{ \begin{array}{rcl} \dot{\hat{x}}_1 & = & 10(y - \hat{x}_1) \\ \dot{\hat{x}}_3 & = & \hat{x}_1 y - \frac{8}{3} \hat{x}_3 \end{array} \right.$$

(partially) synchronizes, i.e., $(\hat{x}_1(t), \hat{x}_3(t)) - (x_1(t), x_3(t)) \to 0 \ (t \to +\infty)$

After $\hat{\Sigma}$ has synchronized, y satisfies:

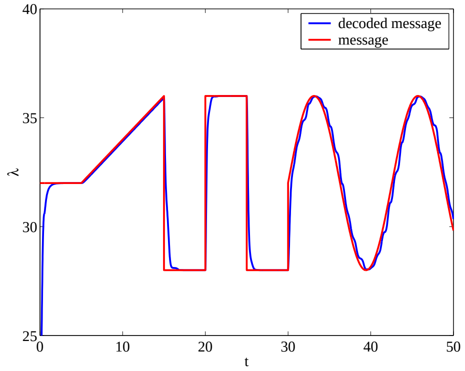
$$\dot{y} = \lambda u_1 - y - u_2, \quad u_1 = \hat{x}_1, \quad u_2 = \hat{x}_1 \hat{x}_3$$

This is a linear system with output y, known inputs u_1,u_2, and linear dependence on the unknown parameter λ.

So linear parameter estimation methods can now be used to estimate λ!

The receiver Σ_R is then e.g. given by (Corron & Hahs, 1997):

$$\Sigma_R \left\{ \begin{array}{rcl} \dot{\hat{x}}_1 & = & 10(y - \hat{x}_1) \\ \dot{\hat{x}}_3 & = & \hat{x}_1 y - \frac{8}{3} \hat{x}_3 \\ \dot{w}_0 & = & (k-1)y - \hat{x}_1 \hat{x}_3 - k w_0, \quad k > 0 \\ \dot{w}_1 & = & \hat{x}_1 - k w_1 \\ \dot{\hat{\lambda}} & = & \frac{q \, \mathrm{sign}(w_1)}{1 + |w_1|} (y - w_0 - w_1 \hat{\lambda}), \quad q > 0 \end{array} \right.$$



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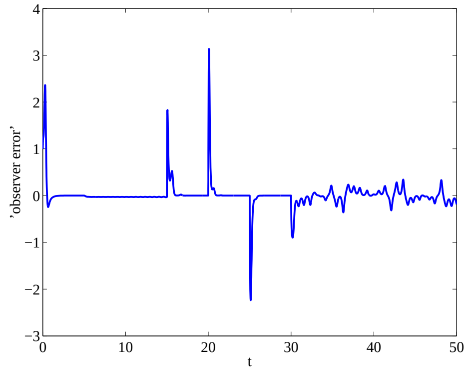

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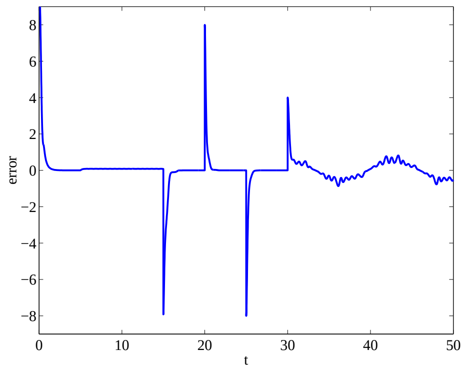

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Some remarks:

- $\hat{x}_1$ and $\hat{x}_3$ need to synchronize with x_1 and x_3 *before* parameter estimation can be achieved.
- However, if λ is (piecewise) constant, it follows from the update law for $\hat{\lambda}$:

$$\dot{\hat{\lambda}} = \frac{q\text{sign}(w_1)}{1+|w_1|}(y - w_0 - w_1\hat{\lambda})$$

  that $w_0 - w_1\hat{\lambda}$ synchronizes with x_2 *after* parameter estimation has been achieved.
- Furthermore, if λ is slowly time-varying, practical synchronization between $w_0 - w_1\hat{\lambda}$ and x_2 will be achieved.
- Thus, the receiver can be viewed as an adaptive (practical) observer for the transmitter.

Synchronization *before* parameter estimation is not necessary! To illustrate this, we consider a Rössler system:

$$\Sigma_T \;:\; \begin{cases} \dot{x}_1 = -x_2 - x_3 \\ \dot{x}_2 = x_1 + \lambda x_2 \\ \dot{x}_3 = 2 + x_3(x_1 - 4) \qquad y = x_3 \end{cases}$$

NB $\quad x_3(0) > 0 \Longrightarrow x_3(t) > 0, \; \forall t > 0.$

New coordinates: $(\xi_1, \xi_2, \xi_3) = (x_1, x_2, \log x_3)$, $\tilde{y} = \xi_3$

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & \lambda & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} + \begin{pmatrix} -e^{\tilde{y}} \\ 0 \\ 2e^{-\tilde{y}} - 4 \end{pmatrix}$$

Receiver (using least squares estimator with exponential forgetting factor):

$$\Sigma_R \begin{cases} \dot{w}_i &= K w_i + L u_i & (i = 0, 1, 2) \\ \hat{y} &= \phi_0(w) + \hat{\lambda}\phi_1(w) & \\ \dot{\hat{\lambda}} &= -\nu \phi_1(w) p (\hat{y} - \tilde{y}) & (\nu > 0) \\ \dot{p} &= -\nu(\phi_1(w)^2 p^2 - \gamma p) & (\gamma > 0) \end{cases}$$

where $u_0 = \log(y)$, $u_1 = -y$, $u_2 = \frac{2}{x_3} - 4$, $K = \mathrm{comp}(k_0, k_1, k_2)$, $L = \mathrm{col}(0, 0, 1)$, $s^3 + k_2 s^2 + k_1 s + k_0$ is Hurwitz, $\phi_0(w) = k_0 w_{01} + (k_1 - 1) w_{02} + k_2 w_{03} + w_{12} + w_{21} + w_{23}$, and $\phi_1(w) = w_{03} - w_{11} - w_{22}$.

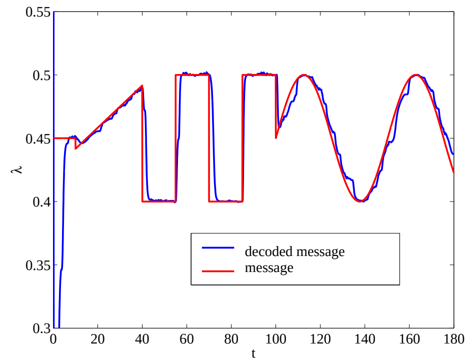
Transmitter:

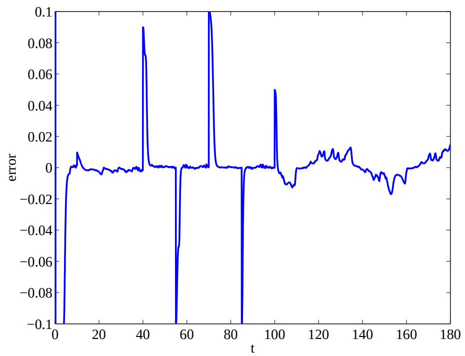
$$\dot{\xi} = A(\lambda)\xi + Bu, \qquad \tilde{y} = C\xi, \qquad u = \Phi(\tilde{y})$$

$$A(\lambda) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & \lambda & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi(\tilde{y}) = \begin{pmatrix} -e^{\tilde{y}} \\ 2e^{-\tilde{y}} - 4 \end{pmatrix}$$

Use linear identification tools!

Chaos helps!





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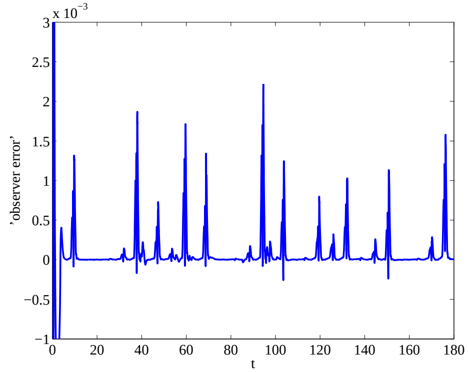

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Update law for $\hat{\lambda}$:

$$\begin{cases} \dot{\hat{\lambda}} &= -\nu\phi_1(w)p(\hat{y}-\tilde{y}) \qquad (\nu > 0) \\ \dot{p} &= -\nu(\phi_1(w)^2p^2 - \gamma p) \qquad (\gamma > 0) \\ \hat{y} &= \phi_0(w) + \hat{\lambda}\phi_1(w) \\ \tilde{y} &= \log(x_3) \end{cases}$$

So if λ is constant and parameter estimation has been achieved:

$$\exp(\phi_0(w) + \hat{\lambda}\phi_1(w)) \text{ synchronizes with } x_3$$

From this, also functions of w synchronizing with x_1 and x_2 can be derived.

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Adaptive observers

Receivers constructed using linear parameter estimation methods may be viewed as adaptive observers.

However, since these receivers have not been constructed as adaptive observers, it is generally not straightforward to obtain the relationship between receiver states and state estimates of the transmitter.

We now give two examples of how adaptive observers may be used as receivers.

Example: Chua circuit

$$\dot{x}_1 = \alpha(-x_1 + x_2 - \varphi_\lambda(x_1, \lambda))$$
$$\dot{x}_2 = x_1 - x_2 + x_3$$
$$\dot{x}_3 = -\beta x_2$$

$$\varphi_\lambda(x_1, \lambda) = \varphi(x_1) + \lambda(t)(|x_1 + 1| - |x_1 - 1|) = m_1 x_1 + (m_2 + \lambda(t))(|x_1 + 1| - |x_1 - 1|)$$

with $m_1 = 5/7, m_2 = -6/7$, $\alpha = 9$, $\beta = 14.286$.

Output: $y = x_1$

Signal $\lambda(t)$: $\lambda(t) = \lambda_0 + \lambda_1 \mathrm{sign}(\sin \omega t)$

Suppose m_1, m_2 are unknown

Adaptive observer:

$$\dot{\hat{x}}_1 = \alpha(-\hat{x}_1 + \hat{x}_2) + c_1(|x_1 + 1| - |x_1 - 1|) + c_2(\hat{x}_1 - x_1)$$
$$\dot{\hat{x}}_2 = \hat{x}_1 - \hat{x}_2 + \hat{x}_3 + 0(\hat{x}_1 - x_1)$$
$$\dot{\hat{x}}_3 = -\beta \hat{x}_2 + 0(\hat{x}_1 - x_1)$$

Adaptation law for $c_1(t), c_2(t)$

$$\dot{c}_1 = -\gamma_1 (x_1 - \hat{x}_1)^2 (|x_1 + 1| - |x_1 - 1|)$$
$$\dot{c}_2 = -\gamma_2 (x_1 - \hat{x}_1)^2$$

γ_1, γ_2 adaptation gains

$$\dot{x}_1 = \alpha(-x_1 + x_2 - \varphi_\lambda(x_1, \lambda))$$
$$\dot{x}_2 = x_1 - x_2 + x_3$$
$$\dot{x}_3 = -\beta x_2$$

$$\varphi_\lambda(x_1, \lambda) = m_1 x_1 + (m_2 + \lambda(t))(|x_1 + 1| - |x_1 - 1|)$$

Suppose m_1 and m_2 are unknown

Adaptive observer:

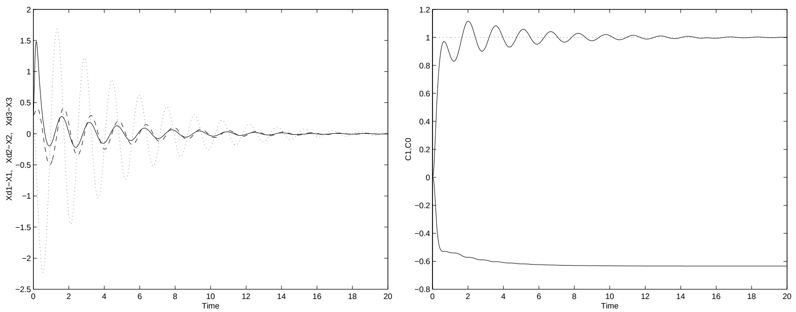
$$\dot{\hat{x}}_1 = \alpha(-\hat{x}_1 + \hat{x}_2) + c_1(|x_1 + 1| - |x_1 - 1|) + c_2(\hat{x}_1 - x_1)$$
$$\dot{\hat{x}}_2 = \hat{x}_1 - \hat{x}_2 + \hat{x}_3 + 0(\hat{x}_1 - x_1)$$
$$\dot{\hat{x}}_3 = -\beta \hat{x}_2 + 0(\hat{x}_1 - x_1)$$

Then due to

- Minimum-phaseness
- Relative degree one
- Linear dependence on unknown parameters

it follows that

- Error vanishes as $t \to \infty$
- $c_1(t)$ converges to the "true signal". Chaos helps!



$\lambda(t) \equiv 0$; Error and adaptation parameters vs time.

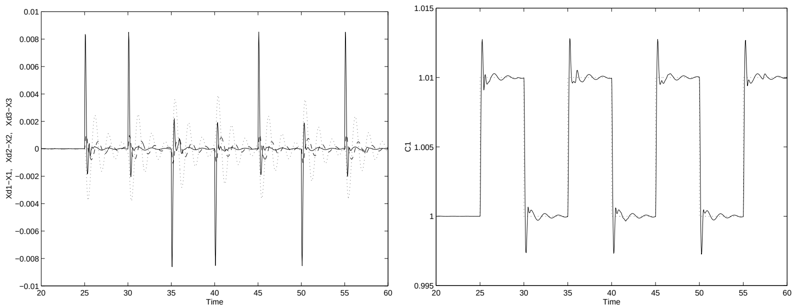
Example: Rössler system.

$$\Sigma_T \ : \ \begin{cases} \dot{x}_1 = -x_2 - x_3 \\ \dot{x}_2 = x_1 + \lambda x_2 \\ \dot{x}_3 = c + x_3(x_1 - b) \qquad y = x_3 \end{cases}$$

$b, c > 0$. Suppose λ is unknown parameter (message).

$$Q(\lambda) = \begin{pmatrix} -\lambda & -1 & 1 \\ 1 & 0 & -\lambda \\ 0 & 0 & 1 \end{pmatrix}.$$

New coordinates: $z = Q(\lambda)\xi$.



$\lambda(t) = \lambda_0 + \lambda_1 \text{sign}(\sin \omega t)$; Error and adaptation parameters vs time.

Suppose λ is unknown parameter (message).

$$Q(\lambda) = \begin{pmatrix} -\lambda & -1 & 1 \\ 1 & 0 & -\lambda \\ 0 & 0 & 1 \end{pmatrix}.$$

New coordinates $z = Q(\lambda)\xi$:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \underbrace{\begin{pmatrix} ce^{-y} - b \\ -e^{y} \\ ce^{-y} - b \end{pmatrix}}_{f_0(y)} + \lambda \underbrace{\begin{pmatrix} e^{y} \\ -ce^{-y} + b \\ y \end{pmatrix}}_{f_1(y)},$$

$$y = z_3 = (0\ 0\ 1)z,$$

Filtered transformation:

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} 0 & k_1 \\ 1 & k_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} y + \begin{pmatrix} e^y \\ -ce^y + b \end{pmatrix},$$

New variables:

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \lambda \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} y$$

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Adaptive observer:

$$\begin{aligned} \begin{pmatrix} \dot{\widehat{\eta}}_1 \\ \dot{\widehat{\eta}}_2 \\ \dot{\widehat{y}} \end{pmatrix} &= \begin{pmatrix} 0 & k_1 & -k_1 k_2 \\ 1 & k_2 & -(k_1 + k_2^2 + 1) \\ 0 & 1 & k_2 \end{pmatrix} \begin{pmatrix} \widehat{\eta}_1 \\ \widehat{\eta}_2 \\ \widehat{y} \end{pmatrix} + \\ & \begin{pmatrix} (k_1+1)(ce^{-y} - b) \\ k_2(ce^{-y} - b) - e^y \\ ce^{-y} - b \end{pmatrix} + \widehat{\theta} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (\xi_2 + y) + \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} (\widehat{y} - y) \\ \dot{\widehat{\theta}} &= -\gamma(\xi_2 + y)(\widehat{y} - y), \quad \gamma > 0 \end{aligned}$$

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In new coordinates:

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & k_1 & -k_1 k_2 \\ 1 & k_2 & -(k_1 + k_2^2 + 1) \\ 0 & 1 & k_2 \end{pmatrix}}_{\bar{A}} \begin{pmatrix} \eta_1 \\ \eta_2 \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} (k_1+1)(ce^{-y} - b) \\ k_2(ce^{-y} - b) - e^y \\ ce^{-y} - b \end{pmatrix}}_{\bar{f}_0} + \lambda \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\bar{B}} \underbrace{(\xi_2 + y)}_{u}$$

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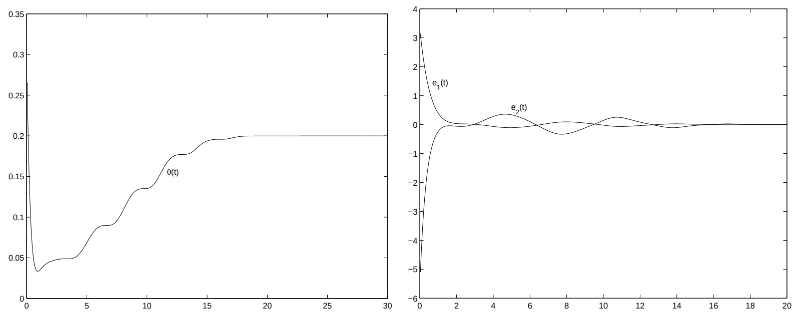
Coordinate transformation depends on $\lambda \Longrightarrow$ to estimate the whole state vector it is required that $\widehat{\theta}(t) \to \lambda$.

Chaos helps!



Convergence of $\widehat{\theta}(t)$ to λ and observation error vs time

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Synchronization and spatial ordering

Coupled systems with local interactions

To model:

- spatially extended systems
- turbulence

Simplified models of high-dimensional systems:

- Spatial structure is modelled by the coupling
- Each free system from the array is low-dimensional

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Contents

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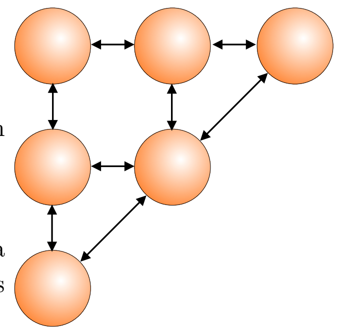
Diffusive cellular networks

Two ways to describe diffusion (A. Turing): PDE or ODE.

Two problems: synchronization via diffusion, generation of oscillations by means of diffusion.



Diffusive cellular networks

$$\begin{cases} \dot{x}_j = f(x_j) + Bu_j & j = 1\ldots k, \quad x_j \in \mathbb{R}^n, \quad u_j, y_j \in \mathbb{R}^m \\ y_j = Cx_j \end{cases} \tag{3}$$

CB is similar to a diagonal matrix with positive entries.

$$u_j = -\gamma_{j1}(y_j - y_1) - \gamma_{j2}(y_j - y_2) - \ldots - \gamma_{jk}(y_j - y_k) \tag{4}$$

with $\gamma_{ji} = \gamma_{ij} \geq 0$

Systems (3) are said to be diffusively coupled.

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Network equations:

$$\begin{cases} \dot{x} = F(x) + (I_k \otimes BC)u \\ u = -(\Gamma \otimes I_m)y \end{cases}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix}, \quad F(x) = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_k) \end{pmatrix}$$

If the network cannot be divided into two or more disconnected networks the matrix Γ has only one zero eigenvalue.

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The coupling matrix

$$\Gamma = \begin{pmatrix} \sum_{i=2}^{k} \gamma_{1i} & -\gamma_{12} & \cdots & -\gamma_{1k} \\ -\gamma_{21} & \sum_{i=1, i\neq 2}^{k} \gamma_{2i} & \cdots & -\gamma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & \cdots & \sum_{i=1}^{k-1} \gamma_{ki} \end{pmatrix}, \tag{5}$$

$\gamma_{ij} = \gamma_{ji} \geq 0$

- Γ is singular (all row sums are zero)
- $\Gamma = \Gamma^T \geq 0$ (Gershgorin theorem)

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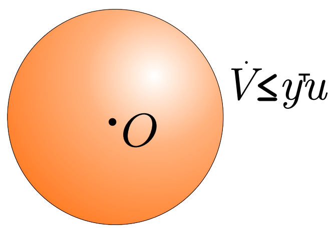
Boundedness of solutions in DCN

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases}$$

- Passivity: $\exists V \geq 0, \quad \dot{V} \leq y^T u$
- Semipassivity: Dissipation inequality is satisfied outside some ball.
- Strict semipassivity: Dissipation inequality is strict.



DCN of strictly semipassive systems with radially unbounded storage function has ultimately bounded solutions.

Example. The Lorenz system

$$\dot{x}_1 = \sigma(y_1 - x_1) + u$$
$$\dot{y}_1 = r x_1 - y_1 - x_1 z_1$$
$$\dot{z}_1 = -b z_1 + x_1 y_1$$

is strictly semipassive from u to $y = x_1$ with the storage function

$$V(x_1, y_1, z_1) = \frac{1}{2}\left(x_1^2 + y_1^2 + (z_1 - \sigma - r)^2\right).$$

$$\dot{V}(x_1, y_1, z_1, u) = yu - H(x_1, y_1, z_1)$$

where

$$H = \underbrace{\sigma x_1^2 + y_1^2 + b\left(z_1 - \frac{\sigma + r}{2}\right)^2}_{\geq 0} - b\frac{(\sigma + r)^2}{4}.$$

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Network equations:

$$\begin{cases} \dot{x}_j = f(x_j) + B u_j \\ y_j = C x_j \end{cases}$$

Nonsingularity of $CB \Longrightarrow$ new coordinates (normal form).

$$\begin{cases} \dot{z}_j = q(z_j, y_j) \\ \dot{y}_j = a(z_j, y_j) + CB u_j \end{cases}$$

Coupling:

$$u = -(\Gamma \otimes I_m) y, \quad u = \mathrm{col}(u_1, \ldots, u_k), y = \mathrm{col}(y_1, \ldots, y_k)$$

Eigenvalues of Γ: $\quad 0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k$

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Convergent systems (Demidovich, 1961)

$$\dot{z} = q(z, y(t)), \quad y(t) \in \mathbb{D}, \quad \mathbb{D} \text{ is } \quad \text{compact}$$

The system is exponentially convergent if

- for any function $y : t \to \mathbb{D}$, $t \in (-\infty, +\infty)$
- $\exists$ a unique bounded limit solution $\bar{z}(t)$ defined on $(-\infty, +\infty)$
- $||z(t) - \bar{z}(t)|| \leq C e^{-\alpha(t - t_0)}$, $\alpha > 0$.

Test for convergence: $\exists P = P^T > 0$, s.t.

$$\frac{1}{2}\left[P\left(\frac{\partial q}{\partial z}(z, w)\right) + \left(\frac{\partial q}{\partial z}(z, w)\right)^T P\right]$$

has negative eigenvalues ($\forall w \in \mathbb{D}$ separated from zero).

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Full synchronization in DCN.

Full synchronization: $x_1(t) = x_2(t) = \ldots = x_k(t)$

Assumptions:

- Strict semipassivity of each system from DCN with radially unbounded storage function
- Exponential convergence of the system

$$\dot{z} = q(z, y(t))$$

Result:

- $\exists \bar{\lambda} > 0$, s.t.
- if $\lambda_2 \geq \bar{\lambda}$ the set $x_1 = x_2 = \ldots = x_k$ contains globally asymptotically stable compact subset.

Example. DCN of Lorenz systems.

$$\begin{cases} \dot{x}_j^1 = \sigma(x_j^2 - x_j^1) + u_j \\ \dot{x}_j^2 = r x_j^1 - x_j^2 - x_j^1 x_j^3 \\ \dot{x}_j^3 = -b x_j^3 + x_j^1 x_j^2 \end{cases}, \qquad y_j = x_j^1$$

$$u = -\Gamma y$$

If the smallest nonzero eigenvalue λ_2 of Γ is large enough

$\Longrightarrow$ full synchronization.

Intermediate regimes?

Partial synchronization.

Global symmetries

Γ contains all information about the coupling

Let Π be a permutation matrix commuting with Γ:

$$\Pi\Gamma - \Gamma\Pi = 0$$

The set

$$\ker(I_{kn} - \Pi \otimes I_n)$$

is invariant.

This set can be described by the equations of the form

$$x_i = x_j$$

partial synchronization if $x_i = x_j$ is stable and/or attractive for some i, j

Partial synchronization

Observation: the set $x_1 = x_2 = \ldots = x_k$ is an invariant linear subspace.

Questions:

- are there any other invariant subspaces?
- how to find them?
- how to prove stability?

Hint:

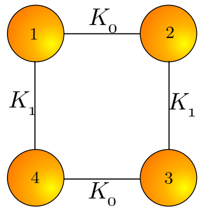
- look for the symmetries

Symmetries:

- Global (depend on the coupling)
- Internal (depend on the properties of free systems)

Example. A ring of four systems.

$$\Gamma = \begin{pmatrix} K_0 + K_1 & -K_0 & 0 & -K_1 \\ -K_0 & K_0 + K_1 & -K_1 & 0 \\ 0 & -K_1 & K_0 + K_1 & -K_0 \\ -K_1 & 0 & -K_0 & K_0 + K_1 \end{pmatrix}$$



Group of permutation matrices: $\Pi_4 = I_4$ and

$$\Pi_1 = \begin{pmatrix} E & O \\ O & E \end{pmatrix}, \ \Pi_2 = \begin{pmatrix} O & I_2 \\ I_2 & O \end{pmatrix}, \ \Pi_3 = \begin{pmatrix} O & E \\ E & O \end{pmatrix}, \ E := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathcal{A}_1 = \{x \in \mathbb{R}^{4n} \ : \ x_1 = x_2, x_3 = x_4\}, \ \mathcal{A}_2 = \{x \in \mathbb{R}^{4n} \ : \ x_1 = x_3, x_2 = x_4\}$$

$$\mathcal{A}_3 = \{x \in \mathbb{R}^{4n} \ : \ x_1 = x_4, x_2 = x_3\}$$

Internal symmetries

Depend on the properties of free system.

Suppose:

- $\exists$ permutation Π, $\Pi\Gamma - \Gamma\Pi = 0$
- $\exists J$, $Jf(x_j) = f(Jx_j)$
- J commutes with BC: $JBC - BCJ = 0$

Then the set

$$\ker(I_{kn} - \Pi \otimes J)$$

is invariant.

Stability of partially synchronized mode

Assumptions:

- Strict semipassivity of each system from DCN with radially unbounded storage function
- Exponential convergence of the system

$$\dot{z} = q(z, y(t))$$

Let γ' be the smallest eigenvalue of $\Gamma|_{\mathrm{range}(I_k - \Pi)}$

Result:

- $\exists \bar{\lambda} > 0$, s.t.
- if $\gamma' \geq \bar{\lambda}$ the set $\ker(I_{kn} - \Pi \otimes I_n)$ contains globally asymptotically stable compact subset.

Example. A ring of four systems (cont'd).

Each cell is described by Lorenz system.

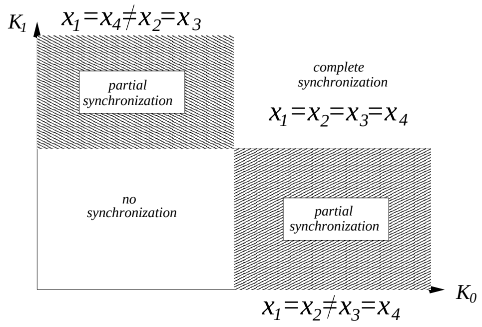
$$\begin{cases} \dot{x}_j^1 = \sigma(x_j^2 - x_j^1) + u_j \\ \dot{x}_j^2 = r x_j^1 - x_j^2 - x_j^1 x_j^3 \\ \dot{x}_j^3 = -b x_j^3 + x_j^1 x_j^2 \end{cases}, \quad y_j = x_j^1, \quad j = 1, \ldots 4, \quad u = -\Gamma y$$

$$J = \mathrm{diag}(-1 \quad -1 \quad 1)$$

Additional invariant sets:

$$\mathcal{A}_1' = \{x \in \mathbb{R}^{12} \ : \ Jx_1 = x_2, Jx_3 = x_4\}$$
$$\mathcal{A}_2' = \{x \in \mathbb{R}^{12} \ : \ Jx_1 = x_3, Jx_2 = x_4\}$$
$$\mathcal{A}_3' = \{x \in \mathbb{R}^{12} \ : \ Jx_1 = x_4, Jx_2 = x_3\}$$
$$\mathcal{A}_4' = \{x \in \mathbb{R}^{12} \ : \ x_i^1 = x_i^2 = 0, \quad i = 1, 2, 3, 4\}.$$

Example. A ring of four Lorenz systems (cont'd)



On topology of DCN

A cellular diffusive network is said to be *regular* if

- All coupling constants are equal: $\gamma_{ij} = \gamma$ for all $i \neq j$
- Each cell is connected to N other cells.

N is the density of DCN.

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Problem statement: Asymptotic behavior of λ_2 when $k \to \infty$ (large DCN).

λ_2 is responsible for synchronization.

For regular networks the following relation is valid

$$\lim_{k\to\infty} \lambda_2(k, N) = 0.$$

Conclusion: in large DCN full synchronization of unstable systems is impossible.

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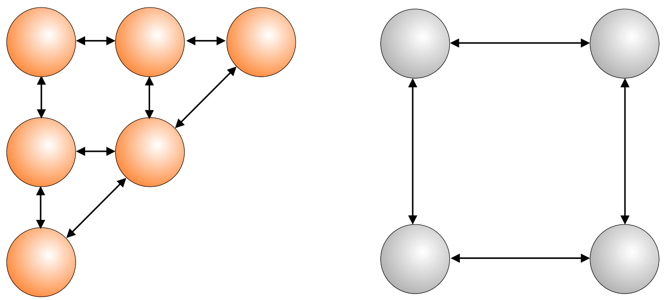

Examples of irregular and regular networks.

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Contents

- Introduction
- An observer view on synchronization
- Communication and synchronization
- Synchronization in diffusive networks
- Controlled synchronization
- Coordination of mechanical systems

Controlled synchronization

Master system : $\dot{x} = f(x), \quad x \in \mathbb{R}^n$
$y = h(x)$

Slave system : $\dot{\hat{x}} = g(\hat{x}, y), \quad \hat{x} \in \mathbb{R}^n$

In general, a given master system and a given slave system will not synchronize. From a control point of view, there are two ways to try to overcome this problem.

- If one is free to choose the slave system, it should be designed as an observer for the master system.
- If the slave system is given beforehand, but there is still some freedom to influence (i.e., control) the slave system, one could try to design a controller to achieve synchronization. This is the Controlled Synchronization Problem.

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Controlled synchronization problem: Find dynamic feedback

$$\begin{aligned} \dot{z} &= k(z, \eta, y) \\ u &= \alpha(z, \eta, y) \end{aligned}$$

such that the closed loop system satisfies:

$$\lim_{t \to +\infty} \|x(t) - \hat{x}(t)\| = 0$$

Sometimes also internal stability required, i.e., the dynamics

$$\begin{aligned} \dot{\hat{x}} &= g(\hat{x}, 0, \alpha(z, \eta, 0)) \\ \dot{z} &= k(z, \eta, 0) \end{aligned}$$

are asymptotically stable.

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Master system : $\dot{x} = f(x), \quad x \in \mathbb{R}^n$
$y = h(x)$

Slave system : $\dot{\hat{x}} = g(\hat{x}, y, u), \quad \hat{x} \in \mathbb{R}^n, \ u \in \mathbb{R}^m$
$\eta = h(\hat{x})$

Controlled synchronization problem: Find dynamic feedback

$$\begin{aligned} \dot{z} &= k(z, \eta, y) \\ u &= \alpha(z, \eta, y) \end{aligned}$$

such that the closed loop system satisfies:

$$\lim_{t \to +\infty} \|x(t) - \hat{x}(t)\| = 0$$

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In fact, the controlled synchronization problem with internally stability can be viewed as (a version of) the *regulator problem.*

So could try to solve the controlled synchronization problem by using methods for solution of the regulator problem.

However, in most applications of chaos synchronization, the master system possesses a chaotic attractor in which several equilibrium points with unstable linearization are embedded.

This means that the Poisson stability hypothesis from the "Byrnes & Isidori solution" to the regulator problem is not met.

Example of class of systems for which controlled synchronization problem can be solved: Lur'e systems.

Master system:

$$\begin{aligned} \dot{x} &= Ax + \Psi(y) \\ y &= Cx \end{aligned}$$

Slave system:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + \Psi(y) + Bu \\ \eta &= C\hat{x} \end{aligned}$$

where (A, B) is stabilizable and (C, A) is detectable.

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Example: Chua circuit.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \underbrace{\begin{pmatrix} -\alpha(m_0 - m_1)\mathrm{sat}(y) \\ 0 \\ 0 \end{pmatrix}}_{\Psi(y)}$$

$$y = x_1, \quad \gamma = -\alpha(m_1 + 1), \alpha = 15.6, \; m_0 = -\tfrac{8}{7}, \; m_1 = -\tfrac{5}{7}, \; \beta = 25$$

Slave system:

$$\dot{\hat{x}} = A\hat{x} + \Psi(y) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u$$

Choose $F = \mathrm{col}(-1 - 15.60)$, $K = (4.36 \;\; 0 \;\; 0)$.

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Master and slave :

$$\begin{aligned} \dot{x} &= Ax + \Psi(y) \\ y &= Cx \\ \dot{\hat{x}} &= A\hat{x} + \Psi(y) + Bu \\ \eta &= C\hat{x} \end{aligned}$$

Controller :

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + K(\tilde{y} - y) + \Psi(y) \\ \dot{\bar{x}} &= A\bar{x} + K(\bar{\eta} - \eta) + \Psi(y) + Bu \\ \tilde{y} &= C\tilde{x} \\ \bar{\eta} &= C\bar{x} \\ u &= F(\tilde{x} - \bar{x}) \end{aligned}$$

with $\sigma(A + BF), \sigma(A + KC) \subset \mathbb{C}$.

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Example: Van der Pol differential equation.

Master system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - (x_1^2 - 1)x_2 \\ y &= x_1 \end{aligned}$$

Slave system:

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + \alpha u \\ \dot{\hat{x}}_2 &= -y - (y^2 - 1)\hat{x}_2 + \beta u \end{aligned}$$

Want to try to achieve synchronization by means of (high gain) static error feedback:

$$u = -c(\hat{x}_1 - x_1)$$

$$\text{Master system}: \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - (x_1^2 - 1)x_2 \\ y &= x_1 \end{aligned}$$

- The origin is the only equilibrium, and it is an unstable focus.
- There is a unique limit cycle C that is (not uniformly) exponentially attracting for all non-zero initial conditions.
- If $\tilde{x}(t)$ is a periodic solution starting on C, T is its period, and $p(t) = \tilde{x}_1^2(t) - 1$, then

$$\bar{p} := \frac{1}{T}\int_0^T p(\tau)d\tau > 0$$

Case 1: $\beta = 0$, $\alpha \neq 0$.

Error dynamics:

$$\dot{e} = \begin{pmatrix} -\alpha c & 1 \\ 0 & -p(t) \end{pmatrix} e$$

Fundamental matrix:

$$\Phi(t, t_0) = \begin{pmatrix} \exp(-\alpha c(t - t_0)) & \psi(t, t_0) \\ 0 & \exp(-\int_{t_0}^t p(\tau)d\tau) \end{pmatrix}$$

Since $\bar{p} > 0$: synchronization if and only if $\alpha c > 0$.

Error dynamics when master system evolves on limit cycle C:

$$\dot{e} = \begin{pmatrix} -\alpha c & 1 \\ -\beta c & -p(t) \end{pmatrix} e$$

with

$$p(t) = \tilde{x}_1^2(t) - 1, \quad \bar{p} := \frac{1}{T}\int_0^T p(\tau)d\tau > 0$$

Linear time-varying differential equation!

Case 2: $\alpha = 0$, $\beta \neq 0$.

Error dynamics:

$$\dot{e} = \begin{pmatrix} 0 & 1 \\ -\beta c & -p(t) \end{pmatrix} e$$

Synchronization *if*

$$\beta c > \left(\frac{\bar{q}}{\bar{p}}\right)^2$$

where

$$\bar{q} = \frac{1}{T}\int_0^T |q(\tau)|d\tau, \quad q(t) = \frac{1}{4}p(t)^2 + \frac{1}{2}\dot{p}(t)$$

Proof involves transformation into Hill equation, results on growth of solutions of Hill equations (Levinson, 1941). Bound is conservative!

Case 3: $\alpha \neq 0$, $\beta \neq 0$.

Error dynamics:

$$\dot{e} = \begin{pmatrix} -\alpha c & 1 \\ -\beta c & -p(t) \end{pmatrix} e$$

Singular perturbations and Tikhonov's Theorem: High-gain feedback with $\alpha c \to +\infty$ works if and only if

$$\frac{\beta}{\alpha} > -\bar{p}$$

High-gain feedback with $\alpha c \to -\infty$ does not work.

Lower bound for c can be given, but is very conservative. Have to take recourse to numerical methods

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What can be learnt from this example?

- Have had to use many example-specific and ad-hoc methods.
- This leads to the conclusion that "genuinely nonlinear" regulation and controlled synchronization is difficult, and that hoping to be able to solve the problem in its full generality seems to be in vain at the outset.
- So should rather concentrate on classes of systems with specific properties.
- One of these classes, fully actuated mechanical systems, will be treated in the next section.

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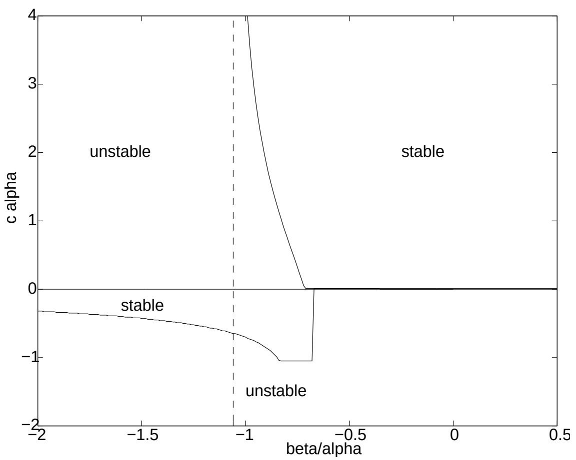

Stability regions in $(\frac{\beta}{\alpha}, c\alpha)$-plane.

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Contents

Coordination of mechanical systems:

- Introduction
- Mutual synchronization controller
- Convergence properties
- Experiments
- Conclusions
- Future extensions

Introduction

Objective

Two or more mutually synchronized robot manipulators

Restrictions

Only position measurements

Motivation

- ✶ Synchronization tasks :
  - mobile platforms (transportation, walking robots),
  - object manipulation (manufacturing industry),
- ✶ Velocity sensor equipment
- ✶ Accessibility on the robot architecture

History

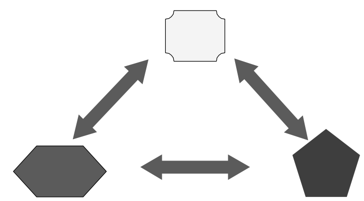
- Huygens (1673): pendulum clocks linked via (flexible) beam
- Rayleigh (1877): nearby organ tubes, tuning forks
- B. van der Pol (1920): electrical-mechanical systems

Definition

- Time conformity
- Certain relations between functionals and/or variables

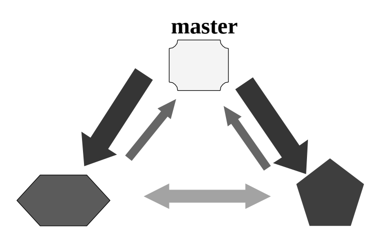
Internal (mutual) synchronization

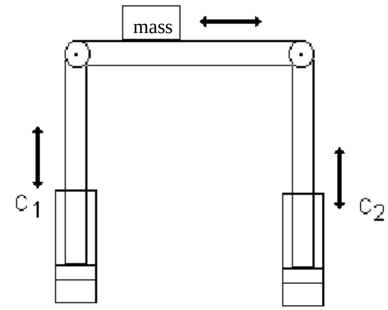
- All objects appear at equal terms
- Synchronous motion as result of interaction/coupling



External synchronization

- One object is more powerful (master)
- Synchronous motion is determined by the master

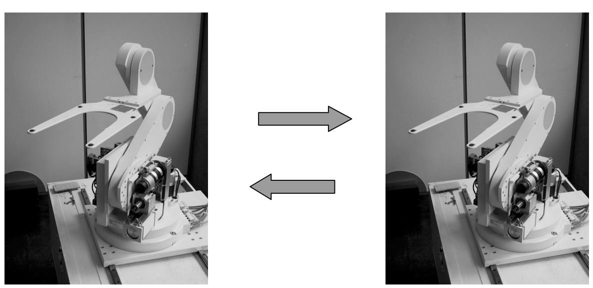




Hydraulic platform

General setup

- n actuated rigid joints
- All joints are revolute



General assumptions

- Only joint position measurements
- Dynamic model and physical parameters are known for all robots
- Desired joint positions, velocities and accelerations are bounded

Synchronization index and functional

$$J_i(q_i,\dot{q}_i) = [q_i^T \quad \dot{q}_i^T]$$

$$f_{i,j} = \| J_i(q_i,\dot{q}_i) - J_j(q_j,\dot{q}_j) \|, \quad i,j = 1,\ldots,p, \quad j \neq i,$$

$$f_{i,i} = \| J_i(q_i,\dot{q}_i) - J_d(q_d,\dot{q}_d) \|, \quad i = 1,\ldots,p$$

Mutual synchronization controller

Rigid joint robot dynamics

$$M_i(q_i)\ddot{q}_i + C_i(q_i,\dot{q}_i)\dot{q}_i + g_i(q_i) = \tau_i \qquad i = 1,\dots,p$$

Ideal feedback control law

$$\tau_i = M_i(q_i)\ddot{q}_{ri} + C_i(q_i,\dot{q}_i)\dot{q}_{ri} + g_i(q_i) - K_{d,i}\dot{s}_i - K_{p,i}s_i$$

Synchronization errors

$$s_i = q_i - q_{ri}, \quad \dot{s}_i = \dot{q}_i - \dot{q}_{ri}$$

$$e_{i,i} = q_i - q_d, \qquad e_{i,j} = q_i - q_j$$

Nominal reference trajectories

$$q_{ri} = q_d - \sum_{j=1, j\neq i}^{p} K_{i,j}(q_i - q_j); \qquad \dot{q}_{ri} = \dot{q}_d - \sum_{j=1, j\neq i}^{p} K_{i,j}(\dot{q}_i - \dot{q}_j)$$

Feedback control law with estimated variables

$$\tau_i = M_i(q_i)\hat{\ddot{q}}_{ri} + C_i(q_i,\hat{\dot{q}}_i)\hat{\dot{q}}_{ri} + g_i(q_i) - K_{d,i}\hat{\dot{s}}_i - K_{p,i}s_i$$

Synchronization errors

$$s_i = q_i - q_{ri}, \quad \hat{\dot{s}}_i = \hat{\dot{q}}_i - \hat{\dot{q}}_{ri}$$

$$e_{i,i} = q_i - q_d, \qquad e_{i,j} = q_i - q_j$$

Nominal reference trajectories

$$q_{ri} = q_d - \sum_{j=1, j\neq i}^{p} K_{i,j}(q_i - q_j); \qquad \hat{\dot{q}}_{ri} = \dot{q}_d - \sum_{j=1, j\neq i}^{p} K_{i,j}(\hat{\dot{q}}_i - \hat{\dot{q}}_j)$$

Observer for slave joint variables

$$\frac{d}{dt}\hat{q}_i = \hat{\dot{q}}_i + \mu_{i,1}\tilde{q}_i$$

$$\frac{d}{dt}\hat{\dot{q}}_i = -M_i(q_i)^{-1}\left(C(q_i,\hat{\dot{q}}_i)\hat{\dot{q}}_i + g_i(q_i) - \tau_i\right) + \mu_{i,2}\tilde{\dot{q}}_i$$

Estimation joint errors

$$\tilde{q}_i := q_i - \hat{q}_i, \quad \tilde{\dot{q}}_i := \dot{q}_i - \hat{\dot{q}}_i$$

Seemingly problem: Algebraic loop !!!

Algebraic loop

$$\frac{d}{dt}\hat{\dot{q}}_i = -M_i(q_i)^{-1}\left(C(q_i,\hat{\dot{q}}_i)\hat{\dot{q}}_i + g_i(q_i) - \tau_i\right) + \mu_{i,2}\tilde{\dot{q}}_i$$

$$\frac{d}{dt}\hat{\dot{q}}_i = -\sum_{j=1, j\neq i}^{p} K_{i,j}\left(\frac{d}{dt}\hat{\dot{q}}_i - \frac{d}{dt}\hat{\dot{q}}_j\right) + \underbrace{\ddot{q}_d - M_i(q_i)^{-1}\left(C(q_i,\hat{\dot{q}}_i)\hat{\dot{s}}_i + K_{d,i}\hat{\dot{s}}_i + K_{p,i}s_i\right) + \mu_{i,2}\tilde{\dot{q}}_i}_{y_i(\ddot{q}_d,\, s_i,\, \hat{\dot{s}}_i,\, q_i,\, \hat{\dot{q}}_i)}$$

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For $i = 1,\dots,p$

$$(I_n + \sum_{j=1,j\neq i}^{p} K_{i,j})\frac{d}{dt}\hat{\dot{q}}_i - \sum_{j=1,j\neq i}^{p} K_{i,j}\frac{d}{dt}\hat{\dot{q}}_j = y_i(\ddot{q}_d, s_i, \dot{s}_i, q_i, \hat{\dot{q}}_i)$$

Such that

$$\underbrace{\begin{bmatrix} I_n + \sum_{j=1,j\neq 1}^{p} K_{1,j} & -K_{1,2} & \cdots & -K_{1,p} \\ -K_{2,1} & I_n + \sum_{j=1,j\neq 2}^{p} K_{2,j} & \cdots & -K_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ -K_{p,1} & -K_{p,2} & \cdots & I_n + \sum_{j=1,j\neq p}^{p} K_{p,j} \end{bmatrix}}_{M_c(K_{i,j})} \begin{bmatrix} \frac{d}{dt}\hat{\dot{q}}_1 \\ \frac{d}{dt}\hat{\dot{q}}_2 \\ \vdots \\ \frac{d}{dt}\hat{\dot{q}}_p \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

$M_c(K_{i,j})$ Nonsingular for any $K_{i,j} \geq 0$!

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Main result

There exist conditions on the minimum eigenvalues of the control gains $K_{p,i}$, $K_{d,i}$ and the observer gains $\mu_{i,1}$, $\mu_{i,2}$ such that

$$s_i \to 0, \quad \dot{s}_i \to 0, \quad \tilde{q}_i \to 0, \quad \tilde{\dot{q}}_i \to 0$$

semi-globally exponentially.

Thus, the robots are semi-globally exponentially synchronized since for $i = 1,\dots,p$, $q_i \to q_j$ and $\dot{q}_i \to \dot{q}_j$ exponentially for any initial condition in the region of convergence.

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- Convergence of $s_i, \dot{s}_i, \tilde{q}_i, \tilde{\dot{q}}_i$

$$V = \frac{1}{2}\sum_{i=1}^{p}\left(\dot{s}_i^T M_i(q_i)\dot{s}_i + s_i^T K_{p,i} s_i\right) + \frac{1}{2}\sum_{i=1}^{p}\begin{bmatrix}\tilde{\dot{q}}_i^T & \tilde{q}_i^T\end{bmatrix}\begin{bmatrix} M_i(q_i) & \eta_i(\tilde{q}_i)I_n \\ \eta_i(\tilde{q}_i)I_n & \mu_{i,2} + \beta_i I_n \end{bmatrix}\begin{bmatrix}\tilde{\dot{q}}_i \\ \tilde{q}_i\end{bmatrix}$$

$$\eta_i(\tilde{q}_i) = \frac{\eta_o}{1 + \|\tilde{q}_i\|}$$

$$\beta_i = \eta_0\mu_{i,1} + 2V_M C_{i,M}(\mu_{i,1} + \eta_0 M_{i,m}^{-1}) - \mu_{i,2}(1 - M_{i,m})$$

Convergence of $s_i, \dot{s}_i$ imply $q_i \to q_j$ and $\dot{q}_i \to \dot{q}_j$!

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$s_i \to 0$ implies in the limit $t \to \infty$ that

$e_{i,i} = q_i - q_d$

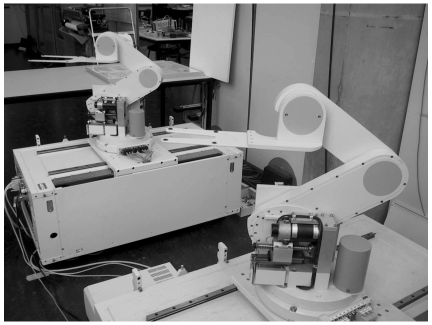
$e_{i,j} = q_i - q_j$

$$\begin{bmatrix} s_1 \\ \vdots \\ s_p \end{bmatrix} = \begin{bmatrix} e_{1,1} + \sum_{j=1,j\neq 1}^{p} K_{1,j}e_{1,j} \\ \vdots \\ e_{p,p} + \sum_{j=1,j\neq p}^{p} K_{p,j}e_{p,j} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} I_n + \sum_{j=1,j\neq 1}^{p} K_{1,j} & -K_{1,2} & \cdots & -K_{1,p} \\ -K_{2,1} & I_n + \sum_{j=1,j\neq 2}^{p} K_{2,j} & \cdots & -K_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ -K_{p,1} & -K_{p,2} & \cdots & I_n + \sum_{j=1,j\neq p}^{p} K_{p,j} \end{bmatrix}}_{M_c(K_{i,j})} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_p \end{bmatrix} = \begin{bmatrix} q_d \\ q_d \\ \vdots \\ q_d \end{bmatrix}$$

Experiments

Two CFT transposer robots



- 4 degrees of freedom (dof)
- sampling frequency: 2 kHz
- encoders: 2000 PPR

Robot dynamics + friction effects

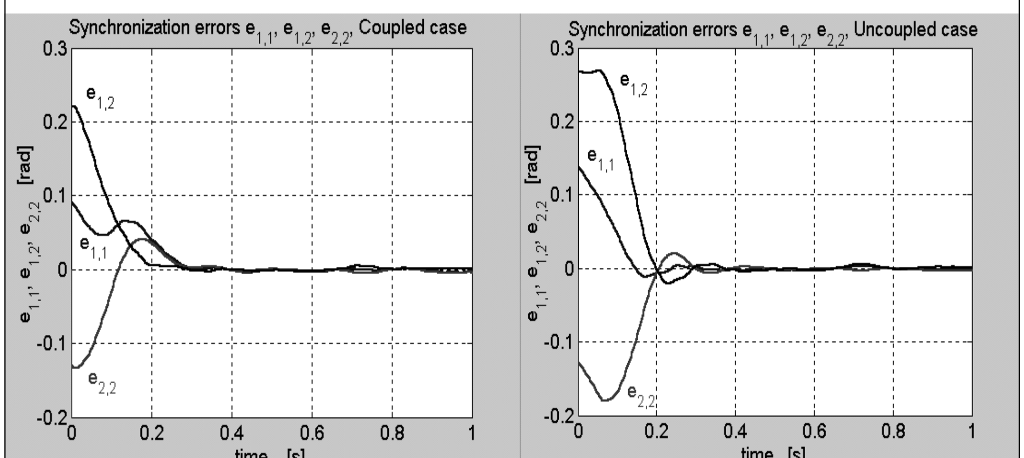
$$M_i(q_i)\ddot{q}_i + C_i(q_i,\dot{q}_i)\dot{q}_i + g_i(q_i) + \tau_f(\dot{q}_i) = \tau_i \qquad i = 1,\ldots,p$$

$$\tau_f(\dot{q}_i) = B_v\dot{q}_i + B_{f1,i}\left(1 - \frac{2}{1+e^{2w_{1,i}\dot{q}_i}}\right) + B_{f2,i}\left(1 - \frac{2}{1+e^{2w_{2,i}\dot{q}_i}}\right)$$

Feedback control law with estimated variables

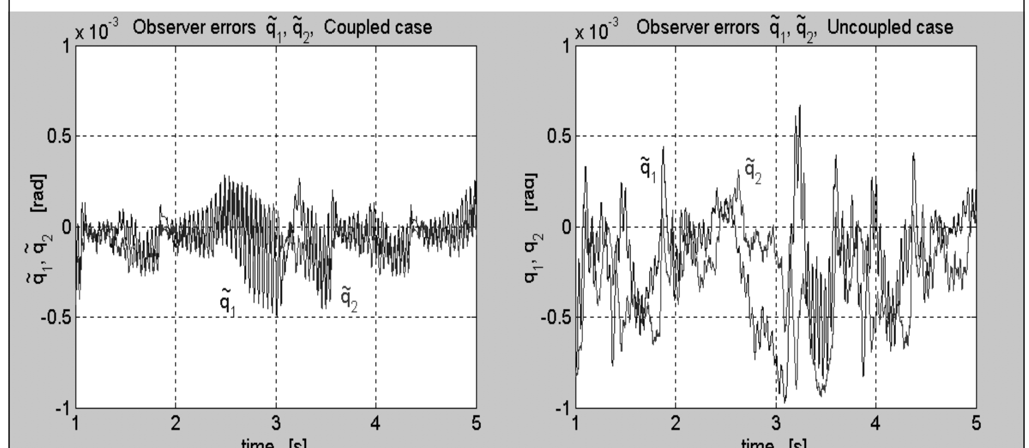
$$\tau_i = M_i(q_i)\hat{\ddot{q}}_{ri} + C_i(q_i,\hat{\dot{q}}_i)\hat{\dot{q}}_{ri} + g_i(q_i) + \tau_f(\hat{\dot{q}}_i) - K_{d,i}\hat{s}_i - K_{p,i}s_i$$

Synchronization errors



$$e_{1,1} = q_1 - q_d, \qquad e_{1,2} = q_1 - q_2, \qquad e_{2,2} = q_2 - q_d$$

Observer errors



$$\tilde{q}_1 = q_1 - \hat{q}_1, \qquad \tilde{q}_2 = q_2 - \hat{q}_2$$

Conclusions

- Semi-global exponential mutual synchronization
- Robustness against noise measurements
- Robustness against disturbances

Future extensions

- Different nominal references:
  - partial synchronization
- Other mechanical systems:
  - mobile systems
  - satellite formations