2.1. Calculus of Variations

©Erik I. Verriest*

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Goal

Find necessary conditions for a *function* $y = u(x)$ to make the integral

$$I = \int_{x_0}^{x_1} F(x, y, y')\, dx \tag{1}$$

stationary, while satisfying the constraints

$$\begin{aligned} y(x_0) &= y_0 \\ y(x_1) &= y_1. \end{aligned} \tag{2}$$

We shall assume that $F(x, y, y')$ has continuous partial derivatives with respect to all three arguments, and that y'' is continuous in (x_0, x_1).

Construction

Let $u(x)$ be the stationary solution and $\eta(x)$ any arbitrary but continuously differentiable function. Set for an arbitrary parameter ϵ, independent of x,

$$y(x) = u(x) + \epsilon\eta(x). \tag{3}$$

Let further $\eta(x)$ satisfy

$$\eta(x_0) = \eta(x_1) = 0. \tag{4}$$

The term $\epsilon\eta(x)$ is called the *variation* of y.

Fundamental Lemma of Calculus of Variations

*School of ECE, Georgia Tech, Atlanta, GA, USA. Presently with SCD, ESAT, KULeuven.

If $\phi(x)$ is continuous in $[x_0, x_1]$, and for all functions $\eta(x)$ satisfying

$$\begin{aligned} \eta(x_0) &= \eta(x_1) = 0 \\ \eta'(x) & \quad \text{continuous in } [x_0, x_1], \end{aligned}$$

we have

$$\int_{x_0}^{x_1} \eta(x)\phi(x)\, dx = 0,$$

then $\phi(x) \equiv 0$ in $[x_0, x_1]$.

Proof: By contradiction, constructing a suitable admissible function $\eta(x)$. See e.g. Forray's *Variational Calculus in Science and Engineering*, McGraw-Hill, 1968.

Notation Convention in Calculus of Variations

The change in $y(x)$ for a *fixed* value of x is called the *variation* of y. It is denoted by

$$\delta y(x) = \epsilon \eta(x)$$

The difference between this and the *differential* of a function should be well understood: The differential of $y(x)$ is the change in y when looking at the same function (or curve) for different values of x.

Substituting (3) in (1) yields

$$I = I(\epsilon) = \int_{x_0}^{x_1} F(x, u + \epsilon\eta, u' + \epsilon\eta')\, dx. \tag{5}$$

Note that I depends on ϵ. Since we assumed that $u(\cdot)$ was the stationary solution, $I(\epsilon)$ is stationary at $\epsilon = 0$. Hence, a *necessary* condition is that

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = 0. \tag{6}$$

So, turning the crank, we obtain

$$\begin{aligned} \frac{dI}{d\epsilon} &= \int_{x_0}^{x_1} \frac{dF(x, u + \epsilon\eta, u' + \epsilon\eta')}{d\epsilon}\, dx \\ &= \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y}\frac{dy}{d\epsilon} + \frac{\partial F}{\partial y'}\frac{dy'}{d\epsilon}\right] dx \\ &= \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial (u + \epsilon\eta)}\eta + \frac{\partial F}{\partial (u' + \epsilon\eta')}\eta'\right] dx, \end{aligned}$$

and finally:

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial u}\eta + \frac{\partial F}{\partial u'}\eta'\right] dx = 0$$

where $F = F(x, u, u')$. Integrating the second term by parts:

$$\int_{x_0}^{x_1} \frac{\partial F}{\partial u'}\eta' = \left[\frac{\partial F}{\partial u'}\eta\right]_{x_0}^{x_1} - \int_{x_0}^{x^1} \eta \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) dx.$$

Hence,

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'}\right] \eta \, dx = 0. \tag{7}$$

But $\eta(x)$ was an *arbitrary* variation in (x_0, x_1), and (7) is necessarily true. From the fundamental lemma of variational calculus, it follows then that

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} = 0.$$

This is *Euler's equation.* Its boundary conditions are

$$\begin{aligned} u(x_0) &= y_0 \\ u(x_1) &= y_1 \end{aligned}$$

Finally, note also that $\delta y' \equiv \epsilon\eta(x)'$. The corresponding change in F is:

$$\begin{aligned} \Delta F &= F(x, u + \epsilon\eta, u' + \epsilon\eta') - F(x, u, u') \\ &= \epsilon\eta\frac{\partial F}{\partial y} + \epsilon\eta'\frac{\partial F}{\partial y'} + \ldots \end{aligned}$$

where the "..." stand for higher order powers of ϵ.
Define thus

$$\delta F = \frac{\partial F}{\partial y}\epsilon\eta + \frac{\partial F}{\partial y'}\epsilon\eta'$$

This notation corresponds again to (but is different from) the differential $dF(x, y)$ of (ordinary) calculus:

$$dF = \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy.$$

Finally, note that from $\delta y = \epsilon\eta(x)$ and $\delta y' = \epsilon\eta(x)'$ we get

$$\frac{d}{dx}[\delta y] = \frac{d}{dx}\epsilon\eta(x) = \epsilon\eta(x)' = \delta y' = \delta\left[\frac{dy}{dx}\right]$$

Thus, the operators δ and $\frac{d}{dx}$ *commute*, where x is the independent variable.