

Application of partial observability for analysis and design of synchronized systems

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Synchronization in identical drive-response systems is a problem that can be cast in an *observer design* framework. In this paper we extend this approach by studying the analysis/design of partial synchronization by means of observer theory. In doing so, we introduce the concept of partial observer—an observer to reconstruct a part of the system state vector. It is also shown how the observability condition can be utilized to analyze the dynamics in an array of coupled identical systems. © 2003 American Institute of Physics. [DOI: 10.1063/1.1499596]

When synchronization sets in, coupled systems oscillate in a coherent way. It is possible to also observe some intermediate regimes characterized by incomplete synchrony which are referred to as partial synchronization. The paper is devoted to two problems: design of partially synchronized systems and analysis of partial synchronization in a network of coupled oscillators. A connection with the so-called observer design problem borrowed from control theory is emphasized.

I. INTRODUCTION

The task of retrieving knowledge about the full state vector of a dynamical system from the measurement of an output signal is commonly known in control theory as the *observer problem*. This is a well-developed problem in control theory, for which a solution exists for linear systems,¹ while only partial results exist for nonlinear systems, see Refs. 2–4, for example.

Observability is also a necessary condition to build a synchronizing copy of a given dynamical system. In fact, a standard approach in solving the observer problem in control theory is to build the observer as an identical copy of the plant (with unknown initial state) modified with an innovation term depending on the difference between the transmitted output from the plant and its prediction derived from the observer. This analogy, introduced in Ref. 5, will be explained in more detail in Sec. II.

This paper deals with the following problem: can observability also be related to other forms of synchronization? After all, full synchronization (defined, briefly, as the asymptotic equality of corresponding state variables of two, or many, identical systems suitably coupled) is not the only possible state of “synchrony,” although it is the one most commonly associated with this term. With specific application to chaotic systems, many different physical situations have been considered to be different forms of synchronized behavior.^{6,7}

In this paper we apply observability techniques to partial synchronization. The latter is a state, typical of coupled cellular networks, for instance, in which some coupled units of a network can show equal corresponding outputs, but not all of them. Here we do not discuss the observability condition applied to phase synchronization and the so-called generalized synchronization. We dealt with the first case in Ref. 8, where we built an observer for a periodically perturbed minimum-phase system. The difficulty in dealing with an observability theory for any general synchronous phenomenon lies first in its definition which, in its more exhaustive formulation, should take into consideration synchrony associated with a functional, where outputs of different systems with eventual temporal shift appear.⁹ We believe it is important to develop an engineering approach to synchronization in its different forms for control applications, where synchronous motion can be induced to ensure the proper functioning of a particular device. Consider, for example, active integrated antennas¹⁰ that can be built as arrays of multiple coupled oscillators to generate circular polarization.¹¹ In robotics, the problem of synchronization is usually referred to as coordination, or cooperation.^{12,13} Another interesting problem is to study (and control) spatiotemporal patterns in an ensemble of coupled systems, for communication purposes.¹⁴

The first problem we address here deals with what is partial coherence between oscillators forming a cellular network. The second problem, instead, can be formulated as follows: given dynamics with output, design an observer that reconstructs a part of its state components.

The paper is organized as follows: in Sec. II we recall some basic facts on the observer design problem. Section III presents an application of the observability condition to analyze the dynamics of a simple cellular network. Section IV deals with the partial synchronization and partial observability in a diffusive cellular network. In Sec. V we introduce the concept of partial observer and the concluding remarks are presented in Sec. VI.

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II. OBSERVABILITY AND DESIGN OF A SYNCHRONOUS SYSTEM

It is well understood that the design of a synchronous system is equivalent to the design of an observer. Consider the case in which system A acts on system B (an identical copy of A) via the transmission of an output signal, a function of the state variables of A . Synchronization analysis deals with finding conditions under which synchrony of A and B is possible. Observer design, instead, addresses the problem of building a suitable system B whose purpose is to estimate or reconstruct the full state vector of A . For this purpose, system B does not need to be an identical copy of A .

Consider a dynamical system given by the following system of differential equations:

$$\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^n, \quad t \in \mathbb{R}_+ \tag{1}$$

with the output signal

$$y(t) = h(x(t)), \quad y \in \mathbb{R}^m, \quad h \in C^\infty \tag{2}$$

available for on-line measurements. Throughout the paper we assume that $m = 1$ and f is smooth, so that existence and uniqueness of solutions of (1) are locally guaranteed. Additionally, we will assume that solutions exist for all $t \in \mathbb{R}_+$ and all initial points x_0 .

Let us denote by $x(t, x_0)$ the solution of Eq. (1) that starts from x_0 as initial condition, and $\dot{x}(t, x_0) = f(x(t, x_0))$.

Definition 1: The system (1) with output (2) is locally observable at x_0 if for all initial conditions x_1, x_2 in some neighborhood U of x_0 , if $h(x(t, x_1)) = h(x(t, x_2))$ for all t such that $x(t, x_1), x(t, x_2) \in U$ implies $x_1 = x_2$. The system (1) with output (2) is called locally observable if it is locally observable at any $x_0 \in \mathbb{R}^n$.

In other words, the output of a locally observable system always sees the evolution of different points as different.

The observation space \mathcal{O} is the linear space over \mathbb{R} of the functions $L_f^k h(x)$, $k = 0, 1, 2, \dots$, where $L_f^0 h(x) = h(x)$, $L_f^{k+1} h(x) = L_f(L_f^k h(x))$, and $L_f h(x) = \sum_i (\partial h(x) / \partial x_i) f_i(x)$. $L_f h(x)$ denotes the directional derivative of h in the direction of f , or Lie derivative of h with respect to the vector field f . Consider the observability codistribution

$$d\mathcal{O}(x) = \text{span}\{Dh(x), DL_f h(x), \dots\},$$

where D denotes the Jacobian. As a standard result from nonlinear control we mention a theorem which claims that the condition

$$\dim d\mathcal{O}(x) = n \tag{3}$$

is equivalent to the local observability of system (1) at x .⁴ For linear dynamics, i.e., when Eq. (1) has the form $f(x) = Ax$, and the output (2) is $h(x) = Cx$, $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{1 \times n}$, condition (3) reduces to

$$\text{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n. \tag{4}$$

In this case the pair (A, C) is said to be *observable*, and it is possible to find a matrix K , such that the matrix $A - KC$ has

arbitrary prespecified eigenvalues, and particularly $A - KC$ can be made Hurwitz by suitably choosing the output injection matrix K .

As the reader may expect, in the case of nonlinear dynamics there are no general results on the existence of an observer, nor does the fulfillment of the observability condition (3) imply the existence of an observer that synchronizes with (1). If the output function is linear, a similar condition as in Eq. (4) can be formulated, with the Jacobian matrix of f in Eq. (1) replacing the matrix A . Conditions (3), or (4), will therefore be state dependent, so there will be zones of the phase space in which condition (3) is violated.

In order to apply the observer approach we need to construct an observer such that the error dynamics between the observer and the plant goes to zero asymptotically. The simplest way to do this is to choose observers that have linearizable error dynamics under output rescaling. Consider for example the Rössler system, $x = (x_1, x_2, x_3)$,

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3, \\ \dot{x}_2 &= x_1 + ax_2, \\ \dot{x}_3 &= c + x_3(x_1 - b) \end{aligned} \tag{5}$$

with positive parameters a, b, c and measured output

$$h(x) = (0 \ 0 \ 1)x = x_3.$$

Using results developed in Ref. 3 one can see that the nonlinear system (5) with the output $y = x_3$ is globally diffeomorphic to a system in Lur'e form (linear system plus output-dependent nonlinearity) via an appropriate output rescaling. This can be achieved considering the following coordinate change: $\xi_1 = x_1, \xi_2 = x_2, \xi_3 = \log x_3$. This coordinate change is well defined since, if $x_3(t_0) > 0$, it follows from the third equation in (5) that $x_3(t) > 0$ for all $t > t_0$. In the new coordinates the system (5) with output x_3 takes the form

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & a & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} + \begin{pmatrix} -e^\eta \\ 0 \\ -b + ce^{-\eta} \end{pmatrix}, \tag{6}$$

$$\eta = (0 \ 0 \ 1)\xi = \xi_3,$$

which is in Lur'e form, with a linear observable part plus nonlinearity which depends only on the measured output $\eta = \xi_3$. As a natural candidate for an observer for the system (6) one can take the following system:

$$\begin{aligned} \begin{pmatrix} \dot{\hat{\xi}}_1 \\ \dot{\hat{\xi}}_2 \\ \dot{\hat{\xi}}_3 \end{pmatrix} &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & a & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \hat{\xi}_3 \end{pmatrix} \\ &+ \begin{pmatrix} -e^\eta \\ 0 \\ -b + ce^{-\eta} \end{pmatrix} + \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} (\eta - \hat{\eta}), \end{aligned} \tag{7}$$

where $\hat{\eta} = (0 \ 0 \ 1)\hat{\xi} = \hat{\xi}_3$ and K_1, K_2, K_3 are the gain coefficients which can be chosen to provide the error system with arbitrarily desirable dynamics. Clearly, system (7) synchro-

nizes with Eq. (6) and the observer (7) can afterwards be transformed in terms of the original dynamics (5).

The theory of nonlinear observers and the considered example highlight that the observer viewpoint is very convenient for the problem of design of (fully) synchronized systems.

III. OBSERVABILITY AND ANALYSIS OF A SYNCHRONOUS SYSTEM

As we previously mentioned, there may be regions of the phase space where the observability condition (3) is not fulfilled. It is interesting, therefore, to ask how special these regions are. Consider the special case of this simple scheme of two coupled identical systems

$$\begin{aligned} \dot{x}_1 &= f(x_1) - K(x_1 - x_2), \\ \dot{x}_2 &= f(x_2) - K(x_2 - x_1). \end{aligned} \tag{8}$$

It is easily recognized that $x_1 = x_2$ is an invariant manifold for Eq. (8). It is customary in investigating the stability of this invariant manifold to decompose the whole phase space in a subspace containing the invariant manifold, and the subspace transverse to it. In a more mathematical setting, in the coordinates

$$v = \frac{1}{2}(x_1 + x_2), \quad w = \frac{1}{2}(x_1 - x_2)$$

the system takes the form, for $w \approx 0$,

$$\begin{aligned} \dot{v} &= f(v), \\ \dot{w} &= (Df(v) - 2K)w \end{aligned} \tag{9}$$

that is in a cascade form. The importance of this structure will be further highlighted in Sec. V. From the second equation in Eq. (9) one can see that $\dot{w} = 0$ if $w = 0$, that is, $w = 0$ is, in the new coordinates, the previous invariant manifold. If the output of system (8) is a function of v coordinates only, $y = h(v) = h(x_1 + x_2)$, system (9) cannot be fully observable. All Lie derivatives of y do not contain terms dependent on w , hence condition (3) is not fulfilled. By measuring v , for trajectories on the invariant manifold, we gain no information on w . If the output of Eq. (8) is, instead, a function of w coordinates only, $y = h(w) = h(x_1 - x_2)$ with, specifically, $h(0) = 0$, the observability condition is not satisfied whenever $w = 0$, because the output becomes identically zero, regardless of the dynamics of v . The observability condition (3) is not fulfilled when the trajectory lies on an invariant manifold, since the dynamics is confined on a lower dimensional subset of the full state space.

While the fulfillment of condition (3) is helpful in designing a synchronizing system for Eq. (1), the converse situation is helpful to detect the presence of invariant manifolds. At the same time, it tells us whether the system can be rewritten in a cascade structure, even if in the new coordinates the system might not have direct physical interpretation.

The examples we introduce now on partial synchronization and partial observability will help to clarify these points. To analyze observability properties of networks of partially synchronized systems, we first present some results related to partial synchronization in diffusive networks.

IV. PARTIAL SYNCHRONIZATION

As the name suggests, partial synchronization is a state in which there is some (but not full) coherence between coupled oscillators. The higher the number of oscillators that form a network, the richer the pattern of possible synchronous states may be (see Ref. 15, for example), hence it is important to study properties of the network itself, in order to predict some of its possible synchronous states.

Consider a *diffusive cellular network* of k coupled identical dynamical systems of the form

$$\begin{aligned} \dot{x}_j &= f(x_j) + B u_j, \\ y_j &= C x_j, \end{aligned} \tag{10}$$

where $j = 1, \dots, k$, $x_j(t) \in \mathbb{R}^n$ is the state of the j th system, $u_j(t) \in \mathbb{R}^m$ and $y_j(t) \in \mathbb{R}^m$ are, respectively, the input and the output of the j th system, and B, C are constant matrices of appropriate dimension. In this representation we can say that the k systems (10) are *diffusively coupled* if the matrix CB is similar to a positive definite matrix, and the k systems are interconnected through mutual linear output coupling,

$$u_j = -\gamma_{j1}(y_j - y_1) - \gamma_{j2}(y_j - y_2) - \dots - \gamma_{jk}(y_j - y_k), \tag{11}$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$ are constants such that $\sum_{j \neq i}^k \gamma_{ji} > 0$ for all $i = 1, \dots, k$. Without loss of generality we can assume that CB is positive definite. Define the symmetric $k \times k$ matrix Γ as

$$\Gamma = \begin{pmatrix} \sum_{i=2}^k \gamma_{1i} & -\gamma_{12} & \dots & -\gamma_{1k} \\ -\gamma_{21} & \sum_{i=1, i \neq 2}^k \gamma_{2i} & \dots & -\gamma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & \dots & \sum_{i=1}^{k-1} \gamma_{ki} \end{pmatrix}, \tag{12}$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$ and all row sums are zero. The matrix Γ is symmetric and therefore all its eigenvalues are real. With the definition (12), the feedback (11) can be rewritten in the more compact form

$$u = -(\Gamma \otimes I_m)y,$$

where $y = \text{col}(y_1, \dots, y_k)$, $u = \text{col}(u_1, \dots, u_k) \in \mathbb{R}^{km}$.

The closed loop system (10) and (11) can show full synchrony, i.e., a state characterized by the equality of all coordinates of all systems forming the network: $x_1 = x_2 = \dots = x_k$, but it is also possible that the closed loop system (10) and (11) possesses an invariant manifold described by a number of equations of the form $x_i = x_j$ for some i, j . We define *partial synchronization* as the situation in which the states of *some* systems are identical. Stability of this partial synchronization regime is the subject of this section. If this manifold contains a (globally) asymptotically stable compact subset, this situation will be referred to as (global) partial synchronization. Analysis of possible symmetries of the closed loop system (10) and (11) helps in determining some of its

linear invariant manifolds. Particularly, it is not difficult to prove that if there is a permutation matrix Π commuting with Γ then the set

$$\ker(I_{kn} - \Pi \otimes I_n) \tag{13}$$

is a linear invariant manifold for the system (10) and (11). From this result it is clear that the set of symmetries for the networks defined by the matrices Π gives rise to a number of linear invariant manifolds for the closed loop system. This set of symmetries is called *global* symmetries, since they are independent of the equations that model a particular unit of the network. Along with the global symmetries it is useful to consider *internal* symmetries too, that are associated with the systems modeling the units of the network. In this paper, however, we are going to consider only global symmetries, while a more general approach can be found in our forthcoming paper.¹⁹

To obtain a stability criterion for the compact invariant subset of Eq. (13) it is convenient first to rewrite the system (10) in a different coordinate system. Let us differentiate y_j ,

$$\dot{y}_j = Cf(x_j) + CBu_j.$$

Then, choosing some $n - m$ coordinates z_j complementary to y_j it is possible to rewrite the system (10) in the form

$$\begin{aligned} \dot{z}_j &= q(z_j, y_j), \\ \dot{y}_j &= a(z_j, y_j) + CBu_j, \end{aligned} \tag{14}$$

where $z_j \in \mathbb{R}^{n-m}$, and q and a are some vector functions. It is important to emphasize that the coordinate change $x_j \rightarrow \text{col}(z_j, y_j)$ is linear. Note that, owing to the linear input-output relations, this transformation is globally defined. This transformation is explicitly computed in, for example, Ref. 17. Now we can formulate the main result of this section.

Theorem 1: *Suppose Π is a permutation matrix commuting with Γ . Let γ' be the minimal eigenvalue of Γ under restriction that the eigenvectors of Γ are taken from the set $\text{range}(I_k - \Pi)$. Suppose the following.*

(i) *There exists a non-negative radially unbounded function V satisfying the following conditions:*

$$\frac{\partial V}{\partial x} f(x) \leq -H(x), \quad \frac{\partial V}{\partial x} B = x^T C^T$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n-m}$ and the function H is positive outside some compact set in \mathbb{R}^n .

(ii) *There exists a positive definite matrix P such that all eigenvalues $\lambda_i(Q)$ of the symmetric matrix*

$$Q(z, w) = \frac{1}{2} \left[P \left(\frac{\partial q}{\partial z}(z, w) \right) + \left(\frac{\partial q}{\partial z}(z, w) \right)^T P \right]$$

are negative and separated from zero, i.e., there is $\delta > 0$ such that $\lambda_i(Q) \leq -\delta < 0$ with $i = 1, \dots, n - m$, for all $z, w \in \mathbb{R}^{n-m}$. Then there exists a positive $\bar{\gamma}$ such that if $\gamma' > \bar{\gamma}$ the set $\ker(I_{kn} - \Pi \otimes I_n)$ contains a globally asymptotically stable compact subset.

The first assumption of the theorem ensures ultimate boundedness, of the solutions of the closed loop system and is referred to as *semipassivity condition* (see, e.g., Refs. 16–18). The second assumption requires some sort of stability of

some internal dynamics of system (14). It guarantees that the z dynamics of system (14) is exponentially convergent (see, e.g., Refs. 16 and 18). The proof of this theorem can be found in our forthcoming paper.¹⁹ As an illustrative example, consider a ring of four coupled Lorenz systems, $x_j = (x_{j,1}, x_{j,2}, x_{j,3})$,

$$\begin{aligned} \dot{x}_{j,1} &= \sigma(x_{j,2} - x_{j,1}) + u_j, \\ \dot{x}_{j,2} &= rx_{j,1} - x_{j,2} - x_{j,1}x_{j,3}, \\ \dot{x}_{j,3} &= -bx_{j,3} + x_{j,1}x_{j,2}, \end{aligned}$$

where $j = 1, \dots, 4$, $\sigma, r, b > 0$ with outputs $y_j = x_{j,1}$ and coupling provided by the inputs u_j ,

$$\begin{aligned} u_1 &= -K_0(y_1 - y_2) - K_1(y_1 - y_4), \\ u_2 &= -K_0(y_2 - y_1) - K_1(y_2 - y_3), \\ u_3 &= -K_0(y_3 - y_4) - K_1(y_3 - y_2), \\ u_4 &= -K_0(y_4 - y_3) - K_1(y_4 - y_1). \end{aligned} \tag{15}$$

The particular geometry of the coupling defines the following coupling matrix:

$$\Gamma = \begin{pmatrix} K_0 + K_1 & -K_0 & 0 & -K_1 \\ -K_0 & K_0 + K_1 & -K_1 & 0 \\ 0 & -K_1 & K_0 + K_1 & -K_0 \\ -K_1 & 0 & -K_0 & K_0 + K_1 \end{pmatrix}. \tag{16}$$

The four permutation matrices for which $\Pi\Gamma = \Gamma\Pi$ are

$$\begin{aligned} \Pi_1 &= \begin{pmatrix} E & O \\ O & E \end{pmatrix}, & \Pi_2 &= \begin{pmatrix} O & I_2 \\ I_2 & O \end{pmatrix}, \\ \Pi_3 &= \begin{pmatrix} O & E \\ E & O \end{pmatrix}, & \Pi_4 &= I_4, \end{aligned} \tag{17}$$

where we denoted

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and O is the 2×2 zero matrix. Let us analyze what the action of these matrices Π is. The action of Π_1 is to switch *simultaneously* x_1 with x_2 and x_3 with x_4 . One can easily notice that this operation leaves the network unchanged, with respect to its connections. Similar actions are brought upon by Π_2 and Π_3 , while Π_4 is the identity, that leaves everything unchanged.

From expression (13), we derive that the linear invariant manifolds associated with Π_1 , Π_2 and Π_3 in Eq. (16) are, respectively,

$$\begin{aligned} \mathcal{A}_1 &= \{x \in \mathbb{R}^{12}: x_1 = x_2, x_3 = x_4\}, \\ \mathcal{A}_2 &= \{x \in \mathbb{R}^{12}: x_1 = x_3, x_2 = x_4\}, \\ \mathcal{A}_3 &= \{x \in \mathbb{R}^{12}: x_1 = x_4, x_2 = x_3\}. \end{aligned} \tag{18}$$

The intersection of any two of these linear manifolds gives the linear manifold describing full synchronization (i.e., $x_1 = x_2 = x_3 = x_4$).

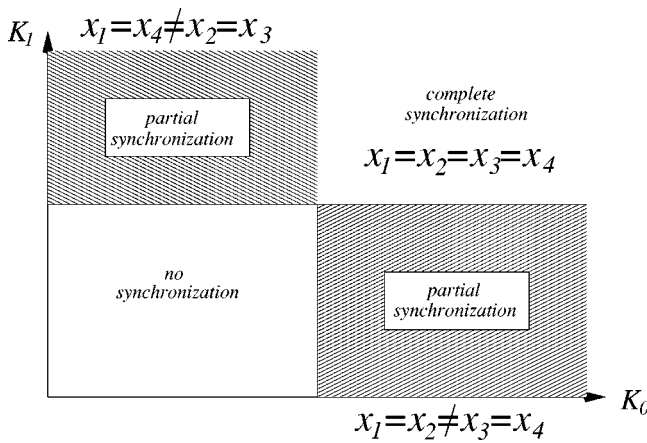


FIG. 1. Synchronization and partial synchronization in a ring of four Lorenz systems.

Once the invariant manifolds in the form (13) are found one has to check conditions (i) and (ii) of the theorem. To check condition (i), consider the following function:

$$V(x_{j,1}, x_{j,2}, x_{j,3}) = \frac{1}{2}((x_{j,1})^2 + (x_{j,2})^2 + (x_{j,3} - \sigma - r)^2).$$

The second equation from condition (i) trivially holds, so it remains to check the first inequality. To this end differentiate V along the trajectories of the free system ($u_j = 0$):

$$\dot{V}(x_{j,1}, x_{j,2}, x_{j,3}, u) = H(x_{j,1}, x_{j,2}, x_{j,3}),$$

where

$$H(x_{j,1}, x_{j,2}, x_{j,3}) = \sigma(x_{j,1})^2 + (x_{j,2})^2 + b \left(x_{j,3} - \frac{\sigma+r}{2} \right)^2 - b \frac{(\sigma+r)^2}{4},$$

so, condition (i) is satisfied. To check condition (ii), one can consider $P = I_2$, for which $Q = \text{diag}(-1-b)$, and condition (ii) is satisfied as well. Note that, for Lorenz system, the transformation $x_j \rightarrow \text{col}(z_j, y_j)$ is simply $y_j = x_{j,1}$ (the output given) and $z_j = \text{col}(x_{j,2}, x_{j,3})$.

Let us analyze the conclusions of the theorem. The eigenvalues of the matrix (16) are $\lambda_1 = 0$, $\lambda_2 = \min\{2K_0, 2K_1\}$, $\lambda_3 = \max\{2K_0, 2K_1\}$, $\lambda_4 = 2(K_0 + K_1)$. Hence, for the permutation described by Π_1 in Eq. (17) we have $\lambda' = 2K_0$. Similarly, $\lambda' = \min\{2K_0, 2K_1\}$ for Π_2 and $\lambda' = 2K_1$ for Π_3 . According to the theorem, for large K_0 and small K_1 one can expect asymptotic stability of a subset of the set \mathcal{A}_1 in Eq. (18). For the permutation Π_3 , for small K_0 and large K_1 , leads to asymptotic stability of a subset of the set \mathcal{A}_3 . Asymptotic stability of the full synchronization occurs for K_0 and K_1 both large enough. The subset of the set \mathcal{A}_2 is stable only as a stable intersection of \mathcal{A}_1 and \mathcal{A}_3 , which describes full synchronization. This situation is schematically shown in Fig. 1.

The observability rank condition. Let us test the observability rank condition on this ring structure of coupled systems. Assume, for simplicity, that the units forming the cells are one dimensional, i.e., $x_j \in \mathbb{R}$. The extension to higher dimension can be treated similarly, but in different notation. The ring of four systems is represented by

$$\dot{x}_j = f(x_j) + Bu_j, \quad j = 1, \dots, 4, \tag{19}$$

with inputs as previously defined in Eq. (15), and choose the output for Eq. (19) as

$$y = \sum_{j=1}^4 y_j = C \sum_{j=1}^4 x_j. \tag{20}$$

Let D stand for the Jacobian, as previously done. The Lie derivatives of the output are given by

$$\begin{aligned} h(x) &= C \sum_{j=1}^4 x_j, \\ L_f h(x) &= C \sum_{j=1}^4 f(x_j), \\ L_f^2 h(x) &= C \sum_{j=1}^4 [Df(x_j)f(x_j) + Df(x_j)Bu_j], \\ L_f^3 h(x) &= \dots \end{aligned}$$

We consider only the explicit formula for the Lie derivatives up to the second. The Jacobians of these derivatives are given by $Dh(x) = C(1111)$, and the Jacobian of the first derivative is a row vector of components

$$\{DL_f h(x)\}_i = CDf(x_i). \tag{21}$$

The second derivative can be written as a row vector of components

$$\{DL_f^2 h(x)\}_i = C \left[\alpha(x_i) + D^2 f(x_i)Bu_i + \sum_{j=1}^4 Df(x_j)B \frac{\partial u_j}{\partial x_i} \right], \tag{22}$$

where $\alpha(x_i)$ is some function of the coordinate x_i only, and it is important to note that $\partial u_j / \partial x_i = -\Gamma_{ji}$, with Γ given in Eq. (16). Let us see what values the row vectors (21) and (22) assume for trajectories on one of the invariant manifolds (18). Consider the manifold \mathcal{A}_1 first. This manifold is defined by the relationships $x_1 = x_2, x_3 = x_4$. For a trajectory lying on \mathcal{A}_1 we have $u_1 = u_2 = -u_3 = -u_4$, and for the last term in Eq. (22), we have, using Eqs. (15) and (16), a row vector of components

$$\begin{aligned} &K_1 [Df(x_4) - Df(x_1)]BC, \\ &K_1 [Df(x_3) - Df(x_2)]BC, \\ &K_1 [Df(x_2) - Df(x_3)]BC, \\ &K_1 [Df(x_1) - Df(x_4)]BC. \end{aligned}$$

Hence, for any trajectory in \mathcal{A}_1 the first and second components of Eqs. (21) and (22) are identical. The third and fourth components are identical too, though different from the first and second. Therefore, the observability condition (3) is not fulfilled whenever a trajectory of Eq. (19) lies on the invariant manifold \mathcal{A}_1 , and the same result holds for all other invariant manifolds in Eq. (18).

The result of this section indicates that in the network of coupled identical systems loss of the observability condition can play a role of necessary condition for the existence of invariant manifolds. However, note that invariance is a property of the dynamics alone, which exists regardless of what output is considered, although the output chosen in Eq. (20) is better suited for their observation.

V. PARTIAL OBSERVERS

We previously mentioned that there is a class of systems for which the observability condition (3) is not satisfied. Therefore, one can pose the following question: how much information can we derive from on-line measurements of the output if the observability condition fails? An alternative way of posing this question is: how wide is the range of functions of the state variables we can observe, for a given output?

Consider the dynamical system given by Eq. (1) with output (2) available for on-line measurements. The problem we want to address is to make an on-line estimation of the signal

$$z(t) = g(x(t)), \quad z \in \mathbb{R}^p, \quad g \in C^\infty. \tag{23}$$

This problem is referred to as the partial observer design problem. It is worth mentioning that the problem differs from the classical reduced observer design problem. Indeed, in the reduced observer design problem, the estimation of $z(t)$ and measurements of $y(t)$ allows the reconstruction of the whole state $x(t)$. Contrary to this, the partial observer design problem requires only the estimation of the signal $z(t)$. In the sequel we will discuss the problem for the scalar case $m = p = 1$.

Although a complete solution of the problem posed in this paper is unavailable, we will discuss a partial answer based on the concept of partial observers (see Ref. 20). To our knowledge, this problem is new in case system (1) with output (2) is not observable (detectable). One possible solution of the problem is by proving that system (1) is diffeomorphic via a differentiable and invertible coordinate change $\phi: x \mapsto (\xi, \zeta)$ to a cascade system of the form

$$\begin{aligned} \dot{\xi} &= f_1(\xi), \\ \dot{\zeta} &= f_2(\xi, \zeta), \end{aligned} \tag{24}$$

where $\xi \in \mathbb{R}^{n_1}, \zeta \in \mathbb{R}^{n_2}, n_1 + n_2 = n$ and, additionally, the pair $(f_1, h \circ \phi)$ is observable (detectable) and in the new coordinates z depends only on ξ : $z(t) = \psi(\xi)$.

In this case it is possible to design an observer for the first (drive) subsystem giving an estimate $\hat{\xi}(t)$ of $\xi(t)$. Then the signal $\psi(\hat{\xi}(t))$ is an estimate for $z(t)$, provided $\xi(t)$ is bounded. An observer for the first (drive) subsystem will then be referred to as a *partial observer*, since it can observe only part of system (24). It is important therefore to state some conditions which ensure the existence of a coordinate transformation such that in the new coordinates system (1) is of the form (24).

Recall the observability condition (3) introduced in Sec. II. A proof is based on the coordinate transformation $s: \mathbb{R}^n \rightarrow \mathbb{R}^n$, where $s(x) = (h(x), L_f h(x), \dots, L_f^{n-1} h(x))$. The observability rank condition ensures that this coordinate transformation is well defined (at least locally). Now suppose that

$$\dim d\mathcal{O}(x) = n_1 < n, \quad n_1 \neq 0 \tag{25}$$

in some neighborhood $\Omega \in \mathbb{R}^n$. In this case, system (1) is not locally observable. However, the mapping $s: \mathbb{R}^n \rightarrow \mathbb{R}^{n_1}$, where $s(x) = (h(x), L_f h(x), \dots, L_f^{n_1-1} h(x))$ allows one to rewrite system (1) locally in Ω in the form (24) where $\xi = s(x)$ and

ζ are some coordinates complementary to ξ . Moreover, the pair (f_1, h) satisfies in this case the observability rank condition (locally in Ω). In this case, condition (25) can be referred to as the partial observability condition, while the concept of partial (local) observability can be introduced similarly to the concept of (local) observability.⁴

Clearly, partial observability is a necessary condition for the (local) design of a partial observer for the first subsystem of Eq. (24), given the output y . Once such an observer has been designed, if the first subsystem has bounded trajectories and, additionally, g depends only on ξ , it is possible to estimate the function z .

A. Example

Consider the following six-dimensional system,

$$\begin{aligned} \dot{x}_1 &= -x_1 - \omega_1 x_2 - e^{x_3} + x_4, \\ \dot{x}_2 &= \omega_1 x_1 + a x_2, \\ \dot{x}_3 &= -c + x_1 + b e^{-x_3}, \\ \dot{x}_4 &= -\omega_2 x_2 - \omega_2 x_5 - e^{x_6}, \\ \dot{x}_5 &= -\omega_1 x_1 + \omega_2 x_4 + a x_5, \\ \dot{x}_6 &= -c + x_4 + b e^{-x_6}, \end{aligned} \tag{26}$$

where a, b, c, ω_1 , and ω_2 are real constants. The local observability condition described in Sec. V can tell us whether there are parts of the dynamics described by Eq. (26) that are unobservable using a specific output. Let it first be $y = h(x) = x_6$. The Lie derivatives of the output $y = h(x) = x_6$ along the trajectories of system (26) are

$$\begin{aligned} h(x) &= x_6, \\ L_f h(x) &= -c + x_4 + b e^{-x_6}, \\ L_f^2 h(x) &= -\omega_2 x_2 - \omega_2 x_5 - e^{x_6} + b c e^{-x_6} \\ &\quad - b x_4 e^{-x_6} - b^2 e^{-2x_6}, \\ L_f^3 h(x) &= -a \omega_2 x_2 - \omega_2^2 x_4 - a \omega_2 x_5 + (c - x_4) e^{x_6} - b \\ &\quad + [b c^2 - b c x_4 + b \omega_2 (x_2 + x_5)] e^{-x_6} \\ &\quad - (b + 3 b^2 c + 2 b^2 x_4) e^{-2x_6} + 2 b^3 e^{-3x_6}, \\ L_f^4 h(x) &= \dots \end{aligned}$$

and the respective Jacobians are

$$\begin{aligned} Dh(x) &= (0, 0, 0, 0, 0, 1), \\ DL_f h(x) &= (0, 0, 0, 1, 0, -b e^{-x_6}), \\ DL_f^2 h(x) &= (0, -\omega_2, 0, -b e^{-x_6}, -\omega_2, *), \\ DL_f^3 h(x) &= (0, -a \omega_2 + b \omega_2 e^{-x_6}, 0, *, -a \omega_2 + b \omega_2 e^{-x_6}, *), \\ DL_f^4 h(x) &= \dots \end{aligned}$$

We do not need to calculate Lie derivatives of the output higher than the third one to realize that the observability codistribution is of dimension at most three. Therefore, there must be a subsystem of Eq. (26) (that evidently includes the output x_6) that acts as a driving for the remaining part, so that the whole system (26) can be rewritten locally in a cascade form.

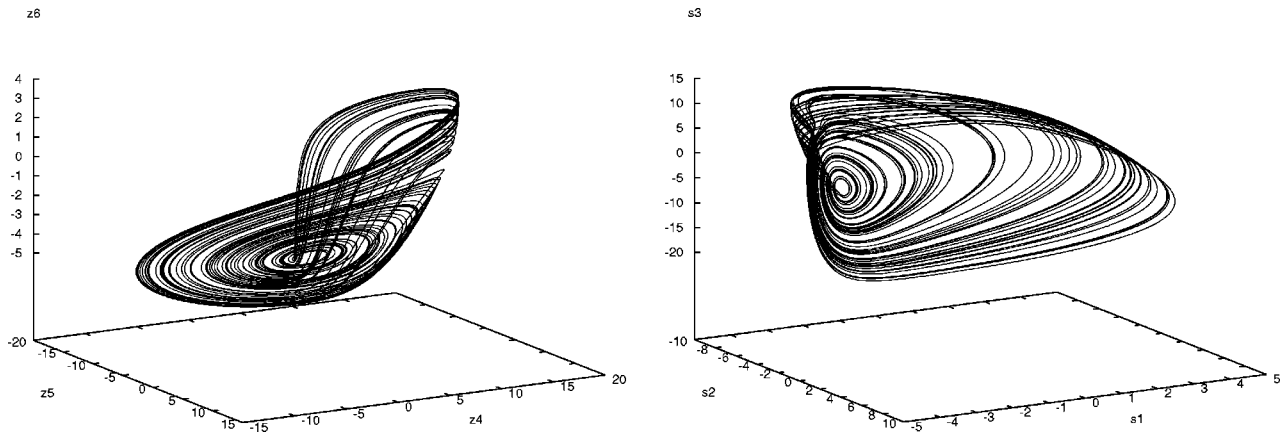


FIG. 2. Left-hand panel. Three-dimensional plot of a trajectory of system (28), in coordinates (z_4, z_5, z_6) , for parameter values $a=0.2, b=0.2, c=9.0, \omega_{1,2}=1 \pm \Delta\omega$, with $\Delta\omega=0.05$, generated with initial conditions $z_4(0)=-2.156, z_5(0)=3.546, z_6(0)=-3.989$. Right-hand panel: Its reconstruction in the coordinates (s_1, s_2, s_3) , from Eq. (27).

A coordinate change that puts system (26) in a cascade form can be explicitly constructed from the Lie derivatives of the output. Let $s_1=h(x)=x_6$ and $s_2=L_f h(x)=-c+x_4+be^{-x_6}$. At this stage we already have that coordinates of x_4 and x_1 of the first system can be represented as $x_6=s_1, x_4=s_2+c-be^{-s_1}$. We can choose the third coordinate as $s_3=L_f^2 h(x)$, i.e., $s_3=-\omega_2(x_2+x_5)-e^{-s_1}$. The term $-\omega_2(x_2+x_5)$ contains only derivatives of x_4 and x_6 , therefore the evolution of (s_1, s_2, s_3) is a system in closed loop form. We can actually choose $s_3=x_2+x_5$, still resulting in a closed loop system, but in a simpler analytical form. The driving part of system (26) is then reconstructed as

$$\begin{aligned} \dot{s}_1 &= s_2, \\ \dot{s}_2 &= \omega_2 s_3 - e^{s_1} - b s_2 e^{-s_1}, \\ \dot{s}_3 &= \omega_2 (s_2 + c - b e^{s_1}) + a s_3, \end{aligned} \tag{27}$$

and, considering auxiliary coordinates for the remaining part in the simple form $s_4=x_1, s_5=x_2$, and $s_6=x_3$, the driven part is expressed as

$$\begin{aligned} \dot{s}_4 &= -s_4 - \omega_1 s_5 - e^{s_6} - c + b e^{-s_1}, \\ \dot{s}_5 &= \omega_1 s_4 + a s_5, \\ \dot{s}_6 &= -c + s_4 + b e^{-s_6}. \end{aligned}$$

This example is better understood when we explain that system (26) had been derived from two coupled Rössler systems as in the form (6),

$$\begin{aligned} \dot{z}_1 &= -\omega_1 z_2 - e^{z_3} - (z_1 - z_4), \\ \dot{z}_2 &= \omega_1 z_1 + a z_2, \\ \dot{z}_3 &= -c + z_1 + b e^{-z_3}, \\ \dot{z}_4 &= -\omega_2 z_5 - e^{z_6}, \\ \dot{z}_5 &= \omega_2 z_4 + a z_5, \\ \dot{z}_6 &= -c + z_4 + b e^{-z_6}, \end{aligned} \tag{28}$$

after the linear change of coordinates that puts $x_5=z_5-z_2$ and leaves all other coordinates unchanged. The two systems (27) and the left-hand part of system (28) are indeed equivalent, related by a smooth (linear in this case) coordinate

change. Figure 2 shows the driving part of the original system (28) (left), and its reconstruction (27) in the s coordinates via the observability condition (right). Details of the simulation are given in the caption of Fig. 2. This example is inspired by a coupled systems model reported in Ref. 6 that can produce chaotic phase synchronization.

B. Linear error dynamics

Note that the driving part (27) is in Lur'e form, hence it admits an observer with linear error dynamics. All results we presented are coordinate-free, but this does not oversimplify the methods for observer design. The best condition is to achieve output linearization, since linear error dynamics always provide asymptotic stability. To illustrate this idea, suppose for a moment that the drive system is represented by the following:

$$\dot{\xi} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \xi + \begin{bmatrix} \alpha_1(\xi_1) \\ \vdots \\ \alpha_{n_1}(\xi_1) \end{bmatrix}, \quad y = \xi_1, \tag{29}$$

where $\alpha_j(\xi_1)$ are some functions of the output $y = \xi_1$. Then the system

$$\begin{aligned} \dot{\hat{\xi}} &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \hat{\xi} + \begin{bmatrix} \alpha_1(\xi_1) \\ \vdots \\ \alpha_{n_1}(\xi_1) \end{bmatrix} + K(\xi_1 - \hat{\xi}_1), \\ \hat{y} &= \hat{\xi}_1 \end{aligned} \tag{30}$$

is a partial observer for Eq. (1) with K such that it ensures fulfillment of the following observation goal:

$$\|\xi(t) - \hat{\xi}(t)\| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

It is important to observe that in this case the error dynamics is linear. Thus it is important to find conditions under which system (1) has a partial observer with a linear error dynamics. To find a solution to this problem, following Ref. 4, we

look for the existence of a transformation $s: \mathbb{R}^n \rightarrow \mathbb{R}^{n_1}$, $s: x \mapsto \xi$ such that in the new coordinates the ξ subsystem takes the form (29) and hence admits the observer (30). The conditions of equivalence via state transformation of systems (1) and (24) with the ξ subsystem in the form (29) are the following:

- (1) Local observability of a part of variables $\dim d\mathcal{O}(x) = n_1$, $\forall x \in \Omega \subset \mathbb{R}^n$.
- (2) There is a vector field r on $\Omega \subset \mathbb{R}^n$ that satisfies $L_r h(x) = L_r L_f h(x) = \dots = L_r L_f^{n_1-2} h(x) = 0$, $L_r L_f^{n_1-1} h(x) = 1$ such that $[r, \text{ad}_f^k r] = 0$, $\forall k = 1, 3, 5, \dots, 2n_1 - 1$.

If, additionally, $g = \psi \circ s$, and ξ is bounded, then the signal $\psi(\hat{\xi})$ is an estimate for the signal z .

VI. CONCLUSIONS

In this paper we presented an observer viewpoint on the partial synchronization problem. While in case of full synchronization the observer design problem is equivalent to the problem of designing a system synchronizing with a given system, in case of partial synchronization the situation is different. Partial observability serves two purposes. Suppose one needs to build an observer for a part of the state vector only; the observability condition can tell us which part is observable for a given output. Alternatively, the search for those regions of the phase space where the observability condition is not met can provide information on some invariant manifolds of the system. With the improvement of numerical techniques to span the observability distribution of a given system for a given output, the possibility of mapping some invariant subspaces of the system under investigation may follow. Additionally, the violation of the observability condition leads to the advantage of being able to rewrite the system in an explicit cascade structure. This property leads to some advantages in modeling and analysis of dynamical systems. It shows that it is possible to find a set of coordinates in which the system is rewritten in a simpler form, although the new governing equations may lose their original physical meaning. To summarize, both approaches have something in common: not fully observable systems are not generic (as follows from the Takens theorem), at the same time the network of identical oscillators is not generic either, since symmetry is involved.

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- ¹T. Kailath, *Linear Systems* (Prentice-Hall, Englewood Cliffs, NJ, 1980).
- ²A. Krener and W. Respondek, "Nonlinear observer with linearizable error dynamics," *SIAM J. Control Optim.* **23**, 47–52 (1985).
- ³K. Tchou and H. Nijmeijer, "On output linearization of observable dynamics," *Contr. Theory Adv. Technol.* **9**, 819–857 (1993).
- ⁴H. Nijmeijer and A. van der Schaft, *Nonlinear Dynamical Control Systems* (Springer, New York, 1990).
- ⁵H. Nijmeijer and I. M. Y. Mareels, "An observer looks at synchronization," *IEEE Trans. Circuits Syst., I: Fundam. Theory Appl.* **44**, 882–890 (1997).
- ⁶A. S. Pikovsky, M. G. Rosenblum, M. A. Zaks, and J. Kurths, "Phase synchronization of regular and chaotic oscillators," in *Handbook of Chaos Control*, edited by H. G. Schuster (Wiley-VCH, Weinheim, 1999).
- ⁷N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. I. Abarbanel, "Generalized synchronization of chaos in directionally coupled chaotic systems," *Phys. Rev. E* **51**, 980–994 (1995).
- ⁸G. Santoboni, A. Yu. Pogromsky, and H. Nijmeijer, "An observer for phase synchronization of chaos," *Phys. Lett. A* **291**, 265–273 (2001).
- ⁹I. I. Blekhnman, A. L. Fradkov, H. Nijmeijer, and A. Yu. Pogromsky, "On self-synchronization and controlled synchronization," *Syst. Control Lett.* **31**, 299–305 (1997).
- ¹⁰Y. Qian and T. Itoh, "Progress in active integrated antennas and their applications," *IEEE Trans. Microwave Theory Tech.* **46**, 1891–1900 (1998).
- ¹¹L. Dussopt and J.-M. Laheurte, "Coupled oscillator array generating circular polarization," *IEEE Microwave Guid. Wave Lett.* **9**, 160–162 (1999).
- ¹²A. Rodrigues-Angeles and H. Nijmeijer, "Coordination of two robot manipulators based on position measurements only," *Int. J. Control* **74**, 1311–1323 (2001).
- ¹³Y.-H. Liu, Y. Xu, and M. Bergerman, "Cooperation control of multiple manipulators with passive joints," *IEEE Trans. Rob. Autom.* **15**, 258–267 (1999).
- ¹⁴N. F. Rulkov, "Images of synchronized chaos: Experiments with circuits," *Chaos* **6**, 262–279 (1996).
- ¹⁵Y. Zhang, G. Hu, H. A. Cerdeira, S. Chen, T. Braun, and Y. Yao, "Partial synchronization and spontaneous spatial ordering in coupled chaotic systems," *Phys. Rev. E* **63**, 026211 (2001).
- ¹⁶A. Yu. Pogromsky, "Passivity based design of synchronizing systems," *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **8**, 295–319 (1998).
- ¹⁷A. Yu. Pogromsky, T. Glad, and H. Nijmeijer, "On diffusion driven oscillations in coupled dynamical systems," *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **9**, 629–644 (1999).
- ¹⁸A. Yu. Pogromsky and H. Nijmeijer, "Cooperative oscillatory behaviour of mutually coupled dynamical systems," *IEEE Trans. Circuits Syst., I: Fundam. Theory Appl.* **48**, 152–162 (2001).
- ¹⁹A. Yu. Pogromsky, G. Santoboni, and H. Nijmeijer, "Partial synchronization: From symmetry towards stability," *Physica D* (to be published).
- ²⁰G. Santoboni, A. Yu. Pogromsky, and H. Nijmeijer, "Partial observers and partial synchronization," *Int. J. Bifurcation Chaos Appl. Sci. Eng.* (to be published).