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A dynamical control view on synchronization

H. Nijmeijer

Department of Mechanical Engineering, Eindhoven University of Technology, PO Box 513, 5600 MB Eindhoven, The Netherlands

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Abstract

Synchronization of complex/chaotic systems is reviewed from a dynamical control perspective. It is shown that notions like observer and feedback control are essential in the problem of how to achieve synchronization between two systems on the basis of partial state measurements of one of the systems. Examples are given to demonstrate the main results.

Keywords: Synchronization; Dynamics; Chaos control; Uncertainty

1. Introduction

Probably one of the earliest detailed accounts on synchronized motion was made by Christiaan Huygens, who around 1650 describes in his notebook [12] an experiment where two identical pendulum clocks are attached to the same (flexible) bar, and these clocks exhibit synchronized motion in a short while in case they are initialized at arbitrary, possibly different phases. The explanation by Huygens is remarkably accurate since by that time the differential calculus needed to describe the clocks' motion was still to be developed. Many other examples of synchronized motion have been described after the 17th century. For instance, Rayleigh describes in his famous treatise "The theory of sound" [25] in 1877 that two organ tubes may produce a synchronized sound provided the outlets are close to each other. Early this century another Dutch scientist, van der Pol, studied synchronization of certain (electrical-) mechanical systems, see [24]. Actually, rotating bodies, or more general rotating mechanical structures form a very important and special class of systems that with or without the interaction through some coupling, exhibit synchronized motion. In fact, synchronization of oscillating physical systems is by today an important subject in some of the major physics journals. An illuminating survey on synchronization of a wide variety of mostly (electrical-) mechanical systems is given in [2]. Also [21] contains a rich class of motivating and illustrative examples of synchronizing systems. The growing interest in synchronization — and the above mentioned surveys are illustrative for this — was probably caused by the paper [22], where among others, secure communication as a potential application has been indicated. Although, sofar it is still questionable whether this application can be fully realized, the Pecora and Carroll paper [22] has formed an impulse for much research along these lines.

On the other hand, for mechanical systems synchronization is of utmost importance as soon as two machines have to cooperate. Typically, robot coordina-

E-mail address: h.nijmeijer@tue.nl (H. Nijmeijer).

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tion, cf. [4] and cooperation of manipulators, see [15] form important illustrations of the same goal, where it is desired that two or more mechanical systems, either identical or different, are asked to work in synchrony.

The purpose of this paper is to address the synchronization problem from a control theory perspective. Control theory is the field in which a systematic study of control systems, together with appropriate controller design(s) is made, see, e.g. [13] as a textbook on linear control theory and [19] for results in nonlinear control. Control theory, and more specifically nonlinear control, forms a powerful framework to formulate and describe various synchronization questions, and the aim of the paper is to present a review of some of the existing tools and methods from nonlinear control in this regard. This includes the obvious problem of how to build a synchronization system for a given system; a problem that is intimately linked with an observer problem in control, see Section 2. In Section 3 various aspects regarding (parameter) uncertainty and noise are reviewed. Especially, in practical applications like communications and coordination these noise and robustness issues are of great importance. Controlled synchronization is the subject of Section 4. Contrary to the setting of Section 2, where for a given system a synchronizing system is sought, here the problem is to achieve synchronization of two system by means of a suitably feedback controller.

This paper does not present formal theorems and proofs but merely develops various problem-solutions through illustrative examples. For detailed formulations and proofs the reader has to consult the appropriate references. Hopefully, the paper initiates further interest in dynamical control methods in the study of synchronization problems.

2. Synchronization and observers

Following [22], we consider the Lorenz system

$$\dot{x}_1 = \sigma(y_1 - x_1), \qquad \dot{y}_1 = rx_1 - y_1 - x_1 z_1,$$
$$\dot{z}_1 = -bz_1 + x_1 y_1. \tag{1}$$

The system (1) is known to exhibit complex or chaotic motions for certain parameters $\sigma, r, b > 0$. With the system (1) viewed as the *transmitter* or *master system*, we introduce the drive signal

$$y = x_1, \tag{2}$$

which can be used at the *receiver*, or slave system, to achieve asymptotic synchronization. This means, as in [22], we take as receiver dynamics

$$\dot{x}_2 = \sigma(y_2 - x_2), \qquad \dot{y}_2 = rx_1 - y_2 - x_1 z_2,$$
$$\dot{z}_2 = -bz_2 + x_1 y_2. \tag{3}$$

Notice that (3) consists of a copy of (1) with state (x_2, y_2, z_2) and where in the (y_2, z_2)-dynamics, the known signal x_1, see (2), is substituted for x_2. Introducing the error variables $e_1 = x_1 - x_2$, $e_2 = y_1 - y_2$, $e_3 = z_1 - z_2$, we obtain the error dynamics

$$\dot{e}_1 = \sigma(e_2 - e_1), \qquad \dot{e}_2 = -e_2 - x_1 e_3,$$
$$\dot{e}_3 = -be_3 + x_1 e_2, \tag{4}$$

which is a linear time-varying system. The stability of $(e_1, e_2, e_3) = (0, 0, 0)$ is straightforwardly checked using the Lyapunov function

$$V(e_1, e_2, e_3) = \frac{1}{\sigma} e_1^2 + e_2^2 + e_3^2 \tag{5}$$

with time-derivative along (4)

$$\dot{V}(e_1, e_2, e_3) = -2(e_1 - \tfrac{1}{2}e_2)^2 - \tfrac{3}{2}e_2^2 - 2be_3^2, \tag{6}$$

showing that (e_1, e_2, e_3) asymptotically (and even exponentially!) converges to $(0, 0, 0)$. In other words, the receiver dynamics (3) asymptotically synchronizes with the chaotic transmitter (1) no matter how (1) and (3) are initialized.

Remark 1. Almost similarly, one can show that the (y_2, z_2) dynamics from (3) — which are independent from x_2 anyway — will synchronize with (y_1, z_1) from (1), using the Lyapunov function $V(e_2, e_3) = e_2^2 + e_3^2$. This also implies that in this manner the state (x_1, y_1, z_1) can be reconstructed from (y_2, z_2) and the known signal x_1.

The synchronization of the transmitter (1) and receiver (3) using the drive signal (2) may at this point seem more a coincidence rather than a structural property. However, as will be argued, this is not the case, but follows in much more generality. In particular, one may cast the foregoing into an *observer problem*. To that end consider the general system

$$\dot{x} = f(x) \tag{7}$$

with $x \in \mathbb{R}^n$, f a smooth vector field, and output (or measurement)

$$y = h(x) \tag{8}$$

for some smooth function h. Note that more generally one may consider (7) and (8) on a manifold with function h mapping into another manifold. The observer problem can be formulated as: given $y(t), t \geq 0$, reconstruct asymptotically $x(t), t \geq 0$.

Remark 2. There is some redundancy in the observer problem in that the output $y = h(x)$ represents known (given) information on the state $x(t)$ which needs not to be estimated. In particular, suppose that the output coincides with one of the components of x (after a coordinate transformation), say $y = x_1$, then it is not necessary to reconstruct x_1, but only the $(n-1)$-dimensional 'state' $(x_2, \ldots, x_n)$. The latter variable is sometimes denoted as $x \pmod y$ and the reduced observer problem thus reads as the question of reconstructing $x(t) \pmod{y(t)}$, given $y(t), t \geq 0$. As an easy example, consider the linear system

$$\dot{x}_1 = y_1, \qquad \dot{y}_1 = ax_1 + by_1 \tag{9}$$

with output

$$y = x_1. \tag{10}$$

Setting for some $k \in \mathbb{R}$,

$$z = y_1 + kx_1, \tag{11}$$

we see that

$$\dot{z} = (b + k)z + (a - bk - k^2)y, \tag{12}$$

and thus we may find a reduced observer as

$$\dot{\tilde{z}} = (b + k)\tilde{z} + (a - bk - k^2)y, \tag{13}$$

provided $b + k < 0$, since $z - \tilde{z} \to 0$ as $t \to \infty$. The state (x_1, y_1) is asymptotically reconstructed from the one-dimensional reduced observer (13) as $(y, \tilde{z} - ky)$.

With the given formulation of the observer problem at hand, the natural question is, how to find, given (7) and (8), a mechanism for reconstructing $x(t), t \geq 0$. Although, in its full generality the answer to the above question is unknown, there are some important cases where a solution can be found. Some of them will be reviewed next. The natural way to approach the observer problem for (7) and (8) is to design another dynamical system driven by the measurements (8)

$$\dot{\tilde{x}} = f(\tilde{x}) + k(\tilde{x}, y), \tag{14}$$

where the y-parameterized vector field k in (14) should be such that $k(\tilde{x}, y) = 0$ if $h(\tilde{x}) = h(x) = y$. The dynamics (14) is called an *observer* for (7) provided that $\tilde{x}(t)$ asymptotically converges to $x(t)$ for any pair of initial conditions $x(0)$ and $\tilde{x}(0)$. The structure of the observer (14) deserves some further attention. One may view (14) as an identical copy of (7) with an 'innovations' term $k(\tilde{x}, y)$ which vanishes in case the estimated output $\tilde{y} = h(\tilde{x})$ coincides with $y = h(x)$. The latter could be phrased as we cannot do better as our measurements allow for. In the Lorenz system (1) and (2) with receiver (3) it is easily checked that the system (3) indeed acts as an observer and can be put in the form (14):

$$\begin{aligned}
\dot{x}_2 &= \sigma(y_2 - x_2) + 0, \\
\dot{y}_2 &= rx_2 - y_2 - x_2 z_2 + (r - z_2)(x_1 - x_2), \\
\dot{z}_2 &= -bz_2 + x_2 y_2 + y_2(x_1 - x_2).
\end{aligned} \tag{15}$$

Also, it is worth noting that (14) is simply a computerized model and no hardware is required in building this system, even if a hardware realization of (7) is given.

Remark 3.

1. Though we restrict attention to observers of the form (14) with dynamics of the same dimension as (7), other possibilities for obtaining suitable estimates for $x(t)$ exist. For instance, the estimate $\tilde{x}(t)$ can arise as a function of a higher dimensional measurement driven dynamics, or even as a solution of an infinite dimensional (pde) system.
2. It should be clear, see also Remark 2 that a reduced order observer should be designed as a

measurement driven dynamics that asymptotically matches with $x(t)$ (mod $y(t)$). Although, some interesting aspects arise, no further attention to reduced observers will be given here.

To illustrate the above observer design, we discuss first the linear observer problem [13]; that is both the dynamics (7) and measurements (8) are assumed to be linear:

$$\dot{x} = Ax, \tag{16}$$

$$y = Cx \tag{17}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ and A and C matrices of corresponding dimensions. An observer in this case should be of the form:

$$\dot{\tilde{x}} = A\tilde{x} + K(Cx - C\tilde{x}), \tag{18}$$

which, setting $e = x - \tilde{x}$, yields the error dynamics

$$\dot{e} = (A - KC)e. \tag{19}$$

Clearly, (18) acts as an observer for (16), or what is the same, $\tilde{x}$ and x asymptotically synchronize, if (19) has $e = 0$ as asymptotically stable equilibrium. The question under what conditions a matrix K can be found, so that (17) is asymptotically stable can be answered using the observability rank condition. The linear system (16) and (17) satisfies the *observability rank condition* if

$$\operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n, \tag{20}$$

which is equivalent to the requirement that the system (16) and (17) is *observable*, i.e., the state $x(t)$ is uniquely determined by $y(t), t \geq 0$. The rank condition (20) is equivalent to the pole placement property, which means that for any symmetric set of n points in $\mathbb{C}$, there exists a real matrix K such that $A - KC$ has these n points as eigenvalues. In particular, it follows that (20) guarantees the existence of an observer (18) (or suitable K) that makes (19) asymptotically stable. In fact, a slightly weaker condition than (20), *detectability*, is required for the stabilizability of (19) with a suitable K. Detectability requires, instead of (20) that A restricted to the largest A-invariant subspace in $\operatorname{Ker} C$ (this subspace is equal to the kernel of the matrix defined in the left-hand side of (20)), should be asymptotically stable. For further details, see [13].

It is clear that the above discussion on synchronization of linear systems cannot directly be used for nonlinear/chaotic systems. On the other hand, there are a number of extensions of the foregoing linear observer design that are relevant for complex nonlinear systems. The first class for which observer design is as simple as in the linear case are the so-called Lur'e systems, which are described as

$$\dot{x} = Ax + \varphi(Cx), \tag{21}$$

$$y = Cx \tag{22}$$

with the pair (A, C) observable, i.e. (20) holds, and φ is a smooth nonlinear vector field depending on y. A synchronizing system (observer) is designed as

$$\dot{\tilde{x}} = A\tilde{x} + \varphi(Cx) + K(Cx - C\tilde{x}), \tag{23}$$

which again produces the error dynamics (19). Notice that the class of systems (21) and (22) only contain nonlinearities in the dynamics that depend upon the measured output y, and which can also be used in the observer (23). Perhaps the best known example of the form (21) and (22) is the Chua circuit:

$$\begin{aligned} \dot{x}_1 &= \alpha(-x_1 + y_1 - \varphi(x_1)), \\ \dot{y}_1 &= x_1 - y_1 + z_1, \\ \dot{z}_1 &= -\lambda y_1, \end{aligned} \tag{24}$$

where $\varphi(x_1) = m_1 x_1 + m_2(|x_1 + 1| - |x_1 - 1|)$ with $m_1 = -\frac{5}{7}, m_2 = -\frac{3}{7}$, and $23 < \lambda < 31, \alpha = 15.6$. Taking as measurements

$$y = x_1, \tag{25}$$

one immediately realizes that this system — which is chaotic and has the so-called double scroll attractor — is of Lur'e type and admits an observer of the form (23), since the corresponding linear part is observable. It is interesting to see that the only nonlinearity in (24) is through the piecewise linear function (nonsmooth) φ, which only depends on the measurements y.

A larger class of systems that admit linear observer design consists of all systems (7) with outputs (8) that possess *linearizable error dynamics*. More precisely, these are systems (7) and (8) that after a suitable state space transformation and output transformation can be brought in Lur'e form (21) and (22). Conditions in terms of f and h that are necessary and sufficient for having linearizable error dynamics are given in [16] in the context of synchronization. As an illustration, consider the hyperchaotic Rössler system, see [1],

$$\begin{aligned}\dot{x}_1 &= -x_2 + ax_1,\\ \dot{x}_i &= x_{i-1} - x_{i+1}, \quad i = 2, \ldots, N-1,\\ \dot{x}_N &= \epsilon + bx_N(x_{N-1} - d)\end{aligned} \tag{26}$$

with $a, b, d, \epsilon \in \mathbb{R}^+$ and N an arbitrary positive integer. The case $N = 3$ corresponds to the usual Rössler system, and when $N = 4$ the system has the so-called hyperchaotic flows, and has two positive Lyapunov-exponents. With (26) we take as output equation

$$y = x_N. \tag{27}$$

It is clear that the solutions of (26) with $x_N(0) > 0$, and which exist for all positive time (no finite escape time!) have $x_N(t) > 0$ for all $t > 0$. Therefore, we may introduce the coordinate transformation $z_1 = x_1, \ldots, z_{N-1} = x_{N-1}, z_N = \ln(x_N)$, and output transformation $\bar{y} = \ln(y)$. Then in the new coordinates the system reads as

$$\begin{aligned}\dot{z}_1 &= -z_2 + az_1,\\ \dot{z}_i &= z_{i-1} - z_{i+1},\\ i &= 2, \ldots, N-2,\\ \dot{z}_{N-1} &= z_{N-2} - \exp(z_N),\\ \dot{z}_N &= bz_{N-1} - bd + \epsilon\exp(-z_N),\end{aligned} \tag{28}$$

$$\bar{y} = z_N. \tag{29}$$

The remarkable fact is that the above system (28) and (29) is again in Lur'e form (21) and (22), and an easy check shows that the linear part is observable, thus allowing for a synchronizing system of the form (23).

The classes of systems for which a successful observer design is possible, sofar all exploit a linear error dynamics. There are, however, other cases where synchronization can be achieved without relying on a 'linearizability' assumption. To that end we return to the system (7) with measurement (8) and we introduce the following assumptions, see [9]:

1. The vector field f in (7) satisfies a global Lipschitz condition on its domain, which as mentioned earlier need not to be $\mathbb{R}^n$.
2. The n functions $h(x), L_f h(x), L_f^2 h(x), \ldots, L_f^{n-1} h(x)$ define new coordinates (globally!). Here, $L^i_f h(x)$ denotes the ith iterated Lie-derivative of the function h in the direction of f.

If both (1) and (2) hold an observer exists of the form

$$\dot{\tilde{x}} = f(\tilde{x}) + K(h(x) - h(\tilde{x})) \tag{30}$$

with K a constant suitable $(n, 1)$-vector. Note that (30) obviously is of the form (14), though some of the entries in K may become very large (high-gain). An illustrative example of a system that fulfills (1) and (2) is formed by the Lorenz-system (1) and (2), when this is restricted to a compact domain. Since it is known that (1) has an attractive compact box, the observer (30) is an interesting alternative for the observer (3).

Besides the above discussed cases for which a synchronizing system can be systematically designed we note that there exist further methods that may be applicable for other classes of systems, like bilinear systems. Also, for certain mechanical systems 'physics-based' observers can be developed, and finally some systems admit a Kalman filter-like observer. But, no general method exists that works for all systems.

3. Uncertainty, robustness and noise

In the previous section, the synchronization problem has been treated under the assumption that the dynamics and output are exactly known. In many cases this is obviously not true and therefore alternative methods are required. We will review here three illustrative examples how one may possibly proceed in such case. The first example to be discussed contains parameter uncertainty in the dynamics (7), i.e.

$$\dot{x} = f(x, p) \tag{31}$$

with p some unknown parameter (vector) or in a communications context, an unknown message. We take again as output

$$y = h(x). \tag{32}$$

Now, in addition to the standard synchronization problem of reconstructing $x(t)$, $t \geq 0$, one may in addition be interested in reconstruction of the parameter p. The latter may be particularly interesting in a communications context where p may represent some (slowly time-varying) signal. The next example illustrates that *adaptive control* (cf. [26]) may form a good approach in such setting. Consider again the Chua circuit

$$\begin{aligned} \dot{x}_1 &= \alpha(-x_1 + y_1 - \varphi(x_1, p)), \\ \dot{y}_1 &= x_1 - y_1 + z_1, \\ \dot{z}_1 &= -\lambda y_1, \end{aligned} \tag{33}$$

where $\varphi(x_1, p) = \varphi(x_1) + p(|x_1 + 1| - |x_1 - 1|) = m_1 x_1 + (m_2 + p)(|x_1 + 1| - |x_1 - 1|)$ with $m_1 = \frac{5}{7}, m_2 = -\frac{6}{7}, \lambda = 14.286$ and $\alpha = 9$. As output, we take

$$y = x_1. \tag{34}$$

The parameter p is assumed to be constant or slowly time-varying, but in practice it may also be a binary time-varying signal. A solution to both the synchronization problem and the parameter estimation problem is given by the following adaptive observer:

$$\begin{aligned} \dot{x}_2 &= \alpha(-x_2 + y_2 - \varphi(x_1)) \\ &\quad + \tilde{p}_1(|x_1 + 1| - |x_1 - 1|) + \tilde{p}_2(x_2 - x_1), \\ \dot{y}_2 &= x_2 - y_2 + z_2, \qquad \dot{z}_2 = -\lambda y_2, \end{aligned} \tag{35}$$

$$\begin{aligned} \dot{\tilde{p}}_1 &= -\gamma_1 (x_1 - x_2)^2, \\ \dot{\tilde{p}}_2 &= -\gamma_2 (x_1 - x_2)(|x_1 + 1| - |x_1 - 1|), \end{aligned} \tag{36}$$

where $\gamma_1, \gamma_2 > 0$ are the positive adaptation gains. It follows, see [8] that again (e_1, e_2, e_3) converges to $(0, 0, 0)$ but also $\tilde{p}_1$ and $\tilde{p}_2$ converge to their true values, and in particular, $\tilde{p}_1$ converges to p, and $\tilde{p}_2$ can be viewed as an observer gain. The key observation in showing this result is actually the fact that the signal $|x_1 + 1| - |x_1 - 1|$ is 'persistently exciting' for the chaotic Chua circuit (33), which among others, means this signal does not converge to some constant value. In case p is a binary signal, the parameter convergence will occur provided the time step in changing p is sufficiently large, see [8] for further details.

The idea of using adaptation mechanisms like in (36) requires that only parametric uncertainties occur. This may be a strong assumption in specific cases and alternatives may be sought. A simple illustration of a *robust* synchronization scheme can be given for a second order (mechanical) system

$$\dot{x}_1 = y_1, \qquad \dot{y}_1 = f(x_1, y_1) \tag{37}$$

with

$$y = x_1. \tag{38}$$

An observer is proposed as

$$\dot{x}_2 = y_2 + k_1(x_1 - x_2), \qquad \dot{y}_2 = k_2(x_1 - x_2) \tag{39}$$

then under the assumption that f in (37) satisfies a global Lipschitz condition one may show that for $k_1, k_2 > 0$ sufficiently large, the error (e_1, e_2) converges to a neighborhood of $(0, 0)$ and moreover, the larger the k_1 and k_2 are selected, the smaller the neighborhood of $(0, 0)$ becomes. In this case, we have the so-called high-gain observer (39) that achieves practical stability of the error (e_1, e_2), and thus the state (x_2, y_2) of (39) asymptotically almost synchronizes with (x_1, y_1), see [18] for further details on the dual problem of robust control of chaotic systems. The implementation of a high-gain observer is simple — no hardware realization of the observer system is build — but it has practical limitations since large values for k_1 and k_2 will amplify measurement errors in the output $y = x_1$. There exist in the control literature a wide range of alternative methods of studying robust observers, and thus robust synchronization; one alternative method can be found in [23], see also [28]. Besides parameter uncertainty or unstructured uncertainty in the dynamics (7) and output (8) the equations may be noisy. Noise may appear for different reasons in (7) and (8), for instance measurement noise or uncertainties in the dynamics. In this case, synchronization becomes even more problematic than in the previous section, and certainly no exact state reconstruction will be possible. Nevertheless, a filtering approach may be very suited in this case and we will illustrate this through an example of a noisy Lorenz-system, see [17] for further details. Consider the Lorenz system with noise

$$\begin{aligned} \dot{x}_1 &= \sigma(y_1 - x_1) + \epsilon_1, \\ \dot{y}_1 &= rx_1 - y_1 - x_1 z_1 + \epsilon_2, \\ \dot{z}_1 &= -bz_1 + x_1 y_1 + \epsilon_3, \end{aligned} \tag{40}$$

and noisy measurements

$$y = x_1 + v, \tag{41}$$

where $(\epsilon_1(t), \epsilon_2(t), \epsilon_3(t)) \sim N(0, Q)$ and $v(t) \sim N(0, R)$ are independent white noise processes. Clearly (40) now represents a set of stochastic differential equations, which due to the nonlinearities are highly nontrivial to solve (numerically). This makes it even more difficult to find a synchronizing system. Instead of a — deterministic — observer one may attempt to use an extended Kalman filter. (Here the word extended refers to the fact that the filter applies to nonlinear equations; the Kalman filter itself applies only to a linear stochastic system with noisy measurements.) The extended Kalman filter reads as

$$\begin{aligned} \dot{x}_2 &= \sigma(y_2 - x_2) + k_1(t)e_1, \\ \dot{y}_2 &= rx_2 - y_2 - x_2 z_2 + k_2(t)e_1, \\ \dot{z}_2 &= -bz_2 + x_2 y_2 + k_3(t)e_1, \end{aligned} \tag{42}$$

where as before, $e_1 = y - x_2 = x_1 + v - x_2$. At this point one may again notice that (42) fits in the structure (14). The crucial point of (42) lies in the way how the gain vector $k(t) = (k_1(t), k_2(t), k_3(t))^{\mathrm{T}}$ is determined. For the filter (42), $k(t)$ is determined via

$$k(t) = P(t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} R^{-1}, \tag{43}$$

where R is the covariance of the measurement noise v and $P(t)$ the solution of the matrix Riccati differential equation

$$\begin{aligned} &\dot{P} = F(t)P + PF(t)^{\mathrm{T}} - PH(t)^{\mathrm{T}} R^{-1} H(t)P + Q, \\ &P(0) = P_0 > 0, \end{aligned} \tag{44}$$

where $F(t) = (\partial f/\partial x)(x_2(t), y_2(t), z_2(t))$ and $H(t) = (\partial h/\partial x)(x_2(t), y_2(t), z_2(t))$ with f and h denoting the right-hand side of (40) and (41). In other words $F(t)$ and $H(t)$ are obtained through linearization along the estimated solution $(x_2(t), y_2(t), z_2(t))$. Although, at this point no complete proof exists which guarantees that in some stochastic sense $(x_2(t), y_2(t), z_2(t))$ converges approximately to $(x_1(t), y_1(t), z_1(t))$, simulations indicate that with suitable initialization, the extended Kalman filter may form an appropriate scheme for synchronization, see [27]. In the discrete-time context, we have recently investigated this in detail for some specific chaotic systems, see [6].

Remark 4.

1. Crucial in [6] is the observation that the chaotic systems under investigation 'live' in a compact region and thus fulfill a Lipschitz condition in this region. It is precisely this fact — which has some similarity with the high gain observer approach in the previous section — that enables a successful extended Kalman filter approach, see [14].
2. It is clear that convergence of the estimate $(x_2(t), y_2(t), z_2(t))$ towards $(x_1(t), y_1(t), z_1(t))$ is at best possible in expectation. The noise in dynamics and measurement prohibit exact asymptotic convergence and therefore simulations based on the filter (42) will become sensitive with respect to the variances Q and R. Likewise, the initialization of the Riccati differential equation (44) is an important design parameter.

4. Controlled synchronization

Synchronization as reviewed in Sections 2 and 3 was merely a property of finding an appropriate mechanism for reconstructing the state of some chaotic system on the basis of some given measurement signal. On the other hand, this could be contrasted with another setting in which both transmitter and receiver dynamics are given, as well as the corresponding output function, and the aim is to find a suitable mechanism to *control* the slave system such that master and slave will asymptotically synchronize. More specifically, assume we have been given the transmitter

$$\dot{x} = f(x) \tag{45}$$

with output

$$y = h(x) \tag{46}$$

together with the receiver dynamics

$$\dot{\tilde{x}} = g(\tilde{x}, u), \tag{47}$$

where we for simplicity assume that x and $\tilde{x}$ both are n-dimensional. The dynamics (47) depends on a control (vector) u, which we assume for the moment to belong in $\mathbb{R}$. The control u is the variable through which we may manipulate or change the dynamics (47), and it is here that we enter the area of nonlinear control, see [19]. Obviously, there exist many controller types that we may use but in the sequel we limit ourselves to the use of a feedback of the form

$$u = \alpha(\tilde{x}, y), \tag{48}$$

where α is a smooth function depending on the state of the receiver and the available measurements of the transmitter. This is, at least at an intuitive level, a natural choice. The closed-loop system (47) and (48) is now described as

$$\dot{\tilde{x}} = g(\tilde{x}\alpha(\tilde{x}, h(x))), \tag{49}$$

and the aim in synchronizing master and slave system now is to find a suitable function α in (48) such that asymptotically $x(t)$ and $\tilde{x}(t)$ coincide. Stated in terms of Section 2 this implies that (49) acts as an observer for (45).

Remark 5. From the foregoing it becomes clear that there are numerous other ways to enforce the synchronization between (45) and (46). In [3] various definitions are given, with perhaps the most general controller being of the form

$$u = \alpha(\tilde{x}, z, y), \qquad \dot{z} = h(z, y), \tag{50}$$

which is in control terminology a dynamic output feedback. Potentially, the introduction of the dynamics in (50) allows for synchronization of (45), (47) and (50), which means that in this case we need not start with systems (45) and (46) of the same dimension.

The general problem of finding, if possible, a suitable output feedback (48) in order that (45) and (49) synchronize is quite difficult. We will illustrate this by means of a relatively simple example of van der Pol systems. Consider as transmitter dynamics the van der Pol system

$$\dot{x}_1 = y_1, \qquad \dot{y}_1 = -x_1 - (x_1^2 - 1)y_1, \tag{51}$$

$$y = x_1, \tag{52}$$

and as receiver we take the 'controlled' van der Pol system

$$\dot{x}_2 = y_2 + \alpha u, \quad \dot{y}_2 = -x_1 - (x_1^2 - 1)y_2 + \beta u. \tag{53}$$

Note that we have exploited the knowledge of x_1 in (53), and also that control in (53) is possible along the direction $(\alpha\beta)^{\mathrm{T}}$. If (53) represents an electrical circuit or physical system it may happen that either $\alpha = 0$ or $\beta = 0$. Typically, the control u is a current (or voltage) or force that acts on the system.

Remark 6. At this point, there is a notable difference with most of the 'control of chaos' literature where often a control parameter is varied as to influence the dynamics, see for instance the OGY paper [20].

To achieve synchronization of (52) and (53) we will use here (high-gain) output error feedback ($c_1 > 0$)

$$u = -c_1(x_1 - x_2) \tag{54}$$

resulting in the error dynamics

$$\begin{aligned} \dot{e}_1 &= -\alpha c e_1 + e_2, \\ \dot{e}_2 &= -\beta c e_1 - (x_1^2 - 1)e_2, \end{aligned} \tag{55}$$

which is a linear time-varying system, in which the time varying signal $(x_1^2 - 1)$ is known, see (52). For the synchronization of (51), (53) and (54) it is required that the error dynamics (55) are asymptotically stable about the equilibrium $(0, 0)$. Already this relatively simple error dynamics require some nontrivial analysis. The most interesting case probably arises if $\alpha = 0$ and $\beta \neq 0$. One can show, see [11] that there exists a constant c_* — which is determined in terms of the transmitter dynamics (51) — such that if the gain $c > c_*$, then the error dynamics (55) are uniformly exponentially stable. In fact, this result follows by transforming (55) to an associated Hill equation and as a result the stability turns out to be rather slow. At the same time, the lower bound from Huijberts et al. [11] may be rather conservative.

The example of controlled synchronization reveals that the problem to find a suitable (output) feedback controller that achieves synchronization of transmitter and receiver will in general become difficult, or even impossible to solve. On the other hand, a systematic analysis that parallels the different cases reviewed in Section 2 may lead to other solutions. For instance, this is true for Lur'e systems, with a transmitter system of the form

$$\dot{x} = Ax + \varphi(Cx), \tag{56}$$

$$y = Cx, \tag{57}$$

and receiver dynamics

$$\tilde{x} = A\tilde{x} + \varphi(Cx) + Bu. \tag{58}$$

It follows that provided the pair (A, C) is detectable, as well as $(A^{\mathrm{T}}, B^{\mathrm{T}})$ is detectable (or equivalently, (A, B) is stabilizable) then there exists a (linear) dynamic output feedback of the form (50) such that the two systems asymptotically synchronize. Recall that detectability of the pair (A, C) requires, instead of the observability condition (20) that the matrix A; restricted to the largest A-invariant subspace in the kernel of C, should be asymptotically stable. For further details and insight in the controlled synchronization problem the reader is referred to [11].

5. Epilogue

We have tried to give a dynamical control view on synchronization. All in all, it is felt that nonlinear control may provide some useful tools to address certain synchronization problems. On the other hand, in many cases, a thorough study of certain time-varying dynamical systems is required and it may be concluded that further research along these lines requires knowledge from both dynamical systems and nonlinear control theory. The review as presented here gives only a partial view on synchronization. There are numerous variants of synchronization defined in the literature, of which one could mention, phase synchronization, partial synchronization and generalized synchronization, see [21] or [3] where a general definition of (controlled) synchronization is proposed. In the study of synchronization several elements from control theory turn out to be relevant. This includes observers (see Section 2), filtering and robustness (Section 3) and feedback control (Section 4), but also further aspects as system inversion, cf. [7] or system identification, cf. [10]. The observer ideas as are put forward here, are quite common in standard control system design. For feedback regulation of an experimental or industrial plant often it is not possible to use state feedback, since the state of the system is partially measured. A standard approach to avoid this problem is to replace in the state feedback controller the state vector by an estimate, which is derived from an observer. Even in simple PD controllers one needs a numerical differentiator (a kind of reduced observer) to obtain the derivative of the output. It should be clear that synchronization problems can be treated in other domains too. In particular, for discrete-time systems various results more or less parallel the material from the foregoing sections. Even for transmitter/receiver dynamics described by partial differential equations one may expect some results along these lines, see, e.g. [5] for a specific example of synchronizing pde's. Likewise, synchronization with time-delayed feedback has also been studied in [5]. Synchronization has numerous potential applications running from coordination problems in robotics to mechanisms for secure communications. Precisely, the latter area was mentioned in [22] as a potential field of application, although sofar much work remains to be done here.

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Partial synchronization: from symmetry towards stability

Alexander Pogromsky, Giovanni Santoboni, Henk Nijmeijer*

Department of Mechanical Engineering, Dynamics and Control Group, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

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Abstract

In this paper we study the existence and stability of linear invariant manifolds in a network of coupled identical dynamical systems. Symmetry under permutation of different units of the network is helpful to construct explicit formulae for linear invariant manifolds of the network, in order to classify them, and to examine their stability through Lyapunov's direct method.

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1. Introduction

The high number of scientific contributions in the field of synchronization of coupled dynamical systems reflects the importance of this subject. The reason for this importance appears to be threefold: synchronization is common in nature, coupled dynamical systems display a very rich phenomenology and, finally, it can find applications.

First of all, many situations can be modelled as ensembles of coupled oscillators. Considering some examples from the natural world, there is evidence of synchrony among pulse-coupled biological oscillators [1], which relates to the observation that thousands of male fireflies can gather in trees and flash at unison (a very nice colour picture of this event can be found in [2]). Synchronous activity has also been observed in many regions of the human brain, relative to behaviour and cognition, where neurons can electrically discharge synchronously in some known frequency ranges [3], a behaviour that is reproduced by different mathematical models [4]. The synchronous firing of cardiac pacemaker cells [5] in human hearts is another example where synchronized motion is reported [6]. Among other evidences of synchronous behaviour in the natural world, one can additionally consider the chorusing of crickets and the metabolic synchrony in yeast cell suspension [7]. A large number of examples of synchronization in nature can be found in [8], and references therein.

The rich phenomenology constitutes another reason for the importance of these studies. Coupled dynamical systems have been shown to give rise to rather complex phenomena not previously observed, especially when

* Corresponding author.
E-mail address: h.nijmeijer@tue.nl (H. Nijmeijer).

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chaotic motion is considered as output of a single dynamical system (for example, see [9,10], for two significant reviews). If the common output of several synchronized dynamical systems is chaotic, there is evidence that as a result of a *blowout bifurcation* (see Definition 3.8 in [11]) a new type of intermittent behaviour, called *on–off intermittency* may appear at the onset of the threshold of attractivity of the synchronized state [12], as well as the presence of very complex domains of attraction, called *riddled basins*, when several attracting solutions exist [13]. These two new phenomena are peculiar to those systems with invariant subsets, as suitably coupled identical dynamical systems are. A more detailed explanation of the onset of attractivity and stability of a synchronized state in coupled chaotic systems can be found in [11,14].

The importance of synchronized motion does not lie only in those situations in which synchronization can be found, but also where synchronous motion can be induced to ensure the proper functioning of a particular device [15,16]. Consider, for example, active integrated antennas [17] that can be built as arrays of multiple coupled oscillators to generate circular polarization [18]. In robotics, the problem of synchronization is usually referred to as coordination, or cooperation [19,20]. The problem appears when two or more robot-manipulators have to operate synchronously, especially in situations when some of them operate in hazardous environment, while others (that serve as reference) may be guided by human operation. Another interesting problem is to study (and control) spatiotemporal patterns in an ensemble of coupled systems. This problem is of interest in connection with many applications in communication engineering [21].

Synchronization is therefore important, so it is especially important to develop criteria that guarantee its *asymptotic* stability, if applications are sought. In this paper we consider networks of identical systems coupled through diffusion, and we give conditions that guarantee asymptotic stability of a particular invariant manifold (a synchronous state) of a given network. Apart from the direct Lyapunov method, if only a local result suffices, the asymptotic stability of the synchronization manifold can be verified via *uniform* asymptotic stability of a linear time-varying system which expresses the dynamics transverse to the synchronization manifold (see Corollary 4.4 in [28]). In [11,28], it was proven that if all transverse Lyapunov exponents are negative, then the attractor lying in the synchronization manifold is transversely asymptotically stable. There is evidence [29] (though not rigorously proven yet) that in this attractor the transverse Lyapunov exponents achieve their supremum on periodic orbits of *low period* that gives a hope to compute the synchronization threshold exactly via computer simulation; for an application of this periodic orbit threshold theory, for example, see [30].

Synchronous motion is most often understood as the equality of corresponding variables of two identical systems. In other words, the trajectories of two (or more) identical systems will follow, after some transient, the same path in time. This situation is not, of course, the only commonly understood situation of synchronization. Other different relationships between coupled systems can be considered synchronous. More generally (and this is the viewpoint taken in [33] to draw a suitable definition of "synchronous behaviour" in a coordinate-independent way), synchronization with respect to some functional happens when this functional has always zero value, for all trajectories of the dynamical system on the synchronous state.

We study the existence and stability of linear invariant manifolds of a network of coupled systems, defined as the equality of some outputs of some systems only, a condition that is nowadays popular in the physical literature as *partial synchronization*. Considering that a network can be formed by an arbitrary number of units, the number of different existing linear invariant manifolds may be large. Therefore, several aspects should be taken into consideration for careful examination.

The first aspect is the *classification* of linear invariant manifolds. As previously mentioned, coupled dynamical systems can show a very rich phenomenology, so it is meaningful to find systematic ways to classify different invariant sets. Zhang et al. [22] reported the result of their studies of a chain of identical Rössler systems, showing that there are several possible synchronous states with different types of correspondences between the variables of different systems.

The second aspect to keep in mind is the possibility of *inclusion* of different invariant manifolds, an aspect studied in [23–25]. Briefly, it is possible that different synchronous states between coupled oscillators will appear by increasing some specified bifurcation parameter. This possibility defines a hierarchy, that arranges different invariant manifolds in a particular order.

Symmetry considerations are helpful to classify several invariant sets, and a possible hierarchy to accommodate them. The symmetry generated by the coupling only has been termed *global*, to distinguish it from the additional symmetries brought upon by the dynamical systems modelling each unit, that has been termed *internal*. This terminology has been introduced in [26] where it is studied how these two groups of symmetries interact.

In this paper we exploit symmetry under permutation of a given network of dynamical systems coupled through diffusion in order to classify some linear invariant manifolds and investigate their stability. More specifically, we see that to any specific symmetry it is associated a linear invariant manifold, and we show how to construct a Lyapunov function to determine its stability, from the same specific symmetry. Therefore, under the conditions formulated in this work, stability in the network descends from its topology. The results we derive on the existence and stability of invariant manifolds hold regardless of the dynamics that takes place in these manifolds.

The paper is organized as follows. The problem statement is explained in Section 2, where the network dynamics is outlined. In Section 3 the association between symmetry and the linear invariant manifolds of the network is discussed. Section 4 begins with some background material from control theory, after which we propose a proof of asymptotic stability of a compact subset of a specified linear invariant manifold. Section 5 shows some relevant examples in connection to the theory.

Throughout the paper we use the following notations. I_k denotes the $k \times k$ identity matrix. The Euclidean norm in $\mathbb{R}^n$ is denoted simply as $|\cdot|$, $|x|^2 = x^\top x$, where $\top$ defines transposition. The notation $\mathrm{col}(x_1, \dots, x_n)$ stands for the column vector composed of the elements $x_1, \dots, x_n$. This notation will also be used in case where the components x_i are vectors again. A function $V : X \to \mathbb{R}_+$ defined on a subset X of $\mathbb{R}^n$, $0 \in X$ is *positive definite* if $V(x) > 0$ for all $x \in X \setminus \{0\}$ and $V(0) = 0$. It is *radially unbounded* (if $X = \mathbb{R}^n$) or *proper* if $V(x) \to \infty$ as $|x| \to \infty$. If a quadratic form $x^\top Px$ with a symmetric matrix $P = P^\top$ is positive definite then the matrix P is called positive definite. For positive definite matrices we use the notation $P > 0$; moreover $P > Q$ means that the matrix $P - Q$ is positive definite. For matrices A and B the notation $A \otimes B$ (the Kronecker product) stands for the matrix composed of submatrices $A_{ij}B$, i.e.

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{pmatrix},$$

where A_{ij}, $i, j = 1, \dots, n$, stands for the ijth entry of the $n \times n$ matrix A.

2. Problem statement

The subject of our research is the existence and stability of partial synchronization regimes in diffusive networks. To make the problem statement clearer, we start our discussion by introducing the concept of diffusive network. In 1976, Smale [31] proposed a model of two interacting cells based on two identical coupled oscillators, and noticed that diffusion, rather counterintuitively, does not necessarily smooth out differences between the two systems' outputs, giving the example of two stable systems that can display oscillations when connected via diffusive coupling. Taking inspiration from Smale's previous research, a *diffusive cellular network* describes a network composed of identical dynamical systems coupled through diffusive coupling that cannot be decomposed into two or more disconnected smaller networks.

To put these statements into a more mathematical description, let us consider k identical systems of the form

$$\dot{x}_j = f(x_j) + Bu_j, \qquad y_j = Cx_j, \tag{1}$$

where f is a smooth vector field, $j = 1, \dots, k$, $x_j(t) \in \mathbb{R}^n$ is the state of the jth system, $u_j(t) \in \mathbb{R}^m$ and $y_j(t) \in \mathbb{R}^m$ are, respectively, the input and the output of the jth system, and B, C are constant matrices of appropriate dimension. This representation of the dynamics of the elements of the network is most common in control theory, and it underlines that a description of a dynamical system is not complete unless inputs and outputs are specified. It also reflects the idea that a dynamical system can be viewed in a model-independent representation, through its input–output characteristics. For instance, interaction among cells, in the example quoted by Smale, can be viewed as a static relationship between inputs and outputs. In this representation we can say that the k systems (1) are *diffusively coupled* if the matrix CB is similar to a positive definite matrix, and the k systems are interconnected through mutual linear output coupling

$$u_j = -\gamma_{j1}(y_j - y_1) - \gamma_{j2}(y_j - y_2) - \cdots - \gamma_{jk}(y_j - y_k), \tag{2}$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$ are constants such that $\sum_{j\neq i}^{k} \gamma_{ji} > 0$ for all $i = 1, \dots, k$. With no loss of generality we assume in the sequel that CB *is* a positive definite matrix. Some results we present in this paper certainly hold for other types of coupling as well, but this special form of coupling, apart from being very naturally linked to the interaction of some real-life systems, allows us to obtain analytical results concerning the stability of some solutions of the whole network.

Define the symmetric $k \times k$ matrix Γ as

$$\Gamma = \begin{pmatrix} \sum_{i=2}^{k} \gamma_{1i} & -\gamma_{12} & \cdots & -\gamma_{1k} \\ -\gamma_{21} & \sum_{i=1, i\neq 2}^{k} \gamma_{2i} & \cdots & -\gamma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & \cdots & \sum_{i=1}^{k-1} \gamma_{ki} \end{pmatrix}, \tag{3}$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$ and all row sums are zero. The matrix Γ is symmetric and therefore all its eigenvalues are real. With the definition (3), the collection of k systems (1) with the feedback (2) can be rewritten in the more compact form

$$\dot{x} = F(x) + (I_k \otimes B)u, \qquad y = (I_k \otimes C)x, \tag{4}$$

with the feedback given by

$$u = -(\Gamma \otimes I_m)y, \tag{5}$$

where we denoted $x = \mathrm{col}(x_1, \dots, x_k)$, $F(x) = \mathrm{col}(f(x_1), \dots, f(x_k)) \in \mathbb{R}^{kn}$, $y = \mathrm{col}(y_1, \dots, y_k)$, and $u = \mathrm{col}(u_1, \dots, u_k) \in \mathbb{R}^{km}$. The matrix Γ in (3) displays some interesting properties: first, it is singular (all row sums are zero), so it has a zero eigenvalue and, according to Gerschgorin's theorem (for example, see [32]), it is positive semidefinite, that is, all its eigenvalues are nonnegative. Additionally, if zero is a simple eigenvalue of Γ, the network of the diffusively coupled systems cannot be divided into two or more disconnected networks.

The matrix Γ is a useful mathematical object in the study of solutions of (4) and (5), because it contains information on the topology of the network, with properties that hold independently of the particular dynamical system $f(\cdot)$ employed to model each of its elements. Consider, for instance

- its size (the number of coupled systems, hence the dimension k of Γ);
- its interconnections (the dimension of $\ker \Gamma$ is the number of disconnected networks);
- the local density of interconnections (the number of systems connected to a specific one, that is, how many nonzero elements in a specific row or column);
- the strength of the interconnections (the constants γ_{ij});
- last but not least, eventual symmetries the network may possess.

Dynamical systems constructed as in (1) and (2), specifically, using identical systems (the same $f(\cdot)$), with identical input and output forms (the same matrices B and C for all k systems), interconnected by diffusive coupling, allow solutions with equality of all (or some) of its states x_j. These situations can be described as fully and partially synchronized. We prefer not to dwell too much here on the suitability of a particular definition of synchronization, as reported in few research papers [33–35], but we simply define *full synchronization* the particular situation in which all states of all systems are identical (i.e. $x_i(t) = x_j(t),\ \forall t,\ \forall i, j = 1, \dots, k$), and with *partial synchronization* all situations in which the states of some systems are identical, but not all of them (i.e. $x_i(t) = x_j(t),\ \forall t$, for some i, j). These are situations in which the overall dynamics of the network takes place on a linear invariant manifold, specified by the equality of some outputs.

All main points have now been introduced in order to formulate a clear problem statement. Can we exploit symmetry in the network to identify its linear invariant manifolds, and benefit from a representation of the system as (1) and (2), and/or (4) and (5), typical for control purposes, in order to give conditions that *guarantee* stability of some selected partial (or the full) synchronized states?

3. Symmetries and invariant manifolds

If a given network possesses a certain symmetry, this symmetry must be present in the matrix Γ. In particular, the network may contain some repeating patterns, when considering the arrangements of the constants γ_{ij}, hence the permutation of some elements will leave the network unchanged. The matricial representation of a permutation σ of the set $\{1, 2, \dots, k\}$ is a permutation matrix $\Pi \in \mathbb{R}^{k\times k}$. Briefly, if $\varepsilon_1, \dots, \varepsilon_k$ denote the columns of I_k, the permutation matrix Π associated with σ is the matrix obtained from I_k by permuting its columns under σ, that is, the columns of Π are $\varepsilon_{\sigma(1)}, \dots, \varepsilon_{\sigma(k)}$. If S_k is the set of all permutations of $\{1, 2, \dots, k\}$ it is possible to prove that the set of all $k \times k$ permutation matrices forms a group that is isomorphic to S_k [36]. Permutation matrices are orthogonal, i.e. $\Pi^\top \Pi = I_k$, and they form a group with respect to the multiplication, so for any two permutation matrices Π_i, Π_j of the same size, $\Pi_i \Pi_j$ is a permutation matrix too.

Rewrite the dynamics of (4) and (5) in the closed loop form

$$\dot{x} = F(x) + Gx, \tag{6}$$

where $G = -(I_k \otimes B)(\Gamma \otimes I_m)(I_k \otimes C) \in \mathbb{R}^{kn\times kn}$, that can be simplified as $G = -\Gamma \otimes BC$. Let us recall here that given a dynamical system as (6), the linear manifold $\mathcal{A}_M = \{x \in \mathbb{R}^{kn} : Mx = 0\}$, with $M \in \mathbb{R}^{kn\times kn}$, is *invariant* if $M\dot{x} = 0$ whenever $Mx = 0$, that is, if at a certain time t_0 a trajectory is on the manifold, $x(t_0) \in \mathcal{A}_M$, then it will remain there for all time, $x(t) \in \mathcal{A}_M$ for all t. The problem can be summarized in the following terms: given G and $F(\cdot)$ find a solution M to

$$MF(x(t_0)) + MGx(t_0) = 0, \tag{7}$$

for all $x(t_0)$ for which $Mx(t_0) = 0$. There is no general solution to this, however, if these objects satisfy certain properties, it is possible to find a class of matrices M that solve (7). A natural way to do this is to exploit the symmetry of the network.

3.1. Global symmetries

In the representation (6), we can establish conditions to identify those permutations that leave a given network invariant. It can be easily derived that Π is a symmetry for the network (6) if

$$(\Pi \otimes I_n)G = G(\Pi \otimes I_n),$$

that is, if Π and Γ commute, $\Pi\Gamma = \Gamma\Pi$. For these symmetries, a first result follows directly from the properties of the permutation matrices. Let $\Sigma = \Pi \otimes I_n$ for simplicity, and assume that, at time t_0 it is $(I_{kn} - \Sigma)x(t_0) = 0$. Consider (6), and suppose that there is a permutation matrix Π commuting with Γ. Hence, also $\Sigma G = G\Sigma$, and since Π is a permutation matrix, it also follows that $\Sigma F(x) = F(\Sigma x)$. If we multiply both sides of (6) by $I_{kn} - \Sigma$, we obtain, at time t_0

$$(I_{kn} - \Sigma)\dot{x}(t_0) = F(x(t_0)) - F(\Sigma x(t_0)) + G(I_{kn} - \Sigma)x(t_0) = 0,$$

because we assumed $(I_{kn} - \Sigma)x(t_0) = 0$. Therefore, it is $(I_{kn} - \Sigma)x(t) = 0$ for all t, and we can reformulate this result as: *given a permutation matrix Π that commutes with Γ, the set*

$$\ker(I_{kn} - \Pi \otimes I_n), \tag{8}$$

is a linear invariant manifold for system (6).

Note that global symmetries have the property of leaving the system Eqs. (4) and (5) invariant. One can see that, applying the linear transformation $x \to \bar{x} = (\Pi \otimes I_n)x$ to the network, if x and y are, respectively, state and output of (4), then $\bar{x}$ and $\bar{y} = (\Pi \otimes I_m)y$ are state and output of the same system. Additionally, if Π and Γ commute, also $\bar{u} = -(\Gamma \otimes I_m)\bar{y}$, where $\bar{u} = (\Pi \otimes I_m)u$. Hence, the triple $(\bar{x}, \bar{y}, \bar{u})$ satisfies the same Eqs. (4) and (5), as the original triple (x, y, u).

3.1.1. Example 1: ring of four coupled oscillators

Consider the example of four coupled systems (1) and (2) in a ring, as shown schematically in Fig. 1. In this figure we have imposed the following symmetry in coupling constants: $\gamma_{12} = \gamma_{34} = K_0$, and $\gamma_{14} = \gamma_{23} = K_1$. The particular geometry of the coupling defines the following coupling matrix:

$$\Gamma = \begin{pmatrix} K_0 + K_1 & -K_0 & 0 & -K_1 \\ -K_0 & K_0 + K_1 & -K_1 & 0 \\ 0 & -K_1 & K_0 + K_1 & -K_0 \\ -K_1 & 0 & -K_0 & K_0 + K_1 \end{pmatrix}. \tag{9}$$

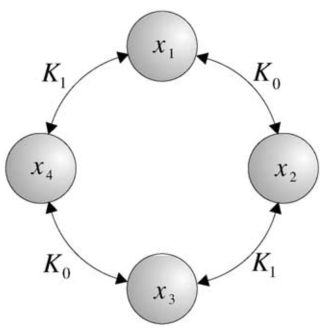

Fig. 1. A network of four coupled identical systems with symmetric coupling at the opposite sides.

The four permutation matrices for which $\Pi\Gamma = \Gamma\Pi$ are

$$\Pi_1 = \begin{pmatrix} E & O \\ O & E \end{pmatrix}, \qquad \Pi_2 = \begin{pmatrix} O & I_2 \\ I_2 & O \end{pmatrix}, \qquad \Pi_3 = \begin{pmatrix} O & E \\ E & O \end{pmatrix}, \qquad \Pi_4 = I_4, \tag{10}$$

where we denoted

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and O is the 2×2 zero matrix. Let us analyse what the action of these matrices Π is. The action of Π_1 is to switch *simultaneously* x_1 with x_2 and x_3 with x_4. One can easily notice from Fig. 1 that this operation leaves the network unchanged, with respect to its connections. Similar actions are brought upon by Π_2 and Π_3, while Π_4 is the identity, that leaves everything unchanged.

From our previous statements, we derive that the linear invariant manifolds associated with Π_1, Π_2 and Π_3 are, respectively

$$\mathcal{A}_1 = \{x \in \mathbb{R}^{4n} : x_1 = x_2, x_3 = x_4\}, \qquad \mathcal{A}_2 = \{x \in \mathbb{R}^{4n} : x_1 = x_3, x_2 = x_4\},$$
$$\mathcal{A}_3 = \{x \in \mathbb{R}^{4n} : x_1 = x_4, x_2 = x_3\}. \tag{11}$$

The intersection of any two of these linear manifolds gives the linear manifold describing full synchronization (i.e. $x_1 = x_2 = x_3 = x_4$). These invariant manifolds (11) descend directly from the symmetries of Γ, hence they exist regardless of the particular form of $f(\cdot)$ chosen in (1).

3.2. *Internal symmetries*

Additional internal symmetries in the differential equations governing the dynamics of the elements of the network will lead to the existence of additional linear invariant manifolds. Consider one uncoupled element of the network, $\dot{x}_j = f(x_j)$, with initial condition $x_j(0)$, generating the particular solution $x_j(t)$. It is easy to see that if

$$Jf(x_j) = f(Jx_j), \tag{12}$$

with $J \in \mathbb{R}^{n\times n}$ constant matrix, then $Jx_j(t)$ is a solution as well, generated by the initial condition $Jx_j(0)$. This property of $f(\cdot)$ defines an additional symmetry to the network, that originates additional invariant manifolds. As in the last argument, we can formulate the following, more general, statement: *suppose there is a permutation matrix Π commuting with Γ, and an $n \times n$ constant matrix J satisfying* (12) *for $f(\cdot)$* in (1), *with J commuting with the $n \times n$ matrix BC. Then the set* $\ker(I_{kn} - \Pi \otimes J)$ *is a linear invariant manifold for system* (6). To prove this statement, let $\Sigma = \Pi \otimes J$ as before, for brevity, and assume $(I_{kn} - \Sigma)x(t_0) = 0$. This argument is the same used for the previous statement, it differs in just replacing I_n with J. If we multiply both sides of (6) by $I_{kn} - \Sigma$ we obtain

$$(I_{kn} - \Sigma)\dot{x}(t_0) = (I_{kn} - \Sigma)F(x(t_0)) + (I_{kn} - \Sigma)Gx(t_0) = F(x(t_0)) - F(\Sigma x(t_0)) + G(I_{kn} - \Sigma)x(t_0) = 0,$$

where we made use of $\Sigma F(x) = F(\Sigma x)$ and assumed that J commutes with BC. Therefore, if $(I_{kn} - \Sigma)x(t_0) = 0$, it will be $(I_{kn} - \Sigma)x(t) = 0$ for all t. A popular case is represented by the *anti-phase synchronization* that may take place in systems whose differential equations are odd functions of the state vector, that is, $f(-x_j) = -f(x_j)$, or $J = -I_n$.

Therefore, for any element Π from the group of global symmetries and for any element J, that commutes with BC, from the group of internal symmetries, the set

$$\ker(I_{kn} - \Pi \otimes J), \tag{13}$$

is a linear invariant manifold for system (6). The representation (13) reflects how two groups of symmetries contribute to the existence of invariant manifolds of a diffusive network.

If there is a positive integer L such that $J^L = I_n$, the J matrices form a finite group, hence the number of invariant manifolds generated by Π's and J's is finite.

3.2.1. *Example 2: four coupled Lorenz systems*

Consider the network introduced in Example 1, and suppose the dynamics of each element of (1) is described by the Lorenz system, with $x_j = (x_{j,1}, x_{j,2}, x_{j,3})^\top \in \mathbb{R}^3$

$$\dot{x}_{j,1} = \sigma(x_{j,2} - x_{j,1}) + u_j, \qquad \dot{x}_{j,2} = rx_{j,1} - x_{j,2} - x_{j,1}x_{j,3}, \qquad \dot{x}_{j,3} = -bx_{j,3} + x_{j,1}x_{j,2}, \tag{14}$$

with output $y_j = x_{j,1}$. For Eq. (14), it is easy to see that if $(x_{j,1}(t), x_{j,2}(t), x_{j,3}(t))$ is a solution, $(-x_{j,1}(t), -x_{j,2}(t), x_{j,3}(t))$ is a solution too, so the matrix

$$J = \mathrm{diag}(\,-1 \quad -1 \quad 1\,), \tag{15}$$

is an internal symmetry for (14). In this example, $B = (\,1 \quad 0 \quad 0\,)^\top$ and $C = (\,1 \quad 0 \quad 0\,)$, and J commutes with BC. The three invariant manifolds given in (11) are associated with the matrices (10) as global symmetries, and (in absence of any information) I_3 as internal symmetry. In addition to (11), the linear invariant manifolds, defined as $\mathcal{A}'_i = \ker(I_{12} - \Pi_i \otimes J)$, associated with Π_i's in (10), and J in (15), are

$$\mathcal{A}'_1 = \{x \in \mathbb{R}^{12} : Jx_1 = x_2, Jx_3 = x_4\}, \qquad \mathcal{A}'_2 = \{x \in \mathbb{R}^{12} : Jx_1 = x_3, Jx_2 = x_4\},$$
$$\mathcal{A}'_3 = \{x \in \mathbb{R}^{12} : Jx_1 = x_4, Jx_2 = x_3\},$$

while the invariant manifold associated with $\Pi_4 = I_4$ is

$$\mathcal{A}'_4 = \{x \in \mathbb{R}^{12} : x_{j,1} = x_{j,2} = 0,\ j = 1, \dots, 4\},$$

and arbitrary $x_{j,3}$, $j = 1, \dots, 4$.

Let us restrict for now with global symmetries only, that is, model-independent symmetries associated with Γ and represented by the matrices Π. For a given network we can "quantify" its symmetry simply by counting how many permutation matrices Π commute with the given Γ. Once the network is given, it is possible, in principle, to find out precisely how many invariant manifolds (associated with permutation symmetry) exist. Symmetry arguments solve the problem of classification of invariant manifolds, and they also give insights on the inclusion problem as well.

4. Stability analysis

In this section we present some important properties of certain classes of control systems. They are properties widely invoked in control theory, and the advantage is in the methodology. Consider systems of the form

$$\dot{x} = f(x, u), \qquad y = h(x), \tag{16}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^m$ is the output, $f(0, 0) = 0$ and f, h are smooth enough to ensure existence and uniqueness of solutions. If the control goal is to find a feedback that stabilizes a solution of system (16), we can find a *class of possible feedbacks* which comply with the specified control goal (e.g. stabilize the system (16)). Any feedback u from this class is a solution to the control problem.

4.1. Passivity and semipassivity

We illustrate this approach by a simple example. Let the control goal be the stabilization of (16) to the origin. Suppose it is possible to find a scalar nonnegative function V defined on $\mathbb{R}^n$, for which $V(0) = 0$, and whose derivative satisfies, along the solutions of (16), the inequality

$$\dot{V}(x, u) = \frac{\partial V(x)}{\partial x} f(x, u) \leq y^\top u. \tag{17}$$

The system (16) is then referred to as a *passive system* (e.g., see [37]), the function V is called the *storage function* and the inequality (17) is called *dissipation inequality*. This inequality immediately defines a *class* of stabilizing feedbacks for the closed loop system (16). Additionally, if we search for the stabilizing feedback in the form $u = \phi(y)$, to prove that a given u belongs to the class of stabilizing feedbacks for (16), we only need to verify the inequality

$$y^\top \phi(y) \leq 0, \tag{18}$$

provided V is positive definite. To prove this statement one can simply consider the storage function V as a Lyapunov function candidate.

It is important to note that for a passive system with a positive definite storage function the dynamics corresponding to the constraint $y \equiv 0$ (that is commonly known as *zero dynamics*) is Lyapunov stable (see (17)). If the zero dynamics is asymptotically stable, systems of this sort are known as *minimum phase*, and it is an important class of systems for application design problems, since they possess some sort of internal stability.

Next, let us consider a different situation. Suppose that the dissipation inequality (17) is satisfied only for x lying outside some ball (or compact set). More rigorously, it can be represented as the following inequality

$$\dot{V}(x, u) \leq y^\top u - H(x), \tag{19}$$

where the function $H : \mathbb{R}^n \to \mathbb{R}$ is nonnegative outside some ball:

$$\exists \rho > 0, \quad \forall |x| \geq \rho \Rightarrow H(x) \geq \varrho(|x|), \tag{20}$$

for some continuous nonnegative function ϱ defined for $|x| \geq \rho$. In this case the system (16) is called *semipassive*. This notion was introduced in [38], while in [39] an equivalent notion was called *quasipassivity*. If the function H is positive outside some ball, i.e. (20) holds for some continuous positive function ϱ, then the system (16) is said to be *strictly semipassive*. In brief, a semipassive system behaves like a passive system for sufficiently large $|x|$.

It is important to observe that the dissipation inequality (19) can be rewritten in an equivalent way as follows:

$$\frac{\partial V}{\partial x} f(x) \leq -H(x), \qquad \frac{\partial V}{\partial x} B = x^\top C^\top.$$

In direct analogy with passive systems, from the inequality (19) one can specify the class of feedbacks which make the semipassive closed loop system (16) *ultimately bounded*, i.e. regardless of the initial conditions, all solutions of the closed loop system approach a compact set in a finite time and this compact set does not depend on the initial conditions.

Indeed, suppose that the system (16) is strictly semipassive and the storage function V satisfying the dissipation inequality (19) is *radially unbounded*, that is $V(x) \to \infty$ when $|x| \to \infty$, then any feedback $u = \phi(y)$ satisfying the inequality (18) makes the closed loop system ultimately bounded. This statement can be proven just by considering the storage function V as a Lyapunov function candidate.

This is an additional benefit of the representation of diffusive network equations as input–output systems. Under strict semipassivity, boundedness of solutions depends only on input–output relations of systems (1) and not on the particular interconnections within the network. The particular constants appearing in the feedback u are not crucial, as long as u satisfies inequality (19). For example, consider the input written in compact form $u = -(\Gamma \otimes I_n)y$ introduced in Section 3. $\Gamma \geq 0$, therefore u is a feedback that ensures boundedness of solutions of the network.

More precisely, *if the systems* (1) *are strictly semipassive with radially unbounded storage functions then all solutions of the coupled system* (1) *and* (2) *exist for all* $t \geq 0$ *and are ultimately bounded*. The technical proof of this statement and more general related results can be found in [38].

4.2. Convergent systems

Consider a dynamical system of the form

$$\dot{z} = q(z, w(t)), \tag{21}$$

with $z \in \mathbb{R}^l$, driven by the external signal $w(t)$ taking values from some compact set. This system is said to be *convergent* [40] if for any bounded signal $w(t)$ defined on the whole time interval $(-\infty, +\infty)$ there is a unique bounded, globally asymptotically stable solution $\bar{z}(t)$ defined on the same interval $(-\infty, +\infty)$, from which it follows

$$\lim_{t\to\infty} |z(t) - \bar{z}(t)| = 0, \tag{22}$$

for all initial conditions. In systems of this type the limit mode of them is solely determined by the external excitation $w(t)$, not by the initial conditions of z. From the existence of a unique mode $\bar{z}(t)$, it obviously descend that two identical copies of convergent system z_1 and z_2 (21) must synchronize, that is, if (22) holds

$$\lim_{t\to\infty} |z_1(t) - z_2(t)| = 0,$$

holds as well. Convergence is then closely related to synchronization, hence it is important to find conditions ensuring it. Recently, an importance of the concept of convergent systems was recognized in control community with a potential application to observer design. In [41] a bit stronger notion was called *incremental global asymptotic stability* (δGAS); therein the necessary and sufficient conditions for δGAS were formulated in terms of the existence of Lyapunov functions. We present here a slight modification of a sufficient condition obtained by Demidovich [40]: *if there is a positive definite symmetric* $l \times l$ matrix P *such that all eigenvalues* $\lambda_i(Q)$ *of the symmetric matrix*

$$Q(z, w) = \frac{1}{2}\left[P\left(\frac{\partial q}{\partial z}(z, w)\right) + \left(\frac{\partial q}{\partial z}(z, w)\right)^{\top} P\right], \tag{23}$$

are negative and separated from zero, i.e. there is $\delta > 0$ *such that*

$$\lambda_i(Q) \leq -\delta < 0, \tag{24}$$

with $i = 1, \dots, l$ *for all* $z, w \in \mathbb{R}^l$, *then system* (21) *is convergent, and there exist a quadratic function* $W(\zeta) = \zeta^{\top} P \zeta$ *satisfying the inequality*

$$\frac{\partial W(z_1 - z_2)}{\partial \zeta}(q(z_1, w) - q(z_2, w)) \leq -\alpha |z_1 - z_2|^2, \tag{25}$$

for some $\alpha > 0$. This condition is a slight modification of the Demidovich theorem on convergent systems in the case $P = I_l$.

4.2.1. Example 3: input–output properties of Lorenz system

Consider the Lorenz system (14)

$$\dot{x}_{j,1} = \sigma(x_{j,2} - x_{j,1}) + u_j, \qquad \dot{x}_{j,2} = rx_{j,1} - x_{j,2} - x_{j,1}x_{j,3}, \qquad \dot{x}_{j,3} = -bx_{j,3} + x_{j,1}x_{j,2},$$

with $\sigma, r, b > 0$. Lorenz system with input u_j and output $y_j = x_{j,1}$ is strictly semipassive. To prove this statement, consider the following storage function candidate:

$$V(x_{j,1}, x_{j,2}, x_{j,3}) = \tfrac{1}{2}((x_{j,1})^2 + (x_{j,2})^2 + (x_{j,3} - \sigma - r)^2).$$

Calculating the derivative of this function along the solutions of the system (14) yields

$$\dot{V}(x_{j,1}, x_{j,2}, x_{j,3}, u) = x_{j,1}u - H(x_{j,1}, x_{j,2}, x_{j,3}),$$

where

$$H(x_{j,1}, x_{j,2}, x_{j,3}) = \sigma(x_{j,1})^2 + (x_{j,2})^2 + b\left(x_{j,3} - \frac{\sigma + r}{2}\right)^2 - b\frac{(\sigma + r)^2}{4}.$$

It is easy to see that the condition $H = 0$ determines the ellipsoid which lies inside the ball

$$\Xi = \{x_{j,1}, x_{j,2}, x_{j,3} : (x_{j,1})^2 + (x_{j,2})^2 + (x_{j,3} - \sigma - r)^2 \leq L^2(\sigma + r)^2\}, \tag{26}$$

in $\mathbb{R}^3$, with L derived in Appendix A, outside which H is strictly positive. This fact proves strict semipassivity of the system (14). Hence, in a diffusive network of any number of Lorenz systems (take for example two systems)

$$\begin{aligned}
&\dot{x}_{1,1} = \sigma(x_{1,2} - x_{1,1}) + u_1, \qquad \dot{x}_{1,2} = rx_{1,1} - x_{1,2} - x_{1,1}x_{1,3}, \qquad \dot{x}_{1,3} = -bx_{1,3} + x_{1,1}x_{1,2},\\
&\dot{x}_{2,1} = \sigma(x_{2,2} - x_{2,1}) + u_2, \qquad \dot{x}_{2,2} = rx_{2,1} - x_{2,2} - x_{2,1}x_{2,3}, \qquad \dot{x}_{2,3} = -bx_{2,3} + x_{2,1}x_{2,2},\\
&u_1 = -\gamma(y_1 - y_2), \quad u_2 = -\gamma(y_2 - y_1),
\end{aligned} \tag{27}$$

with outputs $y_1 = x_{1,1}$ and $y_2 = x_{2,1}$, all solutions are ultimately bounded.

Moreover, the same system, with input u_j and output $y_j = x_{j,1}$, is minimum phase. The dynamics corresponding to the constraint $y \equiv 0$ (*zero dynamics*) is simply

$$\dot{x}_{j,2} = -x_{j,2}, \qquad \dot{x}_{j,3} = -bx_{j,3},$$

which is asymptotically stable. Minimum phaseness and convergence are closely related. If we think of the output y_j as a driving input for the remaining part of Lorenz system, we have

$$\dot{x}_{j,2} = -x_{j,2} + ry_j - y_jx_{j,3}, \qquad \dot{x}_{j,3} = -bx_{j,3} + y_jx_{j,2},$$

which is convergent. Applying Demidovich's result we see that, using $P = I_2$, the matrix $Q(x_j, y_j)$ in (23) is

$$Q(x_j, y_j) = \mathrm{diag}(\,-1 \quad -b\,).$$

One final remark: after examination of these input–output properties of Lorenz system, it is not surprising that driving two identical Lorenz systems with the same $x_{j,1}$ signal leads to synchronization of the driven subsystems, even when their dynamics are chaotic [42,43]. The reader can easily check that using $x_{j,2}$ as drive works well too, and both cases are minimum phase, while using $x_{j,3}$ as input is not! Stability of response systems as referred in [43], with respective numerical evidence for different cases, is exactly convergent dynamics.

4.3. On global asymptotic stability of the partial synchronization manifolds

A permutation matrix Π commuting with Γ defines a linear invariant manifold of system (6), given by (8). This expression stands for a set of linear equations of the form

$$x_i - x_j = 0, \tag{28}$$

for some i and j that can be read off from the nonzero elements of the Π matrix under consideration. Therefore, we can identify a particular manifold associated with a particular matrix Π by the correspondent set $\mathcal{I}_\Pi$ of pairs i, j for which (28) holds.

In this section we are going to investigate asymptotic stability of partial synchronization as asymptotic stability of sets. Due to the converse Lyapunov theorem (e.g., see [27]) the asymptotic stability of a set $\mathcal{A}$ is equivalent to the existence of a scalar smooth nonnegative function V which is zero only on $\mathcal{A}$ and decays along the system trajectories, when not on $\mathcal{A}$. In order to find a Lyapunov function which proves stability of the partial synchronization manifold, one can seek a Lyapunov function candidate as a sum of two functions, the first one dependent on the input–output relations of the systems (1) and the second one dependent on the way the systems interact via coupling. The best way to carry this out is to find a *globally* defined coordinate change that allows us to exploit minimum phaseness.

Let us first differentiate y_j

$$\dot{y}_j = Cf(x_j) + CBu_j.$$

Then, choosing some $n-m$ coordinates z_j complementary to y_j it is possible to rewrite the system (1) in the form

$$\dot{z}_j = q(z_j, y_j), \qquad \dot{y}_j = a(z_j, y_j) + CBu_j, \tag{29}$$

where $z_j \in \mathbb{R}^{n-m}$, and q and a are some vector functions. It is important to emphasize that the coordinate change $x_j \mapsto \mathrm{col}(z_j, y_j)$ can be linear, if CB is nonsingular, and that, owing to the linear input–output relations, this transformation is globally defined. This transformation is explicitly computed in [44]. As the reader may expect, for more complicated input–output relations, this coordinate transformation may not be globally defined. Conditions on the existence of this *normal form* can be found in [37,45]. In the equation for z_j in (29), y_j acts as a forcing input, hence we can apply properties of convergent systems, if the matrix $Q(z, w)$ defined for q in (29) has negative eigenvalues, separated from zero.

The purpose of this section is to prove the following theorem.

Theorem 1. *Let λ' be the minimal eigenvalue of Γ under restriction that the eigenvectors of Γ are taken from the set* $\mathrm{range}(I_k - \Pi)$. *Suppose that*:

1. *Each free system (1) is strictly semipassive with respect to the input u_j and output y_j with a radially unbounded storage function.*
2. *There exists a positive definite matrix P such that inequality (24) holds with some $\delta > 0$ for the matrix Q defined as in (23) for q as in (29).*

Then for all positive semidefinite matrices Γ as in (3) all solutions of the diffusive cellular network (4) and (5) are ultimately bounded and there exists a positive $\bar{\lambda}$ such that if $\lambda' > \bar{\lambda}$ the set $\ker(I_{kn} - \Pi \otimes I_n)$ *contains a globally asymptotically stable compact subset.*

Let us clarify the conditions imposed in the theorem. The first assumption on strict semipassivity ensures the ultimate boundedness of the solutions of the diffusive network. The second assumption requires some sort of stability of some internal dynamics of the system (29). It guarantees that the z-dynamics of system (29) is convergent with y_j as an external signal. This assumption is quite restrictive and is not valid, for example, for Hamiltonian systems, which are conservative. According to the Demidovich result previously mentioned, this assumption implies the existence of a quadratic function $W(\zeta)$ satisfying the inequality (25). From this inequality one can immediately see that, if $q(0,0)=0$, the zero dynamics

$$\dot{z} = q(z, 0), \tag{30}$$

is globally asymptotically stable at the origin. Moreover, the inequality (25) implies that the system (30) has a globally asymptotically stable equilibrium, so the assumption $q(0,0) = 0$ can be always satisfied by means of appropriate coordinate change. Thus the second assumption necessarily implies that the system (29) is *minimum phase*. Recalling Example 3 about the Lorenz system, it means that the signal y_j acting as an input on z_j can make the dynamics z_j convergent. It is worth mentioning, however, that stability of the zero dynamics does not necessarily imply the stability of the free system ($u \equiv 0$). As one can see from the Example 3, the zero dynamics can be asymptotically stable (i.e. the system is minimum phase), while the free system is oscillatory.

In the rest of this section we sketch the proof of Theorem 1. To make the presentation more transparent we omitted some standard technical details which can be found in similar proofs, of related results, presented in [38,46]. Our approach is inspired by the results on feedback-passive systems presented in [47]. In the proof we are mostly focused on the way to find the Lyapunov function guaranteeing stability of the partial synchronization mode. As we previously introduced the notation $y = \mathrm{col}(y_1, \dots, y_k)$, let us denote with $z \in \mathbb{R}^{km}$ the vector $\mathrm{col}(z_1, \dots, z_k)$. Since the derivative of z-variables in (29) does not depend on the coupling, while the derivative of y-variables does, we can search for a Lyapunov function in the form

$$V(z, y) = V_1(z) + V_2(y).$$

There is not a unique way of determine which Lyapunov function is best to investigate stability, although control theory can give guidelines to determine whether a Lyapunov function with a particular form exists. The existence of a Lyapunov function in the form above, structured as a sum of two independent parts is ensured by an important algebraic result known as *frequency theorem*, or *Kalman–Yakubovich–Popov lemma*. This theorem is one of the cornerstones of control theory, since it offers the necessary and sufficient conditions of the solvability of the Lyapunov and Riccati equations. However, further analysis of this theorem is beyond the scope of this paper, therefore we forward the interested reader to [48] for a review. The function V_1 expresses some internal stability properties of each subsystem, which will come from the input–output properties of (1) and (2), while the function V_2 shows how the coupling contributes to the stability of the partial invariant manifolds, where we will explicitly make use of the symmetries of the network, in the representation we examined.

Let us start with the function V_1. Suppose that there is a positive definite radially unbounded function $W(\zeta)$ defined on $\mathbb{R}^{n-m}$ which satisfies the partial differential inequality (25) for all $z_i, z_j \in \mathbb{R}^{n-m}$, $w \in \mathbb{R}^m$. Then we construct the function V_1 as

$$V_1(z) = \sum_{(i,j)\in\mathcal{I}_\Pi} W(z_i - z_j).$$

Along the solutions of the closed loop system, the derivative of $V_1(z)$ satisfies

$$\begin{aligned}\dot{V}_1(z, y) &= \sum_{(i,j)\in\mathcal{I}_\Pi} \frac{\partial W(z_i - z_j)}{\partial \zeta}(q(z_i, y_i) - q(z_j, y_j)) \\ &\le -\alpha \sum_{(i,j)\in\mathcal{I}_\Pi} |z_i - z_j|^2 + \sum_{(i,j)\in\mathcal{I}_\Pi} \frac{\partial W(z_i - z_j)}{\partial \zeta}(q(z_j, y_i) - q(z_j, y_j)).\end{aligned}$$

The next step is to find the second part of the Lyapunov function, the function V_2. It is clear that if $x \in \ker(I_{kn} - \Pi \otimes I_n)$ then necessarily $y \in \ker(I_{km} - \Pi \otimes I_m)$. So, on this invariant manifold, the quantity $\xi(y) = (I_{km} - \Pi \otimes I_m)y$ is identically zero. We can therefore construct the function V_2 as

$$V_2(y) = \frac{1}{2}|\xi(y)|^2 = \frac{1}{2} y^\top (I_{km} - \Pi \otimes I_m)^\top (I_{km} - \Pi \otimes I_m) y = \frac{1}{2} \sum_{(i,j)\in\mathcal{I}_\Pi} |y_i - y_j|^2,$$

that is positive definite with respect to ξ, and zero on the set $\ker(I_{km} - \Pi \otimes I_m)$. Differentiating the function V_2 gives

$$\dot{V}_2(z, y) = \sum_{j=1}^{k} \frac{\partial V_2(y_j)}{\partial y_j} a(z_j, y_j) - U(y),$$

where using commutativity of Π and Γ

$$U(y) = y^\top (I_{km} - \Pi \otimes I_m)^\top (\Gamma \otimes CB)(I_{km} - \Pi \otimes I_m) y.$$

Since $\Gamma \geq 0$ and $CB > 0$ it follows that $U(y) \geq 0$. Moreover, since $\ker \Gamma \subset \ker(I_k - \Pi)$ (the kernel of Γ is a set of vectors with equal coordinates, as we have assumed that the defect of Γ is one) it follows that there is a positive constant κ such that the following inequality holds

$$U(y) \geq \kappa y^\top (I_{km} - \Pi \otimes I_m)^\top (I_{km} - \Pi \otimes I_m) y.$$

The number κ can be easily estimated as $\kappa \geq \lambda' \beta$, where β is the minimal eigenvalue of the matrix CB and λ' is the minimal eigenvalue of Γ under restriction that the eigenvectors of Γ are taken from the set range$(I_k - \Pi)$.

We proceed now to evaluate the derivative of V. From the previous intermediate results it follows that

$$\dot{V}(z, y) \leq -\alpha \sum_{(i,j)\in\mathcal{I}_\Pi} |z_i - z_j|^2 - 2\lambda' \beta V_2(y) + \sum_{j=1}^{k} \frac{\partial V_2(y_j)}{\partial y_j} a(z_j, y_j) + \sum_{(i,j)\in\mathcal{I}_\Pi} \frac{\partial W(z_i - z_j)}{\partial \zeta} (q(z_j, y_i) - q(z_j, y_j)).$$

Note that for any compact set Ω there exist some positive numbers C_1, C_2, C_3 such that the following estimates are valid on Ω:

$$\left| \sum_{j=1}^{k} \frac{\partial V_2(y_j)}{\partial y_j} a(z_j, y_j) \right| = \left| \sum_{(i,j)\in\mathcal{I}_\Pi} (y_i - y_j)^\top (a(z_i, y_i) - a(z_j, y_j)) \right| \leq \left| \sum_{(i,j)\in\mathcal{I}_\Pi} (y_i - y_j)^\top (a(z_i, y_i) - a(z_i, y_j)) \right| + \left| \sum_{(i,j)\in\mathcal{I}_\Pi} (y_i - y_j)^\top (a(z_i, y_j) - a(z_j, y_j)) \right| \leq C_1 V_2(y) + C_2 \sum_{(i,j)\in\mathcal{I}_\Pi} |z_i - z_j||y_i - y_j|,$$

and

$$\left| \sum_{(i,j)\in\mathcal{I}_\Pi} \frac{\partial W(z_i - z_j)}{\partial \zeta} (q(z_j, y_i) - q(z_j, y_j)) \right| \leq C_3 \sum_{(i,j)\in\mathcal{I}_\Pi} |z_i - z_j| \cdot |y_i - y_j|.$$

Now we are going to use strict semipassivity of the systems forming the diffusive network. Recall that strict semipassivity implies ultimate boundedness of all the solutions, that is, all the solutions in a finite time approach some compact set Ω which can be chosen independently on Γ. On this compact set the derivative of V is a quadratic form with respect to $|z_i - z_j|$ and $|y_i - y_j|$. It is clear that, if the value of λ' is large enough (that is, λ' is greater than a positive computable threshold $\bar{\lambda}$), due to inequality (25), the derivative of $V(z, y)$ is nonpositive on this set. After some algebra, an explicit formula for $\bar{\lambda}$ is derived as

$$\bar{\lambda} = \frac{1}{\beta}\left(\frac{C_1}{2} + \frac{(C_2 + C_3)^2}{4\alpha}\right). \tag{31}$$

This argument proves that the set $\ker(I_{kn} - \Pi \otimes I_n)$ contains a globally asymptotically stable compact subset for $\lambda' > \bar{\lambda}$.

4.4. Remarks

Let us explain the result of Theorem 1.

1. Suppose we parameterize the matrix Γ as

$$\Gamma = \mu\Gamma_0,$$

considering the positive parameter μ as a bifurcation parameter. Let λ_j, $j = 1, \dots, k$ be the eigenvalues of Γ arranged in increasing order:

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k.$$

Γ has only one zero eigenvalue, since we have assumed that the network cannot be divided into two or more disconnected networks. Previous results [46], based on similar assumptions to those given in Theorem 1, show that for sufficiently large μ the full synchronization occurs in the sense that the manifold $\mathcal{A} = \{x_j \in \mathbb{R}^n : x_1 = x_2 = \cdots = x_k\}$ contains a bounded closed globally asymptotically stable invariant subset. This situation occurs when the smallest nonzero eigenvalue λ_2 exceeds some computable threshold value.

The result formulated above allows to predict some additional bifurcations on the way to the full synchronization. Suppose that there is a permutation Π commuting with Γ that defines a partial synchronization manifold. For the given Π we can compute the number λ'. This number coincides with one of the nonzero eigenvalues of the matrix Γ, since $\ker\Gamma \subset \ker(I_k - \Pi)$ (the kernel of Γ is a set of vectors with equal coordinates, as we have assumed that the defect of Γ is one). Thus λ' takes value from the set $\{\lambda_2, \dots, \lambda_k\}$. It can happen that λ' exceeds a synchronization threshold, while λ_2 does not. In this case a partial synchronization corresponding to the symmetry Π occurs.

Writing down all the admissible permutation matrices Π_i, $i = 1, \dots, N$ (that define the sets $\mathcal{A}_i$) for the matrix Γ, and computing the corresponding values of ${\lambda'}_i$, $i = 1, \dots, N$, one can predict the possible bifurcations occurring when μ is increasing from zero to infinity. Although the number N of permutation matrices can be large, the route towards full synchrony will show only a limited number of bifurcations, in the following sense. Consider the parametrization $\Gamma = \mu\Gamma_0$. As μ increases, suppose that a first invariant set $\mathcal{A}_1$ contains a globally asymptotically stable compact subset. Further increase of μ may result in an eigenvalue λ' corresponding to a different set $\mathcal{A}_2$ to meet the requirements set in the Theorem. Hence, the sets $\mathcal{A}_1$ and $\mathcal{A}_2$ will both contain globally asymptotically stable compact subsets, that is possible only if intersection $\mathcal{A}_1 \cap \mathcal{A}_2$ contains a globally asymptotically stable compact subset. From further increase of μ it may follow stability of subsequent intersections, so on until the set $\mathcal{A}_1 \cap \mathcal{A}_2 \cap \cdots \cap \mathcal{A}_l$ represents in fact full synchrony. If one considers a highly symmetric network, the number of permutation matrices that leave the network unchanged can be large. However, Theorem 1 predicts a hierarchy with no more than $k-1$ possible bifurcations towards full synchrony, since λ'_i is restricted to be equal to one of $k-1$ positive eigenvalues of the matrix Γ.

2. The proof as constructed before yields the estimate (31) of the threshold beyond which asymptotic stability of the compact subset, is guaranteed. Employing a different Lyapunov function may possibly result in an estimate that is less conservative than (31). A more general Lyapunov function can be chosen, for example, considering V_2 in the proof as

$$V_2(y) = \tfrac{1}{2}\xi(y)^\top P\xi(y), \tag{32}$$

for some positive definite symmetric matrix P. In this case, the sufficient condition for stability is given in terms of the eigenvalues of the matrix $(\Gamma P + P\Gamma)/2$ with all the other restrictions in the Theorem left intact. We illustrate this approach in a forthcoming example that also tests the conservativeness of the estimation.

3. Convergence (assumption 2) is a strict but crucial assumption, since global asymptotic stability is sought but, in case of nonconvergent dynamics, no rigorous results expressed in terms of the eigenvalues of a coupling matrix are available, and asymptotic stability can only be conjectured. Consider, for example, a conjecture on synchronization criteria recently proposed in [49], that can be stated as follows: consider two diffusively coupled networks with coupling matrices Γ' and Γ'', with equal smallest nonzero eigenvalues $\lambda'_2 = \lambda''_2$. Then, conditions on global full synchronization for both networks are equivalent. This conjecture, called Wu–Chua conjecture, is wrong (as also pointed out in [50]), and a specific numerical counterexample can be found in [46]. However, under assumption 2 imposed in Theorem 1, the so called Wu–Chua conjecture is true, and sufficient conditions for synchronization can be formulated in terms of the eigenvalue λ_2.

5. Discussion and examples

5.1. Two coupled Lorenz systems

Consider the Example 3 where two Lorenz systems (27)

$$\begin{aligned}
\dot x_{1,1} &= \sigma(x_{1,2}-x_{1,1})+u_1, & \dot x_{1,2} &= rx_{1,1}-x_{1,2}-x_{1,1}x_{1,3}, & \dot x_{1,3} &= -bx_{1,3}+x_{1,1}x_{1,2},\\
\dot x_{2,1} &= \sigma(x_{2,2}-x_{2,1})+u_2, & \dot x_{2,2} &= rx_{2,1}-x_{2,2}-x_{2,1}x_{2,3}, & \dot x_{2,3} &= -bx_{2,3}+x_{2,1}x_{2,2},
\end{aligned}$$

with outputs $y_1 = x_{1,1}$ and $y_2 = x_{2,1}$ are diffusively coupled with input terms

$$u_1 = -\gamma(y_1-y_2), \quad u_2 = -\gamma(y_2-y_1),$$

respectively. Example 3 shows that Lorenz system has input–output properties that makes it suitable for synchronization via diffusion. How conservative is the estimate (31) for the threshold $\bar\lambda$, beyond which a particular linear invariant manifold contains a globally asymptotically stable subset? Let us consider the stability of the set $\mathcal{A} = \{x_{1,i} = x_{2,i}\}$. In this specific example the transformation to normal form (29) is simply given by $y_j = x_{j,1}$ and $z_j = (x_{j,2}, x_{j,3})$. In the explanatory remarks for the theorem, we suggested that the Lyapunov function

$$V = \tfrac{1}{2}(p(x_{1,1}-x_{2,1})^2 + (x_{1,2}-x_{2,2})^2 + (x_{1,3}-x_{2,3})^2),$$

where p is a positive number, may lead to a less conservative threshold than that obtained via the Lyapunov function used in the proof of the theorem. The time derivative reads

$$\begin{aligned}\dot{V} &= -p(\sigma+2\gamma)e_1^2 + (r+p\sigma-x_{1,3})e_1e_2 + x_{1,2}e_1e_3 - e_2^2 - be_3^2\\ &= -p(\sigma+2\gamma)e_1^2 + (r+p\sigma-x_{2,3})e_1e_2 + x_{2,2}e_1e_3 - e_2^2 - be_3^2\\ &= -p(\sigma+2\gamma)e_1^2 + (r+p\sigma-z)e_1e_2 + ye_1e_3 - e_2^2 - be_3^2,\end{aligned} \tag{33}$$

where $z = \max_j(x_{j,3})$ and $y = x_{i,2}$ for that i, for which the previous maximum is achieved.

This expression is negative definite with respect to e_1, e_2, e_3 if the following inequality is satisfied:

$$p(\sigma+2\gamma) > \frac{1}{4b}(y^2(t) + b(r+p\sigma-z(t))^2). \tag{34}$$

According to Lemma 2 (see Appendix A) it follows that if $2\sigma > b$ then $z(t) \geq 0$. Under assumption that $b \geq 1$ we can derive from Lemma 2 that

$$\limsup_{t\to\infty}\frac{1}{b}(y^2(t) + b(r+p\sigma-z(t))^2) \leq L^2r^2 + p^2\sigma^2 + 2rp\sigma.$$

Therefore the stability condition is

$$2\gamma+\sigma > \frac{L^2r^2 + p^2\sigma^2 + 2rp\sigma}{4p}.$$

Minimizing the right hand side of this inequality with respect to p yields the following stability condition

$$\gamma > \frac{1}{2}\sigma(Mr-1), \qquad M = \frac{L+1}{2}.$$

Using this inequality and the values $\sigma = 10, r = 28, b = 8/3$ we obtain that $L^2 = 16/15$ (see Appendix A), $M = 1.0164$ and the synchronization is globally stable as soon as $\gamma > 137.296$. This estimate serves as an upper bound for the threshold of global asymptotic stability of full synchrony between the two Lorenz systems.

We can obtain a lower estimate of the synchronization threshold by considering that systems (27) also present the additional invariant manifold $\mathcal{A}' = \{x_{1,1} = -x_{2,1}, x_{1,2} = -x_{2,2}, x_{1,3} = x_{2,3}\}$ resulting from the internal symmetry (15) of the Lorenz system discussed in Example 3. This situation is schematically represented in Fig. 2. The picture shows manifolds $\mathcal{A}$ and $\mathcal{A}'$, and the invariant set Ω is the Lorenz attractor in $\mathcal{A}$. Dynamics on the manifold $\mathcal{A}'$ evolves according to

$$\dot{x}_{1,1} = \sigma(x_{1,2} - x_{1,1}) - 2\gamma x_{1,1}, \qquad \dot{x}_{1,2} = rx_{1,1} - x_{1,2} - x_{1,1}x_{1,3}, \qquad \dot{x}_{1,3} = -bx_{1,3} + x_{1,1}x_{1,2}. \tag{35}$$

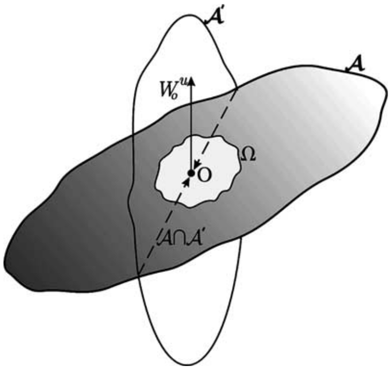

Fig. 2. Schematic representation of the intersection of invariant sets $\mathcal{A}$ and $\mathcal{A}'$.

The intersection $\mathcal{A}' \cap \mathcal{A}$ is the stable manifold of the origin, since any trajectory initialized in this intersection evolves according to

$$\dot{x}_{1,3} = -bx_{1,3}, \qquad \dot{x}_{2,3} = -bx_{2,3},$$

and converges to zero (since $b > 0$).

Consider the well known Lorenz chaotic attractor corresponding to parameters values $\sigma = 10$, $r = 28$, $b = 8/3$. This attractor contains the origin (e.g., see [51]). From this it follows that, if Ω is an attractor in $\mathbb{R}^6$, all unstable manifolds of the origin must lie in $\mathcal{A}$. From (35) one can see that, if

$$\gamma < \tfrac{1}{2}\sigma(r-1),$$

the origin of (35) is unstable so, if $\gamma < 135$, $\mathcal{A}'$ contains an unstable manifold of the origin. Since O is a hyperbolic point, for $\gamma < 135$ Ω cannot be an attractor in $\mathbb{R}^6$. The value $\gamma = 135$ can thus be considered as a lower estimate of the synchronization threshold. Therefore, the threshold for the coupling gain γ which ensures global asymptotic stability of the synchronized state lies in the interval

$$135 = \frac{\sigma(r-1)}{2} \le \bar{\gamma} \le \frac{\sigma(Mr-1)}{2} \approx 137.296. \tag{36}$$

Comparing these two bounds, we can conclude by emphasizing that a threshold that can be derived from the Lyapunov direct method is not so conservative (as often expected when employing Lyapunov functions). Computer simulations show that for γ greater than (about) 5.0, randomly chosen initial conditions converge towards Ω, but from our estimate it follows that the synchronous mode in this case can be attractive *at most* in the Milnor sense [11].

A recent result by Leonov (see Theorem 4 in [52]) quotes that the Lyapunov dimension of the Lorenz chaotic attractor is equal to the Lyapunov dimension of the origin. This is a strong indication that the stability of the origin of (35) is a sufficient condition for *local* asymptotic stability of a compact subset of $\mathcal{A}$ which includes the Lorenz attractor. The interval we report in (36) makes plausible the conjecture that stability of the origin of (35) is actually a sufficient condition for *global* asymptotic stability of the same set.

5.2. Four coupled oscillators

Recall Example 1 of a ring of four coupled oscillators, whose arrangement is schematically shown in Fig. 1. The eigenvalues of the matrix (9) are $\lambda_1 = 0$, $\lambda_2 = \min\{2K_0, 2K_1\}$, $\lambda_3 = \max\{2K_0, 2K_1\}$, $\lambda_4 = 2(K_0 + K_1)$. Hence, for the permutation described by Π_1 in (10) we have $\lambda' = 2K_0$. Similarly, $\lambda' = \min\{2K_0, 2K_1\}$ for Π_2 and $\lambda' = 2K_1$ for Π_3. According to Theorem 1, for large K_0 and small K_1 one can expect asymptotic stability of a subset of the set $\mathcal{A}_1$ in (11). For the permutation Π_3, for small K_0 and large K_1, leads to asymptotic stability of a subset of the set $\mathcal{A}_3$. Asymptotic stability of the full synchronization occurs for K_0 and K_1 both large enough. The subset of the set $\mathcal{A}_2$ is stable only as a stable intersection of $\mathcal{A}_1$ and $\mathcal{A}_3$, which describes full synchronization. Parameterizing Γ by one scalar parameter μ, it follows that the route to full synchrony can be either

$$\text{no synchrony} \to \mathcal{A}_1 \to \mathcal{A}_1 \cap \mathcal{A}_2 \quad (\text{full synchrony}),$$

or

$$\text{no synchrony} \to \mathcal{A}_2 \to \mathcal{A}_2 \cap \mathcal{A}_1 \quad (\text{full synchrony}),$$

depending on the ratio K_0/K_1. The diagram of asymptotic stability of the sets (11) in the (K_0, K_1) parameter space is schematically shown in Fig. 3. It is worth mentioning that this stability diagram is model-independent, that is, any dynamical system $f(\cdot)$ in (1) which satisfies the assumptions of Theorem 1 will yield this diagram, when coupled as in Example 1.

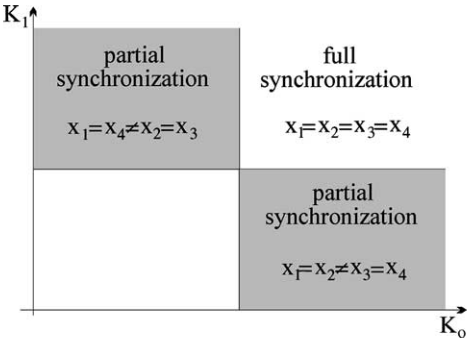

Fig. 3. Stability diagram of different invariant manifolds in the ring of four dynamical systems coupled as in Fig. 1.

5.3. Ring of identically coupled systems

Computer simulations show that, in a ring of k systems (1) and (2), with identical coupling constants (i.e. when all γ_{ij}'s in (2) and (3) are identical), there is no evidence of asymptotically stable partially synchronized modes of oscillation. This evidence may be intuitively correct, considering that this highly symmetric case would not allow for a specific pattern of synchronized motion to arise. However, it is indeed remarkable, since a high number of permutations of the elements of the network are allowed symmetries (i.e. they commute with Γ). In this case, if the common coupling constant is denoted by K, the Γ matrix is a *circulant* matrix

$$\Gamma = K\,\mathrm{circ}(2, -1, 0, \dots, 0, -1),$$

and its eigenvalues can be calculated analytically:

$$\lambda_j = 2K\left(1 - \cos\left(\frac{2\pi j}{k}\right)\right), \qquad j = 1, \dots, k.$$

Properties of circulant matrices can be found in [54], or in the classical Ref. [55]. Among many, it is quoted the property that any circulant matrix can be expressed in powers of the *shift* matrix θ

$$\theta = \mathrm{circ}(0, 1, 0, \dots, 0) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

In this case (note that $\theta^{k-1} = \theta^\top$ and $\theta^k = I_k$) the coupling matrix Γ can be represented in the following form

$$\Gamma = K(2I_k - \theta - \theta^\top).$$

It is not difficult to see that the eigenvalues of θ are the roots of the unity.

In conformity with numerical evidence, Theorem 1 does not predict the asymptotic stability of a partial synchronized mode, but it does not prove its instability either. Since we are looking for *global* asymptotic stability, the invariant manifold under consideration should be invariant under the transformation $x_i \mapsto x_{i+1}$. Therefore among all possible Π's commuting with Γ one has to consider only powers of θ.

According to Theorem 1 the global partial synchronization will happen if, for some integer s, there is a zero eigenvalue of $(1-\theta^s)$ with the eigenvector being the eigenvector for the smallest nonzero eigenvalue of Γ. In other words, there should be a s such that the matrix θ^s has eigenvalue one with the eigenvector being the eigenvector corresponding to the eigenvalue $\exp(2\pi i/k)$ of θ. Since the matrices θ and θ^s have the same eigenvectors, it can happen only in case $s=0$ modulo k. However, in this case $\Pi = I_k$ and $\ker(I_k-\Pi)$ is just the origin.

5.4. Application to systems with different input–output properties

Consider a diffusive network consisting of k coupled Chua's systems:

$$\dot{x}_{j,1} = \alpha(-x_{j,1}+x_{j,2}-\varphi(x_{j,1}))+u_j, \qquad \dot{x}_{j,2} = x_{j,1}-x_{j,2}+x_{j,3}, \qquad \dot{x}_{j,3} = -\beta x_{j,2}, \tag{37}$$

where $\varphi(\xi) = m_1\xi + 0.5(m_1-m_0)(|\xi+1|-|\xi-1|)$, coupled to each other via diffusive coupling

$$u_j = \gamma_{1j}(y_1-y_j)+\cdots+\gamma_{kj}(y_k-y_j),$$

with outputs $y_j = x_{j,1}$. This system is often investigated for parameter values (say, $\alpha = 9.0$, $\beta = 14.28$, $m_0 = -5/7$, $m_1 = -6/7$) which produce a known double-scroll chaotic attractor. System (37) with its input–output relationships, satisfies the convergence condition when $\beta > 0$, but is not semipassive. This statement can be demonstrated by computer simulation: together with a possible chaotic attractor the free system ($u_j \equiv 0$) can have unbounded trajectories. Hence we cannot apply Theorem 1 to Chua's system. Particularly, one cannot draw a conclusion that the partial synchronization manifold contains an asymptotically stable subset that is *compact* whose stability is *global*. However, since system (37) is convergent, predictions of Theorem 1 hold *locally*.

Next we consider a diffusive network of Rössler systems:

$$\dot{x}_j = -y_j - e^{z_j}, \qquad \dot{y}_j = x_j + ay_j + u_j, \qquad \dot{z}_j = ce^{-z_j} + x_j - b,$$

with output y_j, input u_j, $a, b, c > 0$ and the same type of coupling as in the previous example. This system is neither semipassive nor satisfies the convergence condition and, although it is possible to observe both full and partial synchronization in the diffusive network via computer simulation, the synchronization conditions in this case do not depend on only one eigenvalue of the coupling matrix (see [46]). For a study of partial synchronization in a ring of four Rössler systems, see [53].

6. Conclusion

In this paper we have demonstrated an approach, based on second Lyapunov method, to study partial synchronization in diffusively coupled (not necessarily locally) identical dynamical systems. In our approach we considered the diffusive coupling as a *feedback* that allowed us to borrow some useful control techniques. We have considered global symmetries in the network that can be represented by permutation matrices, in order to classify linear invariant manifolds different from the full synchronized state, with respect to each of the permutation matrices that commute with the given Γ matrix that represents the topology of the network. The additional advantage in using symmetry under permutation is that these permutation matrices can be used to construct Lyapunov function candidates for the stability test of the correspondent linear invariant manifolds.

Using this methodology we presented sufficient conditions guaranteeing global asymptotic stability of the partial synchronization regimes. The main limitation of our approach is that we require *minimum phaseness* of each system from the network. As a benefit, the synchronization test is relatively easy to check, at the same time, we believe that a study of the partial synchronization in an diffusive array of nonminimum phase systems may constitute a challenge for the future work.

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Appendix A. Bounds for trajectories of the Lorenz equations

Consider the Lorenz system

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = rx - y - xz, \qquad \dot{z} = -bz + xy, \tag{A.1}$$

with $\sigma, r, b > 0$. Denote

$$L = \begin{cases} 1, & b \leq 2 \\ \dfrac{b}{2\sqrt{b-1}}, & b \geq 2. \end{cases}$$

We are going now to present a well known result (e.g., see [56], Lemmas 5.6.1 and 5.6.2; or Example 2 in [57]).

Lemma 1. *For an arbitrary solution of the system* (A.1) $x(t), y(t), z(t)$, *it follows that*

$$\limsup_{t\to\infty}(y^2(t) + (z(t) - r)^2) \leq L^2 r^2, \tag{A.2}$$

and if, additionally, $2\sigma > b$ *then*

$$\liminf_{t\to\infty}\left(z(t) - \frac{x^2}{2\sigma}\right) \geq 0. \tag{A.3}$$

The proof of this Lemma is based on the following Lyapunov function

$$V(y, z) = \tfrac{1}{2}(y^2 + (z - r)^2),$$

for (A.2) and

$$W(x, z) = \sigma z - \frac{x^2}{2},$$

for the estimate (A.3). For details, see [56].

From this lemma it follows that for any p the following inequality

$$y^2 + (z - p\sigma - r)^2 \leq L^2 r^2 + p^2\sigma^2 + 2rp\sigma, \tag{A.4}$$

holds in the trapping region.

By analogy we can prove the following result.

Lemma 2. *Consider the diffusive network of k Lorenz systems* (14) *with outputs* $y_j = x_{j,1}$ *coupled via the coupling* (2). *If* $2\sigma > b$ *then for any solution* $x_{j,1}(t), x_{j,2}(t), x_{j,3}(t)$, $j = 1, \ldots, k$ *it follows that*

$$\liminf_{t\to\infty}\sum_{j=1}^{k} x_{j,3} \geq 0,$$

$$\limsup_{t\to\infty}(x_{j,2}^2 + (x_{j,3} - r)^2) \le L^2 r^2.$$

The proof of this Lemma follows the same lines as in the previous Lemma with the following Lyapunov functions

$$V_j = \tfrac{1}{2}(x_{j,2}^2 + (x_{j,3} - r)^2),$$

$$W = \sum_{j=1}^{k}\left(\sigma x_{j,3} - \frac{x_{j,1}^2}{2}\right),$$

making use of the fact that the matrix Γ is positive semidefinite.

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System Identification in Communication with Chaotic Systems

Henri Huijberts, *Member, IEEE*, Henk Nijmeijer, *Fellow, IEEE*, and Rob Willems

***Abstract*—Communication using chaotic systems is considered from a control point of view. It is shown that parameter identification methods may be effective in building reconstruction mechanisms, even when a synchronizing system is not available. Three worked examples show the potentials of the proposed method.**

***Index Terms*—Chaotic systems, communication, system identification.**

I. Introduction

In recent years there has been a tremendous interest in studying the behavior of complex systems. Two particularly interesting ideas which have emerged during this time are (chaos) synchronization and chaos control. Recent reviews on these subjects can be found in, for instance, two special issues devoted to the subject, see [12] and [18] [where, in fact, [12] is a follow up of an earlier special issue on the same subject of the same journal ([3])].

Synchronization and controlled synchronization of complex/chaotic systems is a topic that has become popular because of its possible use in communication, see [23] and [22]. Recently, in [19] (motivated by Ding and Ott [7], see also [17], and [25]) a control perspective on synchronization was given which enables us to resolve various synchronization problems as an observer problem. Thus, [19] illustrates, among other things, the benefits of incorporating control theoretic ideas in the study of communication using chaotic systems.

It is the purpose of the present paper to further illustrate these benefits. More specifically, we will look at some problems in communication using chaotic systems for which (standard) synchronization-based schemes may not yield the reconstruction of encoded messages, but that can be resolved using control theoretic ideas. The present paper is an expanded version of the paper [10].

Communication using chaotic systems has received quite some attention in the literature over the last few years (see, e.g., [5], [8], [13], and [31]). In communication using chaotic systems, one considers a transmitter system Σ_T of the form

$$\Sigma_T \begin{cases} \dot{x} = f(x, \lambda), & x \in \mathbb{R}^n \\ y = h(x), & y \in \mathbb{R} \end{cases} \tag{1}$$

where λ is a time-varying message satisfying $\lambda_{\min} \leq \lambda(t) \leq \lambda_{\max}$ $(\forall t)$ and $y \in \mathbb{R}$ is the transmitted signal (i.e., the coded message). It is assumed that the system Σ_T is chaotic (or at least sufficiently complex) for all constant λ satisfying $\lambda_{\min} \leq \lambda \leq \lambda_{\max}$. The task is now to build a receiver system Σ_R that reconstructs the message $\lambda(t)$ from the coded message $y(t)$.

The communication setting as described in (1) obviously is an idealization, since no effects like measurement noise, bandwidth limitations, modeling uncertainties, and the like are considered. Obviously, in a practical setup one has to cope with all such elements. However, this is not the aim of this paper. We will study an ideal communication system (1), and propose a means of reconstructing (slowly time-varying) signals λ from the chaotic transmitted signal y. A short discussion about the more practical issues mentioned will be given in the last section.

If one considers the problem of reconstruction of λ, as described above from a control theoretic point of view, two possible ways to approach the problem come to mind. The first approach is that of system inversion. Interpreting λ in (1) as an input and y as a measurement, one sees that (1) gives a mapping from λ to y. In the problem of system inversion, the task is to find an (asymptotic) inverse of this mapping. This approach will be pursued in future research (note, however, that this idea has also been addressed in [8]). The second approach, that will be followed in this paper and which in a sense was initiated for a particular case by Corron and Hahs in [5], is that of system identification. In system identification, the task is to estimate unknown (possibly slowly time-varying) parameters of a system, based on measurements taken from the system. For linear systems, system identification is well-established (for an overview see, e.g., [28]). In this paper, it will be shown on three examples that these identification methods may be helpful in communication using chaotic systems. Although all three examples concern chaotic and, thus, nonlinear systems, it is possible to use the standard linear identification algorithms once the systems are decomposed and/or transformed properly.

The organization of this paper is as follows. In Section II, we first introduce three examples that illustrate that parameter identification methods may be effective in communication with chaotic systems. In Section III, the essential identification background will be reviewed. In Sections IV–VI, a reconstruction mechanism for each of the three examples will be derived. In

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H. Huijberts is with the Department of Engineering, Queen Mary and Westfield College, London, U.K. (e-mail: H.J.C.Huijberts@qmw.ac.uk).

H. Nijmeijer is with the Faculty of Mechanical Engineering, Eindhoven University of Technology, Eindhoven, The Netherlands (e-mail: H.Nijmeijer@math.tue.nl).

R. Willems is with Statistics Netherlands, Heerlen, The Netherlands (e-mail: rwls@cbs.nl).

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the first example, it will be shown among others that the reconstruction scheme that was proposed by Corron and Hahs in [5] fits well in the identification-based approach to communication. In the last two examples, we will see that the existence of a synchronizing subsystem is not necessary for the existence of a reconstruction mechanism. Rather, one will typically have that (partial) synchronization occurs *after* reconstruction. In Section VI, classes of systems to which identification-based reconstruction schemes may be applied will be indicated. In Section VII, conclusions and a discussion of the proposed schemes will be given.

II. Parameter Identification Methods

In this section, we briefly introduce the so-called equation error identifier that may be used to estimate unknown parameters for linear time-invariant systems.

At first sight, it may seem somewhat strange that parameter identification methods for linear systems may be used for building reconstruction mechanisms in communication with chaotic (and thus nonlinear) systems. Therefore, we will first look at three examples illustrating that indeed linear parameter identification methods may be useful in the design of a reconstruction mechanism. After having introduced these examples, we will review the essential identification background.

Example 1: Consider the following set up for communication using chaotic waveforms that was proposed by Corron and Hahs in [5]. The transmitter is a three-dimensional (3-D) system Σ_T of the form

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3) + g(x_1, x_2, x_3)\lambda \\ \dot{x}_2 = f_2(x_1, x_2, x_3) \\ \dot{x}_3 = f_3(x_1, x_2, x_3) \\ y = x_1 \end{cases} \tag{2}$$

where λ is a message that is mainly slowly time varying (i.e., λ is slowly time varying for most of the time, but may exhibit occasional jumps) and satisfies $\lambda_{\min} \le \lambda(t) \le \lambda_{\max}$ $(\forall t)$. Furthermore, $y \in I\!R$ is the transmitted signal (i.e., the coded message). Also, a second system is considered that has the form

$$\begin{cases} \dot{\hat{x}}_2 = f_2(y, \hat{x}_2, \hat{x}_3) \\ \dot{\hat{x}}_3 = f_3(y, \hat{x}_2, \hat{x}_3). \end{cases} \tag{3}$$

It is assumed that the (x_2, x_3) subsystem in (2) synchronizes with (3) in the sense that for Σ_T, together with the system (3), we have for all initial conditions that

$$\lim_{t\to+\infty}(x_i(t) - \hat{x}_i(t)) = 0, \qquad (i = 2, 3). \tag{4}$$

We now show that the problem of estimating λ may be viewed as a linear parameter identification problem. If one assumes that the systems (2) and (3) have synchronized, the dynamics of y in (2) are given by

$$\dot{y}(t) = u_1(t) + \lambda u_2(t) \tag{5}$$

where

$$\begin{aligned} u_1(t) &:= f_1(y(t), \hat{x}_2(t), \hat{x}_3(t)) \\ u_2(t) &:= g(y(t), \hat{x}_2(t), \hat{x}_3(t)). \end{aligned} \tag{6}$$

We then see that (5) may be interpreted as a linear time-invariant system with output y and inputs u_1, u_2. Our task is now to obtain a mechanism that estimates λ for the linear system (5), based on the measurements y, u_1, u_2. This problem may be interpreted as a linear parameter identification problem and will be treated as such in the sequel.

Note that in the above example the distance between the message λ and the transmitted signal y is small in the sense that already the first time derivative of y explicitly depends on λ (in control theoretic terms, this is expressed by saying that the relative degree (cf. [11]) of y with respect to λ equals 1). As will be argued in the last section, this might be a drawback if one would like to use the above scheme for private communication. Therefore, from the point of view of private communication, it might be worthwhile to consider schemes where the relative degree of y with respect to λ is greater than 1. The following two examples have this property. Furthermore, these examples illustrate that when one considers systems with a relative degree that is greater than one, the assumption of the existence of a synchronizing subsystem will, in general, not be of use any more.

Example 2: In this example, we consider Chua's circuit, which in dimensionless form is described by the equations

$$\begin{cases} \dot{x}_1 = \alpha(-x_1 + x_2 - \phi(x_1)) \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -\lambda x_2 \end{cases} \tag{7}$$

where

$$\phi(x_1) = m_1 x_1 + \frac{m_0 - m_1}{2}(|x_1 + 1| - |x_1 - 1|)$$

and λ is a mainly slowly time-varying message satisfying $23 < \lambda(t) < 31$ $(\forall t)$. For constant λ in this range and $\alpha = 15.6, m_0 = -(8/7), m_1 = -(5/7)$, this system is known to have a so-called double scroll chaotic attractor (see, e.g., [1]). We assume that $y = x_2$ is the transmitted signal. Note that, although it has been shown experimentally that for constant λ the (x_1, x_3)- subsystem synchronizes with the system

$$\begin{cases} \dot{\hat{x}}_1 = \alpha(-\hat{x}_1 + x_2 - \phi(\hat{x}_1)) \\ \dot{\hat{x}}_3 = -\lambda x_2 \end{cases} \tag{8}$$

(see, e.g., [4]), we cannot use this synchronizing subsystem in our reconstruction mechanism, since it explicitly depends on the unknown parameter λ. In order to come up with a reconstruction scheme for λ, we first assume that, besides x_2, we can also measure x_1. The equations for x_2 and x_3 in (7) then have the following form:

$$\begin{cases} \dot{x}_2 = -x_2 + x_3 + u \\ \dot{x}_3 = -\lambda x_2 \\ y = x_2 \end{cases} \tag{9}$$

where we interpret $u := x_1$ as a known input. Thus, (9) has the form of a linear control system depending on an unknown parameter λ, so that again linear parameter estimation methods may be used to obtain a reconstruction mechanism for λ.

In the above example the relative degree (the distance between λ and y) equals two. We can go one step further with a 3-D chaotic transmitter, as is shown in the following example where the relative degree equals three.

Example 3: We consider the following Rössler system:

$$\begin{cases} \dot{x}_1 = -x_2 - x_3 \\ \dot{x}_2 = x_1 + \lambda x_2 \\ \dot{x}_3 = 2 + (x_1 - 4)x_3 \\ y = x_3 \end{cases} \tag{10}$$

where we assume that λ is a mainly slowly time-varying message satisfying $0.3 < \lambda(t) < 0.5\ (\forall t)$ and $x_3(0) > 0$. It is known (see, e.g., [23]) that for (10) the (x_1, x_2) subsystem does not synchronize with the system

$$\begin{cases} \dot{\hat{x}}_1 = -\hat{x}_2 - x_3 \\ \dot{\hat{x}}_2 = \hat{x}_1 + \lambda \hat{x}_2 \end{cases}$$

Thus, in this case no synchronizing subsystem that can be used in a reconstruction mechanism inspired by the scheme in [5] exists. However, it is possible to reconstruct λ based on the measurement y. A first step in this reconstruction is the observation that (10) may be transformed into so-called linearizable error dynamics (see, e.g., [20] and [19]). More specifically, note that, since $x_3(0) > 0$, we have that $x_3(t) > 0\ (\forall t \geq 0)$. Thus, for (10) the coordinate change $\xi_1 = x_1, \xi_2 = x_2, \tilde{y} = \xi_3 = \log x_3$ is well -defined. In these new coordinates, (10) takes the form

$$\begin{aligned} \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{pmatrix} &= \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & \lambda & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{A(\lambda)} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \\ &\quad + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} -e^{\tilde{y}} \\ 2e^{-\tilde{y}} - 4 \end{pmatrix}}_{\Phi(\tilde{y})} \\ \tilde{y} &= \xi_3. \end{aligned} \tag{11}$$

Hence, (11) consists of a linear system $\dot{\xi} = A(\lambda)\xi + Bu$, where the matrix $A(\lambda)$ depends linearly on λ, interconnected with a static nonlinearity $u = \Phi(\tilde{y})$ that only depends on (a function of) the transmitted signal x_3. This means that also in this case linear parameter indentification methods may be used to build a reconstruction mechanism for λ.

Having illustrated the fact that linear parameter identification methods may be effective in communication with chaotic systems, we now describe how a so-called equation error identifier may be obtained. We will restrict ourselves to linear time-invariant systems with one output and two inputs that depend on one unknown parameter. The restriction to systems with only one input and the extension to systems with more than two inputs are straightforward. The exposition is based on [28]. For further details, the reader is referred to this reference.

In the rest of the paper, we use the following notation and terminology. By $\mathbb{R}[s]$, we denote the set of all polynomials in the indeterminate s with real coefficients. Let $a \in \mathbb{R}[s]$. Then there exists an $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{R}$ such that a has the form

$$a(s) = \sum_{j=0}^{n} a_j s^j. \tag{12}$$

If $a_n \neq 0$, we define $\deg(a) := n$. The polynomial a is called monic if $a_n = 1$. Furthermore, a is called Hurwitz if all zeros of a are in the open left-half plane of the complex plane. For a function $f(t)$ that is k times continuously differentiable, we define

$$f^{(k)}(t) := \frac{d^k f}{dt^k}(t).$$

Note that this gives that $f^{(0)}(t) := f(t)$. Let $a \in \mathbb{R}[s]$ of the form (12) be given, and let $f(t)$ be n times continuously differentiable. We then define

$$a\left(\frac{d}{dt}\right) f := \sum_{j=0}^{n} a_j f^{(j)}.$$

We now consider a linear time-invariant system Σ_λ depending on an unknown parameter λ with two inputs and one outputand transfer matrix

$$G_\lambda(s) = \begin{pmatrix} \dfrac{p_\lambda(s)}{q_\lambda(s)} & \dfrac{r_\lambda(s)}{q_\lambda(s)} \end{pmatrix} \tag{13}$$

As is well known (see, e.g., [26]), the fact that the transfer matrix of Σ_λ is given by (13) implies that, given input functions $u_1(t), u_2(t)$, the output $y(t)$ of Σ_λ satisfies the following linear differential equation:

$$q_\lambda\left(\frac{d}{dt}\right) y = p_\lambda\left(\frac{d}{dt}\right) u_1 + r_\lambda\left(\frac{d}{dt}\right) u_2. \tag{14}$$

We make the following assumptions.

- The polynomials $p_\lambda(s), q_\lambda(s), r_\lambda(s)$ depend linearly on λ.
- For all λ, we have that $\deg(q_\lambda) = n$ and q_λ is monic.
- For all λ, we have that $\deg(p_\lambda), \deg(r_\lambda) < n$.

As a consequence of these assumptions, the polynomials $p_\lambda, q_\lambda, r_\lambda$ have the following form:

$$\begin{aligned} p_\lambda(s) &= p_0(s) + p_1(s)\lambda \\ q_\lambda(s) &= q_0(s) + q_1(s)\lambda \\ r_\lambda(s) &= r_0(s) + r_1(s)\lambda \end{aligned} \tag{15}$$

where $p_0, p_1, r_0, r_1, q_0, q_1 \in \mathbb{R}[s]$ have the form

$$\begin{aligned} p_i(s) &= \sum_{j=0}^{n-1} p_{ij} s^j \quad (i = 0, 1) \\ r_i(s) &= \sum_{j=0}^{n-1} r_{ij} s^j \quad (i = 0, 1) \\ q_0(s) &= \sum_{j=0}^{n-1} q_{0j} s^j + s^n \\ q_1(s) &= \sum_{j=0}^{n-1} q_{1j} s^j. \end{aligned} \tag{16}$$

In system identification, the task is now to build an estimator for λ, based on the measurements y, u_1, u_2. Note that in our description of Σ_λ with the transfer matrix $G_\lambda(s)$ we have a description that depends on λ in a nonlinear way, in spite of the fact that the polynomials $p_\lambda, q_\lambda, r_\lambda$ depend on λ in a linear way. In the equation error method, a first step in building a reconstruction mechanism for λ is to obtain a (asymptotic) description of Σ_λ that depends on λ in a linear way. This is achieved as follows. Let $u_1(t)$ and $u_2(t)$ be input signals for Σ_λ, and let $y(t)$ be a corresponding output signal of Σ_λ. Thus, $y(t)$ satisfies the differential equation (14). Let $k \in \mathbb{R}[s]$ be a monic and Hurwitz polynomial with $\deg(k) = n$. Further, let $\tilde{y}(t)$ be a signal satisfying the differential equation

$$k\left(\frac{d}{dt}\right)\tilde{y} = p_\lambda\left(\frac{d}{dt}\right)u_1 + r_\lambda\left(\frac{d}{dt}\right)u_2 + \left[k\left(\frac{d}{dt}\right) - q_\lambda\left(\frac{d}{dt}\right)\right]y. \tag{17}$$

From the above, it follows that $\tilde{y}$ may be interpreted as the output of a linear time-invariant systems with inputs y, u_1, u_2 and transfer matrix

$$H_\lambda(s) = \begin{pmatrix} \frac{k(s) - q_\lambda(s)}{k(s)} & \frac{p_\lambda(s)}{k(s)} & \frac{r_\lambda(s)}{k(s)} \end{pmatrix}. \tag{18}$$

Writing

$$k(s) = \sum_{j=0}^{n-1} k_j s^j + s^n$$

and defining the row vectors

$$\begin{aligned}
p_i^* &:= (p_{i0} \quad \cdots \quad p_{in-1}) \quad (i = 0, 1)\\
p^* &:= p_0^* + p_1^*\lambda\\
q_i^* &:= (q_{i0} \quad \cdots \quad q_{in-1}) \quad (i = 0, 1)\\
q^* &:= q_0^* + q_1^*\lambda\\
r_i^* &:= (r_{i0} \quad \cdots \quad r_{in-1}) \quad (i = 0, 1)\\
r^* &:= r_0^* + r_1^*\lambda\\
k^* &:= (k_0 \quad \cdots \quad k_{n-1})
\end{aligned} \tag{19}$$

a realization ([25]) of $H_\lambda(s)$ is then given by

$$\begin{cases} \dot{\tilde{w}}_0 = K\tilde{w}_0 + Ly\\ \dot{\tilde{w}}_i = K\tilde{w}_i + Lu_i \quad (i = 1, 2)\\ \tilde{y} = (k^* - q^*)\tilde{w}_0 + p^*\tilde{w}_1 + r^*\tilde{w}_2 \end{cases} \tag{20}$$

where

$$K := \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0\\ 0 & 0 & 1 & \cdots & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & & \vdots\\ \vdots & \vdots & & \ddots & \ddots & 0\\ 0 & 0 & \cdots & \cdots & 0 & 1\\ -k_0 & -k_1 & \cdots & \cdots & \cdots & -k_{n-1} \end{pmatrix}$$

and $L := \mathrm{col}(0, 0, \ldots, 0, 1)$.

Now note that from (14), (17) it follows that $\tilde{y}$ in fact satisfies the following differential equation:

$$k\left(\frac{d}{dt}\right)(\tilde{y} - y) = 0. \tag{21}$$

Since k is Hurwitz, this implies that we have

$$\lim_{t\to+\infty}(\tilde{y}(t) - y(t)) = 0 \tag{22}$$

where the convergence is exponential. From this fact and the fact that $H_\lambda(s)$ depends on λ in a linear way, we see that we now have indeed obtained an asymptotic description of Σ_λ that depends on λ in a linear way.

A next step in the procedure to obtain an equation error estimator for λ is to consider a copy of the system (20), where λ is replaced by its estimation $\hat{\lambda}$. Thus, we obtain a system

$$\begin{cases} \dot{w}_0 = Kw_0 + Ly\\ \dot{w}_i = Kw_i + Lu_i \quad (i = 1, 2)\\ \hat{y} = (k^* - q_0^* - q_1^*\hat{\lambda})w_0 + (p_0^* + p_1^*\hat{\lambda})w_1 + (r_0^* + r_1^*\hat{\lambda})w_2 \end{cases}. \tag{23}$$

Making use of (20), (22), (23), it is then straightforwardly shown that

$$\hat{y}(t) - y(t) = \phi(w(t))(\hat{\lambda}(t) - \lambda) + \epsilon(t) \tag{24}$$

where $\epsilon(t)$ tends to zero exponentially for $t \to +\infty$ and $\phi(w)$ is defined by

$$\phi(w) := -q_1^* w_0 + p_1^* w_1 + r_1^* w_2. \tag{25}$$

To (23), an update mechanism for $\hat{\lambda}$ of the following form is added:

$$\dot{\hat{\lambda}} = -\nu\psi(t, w)(\hat{y} - y), \quad \nu > 0. \tag{26}$$

Using (24), it is then easily shown that we have

$$\frac{d}{dt}(\hat{\lambda} - \lambda)^2 = -2\nu\psi(t, w)\phi(w)(\hat{\lambda} - \lambda)^2 - 2\nu\epsilon(t)\psi(t, w)(\hat{\lambda} - \lambda). \tag{27}$$

Exploiting the fact that $\epsilon(t)$ tends to zero exponentially, it may then be shown (see [27] for details) that $\lambda(t) - \lambda \to 0$ $(t \to +\infty)$ exponentially, *if* the following conditions are satisfied:

- $\psi(t, w(t))$ is bounded on $[0, \infty)$;
- $\psi(t, w(t))\phi(w(t)) \geq 0$ on $[0, \infty)$;
- $\psi(t, w(t))\phi(w(t))$ is persistently exciting (P.E.) on $[0, \infty)$, i.e., there exist $\alpha_1, \alpha_2, \delta > 0$ such that for all $t \in [0, \infty)$ we have

$$\alpha_1 \leq \int_t^{t+\delta} \psi(\tau, w(\tau))^2\phi(w(\tau))^2\, d\tau \leq \alpha_2. \tag{28}$$

In the literature, a wide range of possible choices of the function $\psi(t, w)$ is available. It goes without saying that each different choice of ψ will lead to a different estimator with different properties. An estimator that possesses good properties in many cases is the least squares estimator with exponential forgetting factor that is obtained by choosing

$$\psi(t, w) := -\nu\phi(w)p(t), \quad \nu > 0 \tag{29}$$

where the function $p(t)$ satisfies the differential equation

$$\dot{p} = -\nu(\phi(w)^2p^2 - \gamma p), \quad \gamma > 0, \quad p(0) > 0. \tag{30}$$

In the sequel, we will tacitly assume that the signals $\psi(t, w(t))\phi(w(t))$ appearing in our reconstruction mechanisms are P.E. To a degree, this tacit assumption is justified by the fact that it has been shown in [2] that for quite a wide choice of functions $\psi(t, w)$ we will have that $\psi(t, w(t))\phi(w(t))$ is P.E.

when the signals $y(t), u_1(t), u_2(t)$ have a power spectrum that is not concentrated at too few a number of peaks. Since in the applications we will be looking at the signals $y(t), u_1(t), u_2(t)$ will be produced by a chaotic system, it follows from the fact that chaotic systems produce signals with a broad continuous power spectrum (cf. [21]), that indeed $\psi(t, w(t))\phi(w(t))$ may be expected to be P.E.

III. The Corron-Hahs Scheme With Synchronization

We continue with Example 1. The transfer matrix $G_\lambda(s)$ of the system (5) is given by

$$G_\lambda(s) = \begin{pmatrix} \frac{1}{s} & \frac{\lambda}{s} \end{pmatrix}.$$

Thus, we have in the notation of the previous section

$$\begin{aligned} p_\lambda(s) &= 1 \\ q_\lambda(s) &= s \\ r_\lambda(s) &= \lambda. \end{aligned}$$

Letting $\kappa > 0$ we have that the polynomial $k(s) := s + \kappa$ is Hurwitz. Thus, in this case the system (20) has the form

$$\begin{cases} \dot{w}_0 = -\kappa w_0 + y \\ \dot{w}_1 = -\kappa w_1 + u_1 = -\kappa w_1 + f_1(y, \hat{x}_2, \hat{x}_3) \\ \dot{w}_2 = -\kappa w_2 + u_2 = -\kappa w_2 + g(y, \hat{x}_2, \hat{x}_3) \\ \hat{y} = \kappa w_0 + w_1 + \hat{\lambda} w_2 \end{cases}. \tag{31}$$

Furthermore, we have in this case that

$$\phi(w) = w_2. \tag{32}$$

Choosing

$$\psi(t, w) = \frac{\operatorname{sign}(w_2)}{1 + |w_2|} \tag{33}$$

we then obtain the following adaptation law for $\hat{\lambda}$:

$$\dot{\hat{\lambda}} = -\nu \frac{\operatorname{sign}(w_2)}{1 + |w_2|}(\hat{y} - y), \quad \nu > 0. \tag{34}$$

Remark 1: The reconstruction mechanism (31), (34) is not exactly the same as the reconstruction mechanism proposed in [5]. However, if one looks at (31), (33) more closely, one sees that for the reconstruction one does not need to know w_0 and w_1 separately, but that knowledge of the linear combination $\kappa w_0 + w_1$ suffices. Thus, defining

$$\begin{aligned} \tilde{w}_0 &:= \kappa w_0 + w_1 \\ \tilde{w}_1 &:= w_2 \end{aligned}$$

one arrives at the following reconstruction mechanism:

$$\begin{cases} \dot{\tilde{w}}_0 = -\kappa \tilde{w}_0 + \kappa y + f_1(y, \hat{x}_2, \hat{x}_3) \\ \dot{\tilde{w}}_1 = -\kappa \tilde{w}_1 + g(y, \hat{x}_2, \hat{x}_3) \\ \dot{\hat{\lambda}} = -\nu \dfrac{\operatorname{sign}(\tilde{w}_1)}{1 + |\tilde{w}_1|}(\hat{y} - y), \quad \nu > 0 \end{cases} \tag{35}$$

which is exactly the reconstruction mechanism proposed in [5]. Note, however, that in [5] this reconstruction mechanism was obtained in a different way. Further, in [5] the authors do not require the function $\psi(t, w(t))\phi(w(t))$ to be persistently exciting. However, if one carefully checks the derivation in [5], it turns out that also in [5] this requirement is needed.

IV. Chua's Circuit With Partial Synchronization

In this section, we continue our investigation of the possibility to build a reconstruction scheme for λ for the Chua circuit (7) from Example 2. As we have seen in Example 2, λ may be reconstructed by using linear parameter identification techniques if, besides the transmitted signal $y = x_2$, also the signal x_1 is available for measurement.

It is easily checked that the transfer function $G_\lambda(s)$ of (9) is given by

$$G_\lambda(s) = \frac{s}{s^2 + s + \lambda}.$$

Thus, in the notation of Section II we have in this case

$$\begin{aligned} p_\lambda(s) &= s \\ q_\lambda(s) &= s^2 + s + \lambda. \end{aligned}$$

For (9), the least squares estimator with exponential forgetting factor then takes the following form:

$$\begin{cases} \dot{w}_{01} = w_{02} \\ \dot{w}_{02} = -k_0 w_{01} - k_1 w_{02} + y \\ \qquad = -k_0 w_{01} - k_1 w_{02} + x_2 \\ \dot{w}_{11} = w_{12} \\ \dot{w}_{12} = -k_0 w_{11} - k_1 w_{12} + u \\ \qquad = -k_0 w_{11} - k_1 w_{12} + x_1 \\ \hat{y} = (k_0 - \hat{\lambda}) w_{01} + (k_1 - 1) w_{02} + w_{12} \\ \dot{\hat{\lambda}} = \nu w_{01} p(\hat{y} - y), \quad (\nu > 0) \\ \dot{p} = -\nu(w_{01}^2 p^2 - \gamma p), \quad (\gamma > 0) \end{cases} \tag{36}$$

where $k_0, k_1 \in I\!R$ are such that the polynomial $k(s) := s^2 + k_1 s + k_0$ is Hurwitz.

From the above, it follows that if x_1 could be measured, the reconstruction of λ could be achieved by employing the scheme (36). To achieve reconstruction when x_1 cannot be measured, we add the following estimator of x_1 to our reconstruction scheme:

$$\dot{\hat{x}}_1 = \alpha(-\hat{x}_1 + x_2 - \phi(\hat{x}_1)) \tag{37}$$

and let the reconstruction scheme (36) depend on $\hat{x}_1$ instead of x_1, i.e., we replace the reconstruction scheme (36) by the following reconstruction scheme:

$$\begin{cases} \dot{\bar{w}}_{01} = \bar{w}_{02} \\ \dot{\bar{w}}_{02} = -k_0 \bar{w}_{01} - k_1 \bar{w}_{02} + y \\ \qquad = -k_0 \bar{w}_{01} - k_1 \bar{w}_{02} + x_2 \\ \dot{\bar{w}}_{11} = \bar{w}_{12} \\ \dot{\bar{w}}_{12} = -k_0 \bar{w}_{11} - k_1 \bar{w}_{12} + u \\ \qquad = -k_0 \bar{w}_{11} - k_1 \bar{w}_{12} + \hat{x}_1 \\ \bar{y} = (k_0 - \bar{\lambda}) \bar{w}_{01} + (k_1 - 1) \bar{w}_{02} + \bar{w}_{12} \\ \dot{\bar{\lambda}} = \nu \bar{w}_{01} \bar{p}(\bar{y} - y), \quad (\nu > 0) \\ \dot{\bar{p}} = -\nu(\bar{w}_{01}^2 \bar{p}^2 - \gamma \bar{p}), \quad (\gamma > 0) \end{cases} \tag{38}$$

where now $\bar{\lambda}$ denotes the estimate of λ. We then have the following result that is proved in [32].

Theorem 1: Assume that for (36) we have that

$$\lim_{t\to+\infty}(\hat{\lambda}(t)-\lambda)=0 \tag{39}$$

and that

$$\lim_{t\to+\infty}(\hat{x}_1(t)-x_1(t))=0. \tag{40}$$

Then for (38) we have that

$$\lim_{t\to+\infty}(\bar{\lambda}(t)-\lambda)=0. \tag{41}$$

■

From Theorem 1, it follows that if only the transmitted signal $u=x_2$ can be measured, then λ can be reconstructed, provided $\hat{x}_1(t)$ approaches $x_1(t)$. In [4], it was shown experimentally that this will indeed be the case for constant λ. However, one needs to be somewhat careful here for the following reasons. Define the error signal $e(t):=\hat{x}_1(t)-x_1(t)$. Then, for the parameter values given above, e satisfies the following differential equation:

$$\dot{e}=15.6\left(-\frac{2}{7}e+\frac{3}{7}(\mathrm{sat}(e+x_1)-\mathrm{sat}(x_1))\right) \tag{42}$$

where $\mathrm{sat}(\cdot)$ is the saturation function given by $\mathrm{sat}(x)=(1/2)(|x+1|-|x-1|)$. A first observation is that the equilibrium $e=0$ of (42) is unstable when $x_1(t)\equiv 0$. This implies in particular that when (7) is initialized in the origin, we will not have that e tends to zero. It may be argued that from a practical point of view this is not a serious objection since, in practice, one will have (7) running when communicating. However, the system (7) for the given parameter values is chaotic in the sense of Shil'nikov, as was shown in, e.g., [3]. This implies in particular that the origin is a homoclinic point for (7), which gives by the above that e will also not tend to zero when (7) is initialized on the homoclinic orbit. Further, this implies that when (7) is initialized near the homoclinic orbit, we will at least not have that e will tend to zero quickly. This leads to the conclusion that the best one could hope for is that e will tend to zero quickly for a generic choice of x_1.

Theoretical evidence for the asymptotic stability of $e=0$ for (42) with a generic choice of x_1 is obtained in the following way. Consider in the (x_1,e)-plane the compact set S enclosed by the straight lines $e=-(3/2)(x_1\pm 1), e=-3(x_1\pm 1), e=\pm 3$ (see Fig. 1). Further, consider the function $V(e):=(1/2)e^2$. It may then be shown that $\dot{V}=e\dot{e}\geq 0$ on $S\cup\{x_1-\mathrm{axis}\}$, and $\dot{V}=0$ on $\partial S\cup\{x_1-\mathrm{axis}\}$, while $\dot{V}<0$ outside $S\cup\{x_1-\mathrm{axis}\}$. A first conclusion that may be drawn from this is that $\{e\in I\!R \mid |e|\leq 3\}$ is a globally attracting invariant set of (42) for all x_1. Also, the location of S in the (x_1,e) plane suggests that we will have asymptotic stability of $e=0$ for (42) if the residence time of $x_1(t)$ in the region $|x_1|>1$ is large in comparison with the residence time of $x_1(t)$ in the region $|x_1|\leq 1$. Simulations for constant values of λ between 23 and 31 indicate that (asymptotically) we will have that $|x_1(t)|\leq 1$ for about 20% of the time, while $x_1(t)<-1$ respectively $x_1(t)>1$ for about 40% of the time.

In Fig. 2 the proposed reconstruction scheme is illustrated by means of a simulation. Here, the parameters were chosen as $k_0=256, k_1=32, \nu=800, \gamma=0.001$.

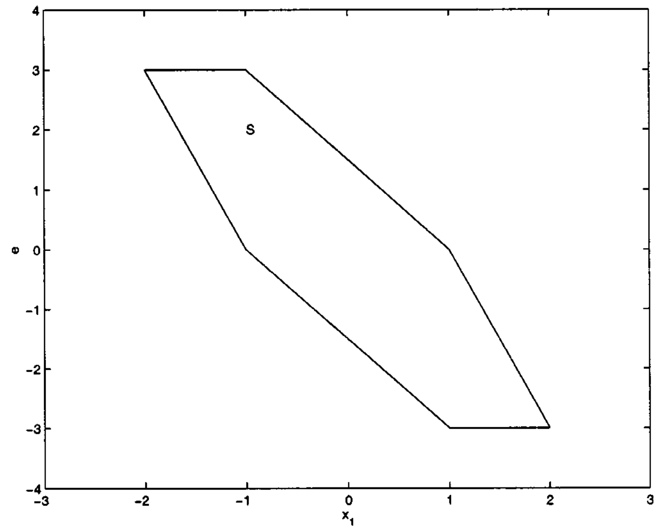

Fig. 1. The set S in the (x_1,e) plane.

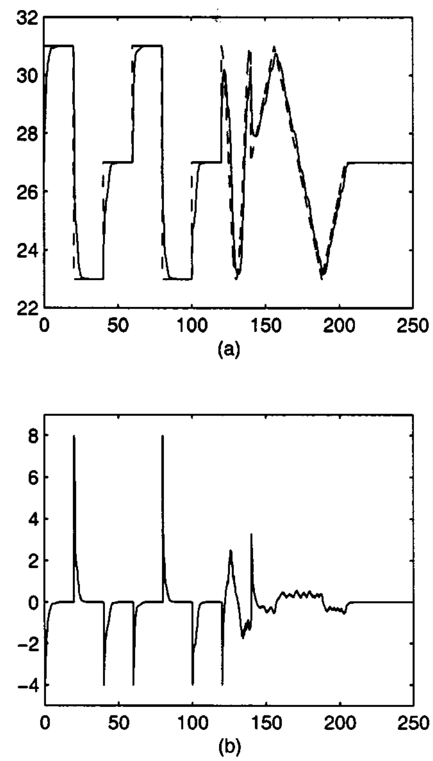

Fig. 2. Simulation results for the Chua system. (a) λ (dashed) and $\hat{\lambda}$ (solid). (b) Estimation error.

In this section, we employed a partially synchronizing subsystem (37) rather than a completely synchronizing subsystem as is often the case in communication using chaotic systems. However, there is also another (partial) synchronization aspect present in the scheme. Namely, it follows that once we have that $\hat{\lambda}=\lambda$ we will have that $\hat{y}\to y$, or, in other words, we will have that $(k_0-\lambda)w_{01}+(k_1-1)w_{02}+w_{12}$ and x_2 will synchronize. Taking time derivatives, this gives in its turn that also $(k_0-k_1\lambda)w_{01}+(k_0-\lambda)w_{02}-k_0w_{11}$ and x_3 will synchronize. Thus, we see that, although our scheme is only based on partial synchronization beforehand, it will also exhibit partial synchronization once λ has been estimated correctly.

V. Rössler System Without Synchronization

In this section, we continue our investigation of the possibility to build a reconstruction scheme for the Rössler system (10) from Example 3. As we have seen in Example 3, λ may be reconstructed by applying linear parameter identification techniques to the transformed system (11).

It is easily checked that the transfer matrix $G_\lambda(s)$ of (11) is given by

$$\begin{pmatrix} \dfrac{s-\lambda}{s^3-\lambda s^2+s} & \dfrac{s^2-\lambda s+1}{s^3-\lambda s^2+s} \end{pmatrix}$$

Thus, in the notation of Section II we have

$$\begin{aligned} p_\lambda(s) &:= s-\lambda \\ q_\lambda(s) &:= s^3-\lambda s^2+s \\ r_\lambda(s) &:= s^2-\lambda s+1. \end{aligned}$$

The least squares estimator with exponential forgetting factor for (11) then takes the following form:

$$\begin{cases} \dot{w}_i = Kw_i + Lu_i, \quad (i=0,1,2) \\ \hat{y} = \phi_0(w) + \hat{\lambda}\phi_1(w) \\ \dot{\hat{\lambda}} = -\nu\phi_1(w)p(\hat{y}-y), \quad (\nu > 0) \\ \dot{p} = -\nu(\phi_1(w)^2p^2 - \gamma p), \quad (\gamma > 0) \end{cases} \tag{43}$$

where $u_0 := \log(x_3), u_1 := -x_3, u_2 := (2/x_3) - 4$

$$K = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_0 & -k_1 & -k_2 \end{pmatrix}, \quad L = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$k_0, k_1, k_2 \in I\!R$ are such that the polynomial $s^3+k_2s^2+k_1s+k_0$ is Hurwitz and $\phi_0(w) := k_0w_{01}+(k_1-1)w_{02}+k_2w_{03}+w_{12}+w_{21}+w_{23}$, $\phi_1(w) := w_{03}-w_{11}-w_{22}$.

In Fig. 3, the proposed reconstruction scheme is illustrated by means of a simulation. Here, the parameters were chosen as $k_0 = 512, k_1 = 192, k_2 = 24, \nu = 800, \gamma = 0.002$.

It may further be shown that, as in Section III, the scheme (43) will exhibit partial synchronization once λ has been estimated correctly.

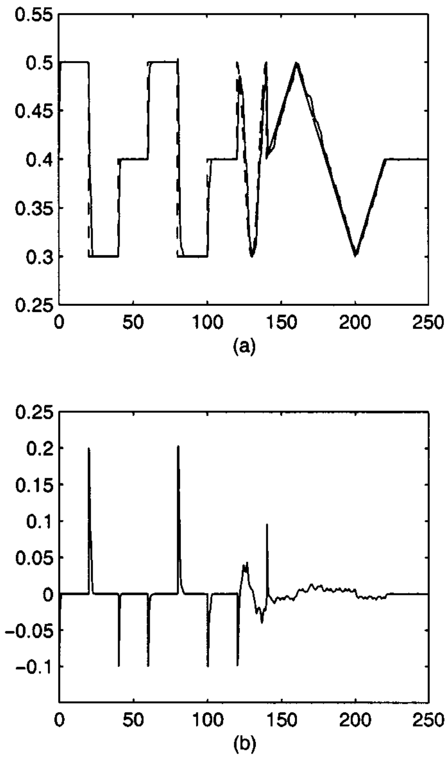

Fig. 3. Simulation results for the Rössler system. (a) λ (dashed) and $\hat{\lambda}$ (solid). (b) Estimation error.

VI. Classes of Transmitters Amenable to Identification Based Reconstruction Schemes

In this section, we briefly indicate two classes of transmitters to which identification based reconstruction schemes may be applied. Further, it is shown that the transmitters treated in the previous sections fit in one of these classes.

A. Partially Linear(izable) Transmitters

A partially linear transmitter is a transmitter of the form

$$\Sigma_T \begin{cases} \dot{x}^1 = A(\lambda)x^1 + B(\lambda)\chi(y,x^2) \\ \dot{x}^2 = f^2(y,x^2) \\ y = C(\lambda)x^1 \end{cases} \tag{44}$$

where $x^1 \in I\!R^q, x^2 \in I\!R^{n-q}, \chi : I\!R \times I\!R^{n-q} \to I\!R^q, f^2 : I\!R \times I\!R^{n-q} \to I\!R^{n-q}$, and $A(\lambda), B(\lambda), C(\lambda)$ are matrices of appropriate dimensions that linearly depend on λ. For Σ_T, we assume the following.

(A1) The x^2 subsystem synchronizes with a copy of itself, i.e., the dynamics

$$\dot{\hat{x}}^2 = f^2(y, \hat{x}^2) \tag{45}$$

satisfy

$$\lim_{t\to+\infty} \|\hat{x}^2(t) - x^2(t)\| = 0 \tag{46}$$

whatever the initial conditions of (44) and (45) are.

(A2) The signals $\chi(y(t), x^2(t))$ are persistently exciting.

If ***(A1)*** and ***(A2)*** are satisfied, a reconstruction mechanism for λ may be obtained by applying standard linear identification techniques to the system

$$\begin{cases} \dot{z} = A(\lambda)z + B(\lambda)u \\ y = C(\lambda)z \end{cases} \tag{47}$$

where $u := \chi(y, \hat{x}^2)$.

Note that the transmitter (2) is a partially linear transmitter with $x^1 := x_1, x^2 := \text{col}(x_2, x_3)$ and

$$A(\lambda) := 0, \quad B(\lambda) := (1 \quad \lambda), \quad C(\lambda) := (1)$$

and

$$\chi(y,x^2) := \begin{pmatrix} f_1(y,x^2) \\ g(y,x^2) \end{pmatrix}, \quad f^2(y,x^2) := \begin{pmatrix} f_2(y,x^2) \\ f_3(y,x^2) \end{pmatrix}.$$

Furthermore, note that also the transmitter (7) is a partially linear transmitter with $x^1 := \text{col}(x_2, x_3), x^2 := x_1$

$$A(\lambda) := \begin{pmatrix} -1 & 1 \\ -\lambda & 0 \end{pmatrix}, \quad B(\lambda) := \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$C(\lambda) := (1 \quad 0)$$

and

$$\chi(y, x^2) := x_1, \quad f^2(y, x^2) := \alpha(-x^2 + y - \phi(x^2)).$$

Next, consider a transmitter Σ_T of the form

$$\Sigma_T \begin{cases} \dot{\xi} = \tilde{f}(\xi, \mu) \\ \tilde{y} = \tilde{h}(\xi) \end{cases} \tag{48}$$

where $\xi \in \mathbb{R}^n$, μ is a mainly slowly time-varying message, and $\tilde{y} \in \mathbb{R}$. This transmitter is called partly linearizable if there exist new coordinates $x(\xi) = \mathrm{col}(x^1(\xi), x^2(\xi))$ with $x^1 \in \mathbb{R}^q, x^2 \in \mathbb{R}^{n-q}$ and invertible mappings $\phi, \psi : \mathbb{R} \to \mathbb{R}$ such that in the new coordinates x and with $y := \phi(\tilde{y}), \lambda := \psi(\mu)$, the transmitter Σ_T takes the form (44). It then follows from the discussion above that also for a partly linearizable transmitter identification based reconstruction schemes may be designed.

To the best of our knowledge, there are no results in the literature that give conditions under which a given transmitter is partly linearizable. The derivation of such conditions remains a topic for further research. It is to be expected that in this derivation results developed in [16] and [9] will be useful.

B. Linearizable Error Dynamics

Linearizable error dynamics are dynamics of the form ([20], [11])

$$\begin{cases} \dot{\xi} = A(\lambda)\xi + B(\lambda)\Phi(\tilde{y}) \\ \tilde{y} = C(\lambda)\xi \end{cases} \tag{49}$$

where $\xi \in \mathbb{R}^n, \tilde{y} \in \mathbb{R}, \Phi : \mathbb{R} \to \mathbb{R}^m, A(\lambda), B(\lambda), C(\lambda)$ are matrices of appropriate dimensions that linearly depend on λ, and $(C(\lambda), A(\lambda))$ is observable (cf. [26]) for all λ. Note that (11) are linearizable error dynamics. If the signals $\Phi(\tilde{y}(t))$ are persistently exciting, a reconstruction mechanism for λ may be obtained by applying standard linear identification techniques to the system

$$\begin{cases} \dot{z} = A(\lambda)z + B(\lambda)u \\ \tilde{y} = C(\lambda)z \end{cases} \tag{50}$$

where $u := \Phi(\tilde{y})$.

Next, consider a transmitter Σ_T of the form

$$\begin{cases} \dot{x} = f(x, \mu) \\ y = h(x) \end{cases} \tag{51}$$

where $x \in \mathbb{R}^n$, μ is a mainly slowly time-varying message, and $y \in \mathbb{R}$. This transmitter is said to admit linearizable error dynamics if there exist new coordinates $\xi(x)$ and invertible mappings $\phi, \psi : \mathbb{R} \to \mathbb{R}$ such that in the new coordinates ξ and with $\tilde{y} := \phi(y), \lambda := \psi(\mu)$, the transmitter Σ_T takes the form (49). It then follow from the discussion above that also for a transmitter that admits linearizable error dynamics, identification based reconstruction schemes may be designed.

For transmitters of the form (51) without parameter dependence [14] (see also [15]) gives conditions under which the transmitter admits linearizable error dynamics. To the best of our knowledge, no conditions are known under which a parameter-dependent transmitter (51) admits linearizable error dynamics. The derivation of such conditions remains a topic for further research.

VII. Conclusions and Discussion

We have studied communication with chaotic systems using ideas from systems and control theory. Since, in general, the unknown message which is to be reconstructed is not available beforehand, insistence on standard synchronization schemes restricts the class of systems that may be employed in designing a viable communication scheme. We therefore propose an adaptive identification scheme that would enable the message reconstruction without explicitly assuming (partial) synchronization. This method forms a generalization of a method developed in [5] and is applicable in a far more general setting than in [5]. It should be noted that the message to be reconstructed has to be mainly slowly time varying, so that the identification scheme is fast enough for the reconstruction. Typically, in communication this will be the case, in particular when dealing with piecewise constant (binary) messages. Two illustrative simulations of the proposed identification schemes are included, together with a discussion of the validity of the imposed conditions. Furthermore, classes of transmitters that are amenable to identification based reconstruction schemes have been identified.

A possible advantage of using chaotic systems for communication is that the transmitted signal y will be a chaotic signal, which implies that it has a broad spectrum. This gives the opportunity to use the chaotic system under consideration for wideband communication (cf. [13]). Furthermore, the fact that the transmitted signal is a chaotic (and thus seemingly random) signal gives the hope that chaotic systems may also be used for private communication. In this respect, the following comparison between the three examples in this paper is in order. As already indicated in Section II, in Example 1 the distance between the message λ and the transmitted signal y is small in the sense that the relative degree (cf. [11]) of y with respect to λ equals 1. This might be a drawback if one would like to use the scheme in Example 1 for private communication since it might mean that λ is not hidden well enough. Indeed, a simple numerical differentiation scheme could be enough to allow eavesdroppers to decode the coded message. Therefore, from the point of view of private communication, it might be worthwhile to consider schemes where the relative degree of y with respect to λ is greater. The schemes considered in Examples 2 and 3 indeed satisfy this property. In Example 2 the relative degree equals two, while in Example 3 the relative degree equals three. Of course, further research as to whether indeed a higher relative degree will enhance the privacy of communication schemes based on chaotic systems is needed. Here, one could investigate to what extent the proposed schemes withstand code breaking mechanisms as described, in e.g., [24], [29], and [30].

As in [5], we have studied communication with chaotic systems in an ideal setting in the sense that our examples are simulation examples where we did not include practical limitations in communications like amplitude attenuation, bandwidth limitations, phase distortion, and channel noise (cf. [27]). All these may effect, to some extent, the idealized outcomes shown in the given simulations. These are topics that are being studied at the moment. Preliminary investigations indicate that for piecewise constant messages, sufficiently small channel noise can be coped with, possibly after having added a filter as described in, e.g., [6], [7], and [31] to the reconstruction mechanism.

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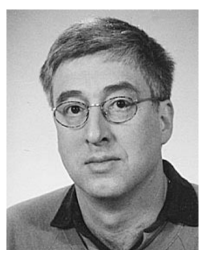
Henri Huijberts (M'00) was born in Schagen, The Netherlands, on December 15, 1962. He received the M.Sc. and Ph.D. degrees from the University of Twente, Enschede, The Netherlands in 1987 and 1991, respectively.

From 1991 until September 1999 he was with the Department of Mathematics and Computing Science, Eindhoven University of Technology, Eindhoven, The Netherlands. Since then, he has been with the Department of Engineering, Queen Mary and Westfield College, London, U.K. His research interests include nonlinear control, dynamical systems, chaos and synchronization.

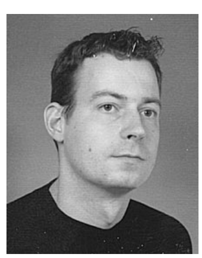
Dr. Huijberts is on the Editorial Board of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL and the *SIAM Journal on Control and Optimization*.



Henk Nijmeijer (M'83, SM'91, F'00) was born in Assen, the Netherlands, on March 16, 1955. He received the M.Sc. and Ph.D. degrees from the University of Groningen, Groningen, the Netherlands, in 1979 and 1983, respectively.

From 1983 to April 2000 he was with the Faculty of Mathematical Sciences, University of Twente, Enschede, the Netherlands. Since May 2000 he has been a Professor at the Faculty of Mechanical Engineering, Eindhoven University of Technology, Eindhoven, the Netherlands. His research interests include nonlinear control and its applications, dynamical systems, chaos, and synchronization.

Dr. Nijmeijer is on the Editorial Board of various journals, including *Automatica*, the *SIAM Journal on Control and Optimization*, the *International Journal of Robust and Nonlinear Control* and the *International Journal of Control*.



Rob Willems was born in Geleen, The Netherlands, on November 30, 1969. He received the M.Sc. degree in applied mathematics from Eindhoven University of Technology, Eindhoven, The Netherlands in 1998.

He is presently with Statistics Netherlands where he works on cleansing of large data sets infected with incorrect and missing data.

20

Discrete-Time Observers and Synchronization

H.J.C. Huijberts,[1] H. Nijmeijer,[2,3] and A. Yu. Pogromsky[3,4]

[1]Faculty of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
hjch@win.tue.nl

[2]Faculty of Mathematical Sciences
University of Twente
P.O. Box 217, 7500 AE Enschede, The Netherlands
h.nijmeijer@math.utwente.nl

[3]Faculty of Mechanical Engineering
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

[4]Institute for Problems of Mechanical Engineering
61, Bolshoy, V.O., 199178, St. Petersburg, Russia
sasha@ccs.ipme.ru

Abstract

The synchronization problem for complex discrete-time systems is revisited from a control perspective and it is argued that the problem may be viewed as an observer problem. It is shown that a solution for the synchronization (observer) problem exists for several classes of systems. Also, by allowing past measurements, a dynamic mechanism for state reconstruction is provided.

20.1 Introduction

Since the work of Pecora and Carroll [18], a huge interest in (chaos) synchronization has arisen. Among others, this is illustrated by the appearance of a number of special issues of journals devoted to the subject, cf. [29, 28, 30]. One clear motivation for this widespread interest lies in the fact that Pecora and Carroll indicated that chaos synchronization might be useful in communications. Although this claim is not yet fully justified, several interesting applications of (chaos) synchronization are envisioned.

Synchronization as it was introduced by Pecora and Carroll has been studied from various viewpoints. Following [18], often a receiver-transmitter (or master-slave) formalism is taken, where typically the receiver system is an exact copy of the transmitter system and the aim is to synchronize the receiver response with that of the transmitter, provided the receiver dynamics are driven by a scalar signal from the transmitter; see [18, 4, 26].

More recently, the above method was recast in an *active-passive decomposition*, see [17], where the decomposition idea has to be understood in the way that part of the transmitter state needs to be transmitted, while the "passive" part then will be derived asymptotically.

Another idea to achieve synchronization between (identical) transmitter and receiver dynamics is to include (linear) feedback of the drive signal in the receiver system; see [16] and [11] where a number of sucessfull experimental settings of this type are discussed.

A third way to achieve synchronization between transmitter and receiver was recently put forward in [6] and essentially advertises the idea of system inversion for (state) synchronization.

Notwithstanding the widespread interest in the synchronization problem, the problem leaves some ambiguity in how to make an active-passive decomposition or how to successfully build an (stable) inverse system. Indeed, this ambiguity disappears when the synchronization problem is viewed as the question of how to reconstruct the full state trajectory of the transmitter system, given some (scalar) drive signal from the transmitter. This is essentially the observer problem from control theory, and has, following the earlier attempts [5, 19, 13], by now obtained a prominent place within recent synchronization literature; see, for instance, [14] and various other observer-based synchronization papers.

The purpose of the present chapter is to revisit the synchronization problem for discrete-time systems using (discrete-time) observers. Synchronization of complex/chaotic discrete-time systems has been the subject of various publications; see, e.g., [2, 1, 7, 21, 27], but only little attention for an observer-based viewpoint exists (see, however, [24, 25] where this viewpoint *is* taken, and [26], which may be interpreted as a particular application of the observer-based viewpoint (although this is not mentioned explicitly in

[26]). One could argue, however, that the synchronization problem for discrete-time systems is as important as the continuous-time counterpart. First, for communications of *binary* signals one can very well base oneself on discrete-time transmitter systems instead of continuous-time transmitters. A second motivation to look at discrete-time synchronization is that many continuous-time models are in the end – for instance, for simulation and implementation – discretized or sampled. A third motivation is that discrete-time dynamics are obtained when one considers the Poincaré map at a suitably defined Poincaré section of a chaotic transmitter system.

As stated, we pursue an observer-based view on (discrete-time) synchronization. Although some clear analogies exist between discrete-time and continuous-time observers, there are various results available in either context which do not admit a proper analogon in the other domain.

This chapter is organized as follows. In Section 20.2, we treat some preliminaries and give our problem statement. Section 20.3 is devoted to nonlinear discrete-time transmitters of a special form, the so-called Lur'e form. It is shown that for this kind of system, the construction of an observer is relatively easy. In Section 20.4, we study the question when a given nonlinear discrete-time transmitter is equivalent to a system in Lur'e form by means of a coordinate transformation. In Section 20.5, we introduce a so-called extended Lur'e form, indicate how observers for transmitters in this extended Lur'e form may be constructed, and give conditions under which a nonlinear discrete-time transmitter may be transformed into an extended Lur'e form. Section 20.6 treats the observer design for perturbed linear transmitters. Section 20.7, finally, contains some conclusions.

20.2 Preliminaries and Problem Statement

Throughout this chapter, we consider discrete-time nonlinear (transmitter) dynamics of the form

$$x(k+1) = f(x(k)), \quad x(0) = x_0 \in \mathbf{R}^n \tag{20.2.1}$$

where the state transition map f is a smooth mapping from $\mathbf{R}^n$ into itself. Note that direct extensions of (20.2.1) are possible by allowing the state to belong to an open subset of $\mathbf{R}^n$ or to a differentiable manifold. The solution $x(k, x_0)$ of (20.2.1) is not directly available, but only an *output* is measured, say

$$y(k) = h(x(k)) \tag{20.2.2}$$

where $y \in \mathbf{R}^p$ and $h : \mathbf{R}^n \to \mathbf{R}^p$ is the smooth output map. Though in the sequel there is no restriction in assuming the transmitted signal $y(k)$

to be p-dimensional, we will for simplicity – and following most work on synchronization – take $p = 1$.

The observer problem for (20.2.1,20.2.2) now deals with the question how to reconstruct the state trajectory $x(k, x_0)$ on the basis of the measurements $y(k)$. A *full observer* (or briefly *observer*) for the system (20.2.1, 20.2.2) is a dynamical system of the form

$$\hat{x}(k+1) = \hat{f}(\hat{x}(k), y(k)), \quad \hat{x}(0) = \hat{x}_0 \in \mathbf{R}^n \tag{20.2.3}$$

where $\hat{x} \in \mathbf{R}^n$, and $\hat{f}$ is a smooth mapping on $\mathbf{R}^n$ parametrized by y, such that the error $e(k) := x(k) - \hat{x}(k)$ asymptotically converges to zero as $k \to \infty$ for all initial conditions x_0 and $\hat{x}_0$. Moreover, we require that if $e(k_0) = 0$ for some k_0, then $e(k) = 0$ for all $k \geq k_0$.

20.3 Systems in Lur'e Form

The problem of observer design in its full generality is a problem that is difficult to solve. Basically, only the observer design problem for linear systems has been solved in its full generality; see [20]. Therefore, we start our survey of possible approaches to observer-based synchronization by considering a class of nonlinear systems that is slightly more general than linear systems, namely, systems in so-called Lur'e form.

Assume that the master dynamics are governed by the following system of difference equations

$$x(k+1) = Ax(k) + \varphi(y(k)), \quad y(k) = Cx(k), \tag{20.3.4}$$

where $x(k) \in \mathbf{R}^n$ is the state, $y(k) \in \mathbf{R}^1$ is the scalar output, $\varphi : \mathbf{R}^1 \to \mathbf{R}^n$ is a smooth function, and A, C are constant matrices of appropriate dimensions. Dynamics of the form (20.3.4) are referred to as dynamics in *Lur'e form*. The question we now pose is, under what conditions is it possible to design an observer for (20.3.4)? As a possible observer candidate one can build a copy of (20.3.4) augmented with so-called *output injection*:

$$\widehat{x}(k+1) = A\widehat{x}(k) + \varphi(y(k)) + L(y(k) - \widehat{y}(k)), \quad \widehat{y}(k) = C\widehat{x}(k), \tag{20.3.5}$$

where $\widehat{x}(k) \in \mathbf{R}^n$ is the estimate of $x(k)$ and L is a $n \times 1$ matrix; see [10].

The solutions of systems (20.3.4) and (20.3.5) will synchronize if for all initial conditions the error $e(k) := x(k) - \widehat{x}(k)$ tends to zero when k tends to infinity. Substracting (20.3.5) from (20.3.4), one can easily see that the error vector $e(k)$ obeys the following linear difference equation:

$$e(k+1) = (A - LC)e(k). \tag{20.3.6}$$

Therefore, if all eigenvalues of $A - LC$ lie in the open unit disc (i.e., the set $\{z \in \mathbf{C} \mid |z| < 1\}$), then (20.3.5) is an observer for (20.3.4). In other words, for the system (20.3.4) the synchronization problem can be reduced to the following question: given A, C, under what conditions does there exist a matrix L such that $A - LC$ has all eigenvalues in the open unit disc? This linear algebraic problem has a simple solution. Namely, a sufficient condition for existence of L is the invertibility of the following linear mapping

$$\mathcal{O}(x) := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x. \tag{20.3.7}$$

In linear control theory, a pair of matrices (C, A) such that $\mathcal{O}(x)$ in (20.3.7) is invertible, is said to be an *observable pair.* Using this terminology we can formulate the following result.

THEOREM 20.1
Assume that the pair (C, A) *is observable. Then the system* (20.3.4) *admits an observer* (20.3.5) *with the exponentially stable linear error dynamics* (20.3.6).

The proof of this result can be found in any textbook on linear control theory (see, e.g., [20]). It is worth mentioning that the proof is constructive. Namely, the linear mapping $\mathcal{O}$ defines a similarity transformation such that the matrix $A - LC$ is similar to the following matrix in Frobenius form:

$$\begin{bmatrix} 0 & \cdots & 0 & a_1 - l_1 \\ 1 & \cdots & 0 & a_2 - l_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & a_n - l_n \end{bmatrix}$$

where $\mathrm{col}(l_1, l_2, \ldots, l_n) = \mathcal{O}(L)$, and the a_i are the coefficients of the characteristic polynomial of A. Since $a_i - l_i$ are the coefficients of the characteristic polynomial of $A - LC$, it is always possible to locate the eigenvalues of $A - LC$ in the open unit disc by means of an appropriate choice of the matrix L.

It is worth mentioning that the condition of observability is, in fact, a sufficient but not necessary condition to allow design of an observer. Namely, the system may have $\mathcal{O}(x)$ of rank lower than n, but at the same time, it may admit an observer. This situation occurs when the so-called unobservable dynamics are exponentially stable. In the control literature, linear systems with exponentially stable unobservable dynamics are referred to as *detectable* (see [20]). In practice it often means that such systems can be transformed to an observable system via model reduction.

Example 20.1

Consider the following discrete-time dynamics in Lur'e form:

$$\begin{aligned} \begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & -\alpha \\ 1 & 1+\alpha \end{bmatrix}}_{A} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -\beta\cos y(k) \end{bmatrix}}_{\varphi(y(k))} \\ y(k) &= \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} z(k) \end{aligned} \tag{20.3.8}$$

where $\alpha, \beta > 0$. In this case, we obtain

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1+\alpha \end{bmatrix} \tag{20.3.9}$$

which clearly is an invertible matrix. Thus, one may construct an observer for (20.3.8) of the following form:

$$\begin{aligned} \begin{bmatrix} \widehat{z}_1(k+1) \\ \widehat{z}_2(k+1) \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & -\alpha \\ 1 & 1+\alpha \end{bmatrix}}_{A} \begin{bmatrix} \widehat{z}_1(k) \\ \widehat{z}_2(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -\beta\cos y(k) \end{bmatrix}}_{\varphi(y(k))} + \\ &\quad +L(y(k) - \widehat{y}(k)) \\ \widehat{y}(k) &= \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \widehat{z}(k) \end{aligned} \tag{20.3.10}$$

where $L = \mathrm{col}(l_1, l_2)$, and l_1 and l_2 are chosen such that all eigenvalues of the matrix

$$A - LC = \begin{bmatrix} 0 & -\alpha - l_1 \\ 1 & 1+\alpha - l_2 \end{bmatrix} \tag{20.3.11}$$

lie in the open unit disc.

20.4 Transformation into Lur'e Form

In the previous section we learned that if the transmitter dynamics are in Lur'e form (20.3.4) and the pair (C, A) is observable, then it is always possible to design a receiver system which synchronizes with (20.3.4).

The result presented in the previous section is very simple. However, the following question remains open: what can we do if the transmitter dynamics are not in the form (20.3.4)? In this section we will present a partial answer to this question.

First of all, notice that the representation (20.3.4) is coordinate dependent. This means that if one rewrites system (20.3.4) in a new coordinate system via a (nonlinear) coordinate change $z = T(x)$, then a new representation of the *same* dynamical system is not necessarily in the form (20.3.4).

By the same token, however, this may also mean that it is possible to transform a system into Lur'e form by means of a nonlinear coordinate change $z = T(x)$. Hence, we arrive at the following problem.

Let a discrete-time system (20.2.1, 20.2.2) with scalar output be given, and assume that $f(0) = 0$, $h(0) = 0$. The problem is to find conditions ensuring existence of an invertible coordinate change $z = T(x)$ such that the system (20.2.1) is locally (or globally) equivalent to the following Lur'e system

$$z(k+1) = Az(k) + \varphi(y(k)), \quad y(k) = Cz(k) \tag{20.4.12}$$

where the pair (C, A) is observable.

As one can see from the problem statement, the coordinate change $z = T(x)$ can be either locally or globally defined (i.e., the inverse mapping T^{-1} can exist on a neighborhood of the origin or everywhere). In the first case the systems (20.2.1, 20.2.2) and (20.4.12) are equivalent if for all k one has that $||x(k)||$ is sufficiently small. In the second case there are no such restrictions.

The following result from [12] gives a (local) solution to the problem.

THEOREM 20.2

A discrete-time system (20.2.1, 20.2.2) *with single output is locally equivalent to a system in Lur'e form* (20.4.12) *with observable pair* (C, A) *via a coordinate change* $z = T(x)$ *if and only if*

(i) *the pair* $(\partial h(0)/\partial x, \partial f(0)/\partial x)$ *is observable,*

(ii) *the Hessian matrix of the function* $h \circ f^n \circ \mathcal{O}^{-1}(s)$ *is diagonal, where*

$x = \mathcal{O}^{-1}(s)$ is the inverse map of

$$\mathcal{O}(x) = \begin{bmatrix} h(x) \\ h \circ f(x) \\ \vdots \\ h \circ f^{n-1}(x) \end{bmatrix}, \tag{20.4.13}$$

with $h \circ f(x) := h(f(x)), f^1 := f, f^j := f \circ f^{j-1}$.

It is important to notice that condition (i) means that the Jacobian $\partial\mathcal{O}(0)/\partial x$ is invertible. In an equivalent form it can be rewritten in the form

$$\dim\left(\operatorname{span}\left\{\frac{\partial h}{\partial x}(0), \frac{\partial h \circ f}{\partial x}(0), \ldots, \frac{\partial h \circ f^{n-1}}{\partial x}(0)\right\}\right) = n.$$

The condition (ii) may be interpreted in the following way. As indicated above, if condition (i) holds, the transformation $s = \mathcal{O}(x)$ is a local diffeomorphism. Thus, s forms a new set of local coordinates for the dynamics (20.2.1) around the origin. It is straightforwardly checked that, in these new coordinates, the system (20.2.1, 20.2.2) takes the form

$$\left\{\begin{array}{rcl} s_1(k+1) & = & s_2(k) \\ & \vdots & \\ s_{n-1}(k+1) & = & s_n(k) \\ s_n(k+1) & = & f_s(s(k)) \\ y(k) & = & s_1(k) \end{array}\right. \tag{20.4.14}$$

where $f_s(s) := h \circ f^n \circ \mathcal{O}^{-1}(s)$. In the literature (see [15]), the form (20.4.14) is referred to as the *observable form* of the system (20.2.1, 20.2.2). Condition (ii) then is equivalent to the local existence of functions $\varphi_1, \cdots, \varphi_n$: $\mathbf{R} \to \mathbf{R}$ such that

$$f_s(s) = \varphi_1(s_1) + \varphi_2(s_2) + \cdots + \varphi_n(s_n). \tag{20.4.15}$$

With the functions $\varphi_1, \cdots, \varphi_n$ at hand, the transformation

$$z_i := s_{n+1-i} - \sum_{k=i+1}^{n} \varphi_k(s_{k-i}) \quad (i = 1, \cdots, n) \tag{20.4.16}$$

then transforms the observable form (20.4.14) into the following Lur'e form:

$$\left\{\begin{array}{rcl} z_1(k+1) & = & \varphi_1(y(k)) \\ z_2(k+1) & = & z_1(k) + \varphi_2(y(k)) \\ & \vdots & \\ z_n(k+1) & = & z_{n-1}(k) + \varphi_n(y(k)) \\ y(k) & = & z_n(k) \end{array}\right. \tag{20.4.17}$$

The mapping $\mathcal{O}$ in (20.4.13) and the observable form play an important role in the observer design for nonlinear discrete-time systems. As one can easily see that, in the linear case, the mapping $\mathcal{O}$ is exactly the linear operator (20.3.7) introduced in the previous section. Since the Jacobian of $\mathcal{O}$ is invertible around $x = 0$ the mapping $\mathcal{O}$ is a local diffeomorphism. If one is interested in finding a coordinate change $z = T(x)$ which is globally defined, it is sufficient to check that $\mathcal{O}$ is a global diffeomorphism from $\mathbf{R}^n$ to $\mathbf{R}^n$ *and* the functions $\varphi_1, \cdots, \varphi_n$ satisfying (20.4.15) exist globally.

Example 20.2

[(*Bouncing ball*)]

Consider the following discrete-time model which describes the bouncing ball system [23, 3]:

$$\begin{cases} x_1(k+1) = x_1(k) + x_2(k) \\ x_2(k+1) = \alpha x_2(k) - \beta \cos(x_1(k) + x_2(k)) \end{cases} \tag{20.4.18}$$

where $x_1(k)$ is the phase of the table at the k-th impact, $x_2(k)$ is proportional to the velocity of the ball at the k-th impact, the parameter α is the coefficient of restitution, and $\beta = 2\omega^2(1+\alpha)A/g$. Here ω is the angular frequency of the table oscillation, A is the corresponding amplitude, and g is the gravitational acceleration. For some values of the parameters the system can exhibit very complex behavior. However, we will show that this is not an obstacle for the design of an observer.

Suppose only the first variable x_1 (the phase) is available for measurement. The question is: can we reconstruct the second variable? Clearly the system (20.4.18) is not in Lur'e form. However, using the theory presented in this section, we will show that there exists a coordinate change that transform (20.4.18) into Lur'e form.

So, we assumed that

$$y(k) = h(x(k)) = x_1(k).$$

Let us check the conditions of Theorem 20.2. A simple calculation gives

$$\frac{\partial h(0)}{\partial x} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \frac{\partial f(0)}{\partial x} = \begin{bmatrix} 1 & 1 \\ 0 & \alpha \end{bmatrix}$$

and this pair is clearly observable. Hence, condition (i) is satisfied.

To check condition (ii), let us find the mapping $\mathcal{O}$. Obviously,

$$\mathcal{O}(x) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x \tag{20.4.19}$$

with $x = \mathrm{col}(x_1, x_2)$. This mapping is linear, it is invertible, and, therefore, it is a global diffeomorphism. Introducing $s = \mathrm{col}(s_1, s_2) := \mathcal{O}(x)$ we see that

$$f_s(s) := h \circ f^2 \circ \mathcal{O}^{-1}(s) = -\alpha s_1 + (1+\alpha)s_2 - \beta \cos s_2$$

and it is clear that the Hessian of this function is diagonal. Thus, condition (ii) is satisfied as well. Note that, in view of (20.4.15), we obtain $f_s(s) = \varphi_1(s_1) + \varphi_2(s_2)$, with $\varphi_1(s_1) := -\alpha s_1$, $\varphi_2(s_2) := (1+\alpha)s_2 - \beta\cos s_2$. Therefore there exists a coordinate change which locally transforms the system (20.4.18) into Lur'e form. Moreover, the mapping $\mathcal{O}$ is a global diffeomorphism *and* the functions φ_1, φ_2 are globally defined, which implies that this coordinate change is, in fact, global.

From (20.4.19) and (20.4.16) we obtain the following coordinate change:

$$\begin{cases} z_1 = -\alpha x_1 + x_2 + \beta \cos x_1 \\ z_2 = x_1 \end{cases} \tag{20.4.20}$$

with the output $y = z_2 = x_1$. In the new coordinate system the original system (20.4.18) has the following form

$$\begin{cases} z_1(k+1) = -\alpha z_2(k) \\ z_2(k+1) = z_1(k) + (1+\alpha)z_2(k) - \beta\cos z_2(k). \end{cases} \tag{20.4.21}$$

Note that the dynamics (20.4.21) are identical to the dynamics (20.3.8). Therefore, an observer for (20.4.21) is given by (20.3.10).

The estimates $\widehat{x}_1, \widehat{x}_2$ for x_1, x_2 are given by the following relations, which immediately follow from (20.4.20)

$$\begin{cases} \widehat{x}_1 = \widehat{z}_2 \\ \widehat{x}_2 = \widehat{z}_1 + \alpha\widehat{z}_2 - \beta\cos\widehat{z}_2 \end{cases} \tag{20.4.22}$$

with $\hat{z}_1, \hat{z}_2$ the observer state for (20.4.21). Moreover, by means of an appropriate choice of l_1, l_2 one can achieve arbitrarily fast convergence of $\widehat{x}(k)$ to $x(k)$.

20.5 Transformation into Extended Lur'e Form

In the previous section we found that if the observability mapping $\mathcal{O}$ is a diffeomorphism and condition (ii) of Theorem 20.2 holds, then there exists a coordinate change transforming the system (20.2.1, 20.2.2) into Lur'e form, which makes the observer design a simple linear algebraic problem. Condition (ii) of Theorem 20.2 is especially restrictive. Therefore, the question arises whether, and in what way, this condition may be relaxed.

To answer this question, we will assume, in this section, that at time k not only $y(k)$ but also the past output measurements $y(k-1), \cdots, y(k-N)$

for some $N > 0$ are available. We first consider nonlinear dynamics of the following form:

$$\left\{ \begin{array}{rcl} x(k+1) & = & Ax(k) + \varphi(y(k), y(k-1), \cdots, y(k-N)) \\ y(k) & = & Cx(k) \end{array} \right. \qquad (20.5.23)$$

where $x(k) \in \mathbf{R}^n$, $y(k) \in \mathbf{R}^1$, $\varphi : \mathbf{R}^{N+1} \rightarrow \mathbf{R}^n$ is a smooth mapping, and A, C are matrices of appropriate dimensions. Note that the dynamics (20.5.23) for $N = 0$ are just the dynamics (20.3.4). Therefore, we refer to dynamics of the form (20.5.23) as dynamics in *extended Lur'e form with buffer N*. Assume that the pair (C, A) is observable. As we have seen in Section 20.3 there then exists a matrix L such that all eigenvalues of $A - LC$ lie in the open unit disc. Along the same lines as in Section 20.3, it may then be shown that the following dynamics are an observer for (20.5.23):

$$\left\{ \begin{array}{rcl} \widehat{x}(k+1) & = & A\widehat{x}(k) + \varphi(y(k), \cdots, y(k-N)) + L(y(k) - \widehat{y}(k)) \\ \widehat{y}(k) & = & C\widehat{x}(k) \end{array} \right. \qquad (20.5.24)$$

As in Section 20.4, we now ask ourselves the question under what conditions the discrete-time system (20.2.1, 20.2.2) may be transformed into an extended Lur'e form for some $N \geq 0$. The transformations we are going to use here are more general than the transformation in Section 20.4, in the sense that we also allow them to depend on the past output measurements $y(k-1), \cdots, y(k-N)$. More specifically, we will be looking at parametrized transformations $z = T(x, \xi_1, \cdots, \xi_N)$, where $z \in \mathbf{R}^n$, with the property that (locally or globally) there exists a mapping $T^{-1}(\cdot, \xi_1, \cdots, \xi_N) : \mathbf{R}^n \rightarrow \mathbf{R}^n$ parametrized by $(\xi_1, \cdots, \xi_N)$, such that for all $(\xi_1, \cdots, \xi_N)$ we have

$$T(T^{-1}(z, \xi_1, \cdots, \xi_N), \xi_1, \cdots, \xi_N) = z$$

A mapping having this property will be referred to as an *extended coordinate change*. We will then say that the system (20.2.1, 20.2.2) may be transformed into an extended Lur'e form with buffer N if there exists an extended coordinate change $T(\cdot, \xi_1, \cdots, \xi_N) : \mathbf{R}^n \rightarrow \mathbf{R}^n$ parametrized by $(\xi_1, \cdots, \xi_N)$ such that the variable

$$z(k) := T(x(k), y(k-1), \cdots, y(k-N)) \qquad (20.5.25)$$

satisfies (20.5.23), where the pair (C, A) is observable. As pointed out above, one may then build an observer (20.5.24) for $z(k)$ in (20.5.25). From this observer, one then obtains estimates $\widehat{x}(k)$ for $x(k)$ by inverting the extended coordinate change T:

$$\widehat{x}(k) := T^{-1}(\widehat{z}(k), y(k-1), \cdots, y(k-N)) \qquad (20.5.26)$$

The following result from [9] (see also [8]) gives conditions under which a system (20.2.1, 20.2.2) may be transformed into an extended Lur'e form with buffer N.

THEOREM 20.3
Consider a discrete-time system (20.2.1, 20.2.2), *and assume that the mapping $\mathcal{O}$ in* (20.4.13) *is a local diffeomorphism. Let $N \in \{0, \cdots, n-1\}$ be given. Then* (20.2.1, 20.2.2) *may be locally transformed into an extended Lur'e form with buffer N if and only if there locally exist functions $\varphi_{N+1}, \cdots, \varphi_n : \mathbf{R}^{N+1} \to \mathbf{R}$ such that the function f_s in the observable form* (20.4.14) *satisfies*

$$f_s(s_1, \cdots, s_n) = \sum_{i=N+1}^{n} \varphi_i(s_i, \cdots, s_{i-N}) \tag{20.5.27}$$

The proof of the above theorem is constructive. Namely, assume that functions $\varphi_{N+1}, \cdots, \varphi_n$ satisfying (20.5.27) exist, and define an extended coordinate change by

$$z_i := \begin{cases} s_{n-i+1} - \sum_{j=N+1}^{n} \varphi_j(s_{j-i}, \cdots, s_{j-i-N}) & (i = 1, \cdots, N-1) \\ \\ s_{n-i+1} - \sum_{j=i+1}^{n} \varphi_j(s_{j-i}, \cdots, s_{j-i-N}) & (i = N, \cdots, n) \end{cases} \tag{20.5.28}$$

It is then straightforwardly checked that in these new extended coordinates the observable form (20.4.14) takes the following extended Lur'e form:

$$\left\{ \begin{array}{rcl} z_1(k+1) & = & 0 \\ z_2(k+1) & = & z_1(k) \\ & \vdots & \\ z_N(k+1) & = & z_{N-1}(k) \\ z_{N+1}(k+1) & = & z_N(k) + \varphi_{N+1}(y(k), \cdots, y(k-N)) \\ & \vdots & \\ z_n(k+1) & = & z_{n-1}(k) + \varphi_n(y(k), \cdots, y(k-N)) \\ y(k) & = & z_n(k) \end{array} \right. \tag{20.5.29}$$

Theorem 20.3 gives necessary and sufficient conditions for the local existence of an extended Lur'e form with buffer N for (20.2.1, 20.2.2). For global existence of an extended Lur'e form with buffer N, the mapping $\mathcal{O}$ in (20.4.13) needs to be a global diffeorphism, and the mappings $\varphi_{N+1}, \cdots, \varphi_n$ satisfying (20.5.27) need to exist globally.

It is easily checked that for $N = n-1$, condition (20.5.27) is always satisfied globally. Thus, we have a system (20.2.1, 20.2.2) for which the mapping $\mathcal{O}$ in (20.4.13) is a local (global) diffeomorphism that may always be locally (globally) transformed into an extended Lur'e form with buffer $n-1$.

20.6 Observers for Perturbed Linear Systems

So far the design procedure for observers has been based on the assumption that for the discrete-time system under consideration the mapping $\mathcal{O}$ in (20.4.13) is a (local or global) diffeomorphism. In the sequel, we consider a particular class of systems for which this might not be the case. Namely, we consider systems of the form

$$\left\{ \begin{array}{rcl} x(k+1) & = & Ax(k) + Bf(x(k)) \\ y(k) & = & Cx(k) \end{array} \right. \tag{20.6.30}$$

where $x(k) \in \mathbf{R}^n$ is the state, $y(k) \in \mathbf{R}^1$ is the scalar output, the function $f : \mathbf{R}^n \to \mathbf{R}^1$ is smooth, A, B, C are matrices of appropriate dimensions, and the pair (C, A) is observable. Clearly, depending on the specific structure of f and B, the system (20.6.30) may have a mapping $\mathcal{O}$ that is not a diffeomorphism. Nevertheless, we may derive conditions on (20.6.30) that guarantee the existence of an observer.

Define the rational function $G(s)$ by

$$G(s) := C(sI - A)^{-1}B\,. \tag{20.6.31}$$

Then $G(s)$ has the form $G(s) = \frac{q(s)}{p(s)}$, where q and p are polynomials in s, with $\deg(p) > \deg(q)$. We now assume that $\deg(p) - \deg(q) = 1$. It may be shown that this is equivalent to the fact that $CB \neq 0$. To obtain an observer for (20.6.30), we first define new coordinates in the following way. Since $CB \neq 0$, there exists an $(n-1) \times n$ matrix N such that $NB = 0$ and the matrix $S := \left[\begin{array}{cc} C^T & N^T \end{array} \right]^T$ is invertible. Thus, $(\xi, z) := (Cx, Nx)$ forms a new set of coordinates for (20.6.30). It is straightforwardly checked that in these new coordinates the system (20.6.30) takes the form

$$\left\{ \begin{array}{rcl} \xi(k+1) & = & \bar{f}(\xi(k), z(k)) \\ z(k+1) & = & A_1\xi(k) + A_2 z(k) \\ y(k) & = & \xi(k) \end{array} \right. \tag{20.6.32}$$

where

$$\bar{f}(\xi, z) = C\left[AS^{-1}\left[\begin{array}{c} \xi \\ z \end{array}\right] + CBf\left(S^{-1}\left[\begin{array}{c} \xi \\ z \end{array}\right]\right)\right]$$

and

$$\left[\begin{array}{cc} A_1 & A_2 \end{array}\right] = NAS^{-1}$$

We now assume the following:

A1 The mapping $\bar{f}$ in (20.6.32) is globally Lipschitz with respect to z, i.e., there exists an $L > 0$ such that

$$(\forall \xi \in \mathbf{R})(\forall z, \bar{z} \in \mathbf{R}^{n-1})(|\bar{f}(\xi, \bar{z}) - \bar{f}(\xi, z)| < L\|\bar{z} - z\|)$$

A2 All zeros of the polynomial $q(s)$ are located in the open unit disc.

As an observer candidate we take the following system:

$$\left\{ \begin{array}{rcl} \widehat{\xi}(k+1) & = & \bar{f}(y(k), \widehat{z}(k)) \\ \widehat{z}(k+1) & = & A_1 y(k) + A_2 \widehat{z}(k) \end{array} \right. \tag{20.6.33}$$

We then have the following result.

THEOREM 20.4
Assume that for (20.6.30) *we have that the pair* (C, A) *is observable, that* $CB \neq 0$, *and that assumptions* **A1** *and* **A2** *hold. Then* (20.6.33) *is an observer for* (20.6.32).

PROOF Defining the error signals

$$e_\xi(k) := \xi(k) - \widehat{\xi}(k), \quad e_z(k) := z(k) - \widehat{z}(k),$$

we obtain the following error equations:

$$\left\{ \begin{array}{rcl} e_\xi(k+1) & = & \bar{f}(\xi(k), e_z(k) + \widehat{z}(k)) - \bar{f}(\xi(k), \widehat{z}(k)) \\ e_z(k+1) & = & A_2 e_z(k) \end{array} \right. \tag{20.6.34}$$

It is easily checked that assumption **A2** implies that all eigenvalues of A_2 are in the open unit disc. This implies, on its turn, that there exist $\gamma > 0, 0 < \lambda < 1$ such that $e_z(k)$ satisfies

$$\|e_z(k)\| \leq \gamma \lambda^k \|e_z(0)\| \tag{20.6.35}$$

Using assumption **A1** and (20.6.35), we then obtain

$$\begin{array}{l} |e_\xi(k)| = |\bar{f}(\xi(k-1), e_z(k-1) + \widehat{z}(k-1)) - \bar{f}(\xi(k-1), \widehat{z}(k-1))| \\ \qquad < L\|e_z(k-1)\| \leq L\gamma\lambda^{k-1}\|e_z(0)\| \end{array} \tag{20.6.36}$$

Since $0 < \lambda < 1$, it follows from (20.6.35) and (20.6.36) that $e_\xi(k), e_z(k) \to 0$ for $k \to +\infty$, and thus (20.6.33) is an observer for (20.6.32).

Remark A.
The result in this section may be generalized to systems (20.6.30) for which we have $\deg(p) - \deg(q) > 1$. This generalization will be given in a forthcoming paper.

20.7 Conclusions

Following a similar line of research as in [14] we develop an observer perspective on the synchronization problem for nonlinear (complex) discrete-time systems. For several classes of discrete-time systems it is shown that a suitable observer can be found. In case such an observer does not exist, or cannot be found analytically, we propose to use an extended observer. The latter method follows [8] (see also [9]), and presents an observer that also uses past measurements and can be applied under fairly general conditions. Like the continuous-time paper [14], it seems that control theory might be a very valuable tool in the study of synchronization.

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Coordination of two robot manipulators based on position measurements only

A. RODRIGUEZ-ANGELES†* and H. NIJMEIJER†

In this note we propose a controller that solves the problem of coordination of two (or more) robots, under a master–slave scheme, in the case when only position measurements are available. The controller consists of a feedback control law, and two non-linear observers. It is shown that the controller yields ultimate uniformly boundedness of the closed loop errors, a relation between this bound and the gains on the controller is established. Simulation results on two two-link robot systems show the predicted convergence performance.

1. Introduction

Synchronization, coordination, and cooperation are intimately linked subjects, and sometimes they are used as synonymous to describe the same kind of behaviour, mainly in mechanical systems. Nowadays, there are several papers related to synchronization of rotating bodies and electrical–mechanical systems (see for instance Blekhman *et al*. 1995, Huijberts *et al*. 2000), and communication systems (Pecora and Carroll 1990). Rotating mechanical structures form a very important and special class of systems that, with or without the interaction through some coupling, exhibit synchronized motion. On the other hand, for mechanical systems synchronization is of great importance as soon as two machines have to cooperate. This cooperative behaviour gives flexibility and dexterity that cannot be achieved by an individual system, e.g. multi-finger robot hands, multi-robot systems, multi-actuated platforms.

Typically robot coordination, and cooperation of manipulators (see Liu *et al*. 1997, 1999, Brunt 1998), form important illustrations of the same goal, where it is desired that two or more mechanical systems, either identical or different, are asked to work in synchrony. In robot coordination the basic problem is to ascertain synchronous motion of two (or more) robotic systems. This is obviously a control problem where, at least for one of the robots, a suitable feedback controller has to be designed such that this robot (slave) follows the other robot (master). This problem is further complicated by the fact that frequently only position measurements of both master and slave robots are available. This partial access to the state of the system has been the reason for developing model-based observers which are integrated in the feedback control loop.

In practice, robot manipulators are equipped with high precision position sensors, such as encoders. Meanwhile new technologies have been design for measuring velocities, e.g. brushless AC motors with digital servo-drivers. Nevertheless such technologies are not yet common in applications. Therefore, velocity measurements are often obtained by means of tachometers which are contaminated by noise. Moreover, velocity sensoring equipment is frequently omitted due to the savings in cost, volume and weight that can be obtained. For these reasons, a number of model-based robot control methods have been proposed (Nicosia and Tomei 1990, Canudas *et al*. 1992). In these methods a velocity observer is integrated in the control loop, although exact knowledge of the non-linear robot dynamics is assumed, which in practice is generally not available. To overcome this drawback, robust tracking controllers only based on position measurements have been proposed (Canudas and Fixot 1991, Berghuis and Nijmeijer 1994, Wong Lee and Khalil 1997). However, all the aforementioned papers deal with the tracking control problem and not with the robot coordination problem.

The problem of coordinating (synchronizing) physical systems, can be seen as tracking between two physical systems. Although it seems to be a straightforward extension of classical tracking controllers, this problem implies challenges that are not considered in the design of tracking controllers. Most of the tracking controllers are based on full knowledge of the desired reference to be tracked, and no one predicts what would happen in the case of partial knowledge of the reference signal, or how to deal with it.

In this paper we present a novel approach for the coordination of two robot manipulators, assuming only position measurements of both robots. Partial knowledge of the reference signals (master trajectories) and the working signals (slave variables) demand the reconstruction of the missing required signals. We solve this problem by using (non-linear) model-based observers. The estimated variables are used in a feedback loop such that the overall coordinating controller, i.e. feedback control plus the observers, guarantees synchronization of the slave and master robots. Of course, other ways of estimating velocity signals, like numerical

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* Author for correspondence. e-mail: A.Rodriguez@tue.nl
† Eindhoven University of Technology, Department of Mechanical Engineering, P.O. Box 513, 5600 MB Eindhoven, The Netherlands.

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differentiation or low pass filters, are available, and in principle such alternatives could be used in the developed control scheme. However, our aim is to provide a systematic way of proving the ultimate uniform semi-global boundedness of the master–slave system. It seems plausible that in a similar manner the same result can be shown with an alternative velocity-estimated control scheme. We leave this to the interested reader.

The general set-up to be considered is as follows. Consider two fully actuated robot manipulators with n joints each. One of these robots (master) is driven by an input torque $\tau_m(\cdot)$ that, in the ideal case, ensures convergence of the joint variables $q_m, \dot{q}_m \in \mathbb{R}^n$ to a desired trajectory $q_d, \dot{q}_d \in \mathbb{R}^n$. However, the input torque τ_m is unknown, at least for the controller design of the second robot (slave), as well as the joint velocity and acceleration variables $\dot{q}_m, \ddot{q}_m$. Under these assumptions, the goal is to design a control law $\tau_s(\cdot)$ for the slave robot such that its joint variables $q_s, \dot{q}_s \in \mathbb{R}^n$ synchronize with the variables $q_m, \dot{q}_m$ of the master robot. Also, we assume that the joint velocities and accelerations $\dot{q}_s, \ddot{q}_s$ are not available. Therefore, the control law τ_s that is to be designed can only depend on position measurements of both robots, i.e. q_m, q_s, and estimated values of the joint velocities and accelerations $\dot{q}_m, \ddot{q}_m, \dot{q}_s, \ddot{q}_s$. Notice that the goal is to follow the trajectories of the master robot $q_m, \dot{q}_m$, and not the desired trajectories $q_d, \dot{q}_d$; therefore knowledge of $q_d, \dot{q}_d$ is not necessary to design the control law τ_s for the slave robot.

This paper is organized as follows. In § 2 the dynamic model of the robot and some of its properties are presented. The feedback control law and the observers for slave and master velocities are proposed in § 3. In § 4 the convergence properties of the closed loop system are examined. In § 5 a simulation study shows the predicted convergence performance. Sections 6 and 7 present some remarks and general conclusions. Throughout this paper standard notation is used, in particular, vector norms are Euclidean, and for matrices the induced norm $\|A\| = \sqrt{\lambda_{\max}(A^{\mathrm{T}}A)}$ is employed, with $\lambda_{\max}(\cdot)$ the maximum eigenvalue. Moreover, for any positive definite matrix A we denote by A_m and A_M its minimum and maximum eigenvalue respectively.

2. Dynamic model of the robot manipulators

Consider a pair of rigid robots, each one with the same number of joints, i.e. $q_i \in \mathbb{R}^n$, where $i = m, s$ identifies the master (m) and slave (s) robot; all the joints are rotational, actuated and, without loss of generality, frictionless. This does not mean, however, that they are identical in their parameters (masses, inertias, etc.).

For each of the robots, the kinetic energy is given by $T_i(q_i, \dot{q}_i) = \frac{1}{2}\dot{q}_i^{\mathrm{T}} M_i(q_i)\dot{q}_i$, $i = m, s$, with $M_i(q_i) \in \mathbb{R}^{n\times n}$ the symmetric, positive-definite inertia matrix, and the potential energy is denoted by $U_i(q_i)$. Hence, applying the Euler–Lagrange formalism (Spong and Vidyasagar 1989) the dynamic model of the robot is given by

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i) = \tau_i \qquad i = m, s \tag{1}$$

where $g_i(q_i) = (\partial/\partial q_i)U_i(q_i) \in \mathbb{R}^n$ denotes the gravity forces, $C_i(q_i, \dot{q}_i)\dot{q}_i \in \mathbb{R}^n$ represents the Coriolis and centrifugal forces, and τ_i denotes the $[n \times 1]$ vector of input torques.

In the subsequent sections we use the following properties.

- If the matrix $C_i(q_i, \dot{q}_i) \in \mathbb{R}^{n\times n}$ is defined using the Christoffel symbols (Spong and Vidyasagar 1989), then the matrix $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew symmetric, i.e.

$$x^{\mathrm{T}}(\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i))x = 0 \qquad \text{for all} \quad x \in \mathbb{R}^n \tag{2}$$

- In addition, for the previous choice of $C_i(q_i, \dot{q}_i)$, it can be written as

$$C_i(q_i, \dot{q}_i) = \begin{bmatrix} \dot{q}_i^{\mathrm{T}} C_{i1}(q_i) \\ \vdots \\ \dot{q}_i^{\mathrm{T}} C_{in}(q_i) \end{bmatrix} \tag{3}$$

  where $C_{ij}(q_i) \in \mathbb{R}^{n\times n}$, $j = 1, \ldots, n$ are symmetric matrices (Craig 1988). It follows that for any scalar α and for all $q_i, x, y, z \in \mathbb{R}^n$

$$\left.\begin{aligned} C_i(q_i, x)y &= C_i(q_i, y)x \\ C_i(q_i, z + \alpha x)y &= C_i(q_i, z)y + \alpha C_i(q_i, x)y \end{aligned}\right\} \tag{4}$$

- $M_i(q_i)$, $C_i(q_i, \dot{q}_i)$ are bounded with respect to q_i (Lewis *et al.* 1993), so

$$0 < M_{i,m} \le \|M_i(q_i)\| \le M_{i,M} \qquad \text{for all} \quad q_i \in \mathbb{R}^n \tag{5}$$

$$\|C_i(q_i, x)\| \le C_{i,M}\|x\| \qquad \text{for all} \quad q_i, x \in \mathbb{R}^n. \tag{6}$$

3. Feedback controller

As stated in § 1, it is assumed that there is no access to $(\dot{q}_m, \ddot{q}_m)$ and $(\dot{q}_s, \ddot{q}_s)$, but only joint positions q_m and q_s can be measured. Therefore, the controller τ_s can only depend on positions measurements (q_m, q_s) and estimated values for the velocities $(\dot{q}_m, \dot{q}_s)$ and accelerations $(\ddot{q}_m, \ddot{q}_s)$.

3.1. *Feedback control law*

With the control law proposed by Paden and Panja (1988) in mind, and under the assumptions that the estimated values are available and the non-linearities and parameters of the slave robot are known, we propose the controller τ_s for the slave robot as

$$\tau_s = M_s(q_s)\hat{\ddot{q}}_m + C_s(q_s, \hat{\dot{q}}_s)\hat{\dot{q}}_m + g_s(q_s) - K_d\hat{\dot{e}}_s - K_p e_s \tag{7}$$

were $\hat{\dot{q}}_s, \hat{\dot{e}}_s, \hat{\dot{q}}_m, \hat{\ddot{q}}_m \in \mathbb{R}^n$ represent the estimates of $\dot{q}_s, \dot{e}_s, \dot{q}_m$ and $\ddot{q}_m$ respectively, the tracking errors $e_s, \dot{e}_s \in \mathbb{R}^n$ are defined by

$$e_s := q_s - q_m, \qquad \dot{e}_s := \dot{q}_s - \dot{q}_m \tag{8}$$

$M_s(q_s)$, $C_s(q_s, \hat{\dot{q}}_s)$ and $g_s(q_s)$ are defined as in equation (1), and $K_p, K_d \in \mathbb{R}^{n\times n}$ are positive definite gain matrices.

3.2. *An observer for the tracking errors* $(e_s, \dot{e}_s)$

Estimated values for the tracking errors $e_s, \dot{e}_s$ (8) are denoted by $\hat{e}_s, \hat{\dot{e}}_s$; these estimated values are obtained by the non-linear Luenberger observer

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_s &= \hat{\dot{e}}_s + \Lambda_1\tilde{e} \\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{\dot{e}}_s &= -M_s(q_s)^{-1}[C_s(q_s, \hat{\dot{q}}_s)\hat{\dot{e}}_s + K_d\hat{\dot{e}}_s + K_p\hat{e}_s] \\ &\quad + \Lambda_2\tilde{e} \end{aligned}\right\} \tag{9}$$

where the estimation position and velocity tracking errors $\tilde{e}, \tilde{\dot{e}}$ are defined by

$$\tilde{e} := e_s - \hat{e}_s, \quad \tilde{\dot{e}} := \dot{e}_s - \hat{\dot{e}}_s \tag{10}$$

and $\Lambda_1, \Lambda_2 \in \mathbb{R}^{n\times n}$ are positive definite gain matrices.

3.3. *An observer for the slave joint variables* $(q_s, \dot{q}_s)$

Let $\hat{q}_s, \hat{\dot{q}}_s$ denote estimated values for $q_s, \dot{q}_s$. To compute these estimated values, we propose the non-linear observer

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\hat{q}_s &= \hat{\dot{q}}_s + L_{p1}\tilde{e}_q \\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{\dot{q}}_s &= -M_s(q_s)^{-1}[C_s(q_s, \hat{\dot{q}}_s)\hat{\dot{e}}_s + K_d\hat{\dot{e}}_s + K_p e_s] \\ &\quad + L_{p2}\tilde{e}_q \end{aligned}\right\} \tag{11}$$

where the estimation position and velocity errors $\tilde{e}_q$ and $\tilde{\dot{e}}_q$ are defined by

$$\tilde{e}_q := q_s - \hat{q}_s \qquad \tilde{\dot{e}}_q := \dot{q}_s - \hat{\dot{q}}_s \tag{12}$$

and $L_{p1}, L_{p2} \in \mathbb{R}^{n\times n}$ are positive definite gain matrices.

3.4. *Estimated values for* $\dot{q}_m, \ddot{q}_m$

As stated, the master robot variables $\dot{q}_m, \ddot{q}_m$ are not available, therefore estimated values for $\dot{q}_m, \ddot{q}_m$ are used in τ_s (7). From (8) and the definition of the estimated variables $\hat{e}_s, \hat{\dot{e}}_s, \hat{q}_s, \hat{\dot{q}}_s$, we can consider that estimated values for $q_m, \dot{q}_m, \ddot{q}_m$ are given by

$$\left.\begin{aligned} \hat{q}_m &= \hat{q}_s - \hat{e}_s \\ \hat{\dot{q}}_m &= \hat{\dot{q}}_s - \hat{\dot{e}}_s \\ \hat{\ddot{q}}_m &= \frac{\mathrm{d}}{\mathrm{d}t}(\hat{\dot{q}}_s - \hat{\dot{e}}_s) \end{aligned}\right\} \tag{13}$$

Remark 1: Note that, in (9) and (10) the estimate for $\dot{e}_s$ is given by $\hat{\dot{e}}_s$, not by $\dot{\hat{e}}$. This definition introduces an extra correcting term in $\dot{\tilde{e}}_q$, as it follows from (9) and (10) that

$$\dot{\tilde{e}} = \dot{e}_s - \dot{\hat{e}}_s = \tilde{\dot{e}} - \Lambda_1\tilde{e}$$

The term $\Lambda_1\tilde{e}$ gives faster estimation performance, especially during transients, but it has some negative effects on noise sensitivity, since it amplifies noise measurements on $\tilde{e}$.

The same can be said for observer (11) and the estimation position and velocity errors (12).

4. Ultimate boundedness of the closed loop system

To simplify the stability analysis, we make the following assumptions on the positive definite gain matrices K_p, K_d, L_{p1}, L_{p2}, Λ_1, Λ_2.

Assumption 1: *The gain matrices Λ_1, Λ_2 and L_{p1}, L_{p2} satisfy*

$$\Lambda_1 = L_{p1}, \qquad \Lambda_2 = L_{p2} \tag{14}$$

Assumption 2: *The gains K_p, K_d, L_{p1}, L_{p2} are symmetric matrices.*

In addition, the following assumption is required.

Assumption 3: *The signals $\dot{q}_m(t)$ and $\ddot{q}_m(t)$ are bounded by V_M and A_M, i.e.*

$$V_M = \sup_t \|\dot{q}_m(t)\| \tag{15}$$

$$A_M = \sup_t \|\ddot{q}_m(t)\| \tag{16}$$

In practice, it is often not difficult to obtain on the basis of the desired motion $q_d(t), \dot{q}_d(t)$ and $\ddot{q}_d(t)$ the master robot bounds on $\dot{q}_m(t)$ and $\ddot{q}_m(t)$. Although due to friction effects, tracking errors, etc., the actual motion of the master robot may differ from its desired motion.

Our main result can be formulated as follows.

Theorem 1: *Consider the master and slave robots describe by* (1)*, and the slave robot in closed loop with the control law* (7)*, and both observers* (9) *and* (11)*. Given scalar parameters $\varepsilon_o, \lambda_o, \mu_o, \gamma_o$, such that*

$$\lambda_o > 0, \qquad \mu_o > 0, \qquad \gamma_o > 0, \qquad \varepsilon_o > 0, \tag{17}$$

and if the gain matrices K_d, K_p, L_{p1}, L_{p2} are chosen such that their minimum eigenvalues satisfy

$$
\begin{aligned}
L_{p2,m} &> \max\ \{\mu_o^2,\ \gamma_o^2, L_{p2q4}, L_{p2q5}, L_{p2q6}\} \\
L_{p1,m} &> \max\ \{2\mu_o, L_{p1q3}, L_{p1q5,a}, L_{p1q5,b}\} \\
K_{p,m} &> \max\ \{K_{pq2}, K_{pq6}\} \\
K_{d,m} &> \ K_{dq1}
\end{aligned} \tag{18}
$$

then, the errors $\dot{e}_s$, e_s, $\widetilde{\dot{e}}$, $\tilde{e}$, $\widetilde{\dot{e}}_q$, $\tilde{e}_q$ in the closed loop system are semi-globally uniformly ultimately bounded. Moreover, this bound can be made small, by a proper choose of $K_{p,m}$ and $L_{p1,M}$. The scalars $L_{p2q4}, L_{p2q5}, L_{p2q6}, L_{p1q3}, L_{p1q5,a}, L_{p1q5,b}, K_{pq2}, K_{pq6}, K_{dq1}$ are defined in Appendix A.

Proof: The proof of the theorem is divided into two steps. First the formulation of the closed loop error dynamics is given in §4.1 and then the stability analysis is presented in §4.2.

4.1. *Closed loop error dynamics*

To simplify the closed loop error dynamics two coordinate transformations are introduced.

Lemma 1: *Consider the tracking errors $(e_s, \dot{e}_s)$, the estimation tracking errors $(\tilde{e}, \widetilde{\dot{e}})$ and the estimation position and velocity errors $(\tilde{e}_q, \widetilde{\dot{e}}_q)$, which are defined by* (8)*,* (10) *and* (12).

Introduce the coordinate transformation defined by

$$
\left.
\begin{aligned}
\tilde{q} &:= \tilde{e} - \tilde{e}_q \\
\dot{\tilde{q}} &:= \widetilde{\dot{e}} - \widetilde{\dot{e}}_q - L_{p1}\tilde{q} \\
\dot{\tilde{e}}_q &:= \widetilde{\dot{e}}_q - L_{p1}\tilde{e}_q
\end{aligned}
\right\} \tag{19}
$$

and

$$
\left.
\begin{aligned}
\bar{q} &:= e_s - \tilde{q} \\
\dot{\bar{q}} &:= \dot{e}_s - \dot{\tilde{q}}
\end{aligned}
\right\} \tag{20}
$$

Define the vectors $x, y \in \mathbb{R}^{6n}$ as

$$
x^{\mathrm{T}} := [\dot{e}_s^{\mathrm{T}} \quad e_s^{\mathrm{T}} \quad \widetilde{\dot{e}}^{\mathrm{T}} \quad \tilde{e}^{\mathrm{T}} \quad \widetilde{\dot{e}}_q^{\mathrm{T}} \quad \tilde{e}_q^{\mathrm{T}}] \tag{21}
$$

$$
y^{\mathrm{T}} := [\dot{\bar{q}}^{\mathrm{T}} \quad \bar{q}^{\mathrm{T}} \quad \dot{\tilde{q}}^{\mathrm{T}} \quad \tilde{q}^{\mathrm{T}} \quad \dot{\tilde{e}}_q^{\mathrm{T}} \quad \tilde{e}_q^{\mathrm{T}}] \tag{22}
$$

then x and y are related by

$$
x = Ty \tag{23}
$$

where

$$
T = \begin{bmatrix}
I & 0 & I & 0 & 0 & 0 \\
0 & I & 0 & I & 0 & 0 \\
0 & 0 & I & L_{p1} & I & L_{p1} \\
0 & 0 & 0 & I & 0 & I \\
0 & 0 & 0 & 0 & I & L_{p1} \\
0 & 0 & 0 & 0 & 0 & I
\end{bmatrix} \tag{24}
$$

Proof: The proof follows from the definition of the coordinate transformations. □

In the new set of error coordinates, the closed loop error dynamics can be formulated as follows.

Lemma 2: *Consider the closed loop system formed by the slave robot* (1)*, the control law* (7)*, and both observers* (9)–(11)*. Then, in the variables defined by* (12)*,* (19)*, and* (20)*, the closed loop error dynamics are given by*

$$
\begin{aligned}
M_s(q_s)\ddot{\bar{q}} &+ C_s(q_s, \dot{q}_s)\dot{\bar{q}} + K_d\dot{\bar{q}} + K_p\bar{q} \\
&= C_s(q_s, \dot{\tilde{e}}_q + L_{p1}\tilde{e}_q)(\dot{\bar{q}} - L_{p1}\tilde{q}) + M_s(q_s)L_{p1}\dot{\tilde{q}} \\
&\quad + C_s(q_s, \dot{q}_s)L_{p1}\tilde{q} + K_d(\dot{\tilde{e}}_q + L_{p1}(\tilde{e}_q + \tilde{q})) \\
&\quad - K_p\tilde{q}
\end{aligned} \tag{25}
$$

$$
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tilde{q} = -M_s(q_s)^{-1}K_p(\tilde{q} + \tilde{e}_q) - L_{p1}\dot{\tilde{q}} - L_{p2}\tilde{q} - \ddot{q}_m \tag{26}
$$

$$
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tilde{e}_q = M_s(q_s)^{-1}[&-K_p(\tilde{q} + \tilde{e}_q) + (C_s(q_s, \dot{\tilde{e}}_q + L_{p1}\tilde{e}_q) \\
&- 2C_s(q_s, \dot{q}_s))(\dot{\tilde{e}}_q + L_{p1}\tilde{e}_q)] \\
&- L_{p1}\dot{\tilde{e}}_q - L_{p2}(\tilde{q} - \tilde{e}_q)
\end{aligned} \tag{27}
$$

Proof: See Appendix B. □

4.2. *Stability of the closed loop error dynamics*

First we introduce a result that supports the stability analysis in the following sections. This result is a modified version of a theorem by Chen and Leitmann (1987) (see also Berghuis and Nijmeijer 1994). It states that a system is uniformly ultimately bounded if it has a Lyapunov function whose time-derivative is negative definite in an annulus of a certain width around the origin.

Lemma 3 (Berghuis and Nijmeijer 1994)**:** *Consider the function $g(\cdot): \mathbb{R} \to \mathbb{R}$*

$$
g(y) = \alpha_0 - \alpha_1 y + \alpha_2 y^2, \qquad y \in \mathbb{R}^+ \tag{28}
$$

where $\alpha_i > 0$, $i = 0, 1, 2$. Then $g(y) < 0$ if $y_1 < y < y_2$, where

$$
\left.
\begin{aligned}
y_1 &= \frac{\alpha_1 - \sqrt{(\alpha_1^2 - 4\alpha_2\alpha_0)}}{2\alpha_2} \\
y_2 &= \frac{\alpha_1 + \sqrt{(\alpha_1^2 - 4\alpha_2\alpha_0)}}{2\alpha_2}
\end{aligned}
\right\} \tag{29}
$$

Proposition 1 (Chen and Leitmann 1987)**:** *Let $x(t) \in \mathbb{R}^n$ be the solution of the differential equation*

$$
\dot{x}(t) = f(x(t), t)
$$

with $f(x(t), t)$ Lipschitz and initial condition $x(t_0) = x_0$, and assume there exists a function $V(x(t), t)$ that satisfies

$$P_m\|x(t)\|^2 \le V(x(t),t) \le P_M\|x(t)\|^2 \tag{30}$$

$$\dot V(x(t),t) \le \|x(t)\| \cdot g(\|x(t)\|) < 0 \quad \text{for all} \quad y_1 < \|x(t)\| < y_2 \tag{31}$$

with P_m and P_M positive constants, $g(\cdot)$ as in (28)*, and y_1, y_2 as in* (29)*. Define $\delta := \sqrt{P_m^{-1}P_M}$. If $y_2 > \delta y_1$, then $x(t)$ is locally uniformly ultimately bounded, that is, given $d_m = \delta y_1$, there exists $d \in (d_m, y_2)$ such that*

$$\|x_0\| \le r \quad \Rightarrow \quad \|x(t)\| \le d \qquad \text{for all} \quad t \ge t_0 + T(d,r)$$

where

$$T(d,r) = \begin{cases} 0 & r \le R \\ \dfrac{P_M r^2 - P_m R^2}{-\alpha_0 R + \alpha_1 R^2 - \alpha_2 R^3} & R < r < \delta^{-1} y_2 \end{cases}$$

and $R = \delta^{-1} d$.

Consider the vector $y \in \mathbb{R}^{6n}$ defined by (22), and take as a candidate Lyapunov function

$$V(y) = \tfrac{1}{2} y^{\mathrm T} P(y) y \tag{32}$$

where $P(y) = P(y)^{\mathrm T}$ is given by

$$P(y) = \begin{bmatrix} \varepsilon_o \begin{bmatrix} M_s(q_s) & \lambda_o M_s(q_s) \\ \lambda_o M_s(q_s) & K_p + \lambda_o K_d \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} I & \mu(\tilde q) I \\ \mu(\tilde q) I & L_{p2} \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} I & \gamma(\tilde e_q) I \\ \gamma(\tilde e_q) I & L_{p2} \end{bmatrix} \end{bmatrix} \tag{33}$$

$\varepsilon_o, \lambda_o \in \mathbb{R}$ are positive constants to be determined, and $\mu(\tilde q), \gamma(\tilde e_q)$ are defined by

$$\mu(\tilde q) := \frac{\mu_o}{1 + \|\tilde q\|}, \qquad \gamma(\tilde e_q) := \frac{\gamma_o}{1 + \|\tilde e_q\|} \tag{34}$$

with $\mu_o, \gamma_o \in \mathbb{R}$ positive constants to be determined; $\mu(\tilde q), \gamma(\tilde e_q)$ are bounded, such that

$$0 < \mu(\tilde q) \le \mu_o \quad \text{and} \quad 0 < \gamma(\tilde e_q) \le \gamma_o \tag{35}$$

Sufficient conditions for positive definiteness of $P(y)$ are

$$K_{d,m} > \lambda_o M_{s,M}, \qquad L_{p2,m} > \max\{\mu_o^2,\ \gamma_o^2\} \tag{36}$$

Therefore, conditions (17) and (18), together with the boundedness from above of $\mu(\tilde q), \gamma(\tilde e_q)$, imply that there exist constants P_m and P_M such that

$$\tfrac{1}{2} P_m \|y\|^2 \le V(y) \le \tfrac{1}{2} P_M \|y\|^2 \tag{37}$$

Along the error dynamics (25)–(27), and using Assumption 2, the time derivative of (32) becomes

$$\dot V(y) = -y^{\mathrm T} Q(y) y + \beta(y, \dot q_s, \ddot q_m) \tag{38}$$

where

$$\begin{aligned} \beta(y,\dot q_s,\ddot q_m) = {} & \varepsilon_o(\dot{\bar q}^{\mathrm T} + \lambda_o \bar q^{\mathrm T}) C_s(q_s, \dot{\tilde e}_q + L_{p1}\tilde e_q)(\dot{\tilde q} - L_{p1}\tilde q) \\ & + \varepsilon_o \dot{\bar q}^{\mathrm T} C_s(q_s,\dot q_s) L_{p1}\tilde q - \varepsilon_o\lambda_o \bar q^{\mathrm T} C_s(q_s,\dot q_s) \\ & \times (\dot{\tilde q} - L_{p1}\tilde q) + (\dot{\tilde e}_q^{\mathrm T} + \gamma \tilde e_q^{\mathrm T}) M_s(q_s)^{-1} \\ & \times (C_s(q_s, \dot{\tilde e}_q + L_{p1}\tilde e_q) - 2C_s(q_s,\dot q_s)) \\ & \times (\dot{\tilde e}_q + L_{p1}\tilde e_q) + \varepsilon_o\lambda_o \dot{\bar q}^{\mathrm T} \dot M_s(q_s)\bar q \\ & + \dot\mu \dot{\tilde q}^{\mathrm T}\tilde q + \dot\gamma \dot{\tilde e}_q^{\mathrm T}\tilde e_q - (\dot{\tilde q}^{\mathrm T} + \mu \tilde q^{\mathrm T})\ddot q_m \end{aligned} \tag{39}$$

and $Q(y) = Q(y)^{\mathrm T}$ is given by

$$Q(y) = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^{\mathrm T} & Q_{22} & Q_{23} \\ Q_{13}^{\mathrm T} & Q_{23}^{\mathrm T} & Q_{33} \end{bmatrix} \tag{40}$$

with the block matrices

$$Q_{11} = \varepsilon_o \begin{bmatrix} K_d - \lambda_o M_s(q_s) & 0 \\ 0 & \lambda_o K_p \end{bmatrix}$$

$$Q_{12} = \frac{\varepsilon_o}{2} \begin{bmatrix} -M_s(q_s) L_{p1} & K_p - K_d L_{p1} \\ -\lambda_o M_s(q_s) L_{p1} & \lambda_o (K_p - K_d L_{p1}) \end{bmatrix}$$

$$Q_{13} = \frac{\varepsilon_o}{2} \begin{bmatrix} -K_d & -K_d L_{p1} \\ -\lambda_o K_d & -\lambda_o K_d L_{p1} \end{bmatrix}$$

$$Q_{22} = \begin{bmatrix} L_{p1} - \mu I & \frac{1}{2}(M_s(q_s)^{-1} K_p + \mu L_{p1}) \\ \frac{1}{2}(M_s(q_s)^{-1} K_p + \mu L_{p1})^{\mathrm T} & \mu(M_s(q_s)^{-1} K_p + L_{p2}) \end{bmatrix}$$

$$Q_{23} = \begin{bmatrix} 0 & \frac{1}{2} M_s(q_s)^{-1} K_p \\ \frac{1}{2}(M_s(q_s)^{-1} K_p + L_{p2}) & \frac{1}{2}((\mu+\gamma) M_s(q_s)^{-1} K_p + \gamma L_{p2}) \end{bmatrix}$$

$$Q_{33} = \begin{bmatrix} L_{p1} - \gamma I & \frac{1}{2}(M_s(q_s)^{-1} K_p + \gamma L_{p1}) \\ \frac{1}{2}(M_s(q_s)^{-1} K_p + \gamma L_{p1})^{\mathrm T} & \gamma(M_s(q_s)^{-1} K_p + L_{p2}) \end{bmatrix}$$

To conclude stability of the variable y defined by (22), we require positive definiteness of $Q(y)$ and boundedness of the term $\beta(y, \dot q_s, \ddot q_m)$ along the closed loop error dynamics. These two requirements are developed in the following sections.

4.2.1. *Boundedness of $\beta(y, \dot q_s, \ddot q_m)$*: First, from the definition of $\mu(\tilde q), \gamma(\tilde e_q)$ (34), it follows that

$$\dot\mu \dot{\tilde q}^{\mathrm T}\tilde q = -\mu\left(\frac{\tilde q^{\mathrm T}\dot{\tilde q}}{1 + \|\tilde q\|}\right)\dot{\tilde q}^{\mathrm T}\tilde q \le \mu\|\dot{\tilde q}\|^2 \tag{41}$$

$$\dot\gamma \dot{\tilde e}_q^{\mathrm T}\tilde e_q = -\gamma\left(\frac{\tilde e_q^{\mathrm T}\dot{\tilde e}_q}{1 + \|\tilde e_q\|}\right)\dot{\tilde e}_q^{\mathrm T}\tilde e_q \le \gamma\|\dot{\tilde e}_q\|^2 \tag{42}$$

Then by boundedness of $\mu(\tilde q), \gamma(\tilde e_q)$ (35) we obtain that

$$\dot\mu \dot{\tilde q}^{\mathrm T}\tilde q \le \mu_o\|\dot{\tilde q}\|^2 \quad \text{and} \quad \dot\gamma \dot{\tilde e}_q^{\mathrm T}\tilde e_q \le \gamma_o\|\dot{\tilde e}_q\|^2 \tag{43}$$

On the other hand, the definition of the tracking errors (8) implies that

$$\dot{q}_s = \dot{e}_s + \dot{q}_m$$

Then, from the definition of $\dot{\bar{q}}$, (20), we obtain a relation between $\dot{q}_s$ and $\dot{\bar{q}}$, which is given by

$$\dot{q}_s = \dot{\bar{q}} + \dot{\tilde{q}} + \dot{q}_m \tag{44}$$

Finally, the definition of the inertia matrix $M_s(q_s)$ implies that

$$\dot{M}_s(q_s) = \frac{\mathrm{d}}{\mathrm{d}t} M_s(q_s) = \frac{\partial M_s(q_s)}{\partial q_s} \dot{q}_s$$

hence, by property (5) and since q_s only appears as argument of sinusoidal functions in $M_s(q_s)$, we can conclude that

$$M_{s,pm} \|\dot{q}_s\| \leq \|\dot{M}_s(q_s)\| \leq M_{s,pM} \|\dot{q}_s\| \tag{45}$$

where

$$M_{s,pm} \leq \left\| \frac{\partial M_s(q_s)}{\partial q_s} \right\| \leq M_{s,pM}$$

Then, from (43)–(45), properties (5) and (6), and taking into account Assumption 3, it follows that $\beta(y, \dot{q}_s, \ddot{q}_m)$ is upperbounded by

$$\begin{aligned}
\beta(y, \dot{q}_s, \ddot{q}_m) \leq & -\varepsilon_o \lambda_o C_{s,M} \|\bar{q}\| (\|\dot{\bar{q}}\| - L_{p1,M} \|\tilde{q}\|) \\
& \times (\|\dot{\bar{q}}\| + \|\tilde{q}\| + V_M) \\
& + \varepsilon_o C_{s,M} L_{p1,M} \|\dot{\bar{q}}\| \|\tilde{q}\| (\|\dot{\bar{q}}\| + \|\tilde{q}\| + V_M) \\
& - A_M (\|\dot{\tilde{q}}\| + \mu_o \|\tilde{q}\|) \\
& + \varepsilon_o C_{s,M} (\|\dot{\bar{q}}\| + \lambda_o \|\bar{q}\|) \\
& \times (\|\dot{\tilde{e}}_q\| (\|\dot{\bar{q}}\| - L_{p1,m} \|\tilde{q}\|) \\
& + \|\tilde{e}_q\| (L_{p1,M} \|\dot{\bar{q}}\| - L_{p1,m}^2 \|\tilde{q}\|)) \\
& - 2 M_{s,m}^{-1} C_{s,M} (\|\dot{\tilde{e}}_q\| + \gamma_o \|\tilde{e}_q\|) \\
& \times (\|\dot{\tilde{e}}_q\| + L_{p1,M} \|\tilde{e}_q\|)(\|\dot{\bar{q}}\| + \|\tilde{q}\| + V_M) \\
& + \varepsilon_o \lambda_o M_{s,pM} \|\dot{\bar{q}}\| \|\bar{q}\| (\|\dot{\bar{q}}\| + \|\tilde{q}\| + V_M) \\
& + \mu_o \|\dot{\tilde{q}}\|^2 + \gamma_o \|\dot{\tilde{e}}_q\|^2 \\
& + M_{s,m}^{-1} C_{s,M} (\|\dot{\tilde{e}}_q\| + \gamma_o \|\tilde{e}_q\|) \\
& \times (\|\dot{\tilde{e}}_q\| + L_{p1,M} \|\tilde{e}_q\|)^2
\end{aligned} \tag{46}$$

where the vector $y_N \in \mathbb{R}^6$ is defined as

$$y_N^{\mathrm{T}} := [\, \|\dot{\bar{q}}\| \quad \|\bar{q}\| \quad \|\dot{\tilde{q}}\| \quad \|\tilde{q}\| \quad \|\dot{\tilde{e}}_q\| \quad \|\tilde{e}_q\| \,] \tag{47}$$

4.2.2. *Negative definiteness of* $\dot{V}(y)$: From the upperbound of $\beta(y, \dot{q}_s, \ddot{q}_m)$ (46), the upperbound of $\mu(\tilde{q})$, $\gamma(\tilde{e}_q)$ (35), and considering y_N defined by (47), it follows that $\dot{V}(y)$ (38) can be upperbounded as

$$\dot{V}(y) \leq \|y_N\| (\alpha_0 - Q_{Nm} \|y_N\| + \alpha_2 \|y_N\|^2) \tag{48}$$

where $Q_{Nm} > 0$ is the minimum eigenvalue of the matrix $Q_N = Q_N^{\mathrm{T}}$

$$Q_N = \begin{bmatrix} Q_{11_N} & Q_{12_N} & Q_{13_N} \\ Q_{12_N}^{\mathrm{T}} & Q_{22_N} & Q_{23_N} \\ Q_{13_N}^{\mathrm{T}} & Q_{23_N}^{\mathrm{T}} & Q_{33_N} \end{bmatrix} \tag{49}$$

with the block matrices

$$Q_{11_N} = \varepsilon_o \begin{bmatrix} K_{d,m} - \lambda_o M_{s,M} & \frac{1}{2}\lambda_o V_M (C_{s,M} - M_{s,pM}) \\ \frac{1}{2}\lambda_o V_M (C_{s,M} - M_{s,pM}) & \lambda_o K_{p,m} \end{bmatrix}$$

$$Q_{12_N} = \frac{\varepsilon_o}{2} \begin{bmatrix} -M_{s,M} L_{p1,M} & K_{p,M} - K_{d,M} L_{p1,M} - C_{s,M} L_{p1,M} V_M \\ -\lambda_o M_{s,M} L_{p1,M} & \lambda_o (K_{p,M} - K_{d,M} L_{p1,M} - C_{s,M} L_{p1,M} V_M) \end{bmatrix}$$

$$Q_{13_N} = \frac{\varepsilon_o}{2} \begin{bmatrix} -K_{d,M} & -K_{d,M} L_{p1,M} \\ -\lambda_o K_{d,M} & -\lambda_o K_{d,M} L_{p1,M} \end{bmatrix}$$

$$Q_{22_N} = \begin{bmatrix} L_{p1,m} - 2\mu_o & \frac{1}{2}(M_{s,m}^{-1} K_{p,M} + \mu_o L_{p1,M}) \\ \frac{1}{2}(M_{s,m}^{-1} K_{p,M} + \mu_o L_{p1,M}) & \mu_o (M_{s,m}^{-1} K_{p,m} + L_{p2,m}) \end{bmatrix}$$

$$Q_{23_N} = \begin{bmatrix} 0 & \frac{1}{2} M_{s,m}^{-1} K_M \\ \frac{1}{2}(M_{s,m}^{-1} K_{p,M} + L_{p2,M}) & \frac{1}{2}((\mu_o + \gamma_o) M_{s,m}^{-1} K_{p,M} + \gamma_o L_{p2,M}) \end{bmatrix}$$

$$Q_{33_N} = \begin{bmatrix} L_{p1,m} - 2\gamma_o + 2M_{s,m}^{-1} C_{s,M} V_M & q_{56} \\ q_{56} & \gamma_o (M_{s,m}^{-1} K_{p,m} + L_{p2,m} + 2M_{s,m}^{-1} C_{s,M} V_M L_{p1,m}) \end{bmatrix}$$

$$q_{56} = \tfrac{1}{2}(M_{s,m}^{-1} K_{p,M} + \gamma_o L_{p1,M}) + M_{s,m}^{-1} C_{s,M} V_M (L_{p1,M} + \gamma_o)$$

and α_0, α_2 are given by

$$\alpha_0 = \left(1 + \sqrt{\mu_o}\right) \sqrt{A_M} \tag{50}$$

$$\begin{aligned}
\alpha_2 = & \sqrt{8 M_{s,m}^{-1} C_{s,M}} \left(\sqrt{\gamma_o} + \sqrt{L_{p1,M}} \right) \\
& + \sqrt{\varepsilon_o C_{s,M}} \left(1 + \sqrt{\lambda_o} \right) \left(L_{p1,M} + 2\sqrt{L_{p1,M}} \right) \\
& + M_{s,m}^{-1} C_{s,M} \Big(5 + \sqrt{\gamma_o + 2L_{p1,M}} \\
& + \sqrt{\gamma_o L_{p1,M} + L_{p1,M}^2 + \gamma_o} + L_{p1,M} \sqrt{\gamma_o} + \sqrt{8 \gamma_o L_{p1,M}} \Big) \\
& + \varepsilon_o \left(\sqrt{C_{s,M}} \left(1 + 2\sqrt{L_{p1,M}} \right) + \sqrt{\lambda_o (M_{s,pM} + C_{s,M})} \right) \\
& + \sqrt{\varepsilon_o \lambda_o} \left(2\sqrt{C_{s,M}} \left(1 + \sqrt{L_{p1,M}} \right) + \sqrt{M_{s,pM} + C_{s,M}} \right)
\end{aligned} \tag{51}$$

If the gains K_d, K_p, L_{p1}, L_{p2} and the constants $\varepsilon_o, \lambda_o, \mu_o, \gamma_o$ satisfy conditions (17) and (18), then Q_N given by (49) is positive definite. Then the right-hand side in (48) corresponds to (31), and together with (37) and Proposition 1, this allows us to conclude uniformly ultimately boundedness of y_N (47) and consequently of y (22). By (23) we therefore can conclude that the original state x, given by (21), is uniformly ultimately bounded.

Moreover, α_2 depends explicitly on $L_{p1,M}$, such that y_2 defined as in Proposition 1, can be made small by a proper choice of $L_{p1,M}$ and thus the upperbound for the closed loop errors $\dot{e}_s, e_s, \dot{\tilde{e}}, \tilde{e}, \dot{\tilde{e}}_q, \tilde{e}_q$ can be made small.

	m (mass) (Kg)	l_c (mass centre) (m)	i (inertia) (Kg m^2)	l (length) (m)
Link 1 (m)	10	0.54	0.02	1.0
Link 2 (m)	7	0.42	0.01	0.8
Link 1 (s)	12	0.6	0.05	1
Link 2 (s)	5	0.5	0.03	0.8

Table 1. Parameters of the master (m) and slave (s) robots.

	Joint 1 (m)	Joint 2 (m)	Joint 1 (s)	Joint 2 (s)
$q(0)$ (rad)	0.8	1	1.8	0.1
$\dot{q}(0)$ (rad/s)	0	0	0	0

Table 2. Joint initial conditions.

Notice that the minimum value for y_2 is given by $Q_{Nm}/(2\alpha_2)$, and recall that Q_N (49) depends on $K_{p,m}$.

On the other hand, a region of attraction is given by

$$B = \left\{ x \in \mathbb{R}^{6n} \quad \middle| \quad \|x\| < \frac{y_2}{\|T\|}\sqrt{\frac{P_m}{P_M}} \right\} \tag{52}$$

where T is given by (24), P_m, P_M are defined by (37), and y_2 as in Proposition 1, with (31) given by (48). Since the size of the region of attraction B (52) is proportional to y_2, this region can be expanded by increasing y_2.

The ultimate boundedness result is due to the absence of measurements of $\ddot{q}_m$, see (48) and (50), therefore we have the following Corollary.

Corollary 1: *If $\ddot{q}_m(t) = 0$ for $t \in (t_2, \infty)$, $t_2 \geq t_0$, and additionally the conditions on Theorem 1 are satisfied, then the control law* (7)*, and both observers* (9) *and* (11) *yields semi-global exponential convergence of the errors $\dot{e}_s$, e_s, $\dot{\tilde{e}}$, $\tilde{e}$, $\dot{e}_q$, $\tilde{e}_q$.*

Proof: From (48) and (50) we have the following. If conditions in Theorem 1 are satisfied and $\ddot{q}_m(t) = 0$ for $t \in (t_2, \infty)$, $t_2 \geq t_0$, then for $t \geq t_2$ (48) reduces to

$$\dot{V}(y) \leq \|y_N\|^2(-Q_{Nm} + \alpha_2\|y_N\|)$$

with $Q_{Nm} > 0$.

On the other hand, the region of attraction (52) guarantees that $Q_{Nm} > \alpha_2\|y_N\|$ and thus $\dot{V}(y)$ can be upperbounded as

$$\dot{V}(y) \leq -\kappa\|y_N\|^2 \qquad \text{for all} \quad t \geq t_2$$

From the last equation and (37), we conclude that there exist some constants $m^*, \rho > 0$, such that

$$\|y(t)\|^2 \leq m^* \mathrm{e}^{-\rho t}\|y(t_2)\|^2 \qquad \text{for all} \quad t \geq t_2$$

by (23) we can conclude the same for x given by (21). □

	Joint 1	Joint 2
$\hat{e}_s(0)$ (rad)	0.5	0.8
$\widehat{\dot{e}}_s(0)$ (rad/s)	0	0
$\hat{q}_s(0)$ (rad)	0.5	0.7
$\widehat{\dot{q}}_s(0)$ (rad/s)	0	0

Table 3. Initial conditions for observers.

Remark 2: An example of Corollary 10 is obtained when the set point regulation of the master robot is considered.

5. Simulations

The master (m) and slave (s) robots are planar manipulators $q_i \in \mathbb{R}^2$, $i = m, s$, with revolute joints, working in the x–z plane. The dynamic model is given in Spong and Vidyasagar (1989) and their parameters are listed in table 1.

The controller for the master robot τ_m is the adaptive control law proposed by Slotine and Li (1987). The desired trajectory for the master robot is given by

$$q_d(t) = \begin{bmatrix} 1 + 0.25\sin(0.5\ t) \\ 0.8 + 0.25\cos(0.5\ t) \end{bmatrix} \qquad \text{(rad)}$$

The initial conditions for both robots and the observers (9) and (11) are listed in tables 2 and 3.

The gain matrices, involved in the controller (7), and both observers (9) and (11), are considered to be of the form kI, where k is a scalar and $I \in \mathbb{R}^{2\times 2}$. The scalars associated with these gain matrices are chosen as follows.

K_p	K_d	Λ_1	Λ_2	L_{p1}	L_{p2}
100	10	50	50	50	50

Table 4. Controller gains.

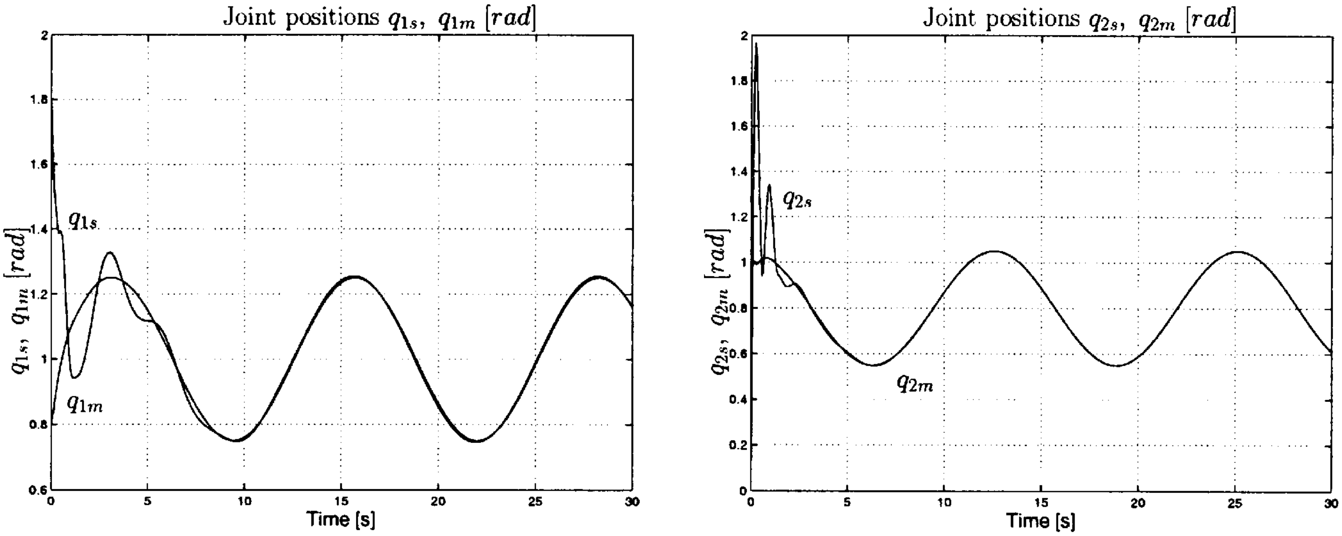

Figure 1. Joint positions q_{1s}, q_{1m} and q_{2s}, q_{2m}.

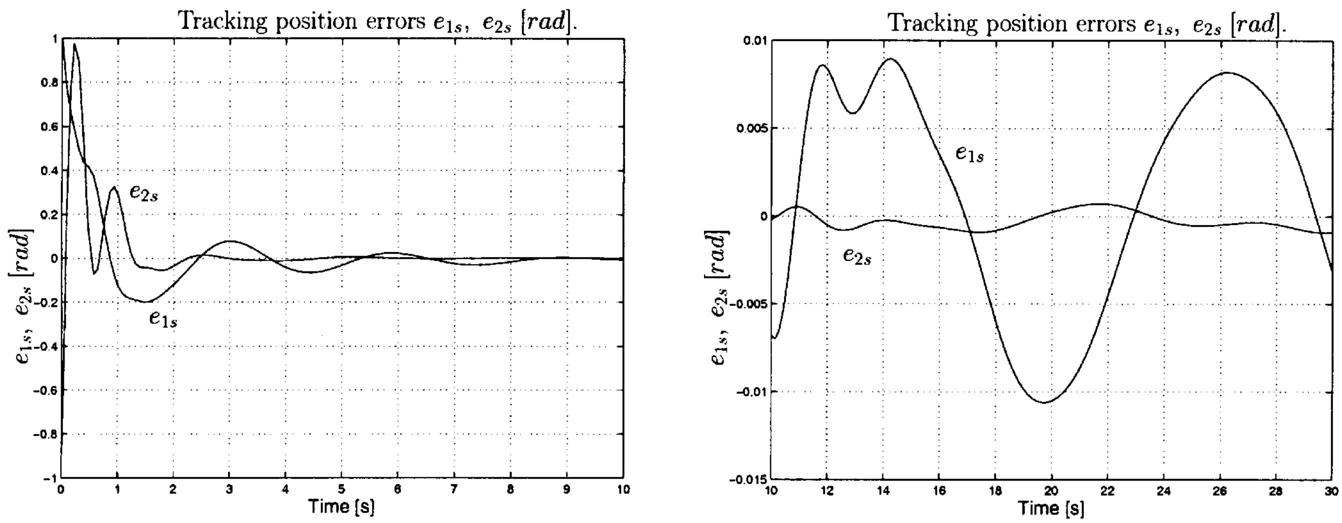

Figure 2. Tracking position errors e_{1s}, e_{2s}.

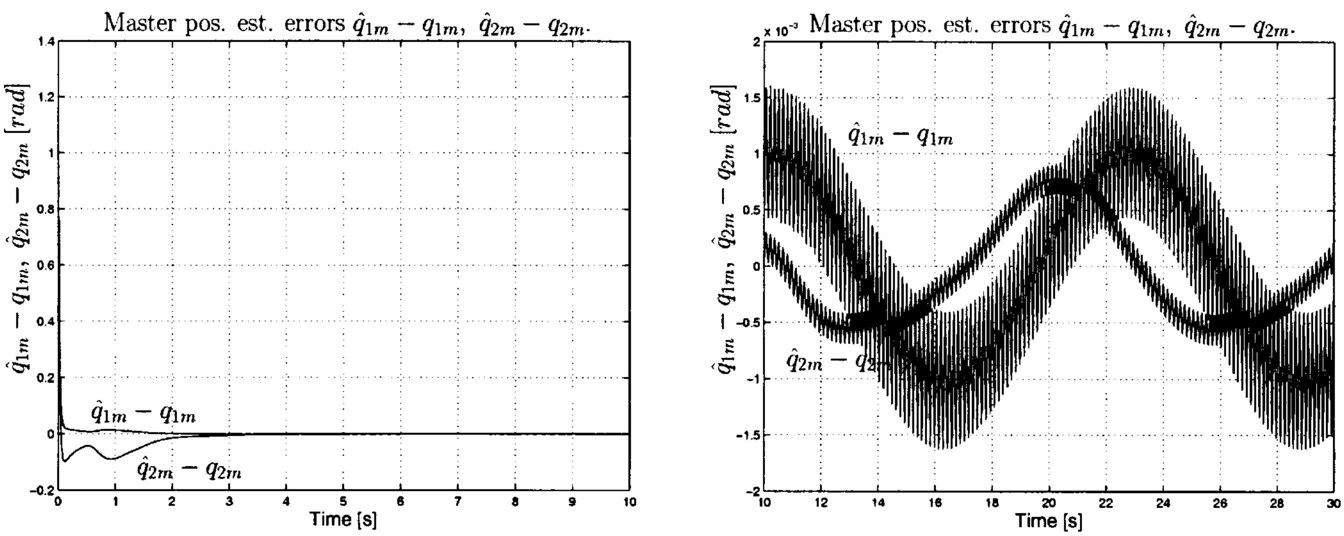

Figure 3. Master position estimation errors $\hat{q}_{1m} - q_{1m}$, $\hat{q}_{2m} - q_{2m}$.

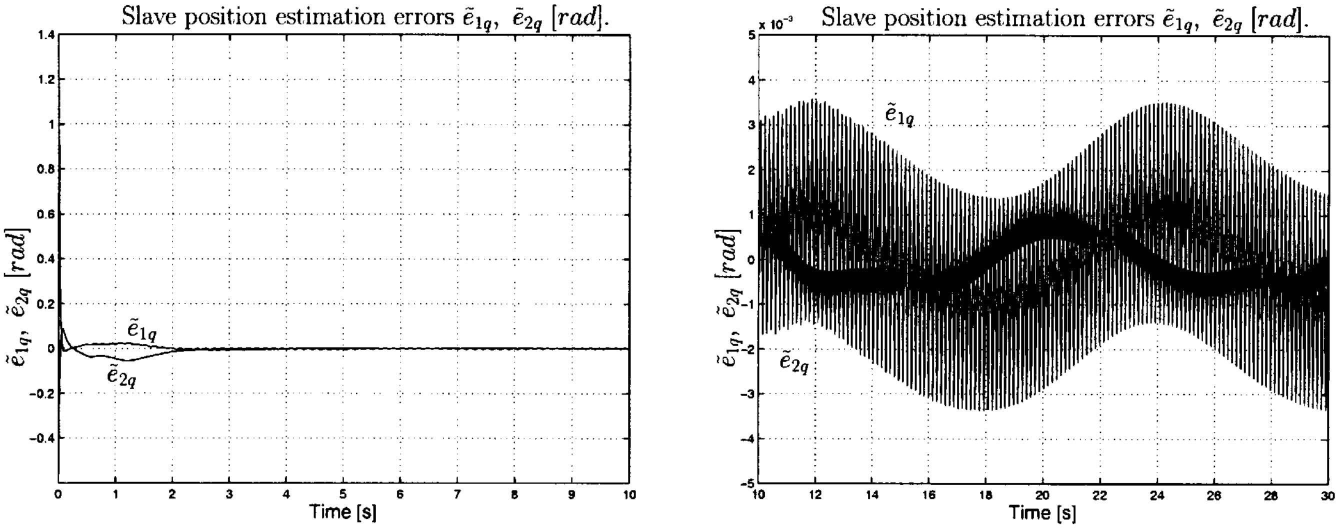

Figure 4. Slave position estimation errors $\tilde{e}_{1q}, \tilde{e}_{2q}$.

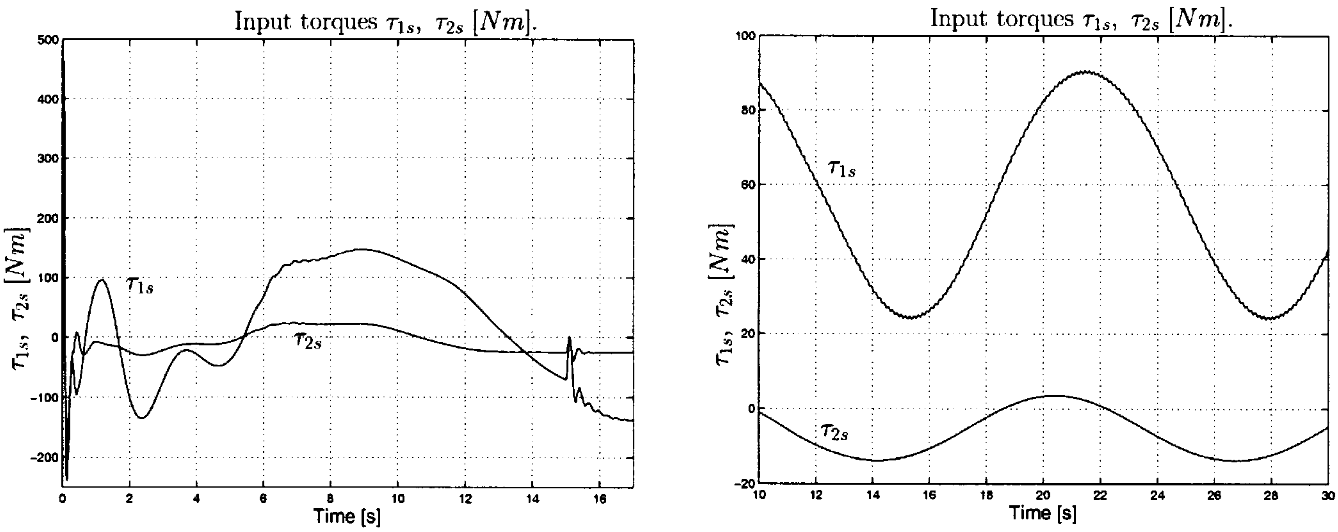

Figure 5. Input torques τ_{1s}, τ_{2s}.

Figure 1 shows convergence between the slave trajectories q_{1s}, q_{2s} and the master trajectories q_{1m}, q_{2m}. However, by spliting the time axes after the ultimate boundedness region has been achieved ($t > 10$), it can be noticed that the tracking errors e_{1s}, e_{2s} are in fact bounded (see figure 2, right). Figures 3 and 4 show that the estimated errors for slave and master joint positions are also bounded. At the same time figures 3 and 4 show fast convergence of the estimation errors during the transient (see Remark 1). The high peaks in the slave input torques τ_{1s}, τ_{2s} (figure 5) compensate the initial tracking errors e_{1s}, e_{2s}, which are 1 (rad) and -0.9 (rad) respectively (see figure 2 and table 2).

On the other hand the simulations were run for different values of the gains, it was observed that by increasing the gains K_p, L_{p1}, the bound of the closed loop system can be made arbitrarily small; at the same time by increasing K_d, the convergence time can be decreased. And thus, we can conclude that the performance showed in the simulations agrees, and moreover it could be predicted, with the stability result obtained in §4.

6. Remarks and discussion

- The proposed control law gives rise to coordination in the joint space. Coordination in the Cartesian space is obtained only if the length of the links of the slave robot are equal to the corresponding links in the master robot.
- In the feedback control (7) and the observer (11) the available signal e_s is used, instead of its esti-

mate $\hat{e}_s$. This is done so as to take advantage of the available information, i.e. the position measurement q_s and the tracking position error e_s. As a result robustness, and better stability and performance at transients are obtained.

- The variables $\tilde{q}, \dot{\tilde{q}}$, defined by (19), can be interpreted as the estimation error in the joint variables of the master robot $q_m, \dot{q}_m$. Therefore, $\tilde{q}, \dot{\tilde{q}}$ give an idea of how good the estimation of the master robot variables can be made based on measured and estimated variables of the slave robot. So, the slave robot, under the proposed controller, can be considered as a physical estimator for the master robot dynamics.
- The uniform ultimately boundedness result is of local nature, with region of attraction (52). This region of attraction and the bound for the closed loop errors depend on y_2 in a proportional way. This is an intrinsic property of the considered method, see Proposition 1 and §4.2.2, and thus a compromise has to be made.

  Nevertheless, the region of attraction mainly depends on the initial estimate errors. Therefore if small initial estimate errors (see observers (9) and (11)) are chosen, then high initial tracking errors can be considered. The high dependency on the estimate errors is the price to be paid for the lack of available measurements or high quality measurements.
- The conditions given by (18) imply relations between the minimum and maximum eigenvalues of L_{p1}, L_{p2}, at the same time Q_{Nm} and α_2 depend on the maximum eigenvalue of L_{p1}. All this relations have to be taken into account to choose the control and observer gains K_d, K_m, L_{p1}, L_{p2}. Nevertheless a study, that is omitted for brevity, shows that $y_2 > \delta y_1$ can always be satisfied.

  On the other hand L_{p2} is also related with the value of P_M, see (37), such that by increasing L_{p2}, P_M also increases. But there is still freedom on the gain K_d, such that the ratio P_m/P_M can be kept far from zero, and thus shrinking of the region of attraction is avoided.
- Even without knowledge of the bounds implied in (17) and (18), the closed loop system can be made uniformly ultimately bounded, by selecting the control gains large enough. However, such high gain implementations are not always desirable in practical circumstances.
- Conditions (18) and the simulation results resemble—without being—high gain observer behaviour. Therefore, we could think that similar results may be obtained by some other techniques, e.g. variable structure control. However more difficult stability conditions and more complicated controls would arise.

  Moreover, the proposed control has the advantage that the control gains can be physically interpreted. Consequently an insight of how they affect the closed loop performance can be obtained, which in general is more difficult to determine for variable structure implementations.
- The controller and observers (7), (9) and (11) are model based, nevertheless the stability analysis allows a straightforward robustness analysis for parametric uncertainties. Because of linearity of the robot dynamical model (1), we have that the parametric uncertainties appear as an additive term in $\dot{V}$, given by (38). And thus, if we consider bounded parametric uncertainties, then this new bounded term appears in (48). So, by retuning the gains we can ensure that $\dot{V}$ is negative definite, such that the convergence properties of the closed loop system are preserved.

  In case of unmodelled dynamics the stability analysis is not straightforward, moreover, it highly depends on the kind of unmodelled dynamic effect.
- A future extension of the proposed technique arises when flexible joint robots are considered. In that case fourth order derivatives of the position are required, which makes the application of numerical differentiation and low pass filters unpractical. On the other hand the master–slave scheme is quite restrictive, nevertheless the extension of the proposed controller for some other schemes seems to be straightforward, such is the case of cooperative schemes, decentralized multirobot systems.
- The proposed control law provides a systematic way of proving stability and boundedness of the closed loop system. This is a drawback of some other schemes for estimating velocities, such as numerical differentiation or low pass filters. For those techniques, in general, do not exist formal stability proofs or a methodology to guarantee stability of the closed loop system.

7. Conclusions

In the present paper we have designed a control scheme for coordination of robot manipulators that requires only position measurements. The control scheme is formed by a feedback controller, which utilizes estimates for the tracking errors, as well as for the velocity and acceleration variables. These estimates are obtain by two non-linear observers. The resulting closed loop system was proved to be semi-globally uniformly ultimately bounded. Also a relation between the bound of the errors and the design parameters was given, which can be used to guarantee the desired tracking accuracy.

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Appendix A

Consider the matrix Q_N given by (49); ΔQ_i represents the determinant of the ith leading minor of Q_N. Sufficient conditions for positive definiteness of Q_N are given by (17) and (18), with $L_{p2q4}, L_{p2q5}, L_{p2q6}, L_{p1q3}, L_{p1q5,a}, L_{p1q5,b}, K_{pq2}, K_{pq6}, K_{dq1}$ given by

$$K_{dq1} = \lambda_o M_{s,M}$$

$$K_{pq2} = \frac{\lambda_o V_M (M_{s,pM} - C_{s,M})^2}{4(K_{d,m} - \lambda_o M_{s,M})}$$

L_{p1q3}: denotes the solution of the equation $\Delta Q_3 = a_1 L_{p1q3} + a_2 = 0$,with a_1, a_2 the resultant coefficients in the factorization of $L_{p1,m}$ in ΔQ_3, and $L_{p1,m}$ substituted by L_{p1q3}.

L_{p2q4}: denotes the solution of the equation $\Delta Q_4 = b_1 L_{p2q4} + b_2 = 0$, with b_1, b_2 the resultant coefficients in the factorization of $L_{p2,m}$ in ΔQ_4, and $L_{p2,m}$ substituted by L_{p2q4}.

$L_{p1q5,a} = 2(\gamma_o - C_{s,M} V_M M_{s,M}^{-1})$.

$L_{p1q5,b}$: denotes the largest solution of the equation $\Delta Q_5 = c_0 + c_1 L_{p1q5} + c_2 L_{p1q5}^2 = 0$, with c_0, c_1, c_2 the resultant coefficients in the factorization of $L_{p1,m}$ in ΔQ_5, and $L_{p1,m}$ substituted by L_{p1q5}.

L_{p2q5}: denotes the solution of the equation $c_2 = r_1 L_{p2q5} + r_2 = 0$, with c_2 as in $L_{p1q5,b}$, r_1, r_2 the resultant coefficients in the factorization of $L_{p2,m}$ in c_2, and $L_{p2,m}$ substituted by L_{p2q5}.

L_{p2q6}: denotes the largest solution of the equation $\Delta Q_6 = t_o + t_1 L_{p2q6} + t_2 L_{p2q6}^2 = 0$, with t_0, t_1, t_2 the resultant coefficients in the factorization of $L_{p2,m}$ in ΔQ_6, and $L_{p2,m}$ substituted by L_{p2q6}.

K_{pq6}: denotes the solution of the equation $t_2 = s_1 K_{pq6} + s_2 = 0$, with t_2 as in L_{p2q6}, s_1, s_2 the resultant coefficients in the factorization of $K_{p,m}$ in t_2, and $K_{p,m}$ substituted by K_{pq6}.

Appendix B

First, we obtain the error dynamics in terms of the tracking errors $(e_s, \dot{e}_s)$, the estimation tracking errors $(\tilde{e}, \widetilde{\dot{e}})$, and the estimation position and velocity errors $(\tilde{e}_q, \widetilde{\dot{e}}_q)$. Second, we consider the coordinate transformation defined by (19) and (20).

B.1. *Tracking error dynamics*

Substitution of τ_s (7) in (1), by adding and subtracting $K_d \dot{e}_s + M_s(q_s)\ddot{q}_m + C_s(q_s, \dot{q}_s)\dot{q}_m$, and considering the tracking errors defined by (8), we obtain that

$$\begin{aligned} M_s(q_s)\ddot{e}_s + C_s(q_s, \dot{q}_s)\dot{e}_s + K_d \dot{e}_s + K_p e_s &= M_s(q_s)(\widehat{\ddot{q}}_m - \ddot{q}_m) + C_s(q_s, \widehat{\dot{q}}_s)\widehat{\dot{q}}_m \\ &\quad - C_s(q_s, \dot{q}_s)\dot{q}_m - K_d(\widehat{\dot{e}}_s - \dot{e}_s) \end{aligned} \tag{53}$$

From (8), (10), (12) and (13), the following equalities can be established

$$\left.\begin{aligned} \hat{q}_m - q_m &= \tilde{e} - \tilde{e}_q \\ \widehat{\dot{q}}_m - \dot{q}_m &= \widetilde{\dot{e}} - \widetilde{\dot{e}}_q \\ \widehat{\ddot{q}}_m - \ddot{q}_m &= \frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{\dot{e}} - \widetilde{\dot{e}}_q) \end{aligned}\right\} \tag{54}$$

Considering (10), (12), (54) and property (4), it follows that

$$\begin{aligned} C_s(q_s, \widehat{\dot{q}}_s)\widehat{\dot{q}}_m - C_s(q_s, \dot{q}_s)\dot{q}_m &= C_s(q_s, \dot{q}_s)\widetilde{\dot{e}} - 2C_s(q_s, \dot{q}_s)\widetilde{\dot{e}}_q + C_s(q_s, \widetilde{\dot{e}}_q)\widetilde{\dot{e}}_q \\ &\quad + C_s(q_s, \widetilde{\dot{e}}_q)\dot{e}_s - C_s(q_s, \widetilde{\dot{e}}_q)\widetilde{\dot{e}} \end{aligned} \tag{55}$$

Substitution of (55) in (53), and considering (10) and (54), yields

$$\begin{aligned} M_s(q_s)\ddot{e}_s + C_s(q_s, \dot{q}_s)\dot{e}_s + K_d \dot{e}_s + K_p e_s &= M_s(q_s)\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{\dot{e}} - \widetilde{\dot{e}}_q) + K_d \widetilde{\dot{e}} \\ &\quad - 2C_s(q_s, \dot{q}_s)\widetilde{\dot{e}}_q + C_s(q_s, \widetilde{\dot{e}}_q)\widetilde{\dot{e}}_q + C_s(q_s, \widetilde{\dot{e}}_q)\dot{e}_s \\ &\quad - C_s(q_s, \widetilde{\dot{e}}_q)\widetilde{\dot{e}} + C_s(q_s, \dot{q}_s)\widetilde{\dot{e}} \end{aligned} \tag{56}$$

B.2. *Estimation tracking error dynamics*

Define states $x_1, x_2 \in \mathbb{R}^n$ as $x_1 := e_s$, $x_2 := \dot{e}_s$, and obtain a state space representation for (56). In the states x_1, x_2 the estimation tracking errors (10) are given by

$$\tilde{e} = x_1 - \hat{e}_s, \qquad \widetilde{\dot{e}} = x_2 - \widehat{\dot{e}}_s \tag{57}$$

Therefore, from the state space representation of (56) and the observer defined by (9), the estimation tracking error dynamics are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{e} = \tilde{\dot{e}} - \Lambda_1\tilde{e}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\dot{e}} = M_s(q_s)^{-1}\Big\{ & -C_s(q_s,\dot{q}_s)x_2 - K_d x_2 - K_p x_1 \\ & + M_s(q_s)\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\dot{e}} - \tilde{\dot{e}}_q) + C_s(q_s,\dot{q}_s)\tilde{\dot{e}} \\ & + C_s(q_s,\hat{\dot{q}}_s)\hat{\dot{e}}_s + (C_s(q_s,\tilde{\dot{e}}_q) - 2C_s(q_s,\dot{q}_s))\tilde{\dot{e}}_q \\ & + C_s(q_s,\tilde{\dot{e}}_q)(x_2 - \tilde{\dot{e}}) + K_d\tilde{\dot{e}} + K_d\hat{\dot{e}}_s + K_p\hat{e}_s\Big\} - \Lambda_2\tilde{e}\end{aligned}$$

Considering (12), (54) and (57) and after a straightforward computation, these equations reduce to

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{e} = \tilde{\dot{e}} - \Lambda_1\tilde{e} \tag{58}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\dot{e}}_q = M_s(q_s)^{-1}\{ & -K_p\tilde{e} - 2C_s(q_s,\dot{q}_s)\tilde{\dot{e}}_q \\ & + C_s(q_s,\tilde{\dot{e}}_q)\tilde{\dot{e}}_q\} - \Lambda_2\tilde{e}\end{aligned} \tag{59}$$

B.3. *Estimation velocity error dynamics*

From the definition of the tracking errors (8), it follows that

$$\ddot{q}_s = \ddot{e}_s + \ddot{q}_m \tag{60}$$

Define states $z_1, z_2 \in \mathbb{R}^n$ as $z_1 := q_s$, $z_2 := \dot{q}_s$, and obtain a state space representation for (56). In the states z_1, z_2 the estimation velocity errors (12) are given by

$$\tilde{e}_q = z_1 - \hat{q}_s, \qquad \tilde{\dot{e}}_q = z_2 - \hat{\dot{q}}_s \tag{61}$$

So, from the state space representation for (56), with states z_1, z_2, and observer (11), the estimation position and velocity error dynamics are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{e}_q = \tilde{\dot{e}}_q - L_{p1}\tilde{e}_q$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\dot{e}}_q = M_s(q_s)^{-1}\Big\{ & -(C_s(q_s,z_2) + K_d)\dot{e}_s - K_p e_s \\ & + M_s(q_s)\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\dot{e}} - \tilde{\dot{e}}_q) + C_s(q_s,z_2)\tilde{\dot{e}} \\ & + (C_s(q_s,\tilde{\dot{e}}_q) - 2C_s(q_s,z_2))\tilde{\dot{e}}_q + C_s(q_s,\tilde{\dot{e}}_q)\dot{e}_s \\ & - C_s(q_s,\tilde{\dot{e}}_q)\tilde{\dot{e}} + C_s(q_s,\hat{\dot{q}}_s)\hat{\dot{e}}_s \\ & + K_d\tilde{\dot{e}} + K_d\hat{\dot{e}}_s + K_p e_s\Big\} - L_{p2}\tilde{e}_q + \ddot{q}_m\end{aligned}$$

considering (12), (54) and (61), these equations reduce to

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{e}_q = \tilde{\dot{e}}_q - L_{p1}\tilde{e}_q \tag{62}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\dot{e}}_q = {} & \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\dot{e}} - \tilde{\dot{e}}_q) + M_s(q_s)^{-1} \\ & \times\{-2C_s(q_s,z_2)\tilde{\dot{e}}_q + C_s(q_s,\tilde{\dot{e}}_q)\tilde{\dot{e}}_q\} \\ & - L_{p2}\tilde{e}_q + \ddot{q}_m\end{aligned} \tag{63}$$

Finally, from (59) and (63), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{e}_q = \tilde{\dot{e}}_q - L_{p1}\tilde{e}_q \tag{64}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\dot{e}} = {} & M_s(q_s)^{-1}\{-2K_p\tilde{e} - 2C_s(q_s,\dot{q}_s)\tilde{\dot{e}}_q + C_s(q_s,\tilde{\dot{e}}_q)\tilde{\dot{e}}_q\} \\ & - 2\Lambda_2\tilde{e} + L_{p2}\tilde{e}_q - \ddot{q}_m\end{aligned} \tag{65}$$

where the fact that $z_2 = \dot{q}_s$ has been used.

B.4. *Coordinate transformations*

Consider the coordinate transformation defined by (19), subtraction of (59) and (64) from (58) and (65) gives rise to the dynamics for $\tilde{q}, \dot{\tilde{q}}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{q} = \tilde{\dot{e}} - \tilde{\dot{e}}_q - L_{p1}\tilde{q}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\dot{e}} - \tilde{\dot{e}}_q) = -M_s(q_s)^{-1}K_p(\tilde{q} + \tilde{e}_q) - L_{p2}\tilde{q} - \ddot{q}_m$$

where Assumption 1 has been used.

From (59) and (64), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{e}_q = \tilde{\dot{e}}_q - L_{p1}\tilde{e}_q$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\dot{e}}_q = M_s(q_s)^{-1}\{ & -K_p(\tilde{q} + \tilde{e}_q) - 2C_s(q_s,\dot{q}_s)\tilde{\dot{e}}_q \\ & + C_s(q_s,\tilde{\dot{e}}_q)\tilde{\dot{e}}_q\} - L_{p2}(\tilde{q} + \tilde{e}_q)\end{aligned}$$

From the last four equations we obtain the error dynamics (26) and (27). And by adding and subtracting $K_p\tilde{q} + C_s(q_s,\dot{q}_s)L_{p1}\tilde{q} + K_d L_{p1}\tilde{q} + M(q_s)(L_{p1}\dot{\tilde{q}} - L_{p1}L_{p1}\tilde{q})$ from (56), and considering the coordinate transformation defined by (19) and (20), it results in (25).

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