Alejandro Rodriguez-Angeles
Phone: +31 40 247 5730
e-mail: A.Rodriguez@tue.nl

Henk Nijmeijer
Phone: +31 40 247 3203
Fax: +31 40 246 1418
e-mail: H.Nijmeijer@tue.nl
Eindhoven University of Technology,
Department of Mechanical Engineering,
P.O. Box 513, 5600 MB Eindhoven,
The Netherlands.

Synchronizing Tracking Control for Flexible Joint Robots via Estimated State Feedback

In this paper, we propose a synchronization controller for flexible joint robots, which are interconnected in a master-slave scheme. The synchronization controller is based on feedback linearization and only requires measurements of the master and slave link positions, since the velocities and accelerations are estimated by means of model-based nonlinear observers. It is shown, using Lyapunov function based stability analysis, that the proposed synchronization controller yields local uniformly ultimately boundedness of the closed loop errors. A tuning gain procedure is presented. The results are supported by simulations in a one degree of freedom master-slave system. [DOI: 10.1115/1.1636197]

1 Introduction

This paper addresses the problem of synchronization of robot manipulators with flexible joints. It is assumed that not all the joint state variables are measured, e.g., link and rotor velocities are unknown, which is a very common situation when joint flexibility is present. In robotic systems synchronization is of great importance as soon as two robots have to cooperate. This cooperative behavior gives manoeuvrability and dexterity that can not be achieved by an individual system, e.g., multi finger robot-hands, multi robot systems, walking robots.

Typically robot coordination and cooperation, see [1,2], and [3], form important illustrations of the same goal, where it is desired that two or more mechanical systems, either identical or different, are asked to work synchronously. This is obviously a control problem that requires the design of suitable controllers to achieve the required synchronous motion. The problem of synchronization of robots, can be seen as tracking between the systems with some additional challenges that are not considered in tracking controllers.

Joint flexibility or joint clasticity, considerably affects the performance of robot manipulators [4], and is a major source of oscillatory behavior. To improve the performance of robot manipulators, joint flexibility has to be taken into account in the modelling and control of such systems. Joint flexibility can be caused by transmission elements such as harmonic drives, belts, or long shafts; and it can be modelled by considering the position and velocity of the motor rotor, and the position and velocity of the link. Therefore, the order of the dynamic model for a flexible joint is twice that of a rigid joint, consequently, the controllers are more complex than those for rigid joint robots. From a modelling point of view, two dynamic models for the flexible joint robot have been considered. In [5] an extended model for flexible joint robots is presented, which includes the full nonlinear dynamic interactions among joint flexibilities and inertial properties of links and actuators. If it is assumed that the kinetic energy of the electrical actuators is due only to their own rotor spinning, then a reduced model is obtained [6]. This reduced model satisfies the conditions of full state linearization and decoupling via static state feedback. Meanwhile it has been proved that the extended model is fully linearizable and decouplable via dynamic state feedback [7,8]. Other techniques like adaptive control, singular perturbations, and robust control have been investigated to design effective controllers for flexible joint robots (see [9,10]).

All the above mentioned controllers assume that all state variables are available, implying the presence of additional sensors in each joint, which in practice is difficult if not impossible. Besides the complexity in the implementation of measuring equipment, velocity measurements are often obtained by means of tachometers, which are often contaminated by noise, or moreover, velocity measuring equipment is frequently omitted due to the savings in cost, volume, and weight that can be obtained. To overcome this problem numerical differentiation, filters and the design of observers have been considered. Numerical differentiation and filters possess the advantage of simplicity in implementation. However they present a reduced bandwidth and in general there is no an analytical method to guarantee that the closed loop system will be stable. On the other hand, observers are in general model-based, and thus require information about the system, and may thus be more difficult to implement. Nevertheless, model-based observers allow, in most cases, a stability proof and a methodology to tune the observer gains, which guarantee a stable closed loop system. In [11] a nonlinear observer based on pseudolinearization techniques has been proposed, a high gain observer is presented in [12], and a semiglobal nonlinear observer is designed in [13]. From an implementation point of view, it is desirable that controllers be designed to require few incasurements as possible. Examples of such philosophy are the controllers for flexible joint robots proposed in [14] and [15], which require only link and rotor position measurements.

In this work, the main goal is to ensure synchronization between two robots, where the robot for which the control will be designed (slave) has flexible joints, and the robot, whose trajectories are to be followed (master), may or may not have flexible joints. This goal is carried out assuming only link position measurements.

Consider two fully actuated robots with n joints each and working in a master-slave scheme, such that the subindexes m, s identify the master and slave robot. Assume that the master robot is driven by a torque $\tau_m(\cdot)$, that in the ideal case, ensures convergence of the link position and velocity q_m, $\dot{q}_m$ to a desired trajectory q_d, $\dot{q}_d$. However, the torque τ_m, the dynamic model and parameters of the master robot, as well as the link velocity and acceleration $\dot{q}_m$, $\ddot{q}_m$, are not available for design of the slave controller $\tau_s(\cdot)$. Also we assume that the slave rotor position θ_s, and the slave link and rotor velocities and accelerations $\dot{q}_s$, $\ddot{q}_s$, $\dot{\theta}_s$, $\ddot{\theta}_s$ are not measured. Therefore, the slave controller τ_s, that is to be designed such that the link variables q_s, $\dot{q}_s \in \mathbb{R}^n$ synchronize with the variables q_m, $\dot{q}_m$, can only depend on link position measurements of both robots, i.e., q_m, q_s, and estimated or re-

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constructed values of θ_s, $\dot{\theta}_s$, $\dot{q}_m$, $\ddot{q}_m$, $\dot{q}_s$, $\ddot{q}_s$. Notice that the goal is to follow the link trajectories of the master robot q_m, $\dot{q}_m$, and no the desired trajectories q_d, $\dot{q}_d$ for the master robot, which may not be realized due to model uncertainties or disturbances, e.g., noise, unknown loads.

A similar setup and goal have been considered in [16] for the case of rigid joint robots. In [16] semiglobal uniform, ultimate boundedness of the synchronization system is proved and a relationship between the controller gains and the ultimate error bounded is obtained. Throughout this paper for matrices the induced norm $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$ is considered, with $\lambda_{\max}(\cdot)$ the maximum eigenvalue. For any positive definite matrix A, A_m and A_M denote its minimum and maximum eigenvalue, respectively.

This paper is organized as follows, Section 2 presents a reduced model for the flexible joint robot and some of its properties. A nominal controller, assuming that all signals θ_s, $\dot{\theta}_s$, q_m, $\dot{q}_m$, q_s, $\dot{q}_s$ are available, is introduced in Section 3. Section 4 presents a version of the nominal controller based on estimated values for θ_s, $\dot{\theta}_s$, $\dot{q}_m$, $\dot{q}_s$. The closed loop system formed by the slave robot and the synchronization controller is obtained in Section 5, and its stability analysis is presented in Section 6. Section 7 summarizes a tuning gain procedure for the gains in the controller and observers. A simulation study is presented in section 8. Some conclusions are given in Section 9.

2 Reduced Dynamic Model of the Flexible Joint Robot

The dynamic model of a flexible joint robot can be obtained by extending the procedures already used for rigid robots [5]. Consider a flexible joint robot, with n rigid links, all joints being flexible, revolute, and actuated by electrical drives. Let $q_i \in \mathbb{R}^n$ be the link positions and $\theta_i \in \mathbb{R}^n$ be the rotor positions, as reflected through the gear ratios, the subindex $i = m$, s identifies the master (m) and slave (s) robot. The difference $q_{ij} - \theta_{ij}$ is the j-th joint deformation, in view of small deformations, joint elasticity is modelled as a linear spring. The rotors of the motors are modelled as balanced uniform bodies having their center of mass on the rotation axis, so that the inertia matrix and the gravity term in the dynamic model are independent from the motor position θ_i. Assuming that the motion of the rotors can be considered as pure rotations with respect to an inertial frame, then the kinetic energy of each rotor is due to its own spinning. Therefore, the inertial coupling between links and rotors can be neglected and a reduced dynamic model is obtained (see [6]). Following [6] and the Lagrangian approach [17], we obtain the dynamic model:

$$M_i(q_i)\ddot{q}_i + N(q_i,\dot{q}_i) + K_i(q_i - \theta_i) = 0 \qquad i = m,s \tag{1}$$

$$J_i\ddot{\theta}_i + K_i(\theta_i - q_i) + B_{v,i}\dot{\theta}_i = \tau_i \tag{2}$$

$$N(q_i,\dot{q}_i) = C_i(q_i,\dot{q}_i)\dot{q}_i + g_i(q_i) \tag{3}$$

where the symmetric positive definite inertia matrix $M_i(q_i) \in \mathbb{R}^{n\times n}$, the Coriolis and centrifugal term $C_i(q_i,\dot{q}_i)\dot{q}_i \in \mathbb{R}^n$, and the gravity term $g_i(q_i) \in \mathbb{R}^n$ are all related to the rigid links, $J_i \in \mathbb{R}^{n\times n}$ is the constant diagonal inertia matrix of the motors, $K_i \in \mathbb{R}^{n\times n}$ is the constant diagonal matrix of the joint stiffness, $B_{v,i} \in \mathbb{R}^{n\times n}$ is the diagonal positive definite viscous friction coefficient matrix, and $\tau_i(\cdot)$ is the n-vector of torques supplied by the motors. The following properties are satisfied by the model (1)–(3)

- If the matrix $C_i(q_i,\dot{q}_i) \in \mathbb{R}^{n\times n}$ is defined using the Christoffel symbols [18], then the matrix $\dot{M}_i(q_i) - 2C_i(q_i,\dot{q}_i)$ is skew symmetric.
- In addition, for the previous choice of the matrix $C_i(q_i,\dot{q}_i)$, the Coriolis and centrifugal term $C_i(q_i,\dot{q}_i)$ can be written as:

$$C_i(q_i,\dot{q}_i) = \begin{bmatrix} \dot{q}_i C_{i1}(q_i) \\ \vdots \\ \dot{q}_i C_{in}(q_i) \end{bmatrix} \tag{4}$$

where $C_{ij}(q_i) \in \mathbb{R}^{n\times n}$ $j = 1, \ldots, n$ are symmetric matrices. It follows that

$$\begin{aligned} C_i(q_i,x)y &= C_i(q_i,y)x \\ C_i(q_i,z+\alpha x)y &= C_i(q_i,z)y + \alpha C_i(q_i,x)y \end{aligned} \tag{5}$$

for any scalar α and for all q_i, x, y, $z \in \mathbb{R}^n$.

- The matrices $M_i(q_i)$, $C_i(q_i,\dot{q}_i)$, $B_{v,i} \in \mathbb{R}^{n\times n}$ are bounded

$$0 < M_{im} \le \|M_i(q_i)\| \le M_{iM} \quad \text{for all} \quad q_i \in \mathbb{R}^n \tag{6}$$

$$\|C_i(q_i,x)\| \le C_{iM}\|x\| \quad \text{for all} \quad q_i, x \in \mathbb{R}^n \tag{7}$$

$$0 < B_{v,im} \le \|B_{v,i}\| \le B_{v,iM} \tag{8}$$

3 Nominal Feedback Controller

Consider that (1)–(3) define the dynamics of two robots working under a master slave scheme. Based on inverse dynamics computation and using De Luca and Lucibello [8], the slave robot with model (1)–(3) can be fully linearized and decoupled via the static feedback control law:

$$\begin{aligned} \tau_s = {} & K_s(\theta_s - q_s) + B_{v,s}\dot{\theta}_s + J_s K_s^{-1}[M_s(q_s)v(t) \\ & + \alpha(q_s,\dot{q}_s,\ddot{q}_s,q_s^{(3)})] \end{aligned} \tag{9}$$

$$\begin{aligned} \alpha(q_s,\dot{q}_s,\ddot{q}_s,q_s^{(3)}) = {} & 2\dot{M}_s(q_s,\dot{q}_s)q_s^{(3)} + (\ddot{M}_s(q_s,\dot{q}_s,\ddot{q}_s) + K_s)\ddot{q}_s \\ & + \ddot{N}(q_s,\dot{q}_s,\ddot{q}_s,q_s^{(3)}) \end{aligned} \tag{10}$$

From (1) it follows that $\ddot{q}_s$, $q_s^{(3)}$ are related to lower order variables q_s, $\dot{q}_s$, $\dot{\theta}_s$ as:

$$\ddot{q}_s = -M_s^{-1}(q_s)(N(q_s,\dot{q}_s) + K_s(q_s - \theta_s)) \tag{11}$$

$$q_s^{(3)} = -M_s^{-1}(q_s)(\dot{M}_s(q_s,\dot{q}_s)\ddot{q}_s + \dot{N}(q_s,\dot{q}_s,\ddot{q}_s) + K_s(\dot{q}_s - \dot{\theta}_s)) \tag{12}$$

Therefore (9) can be written as function of q_s, $\dot{q}_s$, θ_s, $\dot{\theta}_s$, i.e.:

$$\begin{aligned} \tau_s &= K_s(\theta_s - q_s) + \beta(q_s,\dot{q}_s,\theta_s,\dot{\theta}_s) + \varphi(q_s)v(t) \\ \beta(q_s,\dot{q}_s,\theta_s,\dot{\theta}_s) &= B_{v,s}\dot{\theta}_s + J_s K_s^{-1}\alpha(q_s,\dot{q}_s,\ddot{q}_s,q_s^{(3)}) \\ \varphi(q_s) &= J_s K_s^{-1} M_s(q_s) \end{aligned} \tag{13}$$

Applying the controller (9) to the system (1)–(3) yields the linear decoupled closed loop system:

$$q_s^{(4)} = v(t) \tag{14}$$

To ensure synchronization between the slave and the master robot $v(t)$ is proposed as:

$$v(t) = q_m^{(4)} - K_3 e^{(3)} - K_2\ddot{e} - K_1\dot{e} - K_0 e \tag{15}$$

with $K_i \in \mathbb{R}^n$ gain matrices and the synchronization errors defined by:

$$e = q_s - q_m, \qquad \dot{e} = \dot{q}_s - \dot{q}_m \tag{16}$$

where the master position trajectory q_m has to be at least four times differentiable, i.e., $q_m \in C^4$. Clearly, there exist general choices for the gain matrices K_i such that the closed loop systems is stable, but for simplicity and without lost of generality, we assume that the gain matrix K_i is a multiple of the identity matrix, i.e., $K_i = k_i I_n$, $i = 0, 1, 2, 3$, with k_i positive scalars. Then, it is straightforward to conclude that the synchronization error e is exponentially stable, if the scalars k_i, $i = 0, 1, 2, 3$ are chosen such that the polynomial $s^4 + k_3 s^3 + k_2 s^2 + k_1 s + k_0$ is Hurwitz.

4 Feedback Controller Based on Estimated Variables

As stated in Section 1, it is assumed that only the master and slave link positions q_m, q_s are measured, therefore τ_s (9) and $v(t)$ (15) can not be implemented.

Let $\hat{q}_s$, $\hat{\dot{q}}_s$, $\hat{\theta}_s$, $\hat{\dot{\theta}}_s$ denote estimated values for q_s, $\dot{q}_s$, θ_s, $\dot{\theta}_s$, and $v(t)$ given by (15) is implemented based on estimated synchronization errors. Assuming that $\hat{q}_s$, $\hat{\dot{q}}_s$, $\hat{\theta}_s$, $\hat{\dot{\theta}}_s$ are available, the controller (13) can be modified as:

$$\tau_s = K_s(\hat{\theta}_s - \hat{q}_s) + \beta(q_s, \hat{q}_s, \hat{\theta}_s, \hat{\dot{\theta}}_s) + \varphi(q_s)v(t)$$

$$\beta(q_s, \hat{q}_s, \hat{\theta}_s, \hat{\dot{\theta}}_s) = B_{v,s}\hat{\dot{\theta}}_s + J_s K_s^{-1}\alpha(q_s, \hat{\dot{q}}_s, \hat{\ddot{q}}_s, \widehat{q_s^{(3)}}) \quad (17)$$

$$\alpha(q_s, \hat{\dot{q}}_s, \hat{\ddot{q}}_s, \widehat{q_s^{(3)}}) = 2\dot{M}_s(q_s, \hat{\dot{q}}_s)\widehat{q_s^{(3)}} + (\ddot{M}_s(q_s, \hat{\dot{q}}_s, \hat{\ddot{q}}_s) + K_s)\hat{\ddot{q}}_s + \ddot{N}(q_s, \hat{\dot{q}}_s, \hat{\ddot{q}}_s, \widehat{q_s^{(3)}})$$

where according to (11), (12), estimates for $\ddot{q}_s$, $q_s^{(3)}$ are given by

$$\hat{\ddot{q}}_s = -M_s^{-1}(q_s)(N(q_s, \hat{\dot{q}}_s) + K_s(\hat{q}_s - \hat{\theta}_s)) \quad (18)$$

$$\widehat{q_s^{(3)}} = -M_s^{-1}(q_s)(\dot{M}_s(q_s, \hat{\dot{q}}_s)\hat{\ddot{q}}_s + \dot{N}(q_s, \hat{\dot{q}}_s, \hat{\ddot{q}}_s) + K_s(\hat{\dot{q}}_s - \hat{\dot{\theta}}_s)) \quad (19)$$

4.1 An Observer for the Synchronization Errors.

From the work of Berghuis and Nijmeijer [19], we propose the modified controller $v(t)$ (15) as:

$$v = -K_3\dot{w}_2 - K_2\dot{w}_1 - K_1\dot{\hat{e}} - K_0\hat{e} \quad (20)$$

where $\hat{e}$, $\dot{\hat{e}}$, $\dot{w}_1$, $\dot{w}_2$ represent estimates for e, $\dot{e}$, $\ddot{e}$, $e^{(3)}$ respectively. They are obtained by the observer:

$$\begin{aligned} \dot{\hat{e}} &= w_1 + \Gamma_1\tilde{e} \\ \dot{w}_1 &= w_2 + \Gamma_2\tilde{e} \\ \dot{w}_2 &= w_3 + \Gamma_3\tilde{e} \\ \dot{w}_3 &= \Gamma_4\tilde{e} \end{aligned} \quad (21)$$

with $\Gamma_i \in \mathbb{R}^{n\times n}$, $i=1, 2, 3, 4$ diagonal positive definite gain matrices and the estimation synchronization error defined by:

$$\tilde{e} = e - \hat{e} \quad (22)$$

4.2 An Observer for the Slave Variables. q_s, $\dot{q}_s$, θ_s, $\dot{\theta}_s$.

Based on the dynamic model (1)–(3), we propose the nonlinear Luenberger observer:

$$\begin{aligned} \frac{d}{dt}\hat{q}_s &= \hat{\dot{q}}_s + \mu_1\tilde{q} \\ \frac{d}{dt}\hat{\dot{q}}_s &= -M_s^{-1}(q_s)(N(q_s, \hat{\dot{q}}_s) + K_s(\hat{q}_s - \hat{\theta}_s)) + \mu_2\tilde{q} \end{aligned} \quad (23)$$

$$\begin{aligned} \frac{d}{dt}\hat{\theta}_s &= \hat{\dot{\theta}}_s + \mu_3\tilde{q} \\ \frac{d}{dt}\hat{\dot{\theta}}_s &= J_s^{-1}(\tau_s(q_s, \hat{q}_s, \hat{\dot{q}}_s, \hat{\theta}_s, \hat{\dot{\theta}}_s) - K_s(\hat{\theta}_s - \hat{q}_s) - B_{v,s}\hat{\dot{\theta}}_s) + \mu_4\tilde{q} \end{aligned} \quad (24)$$

where $\hat{q}_s$, $\hat{\dot{q}}_s$, $\hat{\theta}_s$, $\hat{\dot{\theta}}_s$ correspond to the estimates of q_s, $\dot{q}_s$, θ_s, $\dot{\theta}_s$, and μ_1, μ_2, μ_3, $\mu_4 \in \mathbb{R}^{n\times n}$ are diagonal positive definite gain matrices. The joint estimation errors are defined by

$$\tilde{q} = q_s - \hat{q}_s, \quad \tilde{\theta} = \theta_s - \hat{\theta}_s \quad (25)$$

5 Closed-Loop System

In this section, the total error dynamics is obtained.

Proposition 1. *Consider the master and slave flexible joint robots, which are described by* (1)–(3), *the slave robot in closed loop with the synchronization controller* τ_s (17), v (20) *and the observers* (21), (23), and (24), then the synchronization closed-loop error dynamics is given by:

$$\begin{aligned} &e^{(4)} + K_3e^{(3)} + K_2\ddot{e} + K_1\dot{e} + K_0e \\ &\quad = q_m^{(4)} + K_3(\tilde{e}^{(3)} + \Gamma_1\ddot{\tilde{e}} + \Gamma_2\dot{\tilde{e}}) + K_2(\ddot{\tilde{e}} + \Gamma_1\dot{\tilde{e}}) + K_1\dot{\tilde{e}} + K_0\tilde{e} \\ &\qquad - M_s^{-1}(q_s)\Phi(q_s, \dot{q}_m, \ddot{q}_m, q_m^{(3)}, \dot{e}, \ddot{e}, e^{(3)}, \tilde{q}, \dot{\tilde{q}}, \tilde{\theta}, \dot{\tilde{\theta}}) \end{aligned} \quad (26)$$

$$\begin{aligned} &\tilde{e}^{(4)} + (\Gamma_1 - K_3)\tilde{e}^{(3)} + (\Gamma_2 - K_3\Gamma_1 - K_2)\ddot{\tilde{e}} \\ &\quad + (\Gamma_3 - K_3\Gamma_2 - K_2\Gamma_1 - K_1)\dot{\tilde{e}} + (\Gamma_4 - K_0)\tilde{e} \\ &\quad = q_m^{(4)} - K_3e^{(3)} - K_2\ddot{e} - K_1\dot{e} - K_0e \\ &\qquad - M_s^{-1}(q_s)\Phi(q_s, \dot{q}_m, \ddot{q}_m, q_m^{(3)}, \dot{e}, \ddot{e}, e^{(3)}, \tilde{q}, \dot{\tilde{q}}, \tilde{\theta}, \dot{\tilde{\theta}}), \end{aligned} \quad (27)$$

$$\begin{aligned} \ddot{\tilde{q}} &= -M_s^{-1}(q_s)((2C_s(q_s, \dot{e} + \dot{q}_m) - C_s(q_s, \dot{\tilde{q}} + \mu_1\tilde{q}))(\dot{\tilde{q}} + \mu_1\tilde{q}) \\ &\quad + K_s(\tilde{q} - \tilde{\theta})) - \mu_1\dot{\tilde{q}} - \mu_2\tilde{q} \\ \ddot{\tilde{\theta}} &= -J_s^{-1}K_s(\tilde{\theta} - \tilde{q}) - J_s^{-1}B_{v,s}(\dot{\tilde{\theta}} + \mu_3\tilde{q}) - \mu_3\dot{\tilde{q}} - \mu_4\tilde{q} \end{aligned} \quad (28)$$

Proof: A sketch of the proof is given in Appendix 1. ■

Notice that Φ can be considered as a disturbance term that affects (26) and (27). Also notice that (26) and (27) present a linear structure, therefore, the following proposition can be formulated.

Proposition 2. *Define states* $x_1, \ldots, x_8 \in \mathbb{R}^n$ *as* $x_1 = e$, $x_2 = \dot{e}$, $x_3 = \ddot{e}$, $x_4 = e^{(3)}$, $x_5 = \tilde{e}$, $x_6 = \dot{\tilde{e}}$, $x_7 = \ddot{\tilde{e}}$, $x_8 = \tilde{e}^{(3)}$, *and* $y \in \mathbb{R}^{4n}$ *as* $y_1 = \tilde{q}$, $y_2 = \dot{\tilde{q}}$, $y_3 = \tilde{\theta}$, $y_4 = \dot{\tilde{\theta}}$. *Such that the state vectors x and y are given by* $x = [x_1^T x_2^T x_3^T x_4^T x_5^T x_6^T x_7^T x_8^T]^T$ *and* $y = [y_1^T y_2^T y_3^T y_4^T]^T$. *Then* (26), (27), *and* (28) *can be written as*:

$$\begin{aligned} \dot{x} &= Ax - BM_s^{-1}(q_s)\Phi(q_s, \dot{q}_m, \ddot{q}_m, q_m^{(3)}, x_2, x_3, x_4, y_1, y_2, y_3, y_4) + Bq_m^{(4)} \\ \dot{y} &= f(q_s, \dot{q}_m, x, y) \end{aligned} \quad (29)$$

$$A = \begin{bmatrix} 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 \\ -K_0 & -K_1 & -K_2 & -K_3 & K_0 & \pi_1 & \pi_2 & K_3 \\ 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n \\ -K_0 & -K_1 & -K_2 & -K_3 & K_0 - \Gamma_4 & \pi_1 - \Gamma_3 & \pi_2 - \Gamma_2 & K_3 - \Gamma_1 \end{bmatrix} \in \mathbb{R}^{8n\times 8n} \quad (30)$$

$$B = [0 \ \ 0 \ \ 0 \ \ I_n \ \ 0 \ \ 0 \ \ 0 \ \ I_n]^T \in \mathbb{R}^{8n\times n} \quad (31)$$

$$\pi_1 = K_3\Gamma_2 + K_2\Gamma_1 + K_1, \quad \pi_2 = K_3\Gamma_1 + K_2$$

with 0, $I_n \in \mathbb{R}^{n\times n}$ *the zero and identity matrices.*

The nonlinear vector function $f(q_s, \dot{q}_m, x, y)$ *is given by*:

$$f(q_s,\dot{q}_m,x,y) = \begin{bmatrix} y_2 \\ -M_s^{-1}(q_s)(2C_s(q_s,x_2+\dot{q}_m) - C_s(q_s,y_2+\mu_1 y_1))(y_2+\mu_1 y_1) \\ -M_s^{-1}(q_s)K_s(y_1-y_3) - \mu_1 y_2 - \mu_2 y_1 \\ y_4 \\ -J_s^{-1}K_s(y_3-y_1) - J_s^{-1}B_{v,s}(y_4+\mu_3 y_1) - \mu_3 y_2 - \mu_4 y_1 \end{bmatrix} \in \mathbb{R}^{4n} \tag{32}$$

Proof: This follows from simple substitution of the states x and y. ■

6 Stability Analysis

The stability analysis is based on a Lyapunov function, whose derivative can be bounded in terms of the closed-loop errors. It is proven that the bound is negative in an annulus around the origin. To derive bounds on the derivative of the Lyapunov function the following assumption is required.

Assumption 3. *The signals* $\dot{q}_m$, $\ddot{q}_m$, $q_m^{(3)}$, $q_m^{(4)}$ *are bounded for all* $t \in [t_0, \infty)$, *therefore there exist* V_M, A_M, D_M *and* E_M *such that*:

$$\begin{aligned} \sup_t \|\dot{q}_m\| &= V_M < \infty, & \sup_t \|\ddot{q}_m\| &= A_M < \infty \\ \sup_t \|q_m^{(3)}\| &= D_M < \infty, & \sup_t \|q_m^{(4)}\| &= E_M < \infty \end{aligned} \tag{33}$$

In practice, it is often not difficult to obtain the master trajectories bounds (33) on the basis of the master desired trajectories $q_d(t)$ and its derivatives, although due to friction effects, tracking errors, etc., the actual motion of the master robot may differ from its desired motion. Also the bounds V_M, A_M, D_M, and E_M can be obtained by considering the structural limitations of the robots, such as maximum velocities and accelerations of the motors. On the other hand, for the sake of simplicity the following assumption is imposed.

Assumption 4. *All the gains in the controller (17), (20), and the observer (21) are a positive multiple of the unit matrix, i.e., of the form* $K = kI$, *where* k *is a positive scalar.*

Based on the above assumptions, the main result of this paper is formulated as follows.

Theorem 5. *Consider the master and slave flexible joint robots, described by (1)–(3), the slave robot in closed loop with the controller* τ_s *(17), and* v *(20) and the observers (21), (23) and (24). Assume that the gain matrices* K_i *and* Γ_j, $i = 0, 1, 2, 3$, $j = 1, 2, 3, 4$ *are chosen such that the matrix A, given by (30), is Hurwitz, and additionally the minimum and maximum eigenvalues of the gains* μ_l, $l = 1, 2, 3, 4$, *i.e.,* μ_{lm} *and* μ_{lM}, *satisfy:*

$$\begin{aligned} \mu_{1m} > \max\{0, 2M_{sm}^{-1}(\lambda_0 - C_M V_M), (2\lambda_0 M_{sm})^{-1} \\ \times(4\lambda_0^2 - M_{sm}K_{sm} - 2C_M V_M(2\lambda_0 + M_{sm}\mu_{1M}) \\ + M_{sm}M_{sM}\mu_{2M})\} \end{aligned} \tag{34}$$

$$\mu_{2m} > \max\{0, -a_{31}^{-1}a_{30}, -a_{41}^{-1}a_{40}\} \tag{35}$$

$$\mu_{3m} > \max\{0, \mu_{33\_}, \mu_{34\_}\}, \quad \mu_{3M} < \min\{\mu_{33}^{-}, \mu_{34}^{-}\} \tag{36}$$

$$\mu_{4m} > \max\{0, J_{sm}^{-1}(4\eta_0^2 - J_{sm}K_{sm} - 2\eta_0 B_{v,sM} + J_{sm}\mu_{4M})\} \tag{37}$$

where M_{sm} *and* J_{sm} *are the minimum eigenvalue of the link and motor rotor inertia matrices, and the scalars* λ_0, η_0, $\mu_{33\_}$, $\mu_{34\_}$, μ_{33}^{-}, μ_{34}^{-}, a_{30}, a_{31}, a_{40}, a_{41} *are defined via the gain tuning procedure of Section 7. Then the synchronization closed-loop errors and the estimation errors are semi-globally uniformly ultimately bounded.*

In particular, it means that ther exist a region of convergence depending on the controller gains, such that if the initial errors at time $t = 0$ *are in this region they will remain bounded for all time* $t \geq 0$ *with a bound smaller that the region of convergence.*

Proof: The proof is divided in three parts, first the candidate Lyapunov function and conditions for positive definitiveness are presented. Second the derivative of the Lyapunov function along (29) is bounded, and finally sufficient conditions for negative definiteness are formulated.

6.1 Lyapunov Function. Consider the synchronization closed-loop error dynamics given by (29), and take as a candidate Lyapunov function:

$$V(q_s, x, y) = x^T P_x x + \frac{1}{2} y^T P_y(q_s, y) y \tag{38}$$

where the positive definite symmetric matrix P_x is the solution of the Lyapunov equation $P_x A + A^T P_x = -Q_x$, for any given symmetric positive definite matrix Q_x, and $P_y(q_s, y)$ is given by:

$$P_y(q_s,y) = \begin{bmatrix} \begin{bmatrix} K_s + 2\lambda_0\mu_1 + \beta_1 I_n & 2\lambda_0 I_n \\ 2\lambda_0 I_n & M_s(q_s) \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} K_s + \mu_4 + \beta_2 I_n & 2\eta(y_3) I_n \\ 2\eta(y_3) I_n & J_s \end{bmatrix} \end{bmatrix}$$

with β_1, β_2 scalars to be determined, and $\eta(y_3)$ defined by:

$$\eta(y_3) := \frac{\eta_0}{1 + \|y_3\|}$$

λ_0, $\eta_0 > 0$ are constant scalars; then for all $\tilde{\theta}$, $\dot{\tilde{\theta}}$ it holds that:

$$0 < \|\eta(y_3)\| < \eta_0, \quad \dot{\eta}(y_3) y_4^T y_3 \leq \eta_0 \|y_4\|^2$$

Existence of P_x is guaranteed if A, which is given by (30), is

Hurwitz. By Assumption 4, a sufficient condition for A being Hurwitz is that the scalar polynomial $\lambda(s)$, given by:

$$\begin{aligned}\lambda(s)=s^8+\gamma_1 s^7+\gamma_2 s^6+\gamma_3 s^5+(\gamma_4+k_1\gamma_1+k_2\gamma_2+k_3\gamma_3)s^4\\ ++(k_0\gamma_1+k_1\gamma_2+k_2\gamma_3+k_3\gamma_4)s^3+(k_0\gamma_2+k_1\gamma_3\\ +k_2\gamma_4)s^2+(k_0\gamma_3+k_1\gamma_4)s+k_0\gamma_4\end{aligned} \tag{39}$$

is Hurwitz. k_i, γ_j are the scalars associated to the gains K_i and Γ_j, $i=0, 1, 2, 3$, $j=1, 2, 3, 4$. Thus by choosing the gains K_i and Γ_j, it can be ensured that the matrix A is Hurwitz. Therefore, there exists a unique positive symmetric matrix P_x, which satisfies $P_xA+A^TP_x=-Q_x$, for any given symmetric positive definite matrix Q_x.

On the other hand, sufficient conditions for positive definiteness of $P_y(q_s,y)$ are:

$$\begin{aligned}\mu_{1m}&>\frac{1}{2\lambda_0 M_{sm}}(4\lambda_0^2-M_{sm}K_{sm}-M_{sm}\beta_1)\\ \mu_{4m}&>\frac{1}{J_{sm}}(4\eta_0^2-J_{sm}K_{sm}-J_{sm}\beta_2)\end{aligned} \tag{40}$$

Finally, positive definiteness of P_x and $P_y(q_s,y)$ imply positive definiteness of $V(q_s,x,y)$ (38). Moreover, there exist positive constants P_m and P_M such that for $\xi^T=[x^T\,y^T]$:

$$\frac{1}{2}P_m\|\xi\|^2\leqslant V(q_s,x,y)\leqslant\frac{1}{2}P_M\|\xi\|^2 \tag{41}$$

6.2 Derivative of the Lyapunov Function. Along the error dynamics (29), the time derivative of V (38) is given by:

$$\begin{aligned}\dot V=&-x^TQ_xx-x^TP_xBM_s^{-1}\Phi-\Phi^TM_s^{-1}B^TP_xx\\ &+y^TP_y(q_s,y)f(x,y)+\frac{1}{2}y^T\dot P_y(q_s,y)y+x^TP_xBq_m^{(4)}\\ &+q_m^{(4)T}B^TP_xx\end{aligned} \tag{42}$$

that can be written as

$$\dot V=-[x\;\;y]^T\begin{bmatrix}Q_x & Q_{xy}\\ Q_{xy}^T & Q_y\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+\Omega(q_s,\dot q_m,\ddot q_m,q_m^{(3)},q_m^{(4)},x,y) \tag{43}$$

with $Q_x\in\mathbb{R}^{8n\times 8n}$ the symmetric positive definite matrix $Q_x=-(P_xA+A^TP_x)$ and $Q_y\in\mathbb{R}^{4n\times 4n}$

$$Q_y=\begin{bmatrix}2\lambda_0(M_s^{-1}K_s+\mu_2) & \alpha_1 & \alpha_2 & \alpha_3\\ \alpha_1^T & M_s\mu_1-2\lambda_0 & \eta(y_3)\mu_3-\frac{1}{2}K_s & \frac{1}{2}J_s\mu_3\\ \alpha_2^T & \left(\eta(y_3)\mu_3-\frac{1}{2}K_s\right)^T & 2\eta(y_3)J_s^{-1}K_s & \alpha_4\\ \alpha_3^T & \frac{1}{2}(J_s\mu_3)^T & \alpha_4^T & B_{v,s}-2\eta(y_3)I_n\end{bmatrix}$$

$$\begin{aligned}\alpha_1&=-\frac{1}{2}(\beta_1I_n+M_s\mu_2)\\ \alpha_2&=-\lambda_0M_s^{-1}K_s+\eta(y_3)(\mu_4+J_s^{-1}B_{v,s}\mu_3-J_s^{-1}K_s)\\ \alpha_3&=\frac{1}{2}(B_{v,s}\mu_3+J_s\mu_4-K_s)\\ \alpha_4&=\eta(y_3)J_s^{-1}B_{v,s}-\frac{1}{2}(\mu_4+\beta_2I_n)\end{aligned} \tag{44}$$

$Q_{xy}\in\mathbb{R}^{8n\times 4n}$ depends on the entries of $P_x\in\mathbb{R}^{8n\times 8n}$, and it is given by:

$$Q_{xy}=\begin{bmatrix}P_{x14}+P_{x18}\\ P_{x24}+P_{x28}\\ P_{x34}+P_{x38}\\ P_{x44}+P_{x48}\\ P_{x45}^T+P_{x58}\\ P_{x46}^T+P_{x68}\\ P_{x47}^T+P_{x78}\\ P_{x48}^T+P_{x88}\end{bmatrix}M_s^{-1}K_sJ_s^{-1}[(B_{v,s}\mu_3-K_s)\;\;0\;\;K_s\;\;B_{v,s}] \tag{45}$$

The scalar function $\Omega(q_s,\dot q_m,\ddot q_m,q_m^{(3)},q_m^{(4)},x,y)$ is locally Lipschitz and is given by:

$$\begin{aligned}\Omega=&-x^TP_xBM_s^{-1}(\Phi+K_sJ_s^{-1}K_s(y_1-y_3)-K_sJ_s^{-1}B_{v,s}(y_4+\mu_3y_1))+x^TP_xBq_m^{(4)}+\\ &-(\Phi+K_sJ_s^{-1}K_s(y_1-y_3)-K_sJ_s^{-1}B_{v,s}(y_4+\mu_3y_1))^TM_s^{-1}B^TP_xx+q_m^{(4)T}B^TP_xx\\ &-(2\lambda_0y_1^TM_s^{-1}+y_2^T)(2C_s(q_s,x_2+\dot q_m)-C_s(q_s,\mu_1y_1))(y_2+\mu_1y_1)\\ &+2\lambda_0y_1^TM_s^{-1}C_s(q_s,y_2)(y_2+\mu_1y_1)+y_2^TC_s(q_s,y_2)\mu_1y_1+2\dot\eta(y_3)y_4^Ty_3\end{aligned} \tag{46}$$

6.2.1 Boundedness of $\dot{V}$. The next lemma synthesizes a general bound for $\dot{V}$ given by (43).

Lemma 6. *Consider $\dot{V}$ given by (43) and introduce the vectors x_N, y_N, and ξ_N:*

$$x_N=[\|x_1\| \ \|x_2\| \ \|x_3\| \ \|x_4\| \ \|x_5\| \ \|x_6\| \ \|x_7\| \ \|x_8\|]^T$$
$$y_N=[\|y_1\| \ \|y_2\| \ \|y_3\| \ \|y_4\|]^T \tag{47}$$
$$\xi_N=[x_N^T \ y_N^T]^T$$

Then $\dot{V}$ given by (43) can be upper-bounded as:

$$\dot{V}\leq -\xi_N^T R_V \xi_N + \Theta(V_M, A_M, D_M, E_M, \xi_N) \tag{48}$$

with V_m, A_m, D_m, E_m given by (33), $\Theta(V_m, A_m, D_m, E_m, \xi_N)$ a scalar function that contains products of at most order 5 in terms of the entries of ξ_N, and the matrix R_V is given by

$$R_V=\begin{bmatrix} R_x & R_{xy}(V_M, A_M, D_M) \\ R_{xy}^T(V_M, A_M, D_M) & R_y(V_m) \end{bmatrix} \tag{49}$$

R_x, $R_{xy}\in\mathbb{R}^{8\times 4}$, $R_y\in\mathbb{R}^{4\times 4}$, Θ and the positive definite matrix $R_x\in\mathbb{R}^{8\times 8}$ as in Appendix 2.

Proof: See Appendix 2. ■

The upperbound of $\dot{V}$ given by (48) can be reduced to a function of the norm of ξ_N.

Lemma 7. *In terms of the vector ξ_N, (47), the upper-bound of $\dot{V}$, (48) can be reduced to*

$$\dot{V}\leq \|\xi_N\|(r_0 - r_1\|\xi_N\| + r_2\|\xi_N\|^2 + r_3\|\xi_N\|^3 + r_4\|\xi_N\|^4) \tag{50}$$

where r_1 is the minimum eigenvalue of the matrix R_V, (49), and the positive scalars $r_0(E_M)$, $r_2(\mu_{1M}, \mu_{3M})$, $r_3(\mu_{1M})$, $r_4(\mu_{1M})$ are determined by $\Theta(V_M, A_M, D_M, E_M, \xi_N)$, with V_M, A_M, D_M, E_M the upperbounds for $\dot{q}_m$, $\ddot{q}_m$, $q_m^{(3)}$, $q_m^{(4)}$, which are defined by (33).

Proof: This follows directly from (48) and the definition of ξ_N. ■

6.2.2 Negative Definiteness of $\dot{V}$. Equation (50) *is an upperbound for $\dot{V}$, such that $\dot{V}$ is negative if and only if r_1 is positive and the vector ξ_N holds $x_{\xi,1}<\|\xi_N\|<x_{\xi,2}$. Where $x_{\xi,1}$ and $x_{\xi,2}$ are the roots of the polynomial*:

$$g(x_\xi)=r_0 - r_1 x_\xi + r_2 x_\xi^2 + r_3 x_\xi^3 + r_4 x_\xi^4 \tag{51}$$

such that $g(x_\xi)<0$ if $x_{\xi,1}<x_\xi<x_{\xi,2}$.

Notice that $x_{\xi,1}$ and $x_{\xi,2}$ determine the region in which $\dot{V}$ is negative definite. Thus $x_{\xi,1}$ and $x_{\xi,2}$ together with the minimum P_m and maximum P_M bounds of the Lyapunov function V, (41), determine the region of convergence of the synchronization closed loop system, similar as in [16]. The scalar r_1 is the minimum eigenvalue of the matrix R_V (49), so, r_1 is positive if and only if R_V is positive definite. The following lemma is useful to prove positive definiteness of R_V.

Lemma 8. *(See [20]) If $L\in\mathbb{R}^{m\times m}$ and $M\in\mathbb{R}^{n\times n}$ are given positive semidefinite matrices and $X\in\mathbb{R}^{m\times n}$, then the symmetric block matrix:*

$$\chi_{LM}=\begin{bmatrix} L & X \\ X^T & M \end{bmatrix} \tag{52}$$

is positive semidefinite if and only if there exists a matrix $C\in\mathbb{R}^{m\times n}$, such that $X=L^{1/2}CM^{1/2}$. If L and M are positive definite, then this criterion is equivalent to:

$$\|L^{-1/2}XM^{-1/2}\|_2\leq 1$$

Moreover χ_{LM} is positive definite if and only if L and M are positive definite and

$$\|L^{-1/2}XM^{-1/2}\|_2<1 \tag{53}$$

Before applying the above lemma, it is required to prove that the matrix R_y is positive definite.

Proposition 9. *Consider $R_y\in\mathbb{R}^{4\times 4}$ the matrix defined in (49), and define scalars β_1, β_2 as*:

$$\begin{aligned}\beta_1&=2C_M V_M(2M_{sm}^{-1}\lambda_0+\mu_{1M})-M_{sM}\mu_{2M}\\ \beta_2&=2\,\eta_0 J_{sm}^{-1}B_{v,sM}-\mu_{4M}\end{aligned} \tag{54}$$

Then sufficient conditions for positive definiteness of R_y are:

1. $\lambda_0>0$
2. $\mu_{1m}>\max\{0, 2M_{sm}^{-1}(\lambda_0 - C_M V_M)\}$
3. $0<\eta_0<\min\{B_{v,sm}/4, \frac{1}{8J_{sM}}(J_{sM}B_{v,sm} + 8J_{sM}J_{sm}^2K_{sm})^{1/2})\}$
4. $\mu_{3m}>\max\{0, \mu_{33\_}, \mu_{34\_}\}$, $\mu_{3M}<\min\{\mu_{33}^-, \mu_{34}^-\}$
5. $\mu_{2m}>\max\{0, -a_{31}^{-1}a_{30}, -a_{41}^{-1}a_{40}\}$

   with $\mu_{33\_}$, $\mu_{34\_}$, μ_{33}^-, μ_{34}^-, a_{30}, a_{31}, a_{40}, a_{41} given in Appendix 3.

Proof: See Appendix 3. ■

The conditions listed in Theorem 5 clearly imply the conditions in the above lemma, therefore it can be ensured that R_y is positive definite and then Lemma 8 can be used.

For R_V, given by (49), the condition (53) can be written as:

$$\|R_x^{-1/2}R_{xy}R_y^{-1/2}\|_2<1$$

Notice that R_{xy} depends on the gains μ_1 and μ_3, but does not depend on μ_2, η_0, λ_0. Then, if the gains μ_1, μ_3 have been chosen according to Proposition 9, it follows that R_{xy} is only determined by P_x, (38), that is determined by Q_x (43). The matrix R_x in R_V, given by (49) is only determine by the positive definite symmetric matrix Q_x, therefore by choosing Q_x it can be ensured that $\det(Q_x)\geq 1$ and thus $\det(R_x)\geq 1$. Also notice that the only entry of R_y which depends on μ_2 is α_0^* thus by choosing the minimum eigenvalue of the gain μ_2 it can be ensured that $\det(R_y)\geq 1$. Therefore, it follows that the entries of $R_x^{-1/2}R_{xy}R_y^{-1/2}$ are small. As a result $\|R_x^{-1/2}R_{xy}R_y^{-1/2}\|_2<1$ can be ensured, and thus R_V, given by (49), is positive definite. Since the condition for R_V being positive definite is given by the minimum eigenvalue of the gain μ_2 then it follows that the minimum eigenvalue of R_V, i.e., r_1, is determined by the minimum eigenvalue of μ_2, that implies that r_1 can be chosen such that it dominates the other terms in (50). To emphasize the last conclusion, notice that r_0, r_2, r_3, r_4 do no depend on the gain μ_2.

Finally, notice that according to Section 6.1, Proposition 9, and the above paragraph, it follows that if conditions in Theorem 5 are fulfilled, then the function V, given by (38), is a Lyapunov function with $\dot{V}<0$ in an annulus around the origin determine by $x_{\xi,1}$, $x_{\xi,2}$, (51), and the minimum P_m and maximum P_M bounds of the Lyapunov function V, (41), see [16]. Therefore the synchronization closed loop errors are uniformly ultimately bounded in such annulus. ■

The ultimate boundedness result is due to the absence of measurements of derivatives of the master trajectory q_m, therefore, we have the following corollary.

Corollary 10. *If set point regulation of the master robot is considered and the master robot controller is able to achieve steady state in finite time, then $q_m^{(4)}(t)=0$ for $t\in(t_2,\infty)$, with $t_2\geq t_0$, the convergence time of the master robot trajectories. If additionally the conditions on Theorem 5 are satisfied, then the controller* (17), (20), *and the observers* (21), (23), and (24) *yield local exponential convergence of the closed-loop errors.*

Proof: If $q_m^{(4)}(t)=0$ for $t\in(t_2,\infty)$, $t_2\geq t_0$, with t_2 the time in which the master robot achieves stationary state, then it implies that the upper-bound for $q_m^{(4)}$ is zero for $t\in(t_2,\infty)$, $t_2\geq t_0$, and thus from Assumption 3 it follows that:

$$E_M=\sup_{t\geq t_2}\|q_m^{(4)}\|=0$$

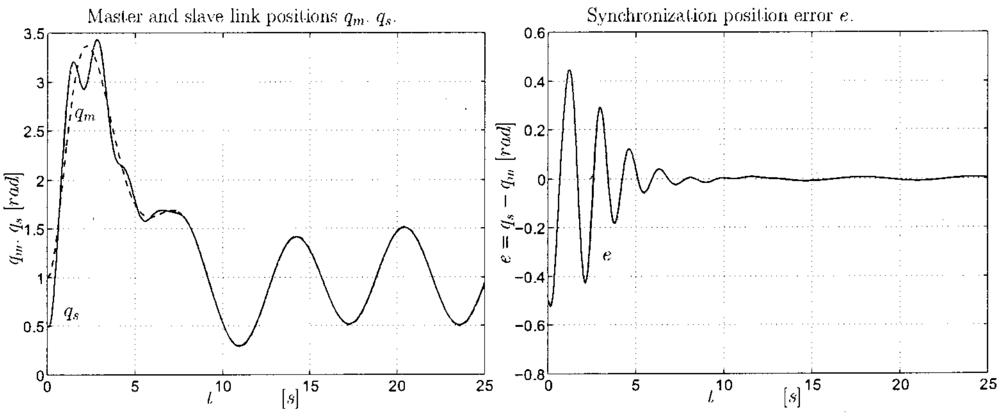

Fig. 1 Master and slave link positions q_m, q_s, and synchronization position error e

so $r_0(E_M)=0$ in (50). Thus, if conditions in Theorem 5 are satisfied, for $t \geqslant t_2$ (50) reduces to:

$$\dot{V} \leqslant \|\xi_N\|^2(-r_1+r_2\|\xi_N\|+r_3\|\xi_N\|^2+r_4\|\xi_N\|^3) \tag{55}$$

with r_1, r_2, r_3, $r_4>0$, and $\dot{V} \leqslant 0$ if $x_{\xi,1}<\|\xi_N\|<x_{\xi,2}$. As a consequence, there exist a positive scalar κ, such that $\dot{V}$ can be upper-bounded as:

$$\dot{V} \leqslant -\kappa\|\xi_N\|^2 \quad \text{for all} \quad t \geqslant t_2, \; x_{\xi,1}<\|\xi_N\|<x_{\xi,2}$$

From the last equation and (41), we conclude that there exist some constants m^*, $\rho>0$, such that:

$$\|\xi_N(t)\|^2 \leqslant m^* e^{-\rho t}\|\xi_N(t_2)\|^2 \quad \text{for all} \quad t \geqslant t_2, x_{\xi,1}<\|\xi_N\|<x_{\xi,2}$$

and thus from $\xi_N(t)$, (47), it follows that the close-loop errors are locally exponentially stable. ∎

Remark 11. *The proposed synchronization controller* (17,20) *is designed to guarantee synchronization between two robots. Nevertheless, it can be used as a tracking controller by taking the desired trajectory $q_d(t)$ as the master robot trajectory $q_m(t)$. In case of tracking the desired trajectory $q_d(t)$ and its derivatives are known, such that $q_m^{(4)}=q_d^{(4)}$ can be included through the control $v(t)$* (20). *In such case, the closed-loop error* (29) *does not depend on $q_m^{(4)}=q_d^{(4)}$, and the stability analysis would result in $\dot{V}$ given by* (55). *Therefore, for tracking of a known desired reference $q_d(t)$ the proposed synchronization controller* (17), (20) *with $\hat{v}(t)$* (20) *modified as*:

$$\hat{v}(t)=q_d^{(4)}(t)-K_3\dot{w}_2-K_2\dot{w}_1-K_1\dot{\hat{e}}-K_0\hat{e}$$

yields semi-global exponential convergence of the closed-loop errors.

7 Design Procedure

The tuning gain procedure can be summarized as follows:

1. Choose the gains K_i and Γ_j, $i=0, 1, 2, 3$, $j=1, 2, 3, 4$ such that $\lambda(s)$ (39) is Hurwitz.
2. Determine the bounds of the physical parameters $M_s(q_s)$, $C_s(q_s,\dot{q}_s)$, $g_s(q_s)$ and their partial derivatives with respect to q_s.
3. Determine the bounds of the master trajectories $\dot{q}_m$, $\ddot{q}_m$, $q_m^{(3)}$, $q_m^{(4)}$.
4. Choose $\lambda_0>0$, $\mu_{1M}>0$, $\mu_{4M}>0$ and a bound for the maximum eigenvalue of μ_2, i.e., μ_{2M}.
5. Choose μ_1, such that $\mu_{1m}>\max\{0, 2M_{sm}^{-1}(\lambda_0 - C_M V_M), (2\lambda_0 M_{sm})^{-1}(4\lambda_0^2 - M_{sm}K_{sm} - 4C_M V_M \lambda_0 - 2C_M V_M M_{sm}\mu_{1M} + M_{sm}M_{sM}\mu_{2M})\}$
6. Choose μ_0, such that $0<\mu_0<\min\{B_{v,sm}/4, \frac{1}{8J_{sM}}(J_{sM}B_{v,sm} + (J_{sM}^2 B_{v,sm}^2 + 8J_{sM}J_{sm}^2 K_{sm})^{1/2})\}$
7. Select μ_3, such that $\mu_{3m}>\max\{0, \mu_{33\_}, \mu_{34\_}\}$, $\mu_{3M} < \min\{\mu_{33}^{-}, \mu_{34}^{-}\}$, see Appendix 3.
8. $\mu_{4m}>\max\{0, J_{sm}^{-1}(4\eta_0^2 - J_{sm}K_{sm} - 2\eta_0 B_{v,sM} + J_{sm}\mu_{4M})\}$
9. Choose Q_x a symmetric positive definite block diagonal matrix, with $n\times n$ block entries, such that $\det(Q_x)\geqslant 1$.
10. Determine P_x such that $P_x A + A^T P_x = -Q_x$
11. Choose μ_2, such that $\mu_{2m}>\max\{0, -a_{31}^{-1}a_{30}, -a_{41}^{-1}a_{40}\}$, see Appendix 3, and μ_{2m} large enough to ensure $\|R_x^{-1/2}R_{xy}R_y^{-1/2}\|_2<1$.

8 Simulations

The slave (s) and master (m) robot considered in the simulation consist of one rigid link with a flexible joint, rotating in a vertical plane. The dynamic model is given by:

$$M_i\ddot{q}_i+K_i(q_i-\theta_i)+\frac{1}{2}m_i g l_i \sin(q_i)=0, \quad i=m, \; s$$

$$J_i\ddot{\theta}_i+K_i(\theta_i-q_i)+B_{v,i}\dot{\theta}_i=\tau_i$$

The master robot parameters are $M_m=0.5$, $K_m=75$, $B_{v,m}=2$, $m_m=1.4$, $l_m=1$, $J_m=0.04$ (all values are in SI units), and its initial conditions are $q_m(0)=1$ rad, $\dot{q}_m(0)=0$ rad/s, $\theta_m(0)=1.1$ rad, $\dot{\theta}_m(0)=0$ rad/s. The master robot is driven by the controller (9) and (15), the gains on (15) are chosen as $k_{0m}=1$, $k_{1m}=3$, $k_{2m}=6$, $k_{3m}=3$. The desired link master trajectory is:

$$q_{md}(t)=1+0.5\sin(t) \;\; [\text{rad}]$$

The slave robot parameters are $M_s=0.4$, $K_s=100$, $B_{v,s}=5$, $m_s=1$, $l_s=1$, $J_s=0.02$. The initial conditions for the slave robot are $q_s(0)=0.5$ rad, $\dot{q}_s(0)=0$ rad/s, $\theta_s(0)=0.51$ rad, $\dot{\theta}_s(0)=0$ rad/s. The initial conditions for the observers (21), (23), and (24) are chosen as $\hat{e}(0)=-0.1$, $w_1(0)=0$, $w_2(0)=0$, $w_3(0)=0$, $\hat{q}_s(0)=0.4$, $\hat{\dot{q}}_s(0)=0$, $\hat{\theta}_s(0)=0.4$, and $\hat{\dot{\theta}}_s(0)=0$.

The scalar gains in τ_s, (20), and the observers (21), (23), and (24) are chosen to be $k_0=65$, $k_1=40$, $k_2=10$, $k_3=4$, $\gamma_1=40$, $\gamma_2=700$, $\gamma_3=4000$, $\gamma_4=1000$, $\mu_1=1$, $\mu_2=5$, $\mu_3=1$, $\mu_4=5$. As it is shown in Fig. 1, the synchronization error between the master and slave link position is stable and bounded after the transient period has finished. The same is concluded for the estimation errors in Figs. 2 and 3.

In agreement with the stability analysis, the simulations have shown that the final bound of the errors depends on the gains μ_2 and K_0, see Section 6.2.2. Meanwhile the transient behavior is mainly determined by μ_1, μ_3 and the gain K_1, this is due to the

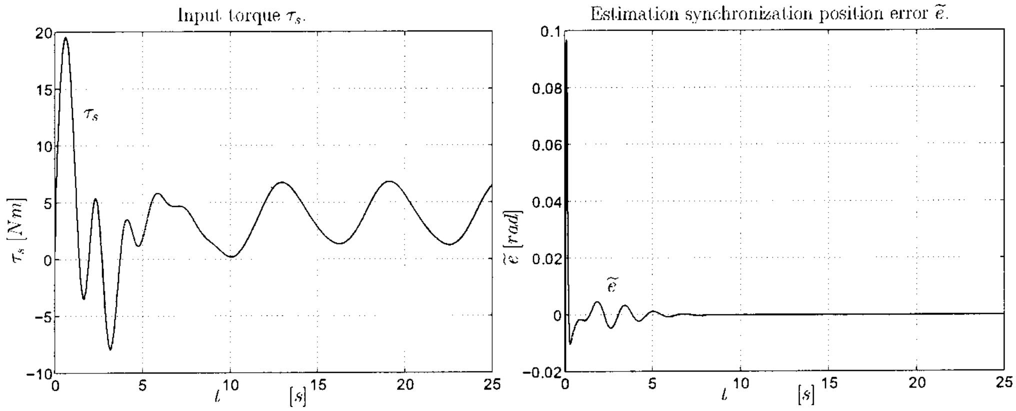

Fig. 2 Input torque τ_s and estimation synchronization position error $\tilde{e}$

fact the K_1 defines the poles of the linear part Ax in (29), and μ_1, μ_3 weight the effect of the estimation errors through the term Φ in (29). Therefore if μ_1, μ_3 are large, as well as the estimation errors, then the term Φ has a large influence in the synchronization error dynamics, this large influence lasts until the estimation errors reach a vicinity around zero.

The simulation study shows that in order to minimize the peaks during the transient period, it is important to tune the gains on v (20) such that the polynomial $\lambda(s)$ (39) corresponds to an overdamped system. At the same time the gains μ_1, μ_3 should be set small to minimize the influence (through the term Φ) of the estimation errors.

9 Conclusions

A synchronization controller for flexible joint robots interconnected on a master-slave scheme, has been proposed. The controller only requires measurements of the master and slave link positions, the velocities and accelerations are estimated by mean of model-based nonlinear observers.

It has been proved that the proposed control law yields local uniformly ultimately boundedess of the closed-loop errors. It is also shown that the final bound of the errors depends on the fourth derivative of the master robot trajectories. A tuning gain procedure to guarantee the stability result has been summarized.

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Appendix 1

Proof of Proposition 1. Consider a state space representation of (1)–(3), with states q_s, $\dot q_s$, θ_s, $\dot\theta_s$, then from the observer (23), (24) and the joint estimation errors (25), it follows that:

$$\begin{aligned}\dot{\tilde q}&=\tilde{\dot q}-\mu_1\tilde q\\ \dot{\tilde{\dot q}}&=-M_s^{-1}(q_s)((2C_s(q_s,\dot q_s)-C_s(q_s,\tilde{\dot q}))\tilde{\dot q}+K_s(\tilde q-\tilde\theta))-\mu_2\tilde q\\ \dot{\tilde\theta}&=\tilde{\dot\theta}-\mu_3\tilde q\\ \dot{\tilde{\dot\theta}}&=-J_s^{-1}K_s(\tilde\theta-\tilde q)-J_s^{-1}B_{v,s}\tilde{\dot\theta}-\mu_4\tilde q\end{aligned} \tag{56}$$

The first and third equation of (56) imply that:

$$\tilde{\dot q}=\dot{\tilde q}+\mu_1\tilde q,\quad \dot{\tilde{\dot q}}=\ddot{\tilde q}-\mu_1\dot{\tilde q} \tag{57}$$

$$\tilde{\dot\theta}=\dot{\tilde\theta}+\mu_3\tilde q,\quad \dot{\tilde{\dot\theta}}=\ddot{\tilde\theta}-\mu_3\dot{\tilde q} \tag{58}$$

therefore (16) and (56) yield the joint estimation error dynamics (28).

Consider the joint estimation errors given by (25), and introduce the variables $\tilde{\ddot q}$, $\widetilde{q^{(3)}}$ as:

$$\tilde{\ddot q}=\ddot q_s-\hat{\ddot q}_s,\quad \widetilde{q^{(3)}}=q_s^{(3)}-\widehat{q_s^{(3)}} \tag{59}$$

Differentiating (1) twice, and by considering (2), τ_s given by (17), and property (4) it follows that:

$$M_s(q_s)(q_s^{(4)}-v(t))+\Psi(q_s,\dot q_s,\ddot q_s,q_s^{(3)},\tilde q,\tilde{\dot q},\tilde{\ddot q},\widetilde{q^{(3)}},\tilde\theta,\tilde{\dot\theta})=0 \tag{60}$$

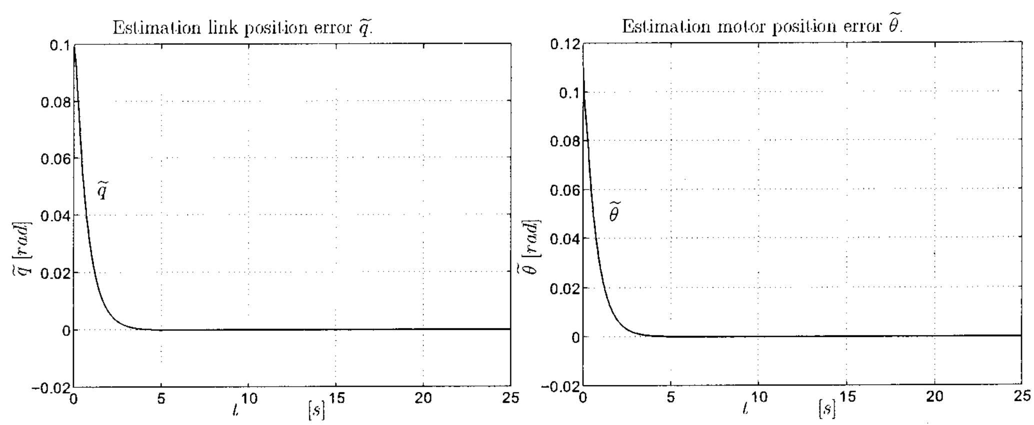

Fig. 3 Estimation link and motor position error $\tilde{q}$, $\tilde{\theta}$

where $\Psi(q_s,\dot q_s,\ddot q_s,q_s^{(3)},\tilde q,\tilde{\dot q},\tilde{\ddot q},\widetilde{q^{(3)}},\tilde\theta,\tilde{\dot\theta})$ represents the mismatch with the desired linearization, which is caused by absence of the high order derivatives of q_s, q_m, θ_s.

According to the definition of $\tilde{\ddot q}$ (59), and by considering (11), (18) and (57) it follows that:

$$\tilde{\ddot q}=-M_s^{-1}(q_s)((2C_s(q_s,\dot q_s)-C_s(q_s,\tilde{\dot q}+\mu_1\tilde q))(\tilde{\dot q}+\mu_1\tilde q)+K_s(\tilde q-\tilde\theta)) \tag{61}$$

In a similar way, but considering (59), (12) and (19), it is obtained that:

$$\widetilde{q^{(3)}}=-M_s^{-1}(q_s)(\dot M_s(q_s,\dot q_s)\ddot q_s-\dot M_s(q_s,\hat q_s)\hat{\ddot q}_s+\dot N(q_s,\dot q_s,\ddot q_s)-\dot N(q_s,\hat q_s,\hat{\ddot q}_s)+K_s(\tilde{\dot q}-\tilde{\dot\theta})) \tag{62}$$

where, by considering property (4) and after a straightforward computation,

$$\dot M_s(q_s,\dot q_s)\ddot q_s-\dot M_s(q_s,\hat q_s)\hat{\ddot q}_s=\frac{\partial M_s(q_s)}{\partial q_s}(\tilde{\dot q}\ddot q_s+(\dot q_s-\tilde{\dot q})\tilde{\ddot q}) \tag{63}$$

$$\begin{aligned}
&\dot N(q_s,\dot q_s,\ddot q_s)-\dot N(q_s,\hat q_s,\hat{\ddot q}_s)\\
&=C_s(q_s,\tilde{\dot q})\ddot q_s+(C_s(q_s,\dot q_s)-C_s(q_s,\tilde{\dot q}))\tilde{\ddot q}+\frac{\partial g_s(q_s)}{\partial q_s}\tilde{\dot q}\\
&+\begin{bmatrix} C_{s1}(q_s)\\ \vdots\\ C_{sn}(q_s)\end{bmatrix}(\tilde{\dot q}\dot q_s+(\ddot q_s-\tilde{\ddot q})\tilde{\dot q})\\
&+\dot q_s\frac{\partial}{\partial q_s}\begin{bmatrix} C_{s1}(q_s)\\ \vdots\\ C_{sn}(q_s)\end{bmatrix}(\tilde{\dot q}\dot q_s+(\dot q_s-\tilde{\dot q})\tilde{\dot q})\\
&+\tilde q\frac{\partial}{\partial q_s}\begin{bmatrix} C_{s1}(q_s)\\ \vdots\\ C_{sn}(q_s)\end{bmatrix}((\dot q_s-\tilde{\dot q})(\dot q_s-\tilde{\dot q}))
\end{aligned} \tag{64}$$

Let $\Phi(q_s,\dot q_m,\ddot q_m,q_m^{(3)},\dot e,\ddot e,e^{(3)},\tilde q,\tilde{\dot q},\tilde\theta,\tilde{\dot\theta})$ denotes the function $\Psi(q_s,\dot q_s,\ddot q_s,q_s^{(3)},\tilde q,\tilde{\dot q},\tilde{\ddot q},\widetilde{q^{(3)}},\tilde\theta,\tilde{\dot\theta})$ after substitution of the relations (57), (58), (61), (62), (63), (64), and (16), it holds that

$$\Phi(q_s,\dot q_m,\ddot q_m,q_m^{(3)},\dot e,\ddot e,e^{(3)},\tilde q,\tilde{\dot q},\tilde\theta,\tilde{\dot\theta})=\Psi(q_s,\dot q_s,\ddot q_s,q_s^{(3)},\tilde q,\tilde{\dot q},\tilde{\ddot q},\widetilde{q^{(3)}},\tilde\theta,\tilde{\dot\theta}) \tag{65}$$

where $\Phi(q_s,\dot q_m,\ddot q_m,q_m^{(3)},\dot e,\ddot e,e^{(3)},\tilde q,\tilde{\dot q},\tilde\theta,\tilde{\dot\theta})$ is the result of a straightforward chain of substitutions and simplifications. Substitution of (65) and (20) in (60), and considering (21), yields the synchronization error dynamics (26).

Consider the observer (21) and the estimation synchronization errors (22), then it follows that:

$$\hat e^{(4)}-\Gamma_1\tilde e^{(3)}-\Gamma_2\tilde{\ddot e}-\Gamma_3\tilde{\dot e}-\Gamma_4\tilde e=0 \tag{66}$$

Subtraction of (66) from the synchronization error dynamics (26), and considering the estimation synchronization errors (22), yields the estimation synchronization error dynamics (27). ■

Appendix 2

Proof of Lemma 6. From the properties of the matrices $M_s(q_s)$, $C_s(q_s,\dot q_s)\dot q_s$, the gravity term $g_s(q_s)$, (see Section 2), and because their nonlinear terms contain only sinusoidal functions of q_s, we have that for all $q_s\in\mathbb{R}^n$, their partial derivatives can be bounded as:

$$\left\|\frac{\partial M_s(q_s)}{\partial q_s}\right\|\leq M_{pM},\quad \left\|\frac{\partial^2 M_s(q_s)}{\partial q_s^2}\right\|\leq M_{ppM}$$

$$\left\|\frac{\partial g_s(q_s)}{\partial q_s}\right\|\leq G_{pM},\quad \left\|\frac{\partial^2 g_s(q_s)}{\partial q_s^2}\right\|\leq G_{ppM}$$

$$\left\|\begin{bmatrix} C_{s1}(q_s)\\ \vdots\\ C_{sn}(q_s)\end{bmatrix}\right\|\leq C_{qM},$$

$$\left\|\frac{\partial}{\partial q_s}\begin{bmatrix} C_{s1}(q_s)\\ \vdots\\ C_{sn}(q_s)\end{bmatrix}\right\|\leq C_{pM},\quad \left\|\frac{\partial^2}{\partial q_s^2}\begin{bmatrix} C_{s1}(q_s)\\ \vdots\\ C_{sn}(q_s)\end{bmatrix}\right\|\leq C_{ppM}.$$

For the sake of simplicity and without loss of generality, let assume that $Q_x=-(P_xA+A^TP_x)$ is a symmetric positive definite block diagonal inatrix, with $n\times n$ block entries, and denote the i-th diagonal $n\times n$ block of Q_x by Q_{xi}. Then from the definition of x_N, y_N, and ξ_N (47), and $\dot V$ given by (43), it follows that the term x^TQ_xx in (43) can be bounded as $x_N^TR_xx_N$, with $R_x\in\mathbb{R}^{8\times 8}$

$$R_x=\mathrm{diag}\{Q_{xiM}\}\quad i=1,\ldots,8 \tag{67}$$

where Q_{xiM} is the maximum eigenvalue of Q_{xi}, and such that positive definiteness of Q_x implies that R_x is positive definite.

From Q_y (44), the term Ω (46), and the bounds of the partial derivatives of $M_s(q_s)$, $C_s(q_s,\dot q_s)\dot q_s$, and $g_s(q_s)$, it follows that y^TQ_yy can be bounded as $y_N^TR_yy_N$, with $R_y\in\mathbb{R}^{4\times 4}$

$$R_y=\begin{bmatrix}\alpha_0^* & \alpha_1^* & \alpha_2^* & \alpha_3^*\\ \alpha_1^* & M_{sm}\mu_{1m}-2\lambda_0+2C_MV_M & \eta_0\mu_{3M}-\frac{1}{2}K_{sM} & \frac{1}{2}J_{sM}\mu_{3M}\\ \alpha_2^* & \eta_0\mu_{3M}-\frac{1}{2}K_{sM} & 2\eta_0J_{sM}^{-1}K_{sm} & \alpha_4^*\\ \alpha_3^* & \frac{1}{2}J_{sM}\mu_{3M} & \alpha_4^* & B_{v,sm}-4\eta_0\end{bmatrix}$$

$$\begin{aligned}
&\alpha_0^*=2\lambda_0(M_{sM}^{-1}(K_{sm}+2C_MV_M\mu_{1m})+\mu_{2m}),\quad \alpha_1^*=C_MV_M(2M_{sm}^{-1}\lambda_0+\mu_{1M})-\frac{1}{2}M_{sM}\mu_{2M}-\frac{1}{2}\beta_1\\
&\alpha_2^*=-\lambda_0M_{sm}^{-1}K_{sM}+\eta_0(\mu_{4M}+J_{sm}^{-1}B_{v,sM}\mu_{3M}-J_{sm}^{-1}K_{sM}),\\
&\alpha_3^*=\frac{1}{2}(B_{v,sM}\mu_{3M}+J_{sM}\mu_{4M}-K_{sM}),\quad \alpha_4^*=\eta_0J_{sm}^{-1}B_{v,sM}-\frac{1}{2}\mu_{4M}-\frac{1}{2}\beta_2
\end{aligned} \tag{68}$$

The matrices R_x and R_y are related to the bounds of the quadratic terms in x and y of $\dot{V}$ (43). Nevertheless, there exist other cross quadratic terms in x and y which come from Q_{xy} (45) and Ω. These cross quadratic terms are bounded, such that, R_{xy} corresponds to the bound of the cross quadratic terms of:

$$\|x^T Q_{xy} x\| + \|x^T P_x B[-M_s^{-1}(\Phi + K_s J_s^{-1} K_s(y_1 - y_3) - K_s J_s^{-1} B_{v,s}(y_4 + \mu_3 y_1)) + q_m^{(4)}]\|$$

Let consider $\Phi + K_s J_s^{-1} K_s(y_1 - y_3) - K_s J_s^{-1} B_{v,s}(y_4 + \mu_3 y_1)$, this term can be bounded as

$$\|\Phi + K_s J_s^{-1} K_s(y_1 - y_3) - K_s J_s^{-1} B_{v,s}(y_4 + \mu_3 y_1)\| \leqslant \Phi_1 + \Phi_r \tag{69}$$

where Φ_1 contains terms of first order in x, y, and Φ_r contains the remaining terms (orders 2, 3 and 4). After a long straightforward computation it is obtained that Φ_1 is given by:

$$\Phi_1 = ((a_1 + a_3)\mu_{1M} + a_2 + a_3\mu_{3M})\|y_1\| + (a_1 + a_3)\|y_2\| + a_2\|y_3\| + a_3\|y_4\|$$

$$\begin{aligned} a_1 = {} & 2M_{sm}^{-1} C_M V_M [A_M(M_{pM} + 2C_{qM}) + G_{pM} + K_{sM} + V_M^2(M_{pM} \\ & + 2C_{pM})] + C_{pM} V_M (6A_M + 2M_{sm}^{-1} C_M V_M^2) + 4C_{ppM} V_M^3 \\ & + 2(A_M^2 + C_{pM} A_M V_M + 2M_{sm}^{-1} C_{pM} C_M V_M^2) \\ & + M_{sm}^{-1} V_M (2M_{pM} + C_M + C_{qM})[G_{pM} + 3C_{pM} V_M^2 + (M_{pM} \\ & + C_M + C_{qM})(A_M + 2M_{sm}^{-1} V_M^2)] + D_M(2M_{pM} + C_M + C_{qM}) \\ & + 2M_{sm}^{-1} C_M V_M A_M (M_{pM} + 2C_{qM}) + 2G_{ppM} + 2M_{ppM} A_M V_M \end{aligned}$$

$$\begin{aligned} a_2 = {} & K_{sm} M_{sm}^{-1} \{2A_M(M_{pM} + 2C_{qM}) + G_{pM} + K_{sM} + V_M^2(M_{pM} \\ & + 2C_{pM})\} + + K_{sm}\{M_{sm}^{-1} C_{pM} V_M (2 + V_M) \\ & + (M_{sm}^{-1} V_M)^2 (M_{pM} + C_M + C_{qM})(2M_{pM} + C_M + C_{qM})\} \end{aligned}$$

$$a_3 = K_{sm} M_{sm}^{-1} V_M (2M_{pM} + C_M + C_{qM})$$

Therefore, the matrix R_{xy} is given by:

$$R_{xy} = M_{sm}^{-1} P_Q [R_1^* \quad a_1 + a_3 \quad a_2 + K_{sM}^2 J_{sm}^{-1} \quad a_3 + K_{sM} J_{sm}^{-1} B_{v,sM}]$$

$$P_Q = \begin{bmatrix} (P_{x14} + P_{x18})_M \\ (P_{x24} + P_{x28})_M \\ (P_{x34} + P_{x38})_M \\ (P_{x44} + P_{x48})_M \\ (P_{x45} + P_{x58})_M \\ (P_{x46} + P_{x68})_M \\ (P_{x47} + P_{x78})_M \\ (P_{x48} + P_{x88})_M \end{bmatrix} \tag{70}$$

$$R_1^* = (a_1 + a_3)\mu_{1M} + a_2 + a_3\mu_{3M} + K_{sM} J_{sm}^{-1}(B_{v,sM}\mu_{3M} - K_{sm})$$

Remark 12. *Notice that a_1, a_2, a_3 are uniquely determined by the physical parameters of the slave robot and the bounds of the master trajectories, and thus they do not depend on the control and observer gains. As a consequence a_1, a_2, a_3 must be evaluated only once*.

At this point all the quadratic terms of $\dot{V}$ have been bounded in terms of R_x, R_y, R_{xy}, and x_N, y_N, ξ_N. Therefore it is only necessary to bound all the remaining terms originated from Ω. From (46) and considering (69), it follows that:

$$\|\Omega(q_s, \dot{q}_m, \ddot{q}_m, q_m^{(3)}, q_m^{(4)}, x, y)\| \leqslant \Theta(V_M, A_M, D_M, E_M, \xi_N) \tag{71}$$

where by considering P_Q, (70), $\Theta(V_M, A_M, D_M, E_M, \xi_N)$ is given by:

$$\begin{aligned} \Theta = {} & 2x_N^T P_Q [M_{sm}^{-1} \Phi r + E_M] + 2\lambda_0 C_M M_{sm}^{-1} \|y_1\| \|y_2\| (\|y_2\| \\ & + \mu_{1M}\|y_1\|) + C_M \mu_{1M} \|y_1\| \|y_2\|^2 + C_M(\mu_{1M}\|y_1\| + 2\|x_2\|) \\ & \times (2\lambda_0 M_{sm}^{-1}\|y_1\| + \|y_2\|)(\|y_2\| + \mu_{1M}\|y_1\|) \end{aligned}$$

■

Appendix 3

Proof of Proposition 9. First notice that the definition of β_1, β_2 given by (54), imply that $\alpha_1^* = 0$, $\alpha_4^* = 0$ in R_y (68). Second, let R_{yi} denote the determinant of the i-th leading minor of R_y, then conditions for $R_{yi} > 0$, $i = 1, \ldots, 4$, are given by:

- $R_{y1} > 0$ if $\lambda_0 > 0$, $\mu_{1m} > 0$, and $\mu_{2m} > 0$
- $R_{y2} > 0$ if $\mu_{1m} > 2M_{sm}^{-1}(\lambda_0 - C_M V_M)$
- For R_{y3}, first notice that it can be written as $R_{y3} = a_{31}\mu_{2m} + a_{30}$, with $a_{31} = b_{32}\mu_{3M}^2 + b_{31}\mu_{3M} + b_{30}$, and $b_{32} < 0$. Then $a_{31} > 0$ if $\mu_{33\_} < \mu_{3m}$, $\mu_{3M} < \mu_{33}^-$, where

$$\mu_{33\_}, \mu_{33}^- = \frac{1}{4J_{sM}\eta_0}(2J_{sM}K_{sM} \mp 4(2J_{sM}\eta_0 K_{sM}(M_{sm}\mu_{1m} + 2(C_M V_M - \lambda_0)))^{1/2})$$

  because $a_{31} > 0$, then $\mu_{2m} > -a_{31}^{-1} a_{30}$ implies $R_{y3} > 0$.
- R_{y4} can be written as $R_{y4} = a_{41}\mu_{2m} + a_{40}$, with $a_{41} = b_{42}\mu_{3M}^2 + b_{41}\mu_{3M} + b_{40}$, and $b_{42} < 0$ if η_0 holds:

$$0 < \eta_0 < \min\left\{\frac{B_{v,sm}}{4}, \frac{1}{8J_{sM}}(J_{sM}B_{v,sm} + (J_{sM}^2 B_{v,sm}^2 + 8J_{sM}J_{sm}^2 K_{sm})^{1/2})\right\}$$

Then $b_{42} < 0$ implies that $a_{41} > 0$ if $\mu_{34\_} < \mu_{3m}$, $\mu_{3M} < \mu_{34}^-$, where

$$\begin{aligned} \mu_{34\_}, \ \mu_{34}^- = {} & \frac{1}{2(\eta_0(2J_{sM}(4\eta_0 - B_{v,sm})) - J_{sm}^2 K_{sm})} \\ & \times (2J_{sM}K_{sm}\eta_0(4\eta_0 - B_{v,sm}) \pm (2J_{sM}K_{sm}\eta_0)^{1/2} \\ & \cdot ((4\eta_0 - B_{v,sm})[J_{sm}^2 K_{sM}^2 + 8\eta_0(2\lambda_0 - 2C_M V_M \\ & - M_{sm}\mu_{1m})(J_{sM}^{-1} J_{sm}^2 K_{sm} - 2\eta_0(4\eta_0 - B_{v,sm}))])^{1/2}) \end{aligned}$$

because $a_{41} > 0$, then $\mu_{2m} > -a_{41}^{-1} a_{40}$ implies $R_{y4} > 0$.

If the above conditions are satisfied, then the determinants of all the leading minors of R_y are positive. Therefore from the Sylvester's criterion, it follows that R_y is positive definite. ■

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