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Brief paper

Time scaling for observer design with linearizable error dynamics☆

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Abstract

In this paper, we consider the problem of observer design for dynamical systems with scalar output by linearization of the error dynamics via coordinate change, output injection, and time scaling. We present necessary and sufficient conditions which guarantee the existence of a coordinate change and output-dependent time scaling, such that in the new coordinates and with respect to the new time the system has linear error dynamics.

Keywords: Observers; Time scaling; Output injection; Linearization

1. Introduction

An important control problem studied extensively is that of observer design for a dynamical system with output. A typical approach to this problem is to find a dynamical system (observer) coupled with the observed system by output injection in such a way that the overall system possesses an invariant asymptotically stable set of a specific structure. Although a solution to the observer design problem in its full generality is not known yet, it is clear that the problem statement is coordinate independent and invariant with respect to time scaling and therefore it is natural to seek conditions, ensuring the existence of an observer, that would also be coordinate independent and invariant with respect to time scaling, similar to the corresponding properties of asymptotic stability. The problem of transforming a system to observer form via change of state and output coordinates has been intensively studied during the last 20 years (see e.g. Bestle & Zeitz, 1983; Gauthier, Hammouri, & Othman, 1992; Gauthier & Kupka, 2001; Hammouri & Gauthier, 1988; Krener & Isidori, 1983; Krener & Respondek, 1985; Plestan & Glumineau, 1997; Xia & Gao, 1989). A recent paper by Moya, Ortega, Netto, Praly, and Picó (2001) suggests enlarging the class of systems admitting an observer exploiting the additional freedom of introducing possible time scaling. In this paper we are going to address this problem. A dual problem of linearization, using time scaling, of dynamics with inputs has been considered by Sampei and Furuta (1986), Respondek (1998), and Guay (1999).

Expressing a system in "physical" coordinates is very natural but sometimes it may complicate the observer design based on the second Lyapunov method. At the same time for the system written in other coordinates that design can be much easier to perform. To be more precise, consider dynamics with output given by the following equations:

$$\dot{z} = s(y)Az, \qquad y = Cz, \tag{1}$$

where $z = (z_1, \dots, z_n)$ is the state, y is the scalar output, and $s(y)$ is some nonvanishing positive real-valued function. In this case, it is possible to linearize the dynamics (1) via time scaling of the form $\mathrm{d}\tau = s(y(t))\,\mathrm{d}t$. For the linear system written with respect to the new time τ, the observer design problem can be easily solved using linear techniques. Therefore, it is interesting to find conditions which guarantee that a nonlinear system with scalar output

$$\dot{x} = f(x), \qquad y = h(x)$$

can be put in form (1).

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The problem which we are going to address in this paper is to find a coordinate change, which transforms the dynamics with a scalar output either to form (1) or to form (1) up to an output injection. Equivalently, we are looking for a coordinate change and a time scaling that either linearize the observed dynamics or linearize it up to an output injection. For the related problem of bilinearization up to output injection we refer the reader to Hammouri and Gauthier (1988).

It is worth mentioning that in a recent paper (Moya et al., 2001) it was reported that time scaling can significantly simplify the controller design for the so-called reaction systems. In particular, in Moya et al. (2001) it was shown that the model of a reaction system can be written in a coordinate system suitable for time scaling that significantly simplifies the controller/observer design.

A preliminary version of our results is presented in Respondek, Pogromsky, and Nijmeijer (2001). An alternative approach to the problem of transforming an observed dynamics to the observer form via a change of state-space coordinates and a time rescaling has been independently proposed by Guay (2001). His method, based on the language of differential forms and thus dual to ours, also leads to solving (a series of) ordinary differential equations on the output space.

In this paper, we consider only the case of dynamical systems with scalar output. The more general case of dynamical systems with multiple outputs (and inputs) will be reported elsewhere.

The paper is organized as follows. The problem is formulated in Section 3. In Section 2, we describe a model of a batch reactor whose analysis can be simplified by a natural time rescaling. Section 4 contains the main results. In Section 5, we compare the systems that can be brought to observer form under a state-space diffeomorphism and a time rescaling with those that can be transformed into observer form under state- and output-diffeomorphism. Finally, in Section 6, we come back to the model of a batch reactor of Section 2 and illustrate our results with it.

2. Motivating example

Consider a batch reactor where the following reactions between chemicals A, B, C, D take place:

$$\mathrm{A}\xrightarrow{k_1}\mathrm{B}\xrightarrow{k_3}\mathrm{C},$$

$$\mathrm{A}\xrightarrow{k_2}\mathrm{D}, \tag{2}$$

where k_1, k_2, k_3 are the reaction rates. Suppose that all reactions are endothermic and have first-order kinetics, the reacting mixture is heated by steam, which follows through a jacket around the reactor with a rate Q. Suppose additionally that the activation energies E_i of the three reactions are equal, that is, $E_1 = E_2 = E_3 = E$. Under these assumptions the state model for the reactor appears as follows:

$$\frac{\mathrm{d}c_\mathrm{A}}{\mathrm{d}t} = -(k_1\mathrm{e}^{-E/RT} + k_2\mathrm{e}^{-E/RT})c_\mathrm{A},$$

$$\frac{\mathrm{d}c_\mathrm{B}}{\mathrm{d}t} = k_1\mathrm{e}^{-E/RT}c_\mathrm{A} - k_3\mathrm{e}^{-E/RT}c_\mathrm{B},$$

$$\frac{\mathrm{d}c_\mathrm{C}}{\mathrm{d}t} = k_3\mathrm{e}^{-E/RT}c_\mathrm{B},$$

$$\frac{\mathrm{d}c_\mathrm{D}}{\mathrm{d}t} = k_2\mathrm{e}^{-E/RT}c_\mathrm{A},$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = J_1k_1\mathrm{e}^{-E/RT}c_\mathrm{A} + J_2k_2\mathrm{e}^{-E/RT}c_\mathrm{A} + J_3k_3\mathrm{e}^{-E/RT}c_\mathrm{B} + Q\Delta H_\mathrm{v}/\varrho c_\varrho V,$$

where $c_{\mathrm{A,B,C,D}}$ are the concentrations, $J_i = -\Delta H_{r_i}/\varrho c_\varrho$, $i = 1,2,3$, $Q\Delta H_\mathrm{v} = UA_\mathrm{t}(T_\mathrm{st} - T)$, V is the reaction volume, c_ϱ is the heat capacity of the reaction mixture, ϱ is the density of the reaction mixture, ΔH_v is the enthalpy of the vaporization, U is the heat transfer coefficient, A_t is the area of the heat exchange surface, T_st is the steam temperature and the concentrations $(c_\mathrm{A}, c_\mathrm{B}, c_\mathrm{C}, c_\mathrm{D})$ satisfy the conservation law $c_\mathrm{A} + c_\mathrm{B} + c_\mathrm{C} + c_\mathrm{D} = \mathrm{const}$ which follows from the system equations.

Now consider the following problem: to estimate the concentrations $c_\mathrm{A}, c_\mathrm{B}, c_\mathrm{C}$ and c_D provided the temperature T is measurable. First, we rescale the independent variable:

$$\mathrm{d}\tau = \mathrm{e}^{-E/RT}\,\mathrm{d}t.$$

Then introducing the state vector

$$x = (c_\mathrm{A}, c_\mathrm{B}, c_\mathrm{C}, c_\mathrm{D}, T)^\mathrm{t}$$

and output $y = T$ the system equations can be rewritten in the form

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = Fx + \gamma(y), \tag{3}$$

$$y = Gx, \tag{4}$$

where

$$F = \begin{pmatrix} -k_1 - k_2 & 0 & 0 & 0 & 0 \\ k_1 & -k_3 & 0 & 0 & 0 \\ 0 & k_3 & 0 & 0 & 0 \\ k_2 & 0 & 0 & 0 & 0 \\ J_1k_1 + J_2k_2 & J_3k_3 & 0 & 0 & 0 \end{pmatrix}$$

and $G = (0\ 0\ 0\ 0\ 1)$,

$$\gamma(y) = (0\ 0\ 0\ 0\ UA_\mathrm{t}(T_\mathrm{st} - y)\mathrm{e}^{E/RT}/V\varrho c_\varrho)^\mathrm{t}.$$

It is now clear that system (3), (4) admits an observer, with linear error dynamics, for the largest observable subsystem (which is the $(c_\mathrm{A}, c_\mathrm{B}, T)$-subsystem; see also Section 6).

3. Problem statement

Consider nonlinear observed dynamics of the form

$$\Sigma : \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad y = h(x),$$

where $x(\cdot) \in \mathbb{R}^n$, and $y(\cdot) \in \mathbb{R}$ is the measurement. In this paper, we will deal with the following two questions.

Question 1: when does there exist a (local) diffeomorphism $z = \varphi(x)$ and a time scaling of the form $\mathrm{d}\tau = s(h(x(t)))\,\mathrm{d}t$, where s is a nonvanishing real-valued function, such that Σ becomes

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = Az, \qquad y = Cz,$$

where the pair (A, C) is observable?

Question 2: when does there exist a (local) diffeomorphism $z = \varphi(x)$ and a time scaling of the form $\mathrm{d}\tau = s(h(x(t)))\,\mathrm{d}t$, where s is a nonvanishing real-valued function, such that Σ becomes

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = Az + \gamma(y), \qquad y = Cz,$$

where the pair (A, C) is observable and γ is a vector field whose components depend on $y = Cz$ only? To answer the first question means to characterize nonlinear dynamics that are linearizable via a diffeomorphism and an output-dependent time scaling, while to answer the second question means to characterize dynamics that are linearizable via a diffeomorphism, an output-dependent time scaling, and an output injection.

It is obvious that the first question is equivalent to the following one: when is Σ (locally) equivalent under a diffeomorphism $z = \varphi(x)$ to

$$\frac{\mathrm{d}z}{\mathrm{d}t} = s(y)(Az + Ky), \qquad y = Cz = z_1, \tag{5}$$

where

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad K = \begin{pmatrix} k_1 \\ \vdots \\ \vdots \\ k_n \end{pmatrix},$$

and

$$C = (1, 0, \ldots, 0)$$

and $s(y)$ is a nonvanishing real-valued function?

The second question is obviously equivalent to the following one: when is Σ (locally) equivalent under a diffeomorphism $z = \varphi(x)$ to

$$\frac{\mathrm{d}z}{\mathrm{d}t} = s(y)(Az + \gamma(y)), \qquad y = z_1 = Cz, \tag{6}$$

where

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad \gamma(z_1) = \begin{pmatrix} \gamma_1(z_1) \\ \vdots \\ \vdots \\ \gamma_n(z_1) \end{pmatrix},$$

and

$$C = (1, 0, \ldots, 0)$$

and $s(y)$ is a nonvanishing real-valued function?

A motivation for our study is the following immediate observation. If Σ is equivalent, via a change of coordinates, to (6), then we can construct the following observer (compare e.g. Krener & Isidori, 1983; Krener & Respondek, 1985):

$$\dot{\hat{z}} = s(y)(A\hat{z} + \gamma(y)) + L(y - C\hat{z})$$

yielding the error $e = \hat{z} - z$ that satisfies

$$\dot{e} = s(y)(A + LC)e$$

and thus gives the linear equation

$$\frac{\mathrm{d}e}{\mathrm{d}\tau} = (A + LC)e,$$

with respect to the new time τ.

We will work around a fixed initial condition $x_0 \in \mathbb{R}^n$ and we will assume that the diffeomorphism φ satisfies $\varphi(x_0) = 0$. Clearly, a necessary condition for Σ to be equivalent to one of the above-discussed forms is the following local observability rank condition (see e.g. Isidori, 1989; Nijmeijer & Van der Schaft, 1990):

$$\dim \operatorname{span}\{\mathrm{d}h(x_0), \mathrm{d}L_f h(x_0), \ldots, \mathrm{d}L_f^{n-1} h(x_0)\} = n$$

and we will assume this throughout the paper.

4. Main results

Following Krener and Isidori (1983) (see also Krener & Respondek, 1985) define a vector field g by

$$L_g L_f^j h = \begin{cases} 0 & \text{if } 0 \leqslant j \leqslant n-2, \\ 1 & \text{if } j = n-1. \end{cases} \tag{7}$$

For $j \geqslant 2$ we put

$$l_j = \frac{j(j-1)}{2} + 1.$$

In order to avoid the trivial case, we will assume throughout that $n \geqslant 2$. We have the following results.

Theorem 1. *Σ is, locally around x_0, equivalent under a diffeomorphism $z = \varphi(x)$ to system* (5) *if and only if in a neighborhood of x_0 it satisfies*

(i) $\mathrm{d}L_g L_f^n h = l_n \lambda \mathrm{d}L_f h \bmod \operatorname{span}\{\mathrm{d}h\}$, *for some smooth function λ;*

(ii) $[ad^i_{\bar f}\bar g, ad^j_{\bar f}\bar g] = 0$, *for* $0 \leqslant i < j \leqslant n$, *where* $\bar f = \frac{1}{s} f$, $\bar g = s^{n-1} g$, *and* $s = \exp\sigma$, *with* σ *being a solution of*

$$L_{ad^j_f g}\sigma = \begin{cases} 0 & \text{if } 0 \leqslant j \leqslant n-2, \\ (-1)^{n-1}\lambda & \text{if } j = n-1. \end{cases}$$

Theorem 2. *Σ is, locally around x_0, equivalent under a diffeomorphism $z = \varphi(x)$ to system* (6) *if and only if in a neighborhood of x_0 it satisfies conditions* (i) *and* (ii) *of Theorem* 1 *with i and j in item* (ii) *satisfying $0 \leqslant i < j \leqslant n-1$.*

Remark 1. Notice that the system

$$L_{ad^j_f g}\sigma = \begin{cases} 0 & \text{if } 0 \leqslant j \leqslant n-2, \\ (-1)^{n-1}\lambda & \text{if } j = n-1 \end{cases}$$

of first-order partial differential equations on the state space $\mathbb{R}^n$ is actually a first-order ordinary differential equation on the output space $\mathbb{R}$. For instance, bring the system Σ, via the (local) diffeomorphism $x_i = L_f^{i-1}h$, for $1 \leqslant i \leqslant n$, to its observability normal form

$$\begin{aligned} \dot x_1 &= x_2, \qquad y = x_1, \\ &\vdots \\ \dot x_{n-1} &= x_n, \\ \dot x_n &= f_n(x_1,\dots,x_n). \end{aligned} \tag{8}$$

Then the above system of first-order partial differential equations becomes the ordinary differential equation

$$\frac{\mathrm{d}\sigma(y)}{\mathrm{d}y} = \lambda(y)$$

on the output space $\mathbb{R}$.

Remark 2. In order to avoid redundancy we can take $i = 0$ and $j = 1, 3, \dots, 2n-1$ in (ii) of Theorem 1 and $i = 0$ and $j = 1, 3, \dots, 2n-3$ in (ii) of Theorem 2 (compare Krener & Isidori, 1983).

In small dimensions we can, instead of checking the commutativity of the frame $\{ad^j_{\bar f}\bar g\}$, bring the system into the observability normal form (8) and verify whether the component f_n satisfies the conditions of the following proposition, which can be proved by tedious but straightforward computations.

Proposition 1. *Σ is, locally around x_0, equivalent under a diffeomorphism $z = \varphi(x)$ to the system* (6) *if and only if in a neighborhood of x_0 its observability normal form* (8) *satisfies*:

$$f_2 = a x_2^2 + b x_2 + c$$

if $n = 2$, where a, b, c are any smooth functions of x_1 defined in a neighborhood of x_0 and

$$f_3 = a x_2^3 + b x_2 x_3 + c x_2^2 + d x_3 + e x_2 + f$$

if $n = 3$, where a, b, c, d, e, f are any smooth functions of x_1 defined in a neighborhood of x_0 and satisfying

$$a = \tfrac{1}{4} b' - \tfrac{1}{8} b^2 \quad \text{and} \quad c = d' - \tfrac{1}{2} bd. \tag{9}$$

Proof of Theorem 1. *Sufficiency*. From the definition of g, it follows that σ, which is a solution of

$$L_{ad^j_f g}\sigma = \begin{cases} 0 & \text{if } 0 \leqslant j \leqslant n-2, \\ (-1)^{n-1}\lambda & \text{if } j = n-1, \end{cases}$$

satisfies $\mathrm{d}\sigma \in \operatorname{span}\{\mathrm{d}h\}$. Therefore,

$$ad^j_{\bar f}\bar g = s^{n-j-1} ad^j_f g (\operatorname{mod} \operatorname{span}\{g, \dots, ad_f^{j-1} g\}),$$

for any $0 \leqslant j \leqslant n-1$. We thus conclude that

$$L_{ad^j_{\bar f}\bar g} h = \begin{cases} 0 & \text{if } 0 \leqslant j \leqslant n-2, \\ (-1)^{n-1} & \text{if } j = n-1. \end{cases} \tag{10}$$

It is well known (see e.g. Nijmeijer & Van der Schaft, 1990; Isidori, 1989) that (ii) implies that we can find a local diffeomorphism $z = \varphi(x)$ such that $\bar g = \partial/\partial z_n$ and

$$\bar f(z) = Az + Kz_1,$$

where

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad K = \begin{pmatrix} k_1 \\ \vdots \\ \vdots \\ k_n \end{pmatrix},$$

which simply means that $z = \varphi(x)$ linearizes the control system $\dot x = \bar f(x) + \bar g(x)u$. Moreover, by (10) we have that in z-coordinates, $h = z_1$. Since $f = s(z_1)\bar f$, it follows that in z-coordinates Σ reads as

$$\frac{\mathrm{d}z}{\mathrm{d}t} = s(z_1)(Az + Kz_1), \qquad y = z_1.$$

Necessity. Assume that there exists $z = \varphi(x)$ bringing Σ into

$$\begin{aligned} \dot z_1 &= s(z_1)(z_2 + k_1 z_1), \qquad y = z_1, \\ &\vdots \\ \dot z_{n-1} &= s(z_1)(z_n + k_{n-1} z_1), \\ \dot z_n &= s(z_1) k_n z_1. \end{aligned}$$

We have $h = z_1$, $L_f h = s(z_2 + k_1 z_1)$, and $L_f^2 h = s^2 z_3 + s' s z_2^2 + z_2 a_2(z_1) + b_2(z_1)$, for some smooth functions a_2, b_2, where "$'$" stands for the derivative with respect to z_1. It is straightforward to prove by an induction argument that

$$\begin{aligned} L_f^j h = {} & s^j z_{j+1} + l_j s' s^{j-1} z_2 z_j \\ & + z_j a_j(z_1) + b_j(z_1, \dots, z_{j-1}), \end{aligned}$$

for some smooth functions a_j, b_j. It thus follows that $g = (1/s^{n-1})(\partial/\partial z_n)$ and that

$$L_f^n h = s^n k_n z_1 + l_n s' s^{n-1} z_2 z_n + z_n a_n(z_1) + b_n(z_1, \ldots, z_{n-1}),$$

for some smooth functions a_n, b_n. Hence $L_g L_f^n h = s a_n + l_n s' z_2 + a_n(z_1)$ and

$$\mathrm{d}L_g L_f^n h = l_n s' \, \mathrm{d}z_2 \bmod \operatorname{span}\{\mathrm{d}h\},$$

which gives $\lambda = s'/s$. The system

$$L_{ad_f^j g}\sigma = \begin{cases} 0 & \text{if } 0 \leqslant j \leqslant n-2, \\ (-1)^{n-1}\lambda & \text{if } j = n-1, \end{cases}$$

is thus the ordinary differential equation

$$\sigma'(z_1) = \frac{s'(z_1)}{s(z_1)} = (\log s(z_1))'. \tag{11}$$

We have $\sigma(z_1) = \log s(z_1) + d$, where d is a constant, which gives a 1-parameter family of solutions $s_c(z_1) = c \exp \sigma(z_1)$, where $c \in \mathbb{R}$, $c \neq 0$, is a multiplicative constant. It is clear that $\bar{f} = (1/s_c) f = (1/c) Az$ and $\bar{g} = (s_c)^{n-1} g = c^{n-1} \partial/\partial z_n$ satisfy (ii), for $0 \leqslant i < j \leqslant n$, which, actually, is another way of expressing the fact that $\dot{z} = \bar{f}(z) + \bar{g}(z)u = (1/c)Az + c^{n-1}(\partial/\partial z_n)u$ is a linear system for any $c \in \mathbb{R}$, $c \neq 0$. □

Proof of Theorem 2. The proof follows the same line, the only difference being that the commutation relation (ii), satisfied for $0 \leqslant i < j \leqslant n-1$, is a necessary and sufficient condition for the observed dynamics $\dot{x} = f(x)$ and $y = h(x)$ to be equivalent to the nonlinear observer form $\dot{z} = Az + \gamma(Cz)$, $y = Cz$ (see Krener & Isidori, 1983). □

Notice that in both cases σ is calculated via (11) up to an additive constant, which means that $s = \exp \sigma$ is calculated up to a multiplicative constant. This is in agreement with the obvious observation that if a time rescaling $\mathrm{d}\tau/\mathrm{d}t = s(y)$ works then any rescaling $\mathrm{d}\tau/\mathrm{d}t = cs(y)$, where $c \neq 0$, works as well.

5. Time scaling versus change of output coordinates

We can notice that Theorems 1 and 2 recall analogous results for, respectively, linearization and linearization up to output injection under a state space and output space diffeomorphisms (see Krener & Respondek, 1985). We will compare these two classes of transformations in two and three dimensions for the problem of linearization up to output injection. For the system

$$\Sigma : \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad y = h(x),$$

we look for a (local) diffeomorphism $z = \varphi(x)$ in the state space and a (local) diffeomorphism $y = \psi(\tilde{y})$ in the output space transforming Σ into

$$\frac{\mathrm{d}z}{\mathrm{d}t} = Az + \gamma(Cz), \qquad \tilde{y} = Cz, \tag{12}$$

which, of course, is equivalent to transforming Σ via a state-space diffeomorphism $z = \varphi(x)$ only into

$$\frac{\mathrm{d}z}{\mathrm{d}t} = Az + \gamma(Cz), \qquad y = \psi(Cz). \tag{13}$$

Notice that the function $s(y)$ determining $\bar{f}$ and $\bar{g}$ in Theorems 1 and 2, and defined as $s = \exp \sigma$ in (ii) of Theorem 1, can be equivalently expressed as a solution of the following system of first-order partial differential equations:

$$L_{ad_{\bar{f}}^j g} s = \begin{cases} 0 & \text{if } 0 \leqslant j \leqslant n-2, \\ (-1)^{n-1}\lambda s & \text{if } j = n-1, \end{cases} \tag{14}$$

where λ is defined by (i) of Theorem 1.

In Krener and Respondek (1985) (see also Plestan & Glumineau, 1997; Respondek, 1985) it is proved that Σ is, locally around x_0, equivalent under a state-space diffeomorphism $z = \varphi(x)$ and an output-space diffeomorphism $y = \psi(\tilde{y})$ to system (12) if and only if in a neighborhood of x_0 it satisfies $[ad_f^i \tilde{g}, ad_f^j \tilde{g}] = 0$, for $0 \leqslant i < j \leqslant n-1$, where $\tilde{g} = \tilde{s} g$, with $\tilde{s}$ being a solution of the system (14), in which λ is replaced by $\tilde{\lambda}$ defined by $\mathrm{d}L_g L_f^n h = n\tilde{\lambda}\, \mathrm{d}L_f h \bmod \operatorname{span}\{\mathrm{d}h\}$.

We will show that for the case $n = 2$ the conditions of the above result and of Theorem 2 coincide while for $n = 3$ the classes of systems satisfying the conditions of these two theorems are, in general, different. Observe that the conditions of both theorems claim the existence of a commuting frame $ad_{\bar{f}}^i \bar{g}$, for $0 \leqslant i \leqslant n-1$, in the case of time-rescaling, and $ad_f^i \tilde{g}$, for $0 \leqslant i \leqslant n-1$, in the case of output transformations. There are, however, two differences in constructing these commuting frames. Firstly, the frame is constructed with $\bar{f} = (1/s) f$ and $\bar{g} = s^{n-1} g$, in Theorem 2, and with f and $\tilde{g} = \tilde{s} g$, in the other case. Secondly, although the functions s and $\tilde{s}$ are determined by systems of linear differential equations of the same form, the functions λ and $\tilde{\lambda}$, defining those equations, differ by a multiplicative constant (actually, $\tilde{\lambda} = (l_n/n)\lambda$) and hence s and $\tilde{s}$ are different.

A particular situation takes place for $n = 2$. Firstly, $l_2 = 2$ implying $\lambda = \tilde{\lambda}$ and thus $s = \tilde{s}$. Secondly, in this case, a simple calculation shows that the commutativity of the vector fields $\bar{g}$ and $ad_{\bar{f}} \bar{g}$ is equivalent to the commutativity of $\tilde{g}$ and $ad_f \tilde{g}$.

Starting from $n = 3$, the classes of systems satisfying, respectively, the conditions of Theorem 2 and of the above result of Krener and Respondek (1985) are different. To see this, we discuss the two following examples.

Example 1. Consider the system

$$\begin{aligned} \dot{x}_1 &= \mathrm{e}^{x_1} x_2, \\ \dot{x}_2 &= \mathrm{e}^{x_1} x_3, \\ \dot{x}_3 &= 0, \\ y &= x_1. \end{aligned} \tag{15}$$

Clearly, this system is transformable to the observer form (12), with $\tilde{y} = y$, by the time scaling $d\tau = s(x_1)\,dt$, where $s(x_1) = e^{x_1}$. To confirm this observation, apply Theorem 2. By a direct calculation, condition (i) of Theorem 2 implies that $\lambda = 1$. Therefore,

$$s = ce^{x_1},$$

where $c \in \mathbb{R}$, $c \neq 0$, and thus have

$$\bar{f} = \frac{1}{s} f = c\left(x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}\right)$$

and

$$\bar{g} = s^2 g = c^2 \frac{\partial}{\partial x_3}.$$

The vector fields $\bar{f}$ and $\bar{g}$ are linear and constant, respectively, so no diffeomorphism is needed. Obviously, they satisfy

$$[ad^i_{\bar{f}}\bar{g}, ad^j_{\bar{f}}\bar{g}] = 0,$$

for $0 \leqslant i < j \leqslant 2$.

Now we will show that system (15) cannot be transformed into observer form (12) via a diffeomorphism and a change of output coordinates. To this end, we apply the result of Krener and Respondek (1985) recalled in the beginning of this section. We have $\tilde{\lambda} = \frac{4}{3}$, therefore

$$\tilde{s} = ce^{(4/3)x_1},$$

where $c \in \mathbb{R}$, $c \neq 0$ and we thus obtain

$$\tilde{g} = \tilde{s}^2 g = ce^{-(2/3)x_1} \frac{\partial}{\partial x_3}.$$

A straightforward calculation shows that

$$[ad_f\tilde{g}, ad^2_f\tilde{g}] = \frac{4}{3} x_2 c^2 e^{5/3x_1} \frac{\partial}{\partial x_3} \neq 0.$$

This implies that system (15) cannot be transformed to the observer form (12) via a change of state and outputs coordinates, although as we have shown earlier, it can be transformed to that form via a time rescaling.

The goal of the next example is to show that the converse inclusion between the two classes of systems does not hold either.

Example 2. Consider the system

$$\dot{x}_1 = e^{x_1} x_2,$$

$$\dot{x}_2 = x_3,$$

$$\dot{x}_3 = 0,$$

$$y = x_1. \qquad (16)$$

By a change of state and output coordinates this system is transformable to the observer form (12) (actually to a linear system). To see this, apply the result of Krener and Respondek (1985). We have

$$\tilde{\lambda} = 1.$$

It follows that

$$\tilde{s} = ce^{x_1},$$

where $c \in \mathbb{R}$, $c \neq 0$. Thus we have

$$\tilde{g} = \tilde{s}g = \frac{\partial}{\partial x_3}$$

and hence

$$[\tilde{g}, ad_f\tilde{g}] = [ad_f\tilde{g}, ad^2_f\tilde{g}] = 0.$$

This implies that system (16) can be transformed to the observer (actually, even linear) form (12). Indeed, we can use the transformation

$$z_1 = -e^{-x_1} + 1,$$

$$z_2 = x_2,$$

$$z_3 = x_3,$$

$$\tilde{y} = -e^{-y} + 1.$$

We will now show that system (16) is not transformable to the observer form via a state-space diffeomorphism and a time scaling, that is, it is not transformable via a state-space diffeomorphism to form (6). Condition (i) of Theorem 2 implies that

$$\lambda = \tfrac{3}{4}.$$

Therefore,

$$s = ce^{(3/4)x_1},$$

where $c \in \mathbb{R}$, $c \neq 0$, and we thus have

$$\bar{f} = \frac{1}{s} f = c\left(x_2 e^{(1/4)x_1} \frac{\partial}{\partial x_1} + x_3 e^{-(3/4)x_1} \frac{\partial}{\partial x_2}\right)$$

and

$$\bar{g} = s^2 g = c^2 e^{(1/2)x_1} \frac{\partial}{\partial x_3}.$$

Straightforward calculations shows that

$$[ad_{\bar{f}}\bar{g}, ad^2_{\bar{f}}\bar{g}] = -\frac{3}{4} x_2 c^7 e^{3/4x_1} \frac{\partial}{\partial x_3} \neq 0$$

implying that the system cannot be brought to the observer form via a state-space diffeomorphism and a time rescaling, that is, it cannot be transformed via a state-space diffeomorphism to form (6). The same conclusion can also be deduced directly from Proposition 1.

Note also that for $n=3$ and f_3 as in Proposition 1, system (8) can be linearized up to output injection via state and output transformations provided

$$a = \tfrac{1}{3} b' - \tfrac{1}{9} b^2 \quad \text{and} \quad c = d' - \tfrac{1}{3} bd.$$

(see Krener & Respondek, 1985; Keller, 1987; Phelps, 1991). This indicates that the conditions for linearization up to output injection via state transformation and time rescaling and via both state and output transformation are indeed different.

6. Motivating example—continuation

Now, we will come back to the model of reactor of Section 2 and illustrate our main result, that is Theorem 2, with that model. To start with, notice that system (3), (4), whose observation is $y = T$, is not observable and decomposes into two subsystems: the observable system evolving on three-dimensional (c_A, c_B, T)-space and a completely unobservable system evolving on 2-dimensional (c_C, c_D)-space. This is clear in view of the fact that the evolution of c_A, c_B, and T depends neither on c_C nor on c_D. Due to the mass conservation, $c_A + c_B + c_C + c_D = \text{const}$ and hence, if the sum of initial concentrations is known *a priori*, the system can be defined on a four-dimensional linear manifold that allows to decrease the dimension of the unobservable part. Consider the case when the sum of initial concentrations is unknown. We will show that we can bring the maximal observable subsystem, that is the system

$$\frac{dc_A}{dt} = -(k_1 e^{-E/RT} + k_2 e^{-E/RT})c_A,$$

$$\frac{dc_B}{dt} = k_1 e^{-E/RT} c_A - k_3 e^{-E/RT} c_B,$$

$$\frac{dT}{dt} = J_1 k_1 e^{-E/RT} c_A + J_2 k_2 e^{-E/RT} c_A + J_3 k_3 e^{-E/RT} c_B + Q\Delta H_v / \varrho c_\varrho V,$$

with the observation $y = h = T$, to form (6). The above system can be expressed as

$$\frac{dx^{Obs}}{dt} = e^{-E/RT}(F^{Obs}x^{Obs} + \gamma^{Obs}(T))$$

$$y = C^{Obs}x^{Obs},$$

where F^{Obs} and C^{Obs} are suitable constant matrices (which are, actually, the restrictions of the matrices F and C, respectively, of (3) and (4) to the observable subspace $\mathbb{R}^3$, equipped with the coordinates $x^{Obs} = (c_A, c_B, T)^t$). Denote $C_1 x^{Obs} = C^{Obs}x^{Obs}$, $C_2 x^{Obs} = C^{Obs}F^{Obs}x^{Obs}$, and $C_3 x^{Obs} = C^{Obs}(F^{Obs})^2 x^{Obs}$. Recall that $l_j = (j(j-1)/2) + 1$, for any $j \geqslant 2$. We have

$$L_f^j h = l_j \frac{E}{RT^2} e^{-jE/RT}(C_j x^{Obs})(C_2 x^{Obs}) + e^{-jE/RT} C_{j+1} x^{Obs} + C_j x^{Obs} a_j(C_1 x^{Obs}) + b_j(C_1 x^{Obs}, \ldots, C_{j-1} x^{Obs}),$$

for $2 \leqslant j \leqslant 3$, for some suitable functions a_j and b_j. It follows that the vector field g is uniquely defined by

$$L_g C_1 x^{Obs} = 0,$$

$$L_g C_2 x^{Obs} = 0,$$

$$L_g C_3 x^{Obs} = e^{2E/RT}.$$

Thus we have

$$dL_g L_f^3 h = \frac{4E}{RT^2} e^{-E/RT} d(C_2 x^{Obs}) \bmod \operatorname{span}\{dh\}$$

and

$$dL_f h = e^{-E/RT} d(C_2 x^{Obs}) \bmod \operatorname{span}\{dh\}.$$

Since $l_3 = 4$, by condition (i) of Theorem 2 we have

$$\lambda = \frac{E}{RT^2},$$

which yields the ordinary differential equation on the output space

$$\frac{ds}{dT} = \frac{E}{RT^2} s,$$

whose solution is

$$s(T) = c e^{-E/RT},$$

where $c \in \mathbb{R}$, $c \neq 0$. Clearly, the time rescaling

$$d\tau = s(T)\, dt = c e^{-E/RT}\, dt$$

brings the system to the observer form

$$\frac{dx^{Obs}}{d\tau} = F^{Obs}x^{Obs} + \gamma^{Obs}(T)$$

$$y = T.$$

Obviously, $ad_{\bar{f}}^i \bar{g}$, for $0 \leqslant i \leqslant 2$, form a commuting frame since

$$\bar{f} = \frac{1}{s} f = \frac{1}{c} e^{E/RT} e^{-E/RT}(F^{Obs}x^{Obs} + \gamma^{Obs}(T))$$

and

$$\bar{g} = s^2 g$$

implying that $\bar{g}$ is defined by

$$L_{\bar{g}} C_1 x^{Obs} = 0,$$

$$L_{\bar{g}} C_2 x^{Obs} = 0,$$

$$L_{\bar{g}} C_3 x^{Obs} = C^2.$$

7. Conclusions

In this paper we considered the problem of transforming, via coordinate change and time scaling, dynamics with output to a form which admits an observer with a linear error dynamics. The class of admissible time scalings used in this paper is given by the equation

$$\mathrm{d}\tau = s(y(t))\,\mathrm{d}t,$$

where s is a real-valued positive function which depends only on the output of the observed dynamical system. This condition is necessary for practical implementation of an observer designed by the proposed technique.

We proposed necessary and sufficient conditions ensuring the existence of an appropriate (local) coordinate change and output-dependent time scaling such that in the new coordinates and with respect to the new time the system admits one of the two normal forms (either linear or linear up to a nonlinear additive output injection) for which the observer design problem can be easily solved. Our conditions involve solving one ordinary differential equation on the output space (whose solution actually determines the needed time rescaling) and calculating Lie brackets.

We also compared the class of systems that can be put into observer form under a state-space diffeomorphism and time scaling with that for which we use a state-space and an output-space diffeomorphism. Starting from $n = 3$ those classes are different and a natural problem for future investigations is to consider the group consisting of all those transformations.

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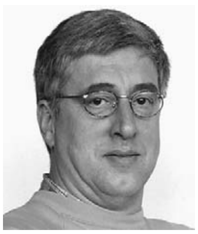

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