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Controlled synchronization of pendula

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***Abstract*— In this paper we design a controller for synchronization problem for two pendula suspended on an elastically supported rigid beam. A relation to Huijgens' experiments as well as the practical motivation are emphasized.**

I. Introduction

The high number of scientific contributions in the field of synchronization reflects the importance of this subject. Synchronization is a fundamental nonlinear phenomenon which was discovered already in the XVIIth century: the Dutch researcher Christian Huijgens discovered that a couple of mechanical clocks hanging from a common support were synchronized [1].

In the field of electrical engineering the burst of interest towards synchronization phenomenon was initiated by pioneering works due to van der Pol [3], [4] who studied frequency locking phenomenon in externally driven nonlinear generator as well as mutual synchronization of two coupled oscillators. The achievements of modern electronics would be impossible without solid synchronization theory developed in last 60 years [5], [6].

In the field of mechanical engineering some of the applications are mentioned in the book on the subject [2]. One of the recent directions in this field is to employ control theory to handle synchronization as a control problem. For example, in robotics the problem of synchronization is usually referred to as coordination, or cooperation [7], [8]. The problem appears when two or more robot-manipulators have to operate synchronously, especially in situations when some of them operate in hazardous environment, while others (that serve as reference) may be guided by human operation.

In this paper we study a controlled synchronization problem for two pendula hanging from an elastically fixed horizontal beam. The problem of synchronization is formulated as controller design problem. The control goal is twofold: first, two pendula should be swung up to the desired level of energy and, second they have to move synchronously in opposite directions. This problem has a practical motivation: during a start-up procedure of various vibrational transporters and mills the synchronous motion of the rotors allows to

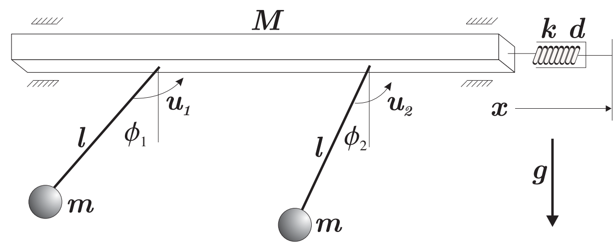

Fig. 1. The setup

avoid resonance oscillations and to reduce energy consumption. The system behaviour considered in this paper is closely related to the phenomenon observed by Huijgens and this relation is emphasized.

The paper is organized as follows. First we formulate the problem statement. Next we analyze the behaviour of the uncontrolled system. The controller is proposed and then investigated in Section 4. Section 5 deals with the local stability analysis of Huijgens phenomenon. The conclusions are formulated in the last section.

II. Problem statement

Consider the system schematically depicted in Fig. 1. The beam of the mass M can move in the horizontal direction with viscous friction with damping coefficient d. One side of the beam is attached to the wall through the spring with elasticity k. The beam supports two identical pendula of the length l and mass m. The torque applied to each pendulum is the control input.

The system equations can be written in a form of Euler-Lagrange equations:

$$\begin{aligned} ml^2\ddot{\phi}_1 + ml\ddot{x}\cos\phi_1 + mgl\sin\phi_1 &= u_1 \\ ml^2\ddot{\phi}_2 + ml\ddot{x}\cos\phi_2 + mgl\sin\phi_2 &= u_2 \\ (M+2m)\ddot{x} + ml\sum_{i=1}^{2}\left(\ddot{\phi}_i\cos\phi_i - \dot{\phi}_i^2\sin\phi_i\right) &= \\ -d\dot{x} - kx, \end{aligned} \quad (1)$$

where $\phi_i \in S^1$ are the angles of the pendulums, $x \in \mathbb{R}^1$ is the horizontal displacement of the beam, and u_1, u_2 are the control inputs.

Introduce the Hamiltonoin function of each pendulum:

$$H(\phi_i, \dot{\phi}_i) = \frac{ml^2\dot{\phi}_i^2}{2} + mgl(1 - \cos\phi_i).$$

The problem which we are going to address in this paper is to design a controller to swing the pendula up to the desired energy level H_* in such a way that the pendula move in opposite directions. Therefore the control objective can be formalized by the following relations:

$$\lim_{t\to\infty} H(\phi_i(t), \dot{\phi}_i(t)) = H_*, \quad i = 1, 2, \qquad (2)$$

$$\lim_{t\to\infty} (\phi_1(t) + \phi_2(t)) = 0. \qquad (3)$$

The first relation implies that the periods of oscillations of each pendulum are identical (*frequency synchronization*), while the second relation indicates a particular case of *phase synchronization*, so called anti-phase synchronization.

Although the problem statement formulated above looks rather artificial it can find an important practical motivation. The system we are going to study is a typical system which models vibrational transporters and mills of several kind [2], [9]. When the rotors of those setups are operating at nominal speed they can demonstrate synchronous phenomenon [2], [9]. At the nominal speed the setup does not consume significant power compared to that during the start-up mode. To decrease the energy consumption during the start-up procedure one can swing up the imbalanced rotors like oscillating pendula until they gain enough potential energy to operate in the rotational mode. One of the main difficulties in applying this technique is the vibration of the beam. This is due to the fact that during the start-up procedure the frequency of the external force applied to the beam

$$F := ml\sum_{i=1}^{2}\left(\ddot{\phi}_i\cos\phi_i - \dot{\phi}_i^2\sin\phi_i\right)$$

can be in resonance with the eigenfrequency of the beam that results in high level of the energy dissipation by the beam. This effect is known to everybody who observed vibrations of the washing machine during first seconds of wringing. On the other hand, if the two (identical) rotors move *synchronously in opposite directions* the resulting force applied to the beam is zero. Therefore, the controller able to achieve the objectives (2,3) can be utilized during the start-up procedure to pass through the resonance and to reduce the power consumption. The main benefit in this case is an opportunity to install lighter and less expensive motors than that used in uncontrolled start-up procedure. The problem of this kind was considered for example in [10], [11], [12], [13].

III. Analysis of the uncontrolled system

We begin our study with the analysis of motion of the uncontrolled system taking $u_i \equiv 0$ in the system equations. To analyze the limit behaviour of this system consider the total energy as a Lyapunov function candidate:

$$\begin{aligned} V &= \frac{m}{2}\sum_{i=1}^{2}\left(\dot{x}^2 + l^2\dot{\phi}_i^2 + 2\dot{x}\dot{\phi}_i l\cos\phi_i\right) \\ &+ \frac{M\dot{x}^2}{2} + mgl\sum_{i=1}^{2}(1-\cos\phi_i) + \frac{kx^2}{2}. \end{aligned}$$

Clearly, $V \geq 0$. Calculating the time derivative of V along the solutions of the uncontrolled system yields

$$\dot{V} = -d\dot{x}^2 \leq 0,$$

so the system can be analyzed by La Salle's invariance principle from which it follows that all the system trajectories tend to the set where

$$\phi_1 = -\phi_2, \quad \dot{\phi}_1 = -\dot{\phi}_2, \quad x = 0, \quad \dot{x} = 0.$$

From this observation one can make a few important conclusions. First, the uncontrolled system can exhibit synchronous behaviour. Clearly, it follows that the Hamiltonian of each pendulum tend to a common limit. However, due to energy dissipation the limit value depends on the initial conditions and particularly, if one initializes the pendula from an identical point, the oscillations will decay. Therefore the uncontrolled system exhibits a behavior which is very close to the desired one, and there is one thing left to the controller – to maintain the energy level for each pendulum.

To demonstrate the behaviour of the uncontrolled system we made a computer simulation for the following system parameters: $M = 10$ kg, $m = 1$ kg, $g = 9.81$ m/sec^2, $l = 1$ m, $d = 20$ N$\cdot$sec/m, $k = 1$ N/m, $\phi_1(0) = 0.5$ rad, $\phi_2(0) = 0$, $\dot{\phi}_1(0) = \dot{\phi}_2(0) = 0$, $x(0) = 0$, $\dot{x}(0) = 0$. The simulations confirm that the system trajectories tend to the synchronous mode where the pendula oscillate in anti-phase, while the oscillations of the beam decay.

IV. Controlled synchronization

In this section we propose a controller to solve the synchronization problem. To this end we assume that all the state variables are available for measurements. The controller basically consists of two loops. The first loop makes the synchronization manifold globally asymptotically stable, while the second loop swing the pendula up to the desired energy level.

To analyze the system behaviour in the coordinates transverse to the desired synchronous mode one can add two first equations of (1). Then to make the synchronous regime asymptotically stable one can try to cancel the terms which depend on $\ddot{x}$ by means of appropriate feedback. It gives a simple hint how to derive the following controller:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (I_2 + \Sigma)^{-1}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \qquad (4)$$

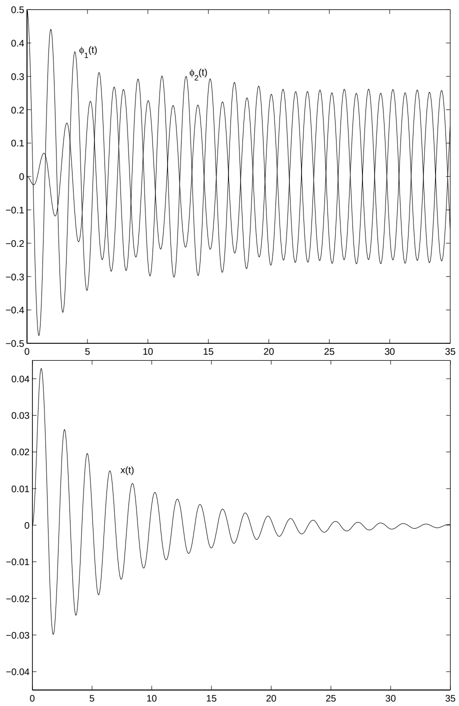

Fig. 2. Synchronization of two pendula in the uncontrolled system. a - oscillations of the pendula; b - oscillations of the beam.

where

$$\Sigma = K \begin{pmatrix} \cos^2\phi_1 & \cos\phi_1\cos\phi_2 \\ \cos\phi_1\cos\phi_2 & \cos^2\phi_2 \end{pmatrix} \geq 0$$

$$K = \frac{m^2 l}{M + m\sin^2\phi_1 + m\sin^2\phi_2}$$

and

$$s_1 = -\gamma\dot{\phi}_1 \left[H(\phi_1, \dot{\phi}_1) + H(\phi_2, \dot{\phi}_2) - 2H_*\right] + w\cos\phi_1 + v$$

$$s_2 = -\gamma\dot{\phi}_2 \left[H(\phi_1, \dot{\phi}_1) + H(\phi_2, \dot{\phi}_2) - 2H_*\right] + w\cos\phi_2 + v$$

where γ is positive gain coefficient and

$$\begin{aligned} w &= K\Big(\frac{g}{2}(\sin 2\phi_1 + \sin 2\phi_2) \\ &+ l(\dot{\phi}_1^2\sin\phi_1 + \dot{\phi}_2^2\sin\phi_2) - d\dot{x} - kx\Big) \end{aligned}$$

and

$$v = \frac{mgl}{2}(\sin\phi_1 + \sin\phi_2) - \lambda_2\sin(\phi_1 + \phi_2) - \lambda_1(\dot{\phi}_1 + \dot{\phi}_2)$$

with positive λ_1, λ_2.

The analysis of the closed loop system then can be performed in two steps. First, one can prove that the set $\phi_1 = -\phi_2, \dot{\phi}_1 = -\dot{\phi}_2$ is globally asymptotically stable. It follows from the equations of the closed loop system with respect to the variable $\phi_1 + \phi_2$:

$$\begin{aligned} &ml^2(\ddot{\phi}_1 + \ddot{\phi}_2) + \lambda_1(\dot{\phi}_1 + \dot{\phi}_2) + \lambda_2\sin(\phi_1 + \phi_2) = \\ &-\gamma(H(\phi_1, \dot{\phi}_1) + H(\phi_1, \dot{\phi}_1) - 2H_*)(\dot{\phi}_1 + \dot{\phi}_2). \end{aligned}$$

Therefore, if $\lambda_1 > 2\gamma H_*$ the set $\dot{\phi}_1 = -\dot{\phi}_2, \ \ \phi_1 = -\phi_2$ is globally asymptotically stable.

There is only one invariant subset of this set, namely, $x \equiv 0$, and hence, the limit dynamics of each pendulum is given by the following equation

$$ml^2\ddot{\phi} + mgl\sin\phi = -2\gamma\dot{\phi}[H(\phi, \dot{\phi}) - H_*]$$

and therefore the control objective

$$\lim_{t\to\infty} H(\dot{\phi}(t), \phi(t)) = H_*$$

is achieved for almost all initial conditions [14].

The above arguments can be summarized as the following statement:

Theorem 1: Suppose $\lambda_1 > 2\gamma H_*$, then in the closed loop system (1,4) the control goal (2,3) is achieved for almost all initial conditions.

To demonstrate the ability of the controller to achieve the control objective we carried out computer simulation for the same system parameters as before with the following parameters of the controller: $H_* = 9.81, \ \lambda_1 = 20, \ \lambda_2 = 4.905, \ \gamma = 0.045$. The results of the simulation are presented in figure 3 for the following initial conditions $\phi_1(0) = 0.1, \ \phi_2(0) = 0.2, \ \dot{\phi}_1(0) = \dot{\phi}_2(0) = 0, \ \ x(0) = 0, \dot{x}(0) = 0$.

As predicted by the theorem there is a set of zero Lebesgue measure of exceptional initial conditions for which the control objective can not be achieved. For example, if one initiate the system at the point where $\phi_1 = \phi_2, \dot{\phi}_1 = \dot{\phi}_2$, the oscillations of both pendula and the beam will decay. However, from practical point of view, it is not difficult to modify the controller to handle this problem.

As we have mentioned, the controlled synchronization can be utilized to avoid resonance oscillations during start-up procedure of speeding up the rotors installed on a common support. To demonstrate this effect we carried out the next simulation. In this case the desired energy level H_* is greater than the critical level $2mgl$, $H_* = 5mgl$. We also decreased the damping coefficient d to make the motion of the beam oscillatory: $d = 5$. The gains γ, λ_1 are decreased as well, $\gamma = 0.005$, $\lambda = 6$. The rest parameters are the same as in the previous simulation. The results are plotted in figure 4. It is seen that during the start-up procedure the rotors oscillate synchronously and the amplitude of the beam oscillations is small.

In the beginning of this section we assumed that all the state variables are available for measurements. This assumption allows to design a simple controller with a relatively simple stability analysis. From the practical point of view, one can impose some additional constraints on the controller, i.e. to avoid the measurements of the beam position/velocity. This problem is definitely feasible but requires a bit more sophisticated analysis.

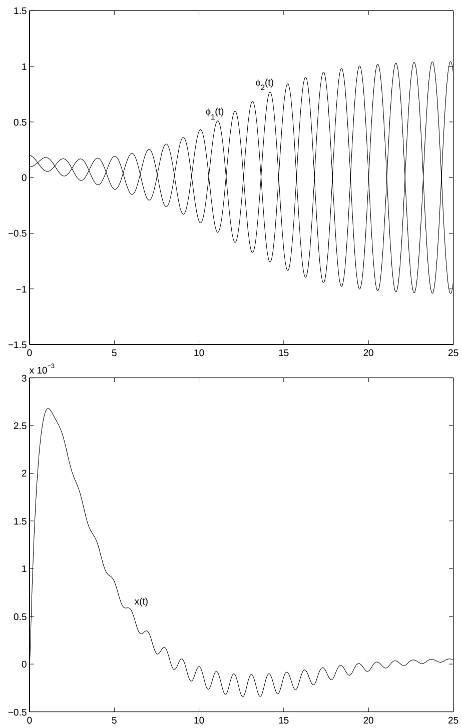

Fig. 3. Synchronization of two pendula in the controlled system. a - oscillations of the pendula; b - oscillations of the beam.

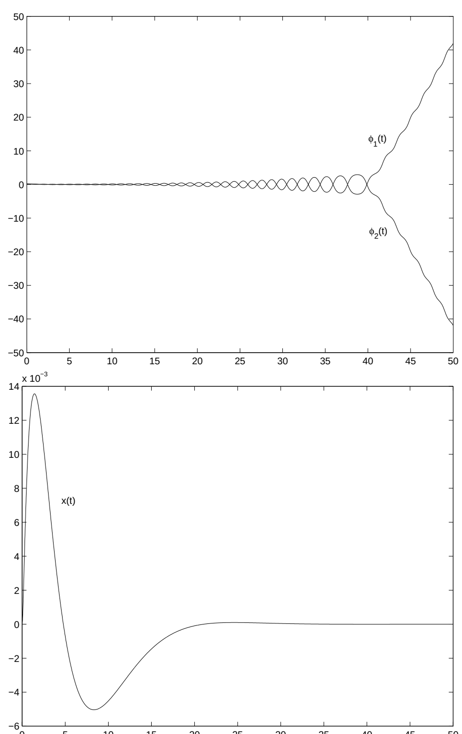

Fig. 4. Synchronization of two imbalanced rotors during the start-up procedure. a - oscillations of the rotors; b - oscillations of the beam.

V. Huijgens' phenomenon

The results of the previous sections are immediately related to the experiment carried out by C. Huijgens in 1665. He detected that a couple of pendulum clocks hanging from a common support had synchronized, i.e. their oscillations coincide perfectly and the pendula moved in opposite directions. Huijgens described in detail such coupled clocks: "In these clocks the lengths of the pendulum was nine inches and its weight one-half pound. The wheels were rotated by the force of weights and were enclosed together with weights in a case which was four feet long. At the bottom of the case was added a lead weight of over one hundred pounds so that the instrument would better maintain a perpendicular orientation when suspended in the ship. Although the motion of the clock was found to be very equal and constant in these experiments, nevertheless we made an effort to perfect it still further in another way as follows ... the result is still greater equality of clocks than before." Some of the original Huijgens' drawings are reproduced in figures 5, 6.

Particularly, Huijgens desribed what is now called "frequency synchronization", i.e. being coupled two oscillators with nonidentical frequencies demonstrate synchronous oscillations with a common frequency. The frequency synchronization of clocks was observed, for example in [2], where the clocks were modelled by Van der Pol equations with slightly different periods of oscillations. Via the averaging technique it was shown that the system of interconnected oscillators possesses an asymptotically stable periodic solution.

In this section we show how to realize the phase synchronization for a very simple yet illustrative model of clocks. The pendulum clocks can be modelled in different ways, see, e.g. [15]. In our clock model we combine together two simple ideas: first, the oscillations of the clock pendulum should be described by equations of the free pendulum with a given level of its energy; second, the model should take into account a mechanism to sustain this level. Then, the simplest model of the pendulum clock is given by the following equation:

$$ml^2\ddot{\phi} + mgl\sin\phi = -\gamma\dot{\phi}[H(\phi,\dot{\phi}) - H_*], \quad \gamma > 0.$$

This equation has orbitally stable periodic solution which corresponds to the motion of the free pendulum with the energy equal to H_*. This limit cycle attracts almost all initial conditions as can be seen from the following relation for the Hamiltonian function $H(\phi,\dot{\phi})$:

$$\dot{H} = -\gamma\dot{\phi}^2(H - H_*).$$

The model of the two clocks hanging from a beam as shown on figure 1 can be thus derived as the system (1) with

$$u_i = -\gamma\dot{\phi}_i[H(\phi_i,\dot{\phi}_i) - H_*], \quad i = 1,2.$$

From the equations of this model it follows that the system has at least two invariant sets $\Omega_1 := \{\dot{\phi}_1 = -\dot{\phi}_2,\ \phi_1 =$

$-\phi_2, x = 0, \dot{x} = 0\}$ and Ω_2 which is a some subset of the set $\{\dot{\phi}_1 = \dot{\phi}_2,\ \phi_1 = \phi_2\}$. Computer simulation shows that both of them can be stable provided the constant H_* is relatively small, while for large values of H_* the system can demonstrate erratic behaviour. Therefore, we perform a local stability analysis for the set Ω_1 assuming that H_* is a small parameter. Under this assumption the system equations are

$$\begin{aligned} ml^2\ddot{\phi}_1 + ml\ddot{x} + mgl\phi_1 &= u_1 \\ ml^2\ddot{\phi}_2 + ml\ddot{x} + mgl\phi_2 &= u_2 \\ (M+2m)\ddot{x} + ml(\ddot{\phi}_1 + \ddot{\phi}_2) &= -d\dot{x} - kx \end{aligned}$$

with

$$u_i = -\gamma\dot{\phi}_i[H(\phi_i, \dot{\phi}_i) - H_*], \quad i = 1, 2$$

and

$$H(\phi, \dot{\phi}) = \frac{ml^2}{2}\dot{\phi}^2 + \frac{mgl}{2}\phi^2.$$

Linearizing this system around the set $H(\phi_1, \dot{\phi}_1) = H(\phi_2, \dot{\phi}_2) = H_*$ yields the following equation (written with a little abuse in notations):

$$\begin{aligned} ml^2(\ddot{\phi}_1 + \ddot{\phi}_2) + 2ml\ddot{x} + mgl(\phi_1 + \phi_2) &= 0 \\ (M+2m)\ddot{x} + ml(\ddot{\phi}_1 + \ddot{\phi}_2) &= -d\dot{x} - kx \end{aligned}$$

and the local stability of the set Ω_1 follows. Therefore synchronous motion of clocks' pendula is asymptotically stable.

VI. Conclusions

In this paper we considered the problem of controlled synchronization of two pendula hanging from a common support. This problem can find an important practical application - to avoid resonance vibration during start up procedure of speeding two imbalanced rotors. We proposed a controller which is able to solve the synchronization problem in such a way that the pendula reach the desired level of energy and they move synchronously in opposite directions. In this case the oscillations of the support beam can be avoided. It is worth mentioning that the solution proposed in this paper is based on the synchronization phenomenon experimentally observed by C. Huijgens in 1665. This example demonstrates that (non)linear control can be utilized in various applied problems related to synchronization.

Acknowledgments

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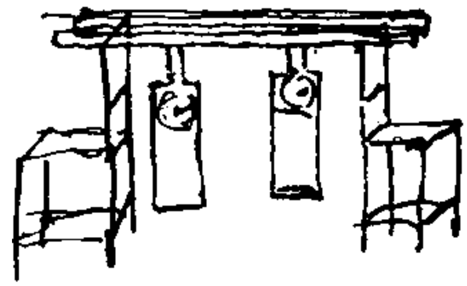

Fig. 5. The Huijgens drawing describing his synchronization experiment

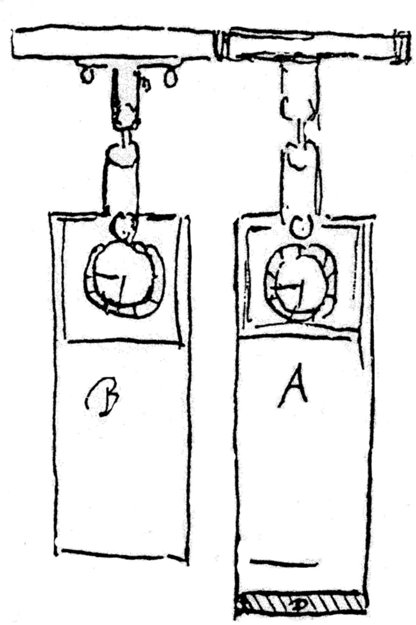

Fig. 6. The Huijgens drawing describing his synchronization experiment

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