Cooperative Oscillatory Behavior of Mutually Coupled Dynamical Systems

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***Abstract*—In this paper, we make a qualitative study of the dynamics of a network of diffusively coupled identical systems. In particular, we derive conditions on the systems and on the coupling strength between the systems that guarantee the global synchronization of the systems. It is shown that the notion of "minimum phaseness" of the individual systems involved is essential in ensuring synchronous behavior in the network when the coupling exceeds a certain computable threshold. On the other hand, it is shown that oscillatory behavior may arise in a network of identical globally asymptotically stable systems in case the isolated systems are nonminimum phase. In addition, we analyze the synchronization or nonsynchronization of the network in terms of its topology; that is, what happens if either the number of couplings and/or systems increases? The results are illustrated by computer simulations of coupled chaotic systems like the Rössler system and the Lorenz system.**

***Index Terms*—Cellular neural networks, oscillatory behavior, passivity, synchronization.**

I. Introduction

RECENTLY, an increasing interest has been devoted to the study of cellular neural networks (CNNs) [4], [20], [23], [24] (see also [19]). A CNN consists of mutually coupled dynamical systems and, due to the interactions, it can exhibit complex behavior even in cases when each cell itself is described by fairly simple differential equations. Among the possible applications of CNNs, we mention the very interesting fields of telecommunication [12] and mathematical biology [9].

Even CNNs consisting of simple cells form, after coupling, high dimensional nonlinear systems, and therefore such CNN's are difficult to study analytically. However, recently, some progress in the qualitative study of CNNs has been made. A thorough analysis of CNNs consisting of cells described by first-order differential equations coupled in linear one-dimensional (1-D) arrays can be found in [20] and [24]. It was shown that, even in such a simple setting, it is possible to observe nontrivial phenomena called local diffusion and global propagation.

In this paper, we are interested in oscillatory behavior in networks composed of identical dynamical systems and coupled in arbitrary arrays via *diffusive coupling*, that is, the systems are mutually coupled through a linear output coupling. In other words, we assume that with the given elementary dynamical system a specific output (read-out map) is given and the coupling between various systems in the network is defined through weighted differences of the form $k(y_i - y_j)$, with k some positive number and y_i, y_j indicating the outputs of the ith and jth systems.

It turns out that, in a network of diffusively coupled identical systems, two structural phenomena can be encountered and analyzed. Namely, on the one hand, we will study the synchronization of all systems in the network, implying that the dynamics of all individual cells will asymptotically match with each other. Of course, it does not imply that the dynamics of each separate system "dies out," but, in contrast, what can happen is that some oscillatory motion in a separate dynamical system will be enforced through the coupling in an identical way in all systems. As an illustration, this happens for instance when a network of identical Lorenz systems is built and the couplings between the separate systems are strong enough.

On the other hand, we will treat the occurrence of asynchronous oscillatory behavior in a diffusively coupled network. In this case, the basic observation is that, through the coupling of the systems, oscillatory motion in each of the systems can be enforced, even if the individual system will stabilize if no coupling exists. Conditions for the presence of oscillatory behavior in a network of diffusively coupled systems are derived in an earlier paper by the authors [16] and are contrasted with the conditions for synchronization.

Given the possible occurrence of one of the two scenarios, a very interesting case arises if the number of cells in the network and/or the number of nonzero interconnections is changed. In particular, we will address the interesting question whether synchronization or diffusion driven instability would be changed when increasing the number of cells and/or interconnections. Some earlier work in this direction can be found in [1] and [27]. In particular, we show that, if the number of cells in the network becomes infinitely large while the coupling strengths are bounded and the number of interconnections between cells does not grow more than linearly with respect to the number of cells, the systems eventually will lose the synchronization property. However, the synchronization can be retained if one allows for a quadratic growth of the number of interconnections.

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The paper is organized as follows. In Section II, we present necessary background material. Section III deals with the synchronization phenomena occurring via diffusion. In Section IV, we briefly discuss the conditions resulting in generation of oscillations via diffusion. Some useful properties of diffusive networks are discussed in Section V. Section VI contains some concluding remarks.

II. Preliminaries

The Euclidean norm in $\mathbb{R}^n$ is denoted simply as $|\cdot|$, $|x|^2 = x^\top x$, where $^\top$ defines transposition. The notation $\mathrm{col}(x_1, x_2, \cdots, x_n)$ stands for the column vector composed of the elements $x_1, \cdots, x_n$. This notation will also be used in the case where the components x_i are vectors again. We will study notions relative to nonempty subsets $\mathcal{A}$, of $\mathbb{R}^n$, $0 \in \mathcal{A}$; for such a set $\mathcal{A}$, $|x|_\mathcal{A} = \mathrm{dist}(x, \mathcal{A}) = \inf_{\eta\in\mathcal{A}} \mathrm{dist}(x, \eta)$ denotes the distance from $x \in \mathbb{R}^n$ to $\mathcal{A}$.

A function $V: X \to \mathbb{R}_+$ defined on a subset X of $\mathbb{R}^n$, $0 \in X$ is *positive definite* if $V(x) > 0$ for all $x \in X\backslash\{0\}$ and $V(0) = 0$. It is *radially unbounded* (if $X = \mathbb{R}^n$) or *proper* if $V(x) \to \infty$ as $|x| \to \infty$. A nonnegative function $V: X \to \mathbb{R}_+$ is said to be *positive definite with respect to the set* $\mathcal{A}$ if $V(x) > 0$ for all $x \in X\backslash\mathcal{A}$ and $V(x) = 0$ for all $x \in \mathcal{A}$. It is *proper with respect to* $\mathcal{A}$ if boundedness of $V(x)$ implies boundedness of $|x|_\mathcal{A}$.

If a quadratic form $x^\top P x$ with a symmetric matrix $P = P^\top$ is positive definite, then the matrix P is called positive definite. For positive definite matrices, we use the notation $P > 0$; moreover, $P > Q$ means that the matrix $P - Q$ is positive definite.

A matrix A for which all eigenvalues have negative real parts is called Hurwitz or stable.

An invariant set $\mathcal{A} \subset \mathbb{R}^n$ for the dynamics $\dot{x} = f(x)$ is said to be *noncritically stable* if it is Lyapunov stable, that is, all solutions starting close enough to $\mathcal{A}$ remain close to $\mathcal{A}$ for all $t > 0$ and additionally $|x(t)|_\mathcal{A} \le Ce^{-\delta t}|x(0)|_\mathcal{A}$ for sufficiently small $|x(0)|_\mathcal{A}$ with positive C, δ.

The system $\dot{x} = f(x)$ is called *Lagrange stable* if all its solutions are bounded. If all solutions eventually end up within a bounded domain which can be chosen independently of the initial conditions, then all solutions are referred to as *ultimately bounded*.

Given a system of autonomous differential equations

$$\dot{x} = F(x) \tag{1}$$

where $x \in \mathbb{R}^n$. We will say that the system (1) is oscillatory in the sense of Yakubovich, or *Y-oscillatory* if any solution of (1) is bounded and for almost all initial conditions the corresponding solution does not tend to a constant (see, e.g., [25]).

For matrices A and B, the notation $A \otimes B$ (the Kronecker product) stands for the matrix composed of submatrices $A_{ij}B$, i.e.,

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{pmatrix} \tag{2}$$

where A_{ij}, $i, j = 1 \cdots n$, stands for the ijth entry of the $n \times n$ matrix A.

Consider the nonlinear time-invariant affine system

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{3}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input which is assumed to be any continuous and (essentially) bounded function of time: $u(\cdot) \in \mathcal{C}^0 \cap \mathcal{L}_\infty$, $y(t) \in \mathbb{R}^m$ is the output; $f: \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $g: \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are smooth enough to ensure existence of solutions in any reasonable sense, e.g., in the sense of Filippov, at least on a finite time interval $0 \le t < T_{x_0, u}$; $h: \mathbb{R}^n \to \mathbb{R}^m$ is the output mapping.

Suppose there exists a nonnegative differentiable[1] *storage function* $V: \mathbb{R}^n \to \mathbb{R}_+$, $V(0) = 0$ and nonnegative continuous function $S: \mathbb{R}^n \to \mathbb{R}_+$, $S(0) = 0$ such that for all admissible inputs u and initial conditions $x(0) = x_0$ and for all time instants $0 \le t < T_{x_0, u}$ the following *dissipation inequality* is valid:

$$\dot{V}(x, u) \le y^\top u - S(x). \tag{4}$$

Then the system (3) is called a *passive system*, see, e.g., [3] and [10]. If, additionally, S is positive definite, then the system (3) is called *strictly passive*.

From the definition of passive systems, one can draw two important conclusions. First, if the strictly passive system (3) possesses some inherent dynamics consistent with the constraint $y = 0$ (which in control theory is referred to as the *zero dynamics*) and V is positive definite, then the origin is an asymptotically stable equilibrium of the zero dynamics. Secondly, if V is positive definite, then the origin is a stable equilibrium of the free system ($u \equiv 0$) (in case of strict passivity the origin is asymptotically stable).

The theory of passive systems plays an important role in modern control theory [3], [14], [26]. In a natural way it extends the notion of positive real linear systems to nonlinear systems.

In this paper we need some weakened version of passivity because we focus our attention on systems exhibiting oscillatory behavior for $u \equiv 0$.

Next we define a *semipassive system*. This notion was introduced in [15]; in [17], an equivalent notion was called *quasipassivity*. Roughly speaking, a semipassive system behaves like a passive system for sufficiently large $|x|$. More precisely, assume that there exists a nonnegative function $V: \mathbb{R}^n \to \mathbb{R}_+$ such that for all admissible inputs, for all initial conditions and for all t for which the corresponding solution of (3) exists, we have

$$\dot{V}(x, u) \le y^\top u - H(x) \tag{5}$$

where the function $H: \mathbb{R}^n \to \mathbb{R}$ is nonnegative outside some ball:

$$\exists \rho > 0, \qquad \forall |x| \ge \rho \Longrightarrow H(x) \ge \varrho(|x|) \tag{6}$$

[1] In fact, the function V needs not necessarily be differentiable. In this case, we will assume that V is locally Lipschitz continuous, i.e., it satisfies a Lipschitz condition on any compact set. Indeed, if $x(t)$ is a bounded solution in the sense of Filippov, then it is an absolutely continuous function of time and therefore $V(x(t))$ is also an absolutely continuous function of time, that is, its time-derivative exists almost everywhere.

for some continuous nonnegative function ϱ defined for $|x| \geq \rho$. If the function H is positive outside some ball, i.e., (6) holds for some continuous positive function ϱ, then the system (3) is said to be *strictly semipassive*.

The concept of semipassivity allows one to find simple conditions which ensure boundedness of the solutions of interconnected systems. Consider k (possibly different) systems of the form (3) as

$$\begin{cases} \dot{x}_j = f_j(x_j) + g_j(x_j)u_j \\ y_j = h_j(x_j) \end{cases} \tag{7}$$

where $j = 1, \cdots, k$.

Define the symmetric $k \times k$ matrix Γ as

$$\Gamma = \begin{pmatrix} \sum_{i=2}^{k} \gamma_{1i} & -\gamma_{12} & \cdots & -\gamma_{1k} \\ -\gamma_{21} & \sum_{i=1,\, i\neq 2}^{k} \gamma_{2i} & \cdots & -\gamma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & \cdots & \sum_{i=1}^{k-1} \gamma_{ki} \end{pmatrix} \tag{8}$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$ and all row sums are zero. The matrix Γ is symmetric and therefore all its eigenvalues are real. Moreover, applying Gerschgorin's theorem about localization of eigenvalues (see, e.g., [22]), one can see that all eigenvalues of Γ are nonnegative, that is, the matrix Γ is positive semidefinite.

The following result gives conditions under which the solutions of the interconnected systems (7) are bounded.

Lemma 1: [16]: Suppose that the systems (7) are semipassive with radially unbounded storage functions $V_j\colon \mathbb{R}^{n_j} \to \mathbb{R}_+$. Then all solutions of the systems (7) in closed loop with the feedback

$$u_j = -\gamma_{j1}(y_j - y_1) - \gamma_{j2}(y_j - y_2) - \cdots - \gamma_{jk}(y_j - y_k) \tag{9}$$

with $\gamma_{ji} = \gamma_{ij} \geq 0$ exist for all $t \geq 0$ and are bounded, that is, the system (7), (9) is Lagrange stable. Moreover, if the systems (7) are strictly semipassive with radially unbounded storage functions V_j: $\mathbb{R}^{n_j} \to \mathbb{R}_+$ then all solutions of the coupled system (7), (9) exist for all $t \geq 0$ and are ultimately bounded.

Next we give a definition of diffusive coupling of k identical systems. This definition was introduced in [16] and was inspired by the paper of Smale on interaction between two cells [21]. We will understand a diffusive medium as a dynamical system consisting of a number of interconnected identical dynamical systems. Each separate system has inputs and outputs of the same dimension. The diffusive coupling is described by a static relation between inputs and outputs. Notice that our approach to describe the diffusion is different from that proposed in [24].

Definition 1: Given the smooth systems

$$\begin{cases} \dot{x}_j = f(x_j) + Bu_j \\ y_j = Cx_j \end{cases} \tag{10}$$

where $j = 1, \cdots, k$, $x_j(t) \in \mathbb{R}^n$ is the state of the jth system, $u_j(t) \in \mathbb{R}^m$ is the input, $y_j(t) \in \mathbb{R}^m$ is the output of the jth system, $f(0) = 0$, and B, C are constant matrices of appropriate dimension. We say that the systems (10) are *diffusively coupled* if the matrix CB is similar to a positive definite matrix and the systems (10) are interconnected by the following feedback:

$$u_j = -\gamma_{j1}(y_j - y_1) - \gamma_{j2}(y_j - y_2) - \cdots - \gamma_{jk}(y_j - y_k) \tag{11}$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$ are constants such that $\sum_{j\neq i}^{k} \gamma_{ji} > 0$ for all $i = 1, \cdots, k$.

Note that if we define the coupling matrix Γ as $\Gamma := (\gamma_{ij})$, $\gamma_{ii} := \sum_{i\neq j}^{k} \gamma_{ij}$, then the feedback (11) can be written in a matrix notation as

$$u = -\Gamma \otimes I_m y.$$

An observation that the use of the Kronecker product makes the study of CNN more convenient is made in [28].

III. Synchronization of Diffusively Coupled Systems

These days, synchronization of dynamical systems is a very popular topic. It attracts the attention of researchers from different fields (see, e.g., the November 1997 special issue of IEEE Transactions on Circuits and Systems—I: Fundamental Theory and Applications). Various definitions of this phenomenon exist and below we define synchronization following [2].

Consider a dynamical system described by the k *interconnected* systems of ordinary differential equations:

$$S_i\colon \dot{x}_i = F_i(x_1, x_2, \ldots, x_k, t), \qquad i = 1, \ldots, k \tag{12}$$

where F_i: $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \times \mathbb{R}_+ \to \mathbb{R}^{n_i}$. A general coordinate-free definition of the problem of controlled synchronization involves a coordinate-free definition of dynamical systems and can be found in [2]. Associated with the systems (12) consider the following function Q:

$$Q(t) = \sum_{i=2}^{k} |x_1(t) - x_i(t)|. \tag{13}$$

We say that solutions of the systems S_i are synchronized if the value of the function Q is identically zero for the solutions of (12).

Definition 2: The solutions $x_1(t), \ldots, x_k(t)$ of the systems $S_1, \ldots, S_k$ with initial conditions $x_1(0), \ldots, x_k(0)$ are called *synchronized* if

$$Q(t) \equiv 0 \tag{14}$$

for all $t \in \mathbb{R}_+$.

The solutions $x_1(t), \cdots, x_k(t)$ of the systems $S_1, \cdots, S_k$ with initial conditions $x_1(0), \cdots, x_k(0)$ are *asymptotically synchronized*, if

$$\lim_{t\to\infty} Q(t) = 0. \tag{15}$$

Asymptotic synchronization can be interpreted as the convergence to the "diagonal" set $\mathcal{A} = \{x_1, \cdots, x_k \in \mathbb{R}^n\colon x_i =$

$x_j;\, i, j = 1, \cdots k\}$. If this is the case for all initial conditions in some open neighborhood of $\mathcal{A}$, then the synchronization is equivalent to the attractivity of $\mathcal{A}$. For practical reasons, it is convenient to consider a stronger version where the set $\mathcal{A}$ is not only attractive but also Lyapunov stable. This case is called *strong synchronization* [8].

Remark 1: Note that attractivity of $\mathcal{A}$ does not necessarily imply Lyapunov stability of $\mathcal{A}$. Indeed, even if $\mathcal{A}$ is a singleton, e.g., $\mathcal{A} = \{0\}$, then even if any solution of the system $\dot{x} = f(x)$, $f(0) = 0$, $x(t) \in \mathbb{R}^n$, $n \geq 2$ starting close to the origin satisfies $x(t) \to 0$ as $t \to \infty$ it does not mean that the origin is asymptotically stable, see [7, Section 40]. △

The case when $\mathcal{A}$ attracts solutions starting from a set of positive Lebesgue measures but $\mathcal{A}$ is not Lyapunov stable is called *weak synchronization* [8]. Although the case of weak synchronization is of some theoretical interest, for practical purposes it is not so important because small disturbances affecting the system can destroy the synchronization. In the sequel, we will study strong synchronization, namely we will present sufficient conditions when $\mathcal{A}$ has a compact asymptotically stable subset.

In what follows, we are interested in the case of identical synchronization of identical systems forming a diffusive network.

In this case, we rewrite the systems (10) in a form which can be obtained from (10) via a linear change of coordinates due to the nonsingularity of CB (see, e.g., [16]):

$$\begin{cases} \dot{z}_j = q(z_j, y_j) \\ \dot{y}_j = a(z_j, y_j) + CBu_j \end{cases} \tag{16}$$

where $z_j(t) \in \mathbb{R}^{n-m}$, q: $\mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^{n-m}$, a: $\mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$.

Let the eigenvalues of the matrix Γ be ordered as: $0 = \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_k$.

Theorem 1: Consider the k smooth diffusively coupled systems (10) and (11), which, because of the nonsingularity of CB, are rewritten as (16) and (11). Assume the following.

A1. The system

$$\begin{cases} \dot{z} = q(z, y) \\ \dot{y} = a(z, y) + CBu \end{cases} \tag{17}$$

is strictly semipassive with respect to the input u and output y with a radially unbounded storage function V: $\mathbb{R}^n \to \mathbb{R}_+$.

A2. There exists a $\mathcal{C}^2$-smooth positive definite function V_0: $\mathbb{R}^{n-m} \to \mathbb{R}_+$ and a positive number α such that the following inequality is satisfied

$$(\nabla V_0(z_1 - z_2))^\top (q(z_1, y_1) - q(z_2, y_1)) \leq -\alpha |z_1 - z_2|^2.$$

for all $z_1, z_2 \in \mathbb{R}^{n-m}$, $y_1 \in \mathbb{R}^m$.

Then, for all positive semidefinite matrices Γ as in (8), all solutions of the closed-loop system (16), (11) are ultimately bounded and there exists a positive $\overline{\gamma}$ such that for all positive semidefinite matrices Γ with eigenvalues $0 = \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_k$ for which $\gamma_2 > \overline{\gamma}$ there exists a globally asymptotically stable compact subset of the diagonal set $\mathcal{A} = \{y_j \in \mathbb{R}^m, z_j \in \mathbb{R}^{(n-m)} : y_i = y_j, z_i = z_j, i, j = 1, \cdots, k\}$.

Proof: The existence and uniqueness of solutions of the closed loop system, at least on a finite time interval, follows from the smoothness of the right-hand side of the closed system. By Lemma 1, all solutions are ultimately bounded and hence exist for all $t \geq 0$:

$$\begin{aligned} &\limsup_{t\to\infty} |z_i(t)| \leq \mathcal{B}_z, \\ &\limsup_{t\to\infty} |y_i(t)| \leq \mathcal{B}_y, \qquad i = 1, \cdots, k. \end{aligned} \tag{18}$$

Moreover, from the proof in [15] and [16], one can see that the bounds $\mathcal{B}_z$, $\mathcal{B}_y$ do not depend on the coupling matrix Γ.

Let $x = \mathrm{col}(x_1, \cdots, x_k) \in \mathbb{R}^{kn}$, $z = \mathrm{col}(z_1, \cdots, z_k) \in \mathbb{R}^{k(n-m)}$, $y = \mathrm{col}(y_1, \cdots, y_k) \in \mathbb{R}^{km}$. Introduce a new set of variables: $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_1 - x_2, \cdots, \tilde{x}_k = x_1 - x_k$. In matrix notation, this change of coordinates can be written as $\tilde{x} = Mx$, where $M = P \otimes I_n$ is the nonsingular matrix $kn \times kn$ with

$$P = \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{1} & -I_{k-1} \end{pmatrix}$$

with $\mathbf{1} = \mathrm{col}(1, \cdots, 1)$ and $\mathbf{0} = \mathrm{col}(0, \cdots, 0)$.

Notice that

$$P\Gamma P^{-1} = \begin{pmatrix} 0 & * \\ \mathbf{0} & \Gamma_1 \end{pmatrix}$$

where the $(k-1) \times (k-1)$ matrix Γ_1 has eigenvalues $\gamma_2, \cdots, \gamma_k$.

Denote $w_1 = \mathrm{col}(\tilde{z}_2, \cdots, \tilde{z}_k)$ and $w_2 = \mathrm{col}(\tilde{y}_2, \cdots, \tilde{y}_k)$, where $\tilde{z}_1 = z_1$, $\tilde{z}_2 = z_1 - z_2, \cdots, \tilde{y}_1 = y_1$, $\tilde{y}_2 = y_1 - y_2 \cdots$ and so on. Assumption A2 implies the existence of a positive definite function V_1: $\mathbb{R}^{(k-1)(n-m)} \to \mathbb{R}_+$ such that

$$\dot{V}_1(w_1, w_2)\Big|_{w2=0} \leq -\alpha_1 |w_1|^2$$

with $\alpha_1 > 0$.

Since $\Gamma = \Gamma^\top$, one can conclude that there exists a nonsingular $(k-1) \times (k-1)$ matrix F such that $F\Gamma_1 F^{-1} = \mathrm{diag}(\gamma_2, \cdots, \gamma_k)$. Introduce new coordinates $\tilde{w}_1 = (F \otimes I_{n-m})w_1$ and $\tilde{w}_2 = (F \otimes I_m)w_2$. Since stability is invariant under a change of coordinates, there exists a positive definite function V_2: $\mathbb{R}^{(k-1)(n-m)} \to \mathbb{R}_+$, $\tilde{w}_1 \mapsto V_2(\tilde{w}_1)$ such that

$$\dot{V}_2(\tilde{w}_1, \tilde{w}_2)\Big|_{\tilde{w}_2=0} \leq -\alpha_2 |\tilde{w}_1|^2$$

for some $\alpha_2 > 0$.

Now consider the following Lyapunov function candidate:

$$V_3(\tilde{w}_1, \tilde{w}_2) = V_2(\tilde{w}_1) + \tfrac{1}{2}\, \tilde{w}_2^\top \tilde{w}_2. \tag{19}$$

Notice that, due to the ultimate boundedness of all solutions and the C^2-smoothness of the function V_2, we have for $|z| \leq \mathcal{B}_z$, $|y| \leq \mathcal{B}_y$

$$|\nabla V_2(\tilde{w}_1)| \leq C_0 |\tilde{w}_1|$$

for some $C_0 > 0$, and, hence, due to smoothness of the right-hand side of the closed-loop system, it follows that

$$\dot{V}_2(\tilde{w}_1, \tilde{w}_2) - \dot{V}_2(\tilde{w}_1, 0) \leq C_1 |\tilde{w}_1| \cdot |\tilde{w}_2|$$

and

$$\frac{1}{2}\frac{d}{dt}\tilde{w}_2^\top \tilde{w}_2 \le C_2|\tilde{w}_1|\cdot|\tilde{w}_2| - (\gamma_2\beta - C_3)|\tilde{w}_2|^2$$

for $|z| \le \mathcal{B}_z$, $|y| \le \mathcal{B}_y$ and for some nonnegative C_1, C_2, C_3 with $\beta > 0$ being the smallest eigenvalue of the matrix CB.

Hence

$$\dot{V}_3 \le -\alpha_2|\tilde{w}_1|^2 + (C_1 + C_2)|\tilde{w}_1|\cdot|\tilde{w}_2| - (\gamma_2\beta - C_3)|\tilde{w}_2|^2.$$

In other words, for sufficiently large $\gamma_2 > \overline{\gamma}$, where

$$\overline{\gamma} = \frac{(C_1+C_2)^2}{4\alpha_2\beta} + \frac{C_3}{\beta}$$

we have for some $\varepsilon > 0$

$$\dot{V}_3 \le -\varepsilon(|\tilde{w}_1|^2 + |\tilde{w}_2|^2). \tag{20}$$

Integrating this inequality over $[0, t)$ yields

$$V_3(\tilde{w}_1(t), \tilde{w}_2(t)) \le V_3(\tilde{w}_1(0), \tilde{w}_2(0))$$

which proves the Lyapunov stability of the set $\tilde{w}_1 = 0$, $\tilde{w}_2 = 0$. Next we prove that this set contains a compact attractive subset which attracts all solutions.

Integrating (20) over $[0, \infty)$ (recall that we have proved that all solutions are bounded and therefore exist on the infinite time interval) yields

$$V_3(\tilde{w}_1(0), \tilde{w}_2(0)) \ge \varepsilon\int_0^\infty (|\tilde{w}_1(t)|^2 + |\tilde{w}_2(t)|^2)\, dt.$$

The left-hand side of this inequality is bounded and the integrand is nonnegative, therefore the integral exists and is finite. Consequently, all solutions of the closed-loop system are bounded, and the right-hand side of the closed-loop system is locally Lipschitz continuous, therefore the right-hand side of the closed-loop system is bounded for any solution, or, equivalently, $\dot{\tilde{w}}_1(t)$, $\dot{\tilde{w}}_2(t)$ are bounded. Hence $\tilde{w}_1(t)$, $\tilde{w}_2(t)$ are uniformly continuous in t and therefore $|\tilde{w}_1(t)|^2 + |\tilde{w}_2(t)|^2$ is uniformly continuous in t as well. So, we have proved that there exists a finite integral (from zero up to infinity) of the uniformly continuous function $|\tilde{w}_1(t)|^2 + |\tilde{w}_2(t)|^2$. According to Barbalat's lemma [18], [19], this function tends to zero, that is $|\tilde{w}_1(t)|^2 + |\tilde{w}_2(t)|^2 \to 0$ as $t \to \infty$. Combining this with (18), we obtain the result. ■

Remark 2: Since Assumption A1 implies ultimate boundedness of all solutions, it is sufficient to require that Assumption A2 is valid only on the compact set $|z| \le \mathcal{B}_z$, $|y| \le \mathcal{B}_y$. △

Let us explain the result of Theorem 1. It claims that under the conditions imposed the diagonal set $\mathcal{A} = \{x_j \in \mathbb{R}^n\colon x_1 = x_2 = \cdots = x_k\}$ contains a bounded closed invariant globally attractive set $\mathcal{A}_1 \subset \mathcal{A}$, that is, the distance between any solution $x(t)$ and this set vanishes with time. Additionally, it claims that this set is Lyapunov stable: the maximum of the distance between $x(t)$ and $\mathcal{A}_1$ depends continuously on the initial distance between $x(0)$ and $\mathcal{A}$.

The result of Theorem 1 can be considered as a generalization of the result presented in [15] to the case of an arbitrary topology in the network.

At this point, it is useful to make some comments. Consider the systems (16). It can be seen that these systems have inherent dynamics consistent with the external constraints $y_1 = y_2 = \cdots = y_k$ governed by the following equations:

$$\begin{cases} \dot{z}_1 = q(z_1, y_1) \\ \vdots \\ \dot{z}_k = q(z_k, y_1) \end{cases} \tag{21}$$

$$\dot{y}_1 = a(z_1, y_1). \tag{22}$$

Moreover, Assumption A2 can be interpreted as follows. The dynamics (21) driven by an admissible $y_1(t)$ has a noncritically stable set $z_1 = z_2 = \cdots = z_k$. Therefore, Assumption A2 is a natural generalization of the notion of noncritical minimum phaseness to the case of stabilization of sets. For the most general characterization of asymptotic stability and robust stability of invariant sets in terms of Lyapunov functions, see [11].

Recall that noncritical minimum phaseness of each subsystem means that the system

$$\dot{z} = q(z, 0)$$

has a noncritically stable zero solution. As one can notice, Assumption A2 is a sufficient condition for noncritical minimum phaseness. A possible characterization of Assumption A2 can be given by the use of concept of convergent systems.

Consider the following system:

$$\dot{z} = q(z, d) \tag{23}$$

where $z(t) \in \mathbb{R}^s$, $d(t) \in \mathbb{D}$, $\mathbb{D}$ is some compact subset of $\mathbb{R}^p$, the function $d\colon \mathbb{R}^1 \to \mathbb{D}$ is assumed to be continuous and the vector field $q\colon \mathbb{R}^s \times \mathbb{D} \to \mathbb{R}^s$ is locally Lipschitz continuous in z and continuous in d.

Following Demidovich [5], we give the following definition:

Definition 3: The system (23) is said to be *convergent* if

i) all solutions $z(t)$ are well defined for all $t \in (-\infty, +\infty)$ and all initial conditions $z(0)$.

ii) there exists a unique globally asymptotically stable solution $\overline{z}(t)$ bounded for all $-\infty < t < \infty$, i.e., for any solution $z(t)$ it follows that

$$\lim_{t\to\infty} |z(t) - \overline{z}(t)| = 0.$$

Moreover, if for all initial conditions from the arbitrary δ-ball centered at $\overline{z}(t_0)$ there are $C > 0$ and $\alpha > 0$ independent of t_0, such that it follows $|z(t) - \overline{z}(t)| \le C\exp(-\alpha(t - t_0))$, we will say that the system (23) is *noncritically convergent*.

If, additionally, the system (23) is convergent for all continuous functions d from the given class $\mathcal{D} = \{d \in \mathcal{C}^0\colon \mathbb{R}^1 \to \mathbb{D}\}$, the system (23) is referred to as *convergent in* $\mathcal{D}$.

According to [5], there exists a simple sufficient condition which guarantees that the system (23) is convergent (see [5, p. 286]; we give a slightly more general result which can be derived from [5] via a linear coordinate change):

Theorem 2: Assume that there exists a positive definite matrix $P = P^\top > 0$ such that all eigenvalues of the symmetric matrix

$$\frac{1}{2}\left[P\left(\frac{\partial q}{\partial z}(z,\zeta)\right)+\left(\frac{\partial q}{\partial z}(z,\zeta)\right)^\top P\right]$$

are negative and separated from zero for all $z \in \mathbb{R}^s$ and $\zeta \in \mathbb{D}$.

Then the system (23) is noncritically convergent in the class $\mathcal{D}$.

The proof of this theorem is based on the calculation of the time derivative of the quadratic Lyapunov function $(z(t)-\overline{z}(t))^\top P(z(t)-\overline{z}(t))$. This quadratic function can be used to check Assumption A2.

Example 1: The following example illustrates Theorem 1 for a network of Lorenz systems.

Consider the following k systems:

$$\begin{cases} \dot{x}_j = \sigma(y_j - x_j) + u_j \\ \dot{y}_j = rx_j - y_j - x_j z_j \\ \dot{z}_j = -bz_j + x_j y_j \end{cases} \tag{24}$$

with $j = 1, \cdots, k$ and

$$u_j = -\gamma_{j1}(x_j - x_1) - \gamma_{j2}(x_j - x_2) - \cdots - \gamma_{jk}(x_j - x_k).$$

We will show that if the eigenvalue γ_2 of the matrix Γ is large enough then the k systems asymptotically synchronize according to

$$\sum_{j=2}^{k} |x_1(t) - x_j(t)| \to 0,$$

$$\sum_{j=2}^{k} |y_1(t) - y_j(t)| \to 0,$$

$$\sum_{j=2}^{k} |z_1(t) - z_j(t)| \to 0$$

as $t \to \infty$.

First we check that the system

$$\begin{cases} \dot{x}_1 = \sigma(y_1 - x_1) + u \\ \dot{y}_1 = rx_1 - y_1 - x_1 z_1 \\ \dot{z}_1 = -bz_1 + x_1 y_1 \end{cases} \tag{25}$$

is strictly semipassive with respect to the input u and output x_1. To this end, consider the smooth function

$$V(x_1, y_1, z_1) = \frac{1}{2}\left(x_1^2 + y_1^2 + (z_1 - \sigma - r)^2\right) \tag{26}$$

Its time derivative with respect to the uncontrolled system ($u(t) \equiv 0$) satisfies

$$\dot{V} = -\sigma x_1^2 - y_1^2 - b\left(z_1 - \frac{\sigma + r}{2}\right)^2 + b\frac{(\sigma + r)^2}{4}.$$

It is seen that $\dot{V} = 0$ determines an ellipsoid outside of which the derivative of V is negative. If K satisfies

$$K^2 = \frac{1}{4} + \frac{b}{4}\max\left\{\frac{1}{\sigma}, 1\right\}$$

then this ellipsoid lies inside the ball

$$\Xi = \{x, y, z \colon x^2 + y^2 + (z - \sigma - r)^2 \le K^2(\sigma + r)^2\} \tag{27}$$

which means that all solutions of the uncontrolled system approach within some finite time the set defined by (27). Calculating the time derivative of V along solutions of the system (25) yields

$$\dot{V}(x_1, y_1, z_1, u) = \dot{V}(x_1, y_1, z_1, 0) + x_1 u.$$

Therefore, the function V is a storage function which proves strict semipassivity of the system (25) from the input u to the output x_1.

Secondly, we find the *zero dynamics* by imposing the external constraints $x_1 = x_j$, $j = 1, \cdots, k$:

$$\begin{cases} \dot{y}_1 = rx_1 - y_1 - x_1 z_1 \\ \dot{z}_1 = -bz_1 + x_1 y_1 \\ \dot{y}_2 = rx_1 - y_2 - x_1 z_2 \\ \dot{z}_2 = -bz_2 + x_1 y_2 \\ \vdots \\ \dot{y}_k = rx_1 - y_k - x_1 z_k \\ \dot{z}_k = -bz_k + x_1 y_k \\ \dot{x}_1 = \sigma(y_1 - x_1). \end{cases} \tag{28}$$

Now we show that the system

$$\begin{cases} \dot{y}_1 = rx_1 - y_1 - x_1 z_1 \\ \dot{z}_1 = -bz_1 + x_1 y_1 \end{cases}$$

is noncritically convergent for any bounded $x_1(t)$. Indeed, the symmetrized Jacobi matrix for this system has two eigenvalues -1 and $-b$ and, therefore, according to Theorem 2, there exists a quadratic function which satisfies Assumption A2 of Theorem 1.

Thus, all the conditions of Theorem 1 are satisfied and so there exists a number $\overline{\gamma}$ such that for sufficiently large $\gamma_2 > \overline{\gamma}$ the system of k diffusively coupled Lorenz systems has an asymptotically stable compact subset of the set $\{x_1 = x_2 = \cdots = x_k, y_1 = y_2 = \cdots = y_k, z_1 = z_2 = \cdots = z_k\}$.

To confirm the theoretical results, we carried out some computer simulations. First consider the case of two Lorenz systems ($k = 2$ with "standard parameters" $\sigma = 10$, $r = 28$, $b = 8/3$) coupled by the following coupling matrix

$$\Gamma' = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}.$$

This matrix has eigenvalues (0, 6) and, as is seen from Fig. 1, it ensures synchronization of two coupled systems [initial conditions are taken as follows: $x_1(0) = 0$, $y_1(0) = 1$, $z_1(0) = 3$, $x_2(0) = 1$, $y_2(0) = 0.1$, $z_2(0) = 3.1$]. Fig. 1 shows that the quantity $(z_1(t) - z_2(t))^2$ decays as a function of time.

Next, a simulation was performed for the case of three Lorenz systems with the coupling matrix defined as follows,

$$\Gamma'' = \begin{pmatrix} 8 & -2 & -6 \\ -2 & 4 & -2 \\ -6 & -2 & 8 \end{pmatrix}.$$

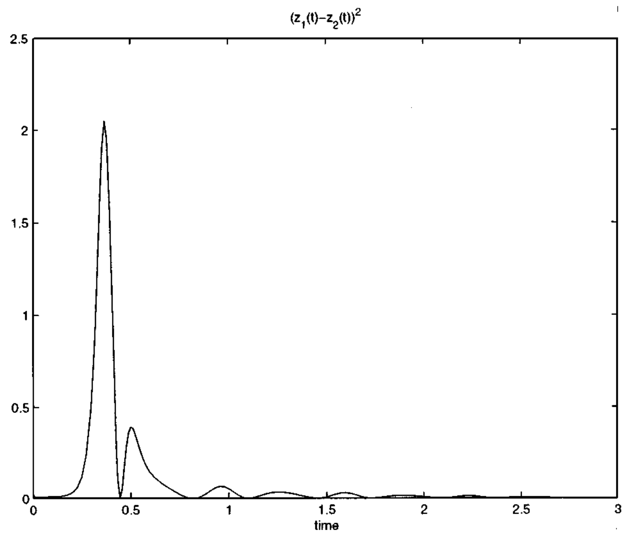

Fig. 1. Synchronization of two Lorenz systems.

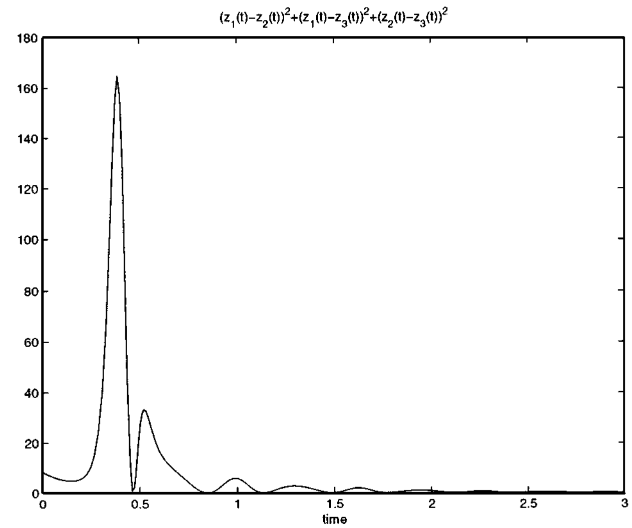

Fig. 2. Synchronization of three Lorenz systems.

Γ'' has as eigenvalues (0, 6, 14) and, according to the theoretical results, it guarantees synchronization since its smallest nonzero eigenvalue is the same as in the previous case. This fact can be observed from Fig. 2. During the simulation, the initial conditions are taken as follows: $x_1(0) = 0$, $y_1(0) = 1$, $z_1(0) = 3$, $x_2(0) = 1$, $y_2(0) = 0.1$, $z_2(0) = 3.1$, $x_1(0) = 0.3$, $y_1(0) = 0$, $z_1(0) = 1$. $\triangle$

IV. On Diffusion-Driven Instability

In the previous section, we considered a phenomenon which can be observed in a network of diffusively coupled minimum phase systems. Even in the case when each separate free system oscillates irregularly, coupled together they may exhibit some kind of synchronization. In this case, synchronization can be considered as a sign of cooperation via diffusion. However, as a result of cooperation via diffusion, arrays of coupled systems may also exhibit asynchronous oscillatory behavior. Namely, assume that each free system in the array of diffusively coupled systems is globally asymptotically stable. A common understanding of diffusion is a smoothening or trivializing process; however, a network of diffusively coupled globally asymptotically stable systems may demonstrate oscillatory behavior.

In [16], an explicit construction of diffusively coupled globally asymptotically stable systems being oscillatory when interconnected is given. One of the possible motivations of this problem lies in the field of mathematical biology (see [21]). In some sense, the problem is opposite to the synchronization problem. Namely, a network of diffusively coupled globally asymptotically stable systems will be oscillatory if all trajectories are bounded and the whole system has a unique hyperbolic equilibrium. Therefore, to prove oscillatory behavior in the diffusive network, we need to prove instability of the unique equilibrium. This is contrary to the proof of stability in the case of synchronization. The boundedness of trajectories can be established via the semipassivity property. To ensure that the origin becomes unstable for all γ_k greater than some threshold value (γ_k is the largest eigenvalue of the coupling matrix Γ), it is sufficient to require that the dynamics consistent with the constraint $\Gamma \otimes I_m y = 0$ are locally asymptotically noncritically unstable, or each system in the diffusive network is hyperbolically nonminimum phase. Additional conditions should be checked to prove that the origin is a unique equilibrium for all (or admissible) coupling matrices Γ (for details, see [16]).

We illustrate a possibility of diffusion-driven instability by an example of diffusively coupled systems of third order. This is the minimal order of systems which can become oscillatory via diffusion [16].

Example 2: Consider the following k diffusively coupled systems

$$\begin{cases} \dot{x}_1 = Ax_1(1 + |x_1|^2) + Bu_1 \\ \dot{x}_2 = Ax_2(1 + |x_2|^2) + Bu_2 \\ \vdots \\ \dot{x}_k = Ax_k(1 + |x_k|^2) + Bu_k \end{cases}$$

where $x_j \in \mathbb{R}^3$, $y_j = Cx_j$,

$$u_j = -\gamma_{j1}(y_j - y_1) - \gamma_{j2}(y_j - y_2) - \cdots - \gamma_{jk}(y_j - y_k)$$

and

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -4 & 2 & 1{-}3 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = (0\;0\;1).$$

It is easy to see that each free system ($u_i \equiv 0$) is globally asymptotically stable (it follows from the fact that the matrix A is Hurwitz).

Let $\Gamma = (\gamma_{ij})$, $\gamma_{ii} := \sum_{i \neq j}^{k} \gamma_{ij}$ be the coupling matrix with eigenvalues $\gamma_1 \leq \cdots \leq \gamma_k$. Then, with γ_k sufficiently large, the coupled system becomes Y-oscillatory. The key idea behind this example is (see [16]) that the individual systems are *not* minimum phase.

We can demonstrate this example by computer simulation. Consider three systems coupled by the following coupling matrix:

$$\Gamma'' = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

It is seen from Fig. 3 that this coupling ensures oscillatory behavior of the network despite the fact that each free system is globally asymptotically stable. Initial conditions are taken as $x_1(0) = (1, 0, 0)^\top, x_2(0) = (0, 1, 0)^\top\ x_3(0) = (0, 0, 1)^\top.\triangle$

The above results allow for a better understanding of different oscillatory phenomena occurring as a result of cooperation via diffusion. Loosely speaking, in an array of diffusively coupled minimum phase systems, one can expect the existence of synchronous properties while, in contrast, an array of diffusively coupled nonminimum phase systems may exhibit oscillatory behavior even when each free system is globally asymptotically stable.

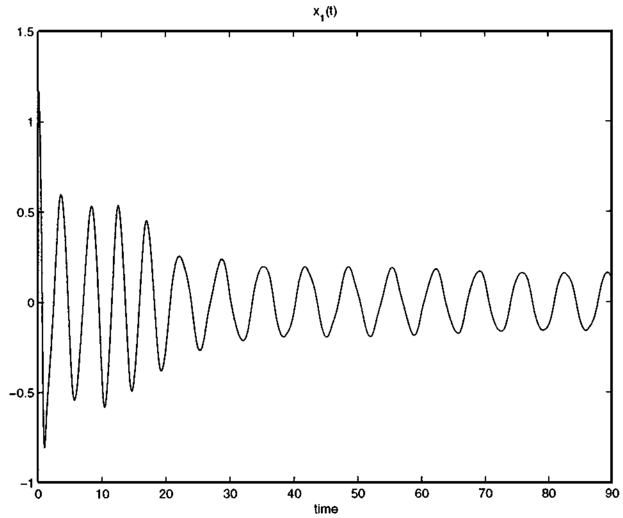

Fig. 3. Oscillatory behavior in the network of nonminimum phase systems.

V. The Topology of Diffusive Networks

In the previous sections, we discussed some oscillatory phenomena occurring as a result of diffusive interaction between identical subsystems. We presented results which are essentially based on some properties of the matrix Γ which describes the topology of the interconnections. In this section, we will discuss the synchronization or nonsynchronizing oscillatory behavior of the diffusively coupled systems in relation to the topology of the coupling. Clearly, a crucial role in this regard is played by the coupling matrix Γ, and more specifically, by the eigenvalues γ_2 (in case of minimum phase systems, see Section III) and γ_k (in the case of nonminimum phase systems, see Section IV).

Definition 4: A system consisting of k diffusively coupled systems is said to be a *cellular diffusive network* if it cannot be decomposed into two or more disconnected subsystems.

Definition 5: The maximal number N of cells connected to one cell in a diffusively coupled array of systems is called the *density* of the cellular network.

Definition 6: A cellular diffusive network is said to be *regular* if

1) All coupling constants are equal: $\gamma_{ij} = \gamma$ for all $i \neq j$
2) Each cell is connected to N other cells.

Notice that we did not impose other restrictions on the topology of the interconnections, for example symmetry of the interconnections (e.g., cyclic or rosette-like structures). Note that regular networks can model very complex structures including isotropic or anisotropic media. It is worth mentioning, however, that some symmetry in the topology can generate very interesting properties of solutions bifurcated via a Poincaré–Andronov–Hopf bifurcation which leads to oscillations in coupled systems [6]. Moreover, in some particular cases, e.g., for cyclic one-dimensional arrays, for rosette-like structures, the matrix Γ has a special structure (it turns out to be a *cyclic matrix*) for which all eigenvalues can be found analytically [9].

The numbers k (i.e., number of cells) and N (i.e., maximal number of connections at each cell) in no way define the complete topology of the network. Many different structures of the network correspond to the same k and N. As before, denote the eigenvalues of the matrix Γ as $0 = \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_k$. For any given k and N, the largest possible γ_2 subject to different topologies will be denoted as $\gamma_2(k, N)$ while the smallest possible γ_k will be denoted as $\gamma_k(k, N)$.

As we have seen in the previous sections, stability analysis in diffusive networks essentially depends on the eigenvalues γ_2 and γ_k of the matrix Γ which describes the topology of the interconnections. Therefore, in the design of diffusive networks, the following discrete optimization problems are of interest.

Given N and k, find the structure of a (regular) network which maximizes $\gamma_2(k, N)$ [maximizes $\gamma_k(k, N)$] under the constraint that all nonzero γ_{ij}, $i \neq j$ are bounded from below and above by given constants.

In general, an analytic solution to these problems is unknown. However, using methods of discrete programming, one can find solutions based on numerical computation. Clearly, the computational complexity increases significantly with large k. In what follows, we will present a solution for particular cases and then we will focus attention to the asymptotic behavior of $\gamma_2(k, N)$ and $\gamma_k(k, N)$ when k tends to infinity.

Example 3: Consider the following problem. Given the density N, find k and a structure for a regular network such that $\gamma_2(k, N)$ is maximal possible for all k. A solution to this problem is trivial: $k = N + 1$. It corresponds to the "all to all" structure. In this case, the matrix Γ has $k - 1 = N$ eigenvalues equal to $(N + 1)\gamma$ and one zero eigenvalue. Since $\mathrm{tr}\Gamma = N(N + 1)\gamma$, the solution is optimal. $\triangle$

Example 4: Now consider a similar problem. Given N, find k and a structure of regular network such that $\gamma_k(k, N)$ is maximal possible for all k. A solution is $k = 2N$. Take two clusters consisting of N cells and connect each cell from the first cluster to each cell from the second cluster. The matrix Γ in this case has one zero eigenvalue, one eigenvalue equal to $2N\gamma$ and $2(N-1)$ eigenvalues equal to $N\gamma$. According to Gerschgorin's theorem [22], $2N\gamma$ is the maximal possible eigenvalue for regular networks of density N for arbitrary k. Therefore, the solution is optimal. $\triangle$

Next we will investigate the asymptotic behavior of $\gamma_2(k, N)$ when $k \to \infty$. We are able to establish the following fact for regular networks.

Theorem 3: For regular networks, the following relation is valid:

$$\lim_{k\to\infty} \gamma_2(k, N) = 0.$$

Proof: For simplicity, take $\gamma = 1$. Consider the vector $z_1 \in \mathbb{R}^k$

$$z_1 = \left(1\Big/\sqrt{k-1},\ 1\Big/\sqrt{k-1},\ \cdots, 1\Big/\sqrt{k-1}, \Big/\ 0\right)^\top.$$

Clearly, $|z_1| = 1$. Moreover, it follows that $z_1^\top \Gamma z_1 = N(k-1)^{-1}$ (the sum of $k-N-1$ rows is zero, while the sum of the other rows is 1). Consider the vector $z_2 \in \mathbb{R}^k$

$$z_2 = \left(1\Big/\sqrt{k}\,, 1\Big/\sqrt{k}\,, \cdots, 1\Big/\sqrt{k}\right)^\top.$$

As before, $|z_2| = 1$. Moreover z_2 is the eigenvector corresponding to the zero eigenvalue of Γ, i.e., $z_2^\top \Gamma z_2 = 0$. According to Fischer's Theorem (see Appendix)

$$\gamma_2 = \min_{\dim \mathcal{X}=2}\ \max_{z\in\mathcal{X},\, |z|=1} z^\top \Gamma z.$$

Let z be a linear combination of z_1, z_2 such that $|z| = 1$. Denote by $\mathbb{D}$ a vector space spanned by z_1, z_2. Clearly,

$$\gamma_2 \le \max_{z\in\mathbb{D},\, |z|=1} z^\top \Gamma z.$$

Since $z_2^\top \Gamma z_2 = 0$ and $z_1^\top \Gamma z_1 = N(k-1)^{-1}$, it follows that $\gamma_2(k, N) \to 0$ as $k \to \infty$. ■

Remark 3: The theorem remains true for arbitrary cellular networks if we assume that its density and maximal value of the coupling constants γ_{ij}, $i \ne j$ are bounded for $k \to \infty$. △

The theorem claims that if the network density is constant and the number of cells is increasing, then the second eigenvalue of the topology matrix tends to zero regardless of the topology of the network.

In other words, if the total number of interconnections in the network grows at most *linearly* with respect to the number of cells (preserving the fact that each cell is connected with no more than N cells and N as well as diffusive factors γ_{ij} do not depend on k), then the eigenvalue γ_2 of the matrix Γ which is "responsible" for synchronization decays as k goes to infinity. This fact significantly restricts possible synchronous modes in large diffusive networks. For example, if the regular network consisting of Lorenz systems studied in Example 1 grows, then for any given γ and N there exist a $\bar{k}$ such that there is no synchronization as soon as $k \ge \bar{k}$. This fact explains the computer simulations carried out in [12].

Moreover, using the same technique as in the proof of the previous theorem, it is possible to show that zero is an accumulation point in the spectrum of the matrix Γ when k increases. At the same time, if one allows the number of interconnections to grow quadratically with respect to the number of cells in regular networks (this is the case for example in "all to all" structures), then the smallest nonzero eigenvalue grows with k and the coupling gain γ which ensures synchronization decays as k^{-1} (see [15, Proposition 1]).

We illustrate Theorem 3 with the following example.

Example 5: Consider a network with the $k \times k$ coupling matrix given by

$$\Gamma = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

This coupling matrix corresponds to a ring structure of a diffusive network consisting of k cells when each cell is connected with its two neighbors. The matrix Γ in this case is *cyclic* and its eigenvalues can be calculated analytically (see, e.g., [9]):

$$\gamma_j = 2\left(1 - \cos\left(\frac{2\pi j}{k}\right)\right), \qquad j = 1, \cdots, k.$$

It is clear that zero is an accumulation point in the spectrum of Γ as k tends to infinity. △

The situation with the largest eigenvalue of the matrix Γ which is "responsible" for generation of oscillations in diffusive networks is different. Indeed, the trace of the matrix Γ grows linearly with respect to k and therefore the largest eigenvalue $\gamma_k(k, N)$ is separated from zero. In other words, if N and the diffusive factors between cells do not depend on k then diffusion driven oscillations may occur in arbitrarily large networks.

Example 6: The last conclusion allows us to establish that the statement conjectured by Wu and Chua [29] is wrong. Given two diffusive networks with coupling matrices Γ' and Γ'' with equal smallest nonzero eigenvalues $\gamma_2' = \gamma_2''$. The Wu–Chua conjecture claims that the conditions of global (identical) synchronization for the first network are equivalent to the conditions of global synchronization in the second network. Theorem 1 gives sufficient conditions under which this statement is true: we derived conditions ensuring synchronization which depend only on smallest nonzero eigenvalues of the coupling matrix. Particularly, condition A3 is close to a necessary condition under which the Wu–Chua conjecture is true. Indeed, in [16] it is shown that if $\dot{z} = q(z, 0)$ is locally hyperbolically unstable (and hence A3 is not satisfied), then if the largest eigenvalue of the coupling matrix exceeds some threshold value then for some i, j $|x_i(t) - x_j(t)|$ does not tend to zero. However, minimum phaseness is not a necessary condition for synchronization if one allows for *all* eigenvalues of the coupling matrix to lie in some region. Since it is possible to design a system consisting of two diffusively coupled subsystems which are synchronized only if the coupling strength lies in some region (e.g., when each system is hyperbolically nonminimum phase), and according to the Wu–Chua conjecture, the largest eigenvalues of Γ' and Γ''

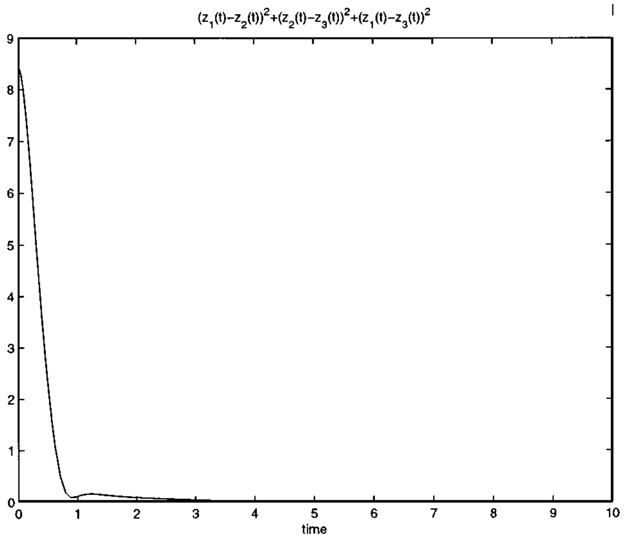

Fig. 4. Synchronization for the first type of coupling [see (30)].

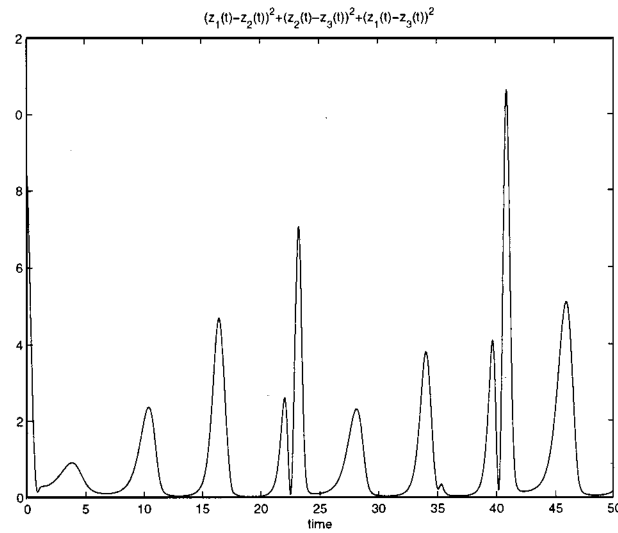

Fig. 5. Absence of synchronization for the second type of coupling [see (31)].

are admitted to be different, in the general case the Wu–Chua conjecture is not true.

Let us demonstrate this idea by computer simulation. Consider a diffusive network consisting of three identical systems:

$$\begin{cases} \dot{x}_i = -y_i - e^{z_i} \\ \dot{y}_i = x_i + a y_i + u_i \\ \dot{z}_i = c e^{-z_i} + x_i - b \end{cases} \tag{29}$$

where $i = 1, 2, 3$ and a, b, c are positive parameters. Note that the coordinate change $\xi_i = e^{z_i}$ transforms each system into the Rössler system (this coordinate transformation is very useful in the design of observer-based synchronization schemes [13]). The coupling is supposed to be determined by the following relation:

$$u = -\Gamma y$$

with $u = (u_1, u_2, u_3)^\top$ and $y = (y_1, y_2, y_3)^\top$. Consider two coupling matrices

$$\Gamma' = \tfrac{2}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{30}$$

and

$$\Gamma'' = \frac{2}{3} \begin{pmatrix} 81 & -1 & -80 \\ -1 & 2 & -1 \\ -80 & -1 & 81 \end{pmatrix}. \tag{31}$$

The matrix Γ' has eigenvalues (0, 2, 2) and the matrix Γ'' has eigenvalues (0, 2, 322/3). Computer simulations show that although the matrices Γ' and Γ'' have equal smallest nonzero eigenvalues the first coupling ensures synchronization (at least for some initial data) while the second coupling cannot provide synchronization with the same initial data. Fig. 4 shows the decay of the value $(z_1(t)-z_2(t))^2 + (z_2(t)-z_3(t))^2 + (z_1(t)-z_3(t))^2$ versus time with $a = 0.2$, $b = 5.7$, $c = 0.2$ for the first type of coupling (30). Fig. 5 illustrates that there is no synchronization for the second type of coupling [see (31)]. Initial conditions were taken in both cases as $x_1(0) = 0$, $x_2(0) = 1$, $x_3(0) = 0.3$, $y_1(0) = 0$, $y_2(0) = 0.1$, $y_3(0) = 0$, $z_1(0) = 3$, $z_2(0) = 3.1$, $z_3(0) = 1$. △

VI. Conclusions

In this paper, we presented analytical tools for the study of oscillatory behavior in arrays of diffusively coupled systems with an arbitrary topology of interconnections. The dynamics of the network are essentially based on the stability property with respect to sets of the constrained dynamics usually referred to as the zero dynamics. In the case that this dynamics has a noncritically asymptotically stable compact set consistent with $x_1 = x_2 = \cdots = x_k$, the whole system has a tendency to synchronization. Instability of the zero dynamics, in turn, leads to the generation of oscillations in diffusive networks.

We also have shown that in growing networks the growth rate of the number of interconnection is essential for synchronization but not so important for the generation of oscillations via diffusion in large networks.

Appendix
Fischer's Theorem

Theorem 4 [22]: Let an Hermitian $n \times n$ matrix H have eigenvalues $\lambda_1 \le \cdots \le \lambda_n$. Then

$$\lambda_i = \min_{\dim X = i} \max_{x \in X,\, x^H x = 1} x^H H x$$

and

$$\lambda_i = \max_{\dim X = n-i+1} \min_{x \in X,\, x^H x = 1} x^H H x$$

where x^H stands for the Hermitian transpose of x.

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