Control of partial differential equations

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Contents

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1 Introduction

A control system is a dynamical system on which one can act by using suitable *controls*. In this article, the dynamical model is modeled by partial differential equations of the following type

$$\dot{y} = f(y, u). \tag{1.1}$$

The variable y is the state and belongs to some space $\mathcal{Y}$. The variable u is the control and belongs to some space $\mathcal{U}$. In this article, the space $\mathcal{Y}$ is of infinite dimension and the differential equation (1.1) is a partial differential equation.

There are a lot of problems that appear when studying a control system. But the most common one is the *controllability* problem, which is, roughly speaking, the following one. Let us give two states. Is it possible to steer the control system from the first one to the second one? In the framework of (1.1), this means that, given the state $a \in \mathcal{Y}$ and the state $b \in \mathcal{Y}$, does there exits a map $u : [0, T] \to \mathcal{U}$ such that the solution of the Cauchy problem $\dot{y} = f(y, u(t))$, $y(0) = a$, satisfies $y(T) = b$. If the answer is yes whatever are the given states, the control system is said to be controllable. If $T > 0$ can be arbitrary small one speaks of small-time controllability. If the two given states and the control are restricted to be close to an equilibrium one speaks of local controllability at this equilibrium. (An equilibrium of the control system is a point $(y_e, u_e) \in \mathcal{Y} \times \mathcal{U}$ such that $f(y_e, u_e) = 0$). If, moreover, the time T is small, one speaks of small-time local controllability.

2 Examples of control systems modeled by PDE's

Let us give some examples on control systems modeled by PDE's.

2.1 A transport equation

This is this the simplest control system modeled by PDE's. It is the following one

$$y_t + y_x = 0,\ x \in (0, L), \tag{2.1}$$

$$y(t, 0) = u(t), \tag{2.2}$$

where, at time $t \in (0, T)$, the control is $u(t) \in \mathbb{R}$, the state is $y(t, \cdot) : (0, L) \to \mathbb{R}$ and L is a given positive real number.

2.2 A Korteweg-de Vries equation

Let $L > 0$ be given. Our Korteweg-de Vries control system is

$$y_t + y_x + y_{xxx} + yy_x = 0,\ t \in (0,T),\ x \in (0,L), \tag{2.3}$$

$$y(t,0) = y(t,L) = 0,\ y_x(t,L) = u(t),\ t \in (0,T), \tag{2.4}$$

where, at time $t \in [0,T]$, the control is $u(t) \in \mathbb{R}$ and the state is $y(t,\cdot) : (0,L) \mapsto \mathbb{R}$. Equation (2.3) is a Korteweg-de Vries equation, which serves to model various physical phenomena, for example, the propagation of small amplitude long water waves in a uniform channel (see, e.g., [45, Section 4.4, pages 155–157] or [133, Section 13.11]). Let us recall that Jerry Bona and Ragnar Winther pointed out in [24] that the term y_x in (2.3) has to be added to model the water waves when x denotes the spatial coordinate in a *fixed* frame. It is also interesting to consider the linearized control system around the trajectory $(\bar{y}, \bar{u}) := (0,0)$, i.e. the following linear control system:

$$y_t + y_x + y_{xxx} = 0,\ x \in (0,L), \tag{2.5}$$

$$y(t,0) = y(t,L) = 0,\ y_x(t,L) = u(t), \tag{2.6}$$

where, at time t, the control is $u(t) \in \mathbb{R}$ and the state is $y(t,\cdot) : (0,L) \mapsto \mathbb{R}$.

2.3 A heat equation

Let Ω be a nonempty bounded open set of $\mathbb{R}^l$ and let ω be a nonempty open subset of Ω. We consider the following linear control system:

$$y_t - \Delta y = u(t,x),\ x \in \Omega,$$

$$y = 0 \text{ on } (0,T) \times \partial\Omega,$$

where, at time t, the state is $y(t,\cdot) : \Omega \to \mathbb{R}$ and the control is $u(t,\cdot) : \Omega \to \mathbb{R}$. We require that

$$u(\cdot,x) = 0,\ x \in \Omega \setminus \omega.$$

Hence we consider the case of "internal control" (in contrast with the above examples where the control was on the boundary of the domain).

2.4 A water-tank control system

We consider a 1-D tank containing an inviscid incompressible irrotational fluid. The tank is subject to one-dimensional horizontal moves. We assume that the horizontal acceleration of the tank is small compared to the gravity constant and that the height of the fluid is small compared to the length of the tank. These physical considerations motivate the use of the Saint-Venant equations [123] (also called shallow water equations) to describe the motion of the fluid; see e.g. [45, Sec. 4.2]. Hence the considered dynamics equations are (see the paper [46] by François Dubois, Nicolas

Petit and Pierre Rouchon)

$$H_t(t,x) + (Hv)_x(t,x) = 0,\ x \in [0,L], \tag{2.7}$$

$$v_t(t,x) + \left(gH + \frac{v^2}{2}\right)_x (t,x) = -u(t),\ x \in [0,L], \tag{2.8}$$

$$v(t,0) = v(t,L) = 0, \tag{2.9}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t}(t) = u(t), \tag{2.10}$$

$$\frac{\mathrm{d}D}{\mathrm{d}t}(t) = s(t), \tag{2.11}$$

where (see Figure 1),

- L is the length of the 1-D tank,
- $H(t,x)$ is the height of the fluid at time t and at the position $x \in [0,L]$,
- $v(t,x)$ is the horizontal water velocity of the fluid *in a referential attached to the tank* at time t and at the position $x \in [0,L]$ (in the shallow water model, all the points on the same vertical have the same horizontal velocity),
- $u(t)$ is the horizontal acceleration of the tank in the absolute referential,
- g is the gravity constant,
- s is the horizontal velocity of the tank,
- D is the horizontal displacement of the tank.

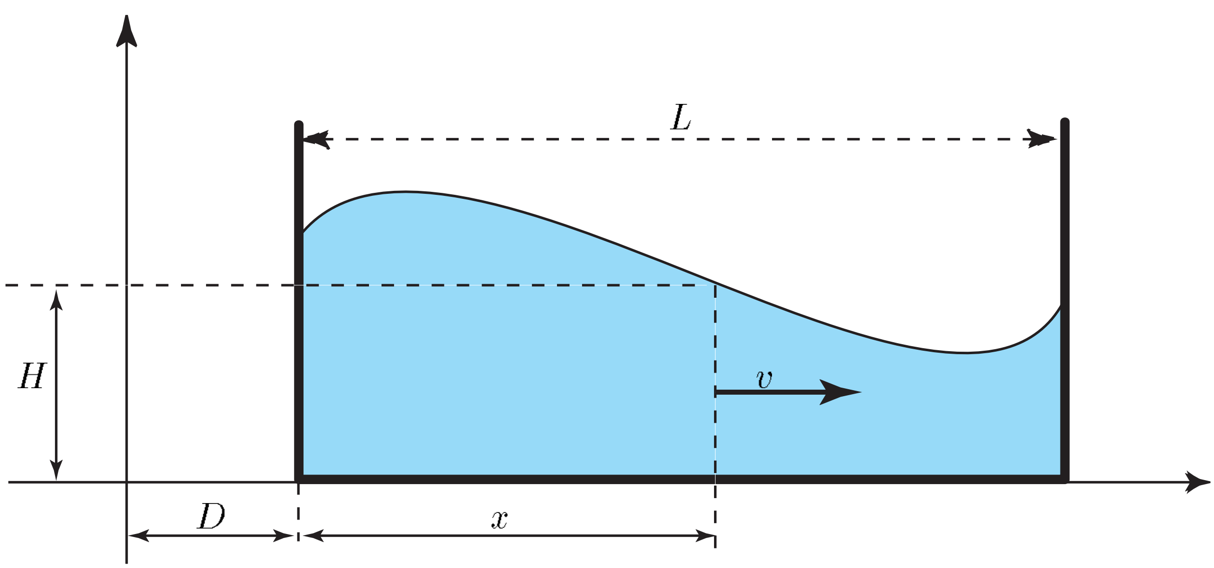

Figure 1: Fluid in the 1-D tank.

This is a control system where, at time t,

- the state is $Y(t) = (H(t,\cdot), v(t,\cdot), s(t), D(t))$,
- the control is $u(t) \in \mathbb{R}$.

2.5 A Schrödinger equation

Let $I = (-1, 1)$ and let $T > 0$. We consider the Schrödinger control system

$$\psi_t = i\psi_{xx} + iu(t)x\psi,\ (t,x) \in (0,T) \times I, \tag{2.12}$$

$$\psi(t,-1) = \psi(t,1) = 0,\ t \in (0,T), \tag{2.13}$$

$$\dot{S}(t) = u(t),\ \dot{D}(t) = S(t),\ t \in (0,T). \tag{2.14}$$

This is a control system, where, at time $t \in [0,T]$,

- the state is $(\psi(t,\cdot), S(t), D(t)) \in L^2(I;\mathbb{C}) \times \mathbb{R} \times \mathbb{R}$ with $\int_I |\psi(t,x)|^2 dx = 1$,
- the control is $u(t) \in \mathbb{R}$.

This system has been introduced by Pierre Rouchon in [118]. It models a nonrelativistic charged particle in a 1-D moving infinite square potential well. At time t, $\psi(t,\cdot)$ is the wave function of the particle in a frame attached to the potential well, $S(t)$ is the speed of the potential well and $D(t)$ is the displacement of the potential well. The control $u(t)$ is the acceleration of the potential well at time t. (For other related control models in quantum chemistry, let us mention the paper [96] by Claude Le Bris and the references therein.)

2.6 Euler equations of incompressible fluids

Let us introduce some notations. Let $l \in \{2,3\}$ and let Ω be a bounded nonempty connected open subset of $\mathbb{R}^l$ of class C^∞. Let Γ_0 be a nonempty open subset of $\Gamma := \partial\Omega$. The set Γ_0 is the part of the boundary Γ on which the control acts. The fluid that we consider is incompressible, so that the velocity field y satisfies

$$\mathrm{div}\ y = 0.$$

On the part of the boundary $\Gamma \setminus \Gamma_0$ where there is no control, the fluid does not cross the boundary: it satisfies

$$y \cdot n = 0 \text{ on } \Gamma \setminus \Gamma_0, \tag{2.15}$$

where n denotes the outward unit normal vector field on Γ. The control system of inviscid incompressible fluids is

$$\frac{\partial y}{\partial t} + (y \cdot \nabla)y + \nabla p = 0 \text{ in } (0,T) \times \Omega, \tag{2.16}$$

$$\mathrm{div}\ y = 0 \text{ in } (0,T) \times \Omega, \tag{2.17}$$

$$y(t,x) \cdot n(x) = 0,\ \forall (t,x) \in (0,T) \times (\Gamma \setminus \Gamma_0). \tag{2.18}$$

In (2.16) and throughout the whole article, for $A : \Omega \to \mathbb{R}^l$ and $B : \Omega \to \mathbb{R}^l$, $(A \cdot \nabla)B : \Omega \to \mathbb{R}^l$ is defined by

$$((A \cdot \nabla)B)^k := \sum_{j=1}^{l} A^j \frac{\partial B^k}{\partial x_j},\ \forall k \in \{1, \ldots, l\}.$$

In the control system (2.16)-(2.17)-(2.18) the state at time $t \in [0,T]$ is $y(t,\cdot)$.

Remark 1 *In this formulation of the control system associated to the Euler equations, the control does not appear explicitly. One can take, for example, $y \cdot n$ on Γ with $\int_\Gamma y \cdot nds = 0$ and*

1. *If $l = 2$, curl y on Γ for the incoming flow (i.e. at the points $(t,x) \in [0,T] \times \Gamma$ such that $y(t,\cdot) \cdot n(x) < 0$)*
2. *If $l = 3$, the tangential component of curl y on Γ for the incoming flow.*

2.7 Navier-Stokes of incompressible fluids

In this section the incompressible is now viscous. Equation (2.16) is replaced by

$$\frac{\partial y}{\partial t} - \nu \Delta y + (y \cdot \nabla) y + \nabla p = 0 \text{ in } (0,T) \times \Omega, \tag{2.19}$$

where $\nu > 0$ is the viscosity of the fluid (a positive real number independent of y: it depends only on the considered incompressible fluid). Since (2.19) contains a spatial second order partial differential term (namely the Laplacian Δy of y), the boundary condition (2.18) is no longer sufficient. One add a "wall law". Two wall laws are classical

- The Stokes noslip boundary condition:

  $$y = 0 \text{ on } (0,T) \times (\Gamma \setminus \Gamma_0), \tag{2.20}$$

  which implies (2.18).

- The Navier slip boundary condition: Besides (2.18), one also requires

  $$\overline{\sigma} y \cdot \tau + (1 - \overline{\sigma}) \sum_{i=1,j=1}^{i=l,j=l} n^i \left(\frac{\partial y^i}{\partial x^j} + \frac{\partial y^j}{\partial x^i} \right) \tau^j = 0 \text{ on } (0,T) \times (\Gamma \setminus \Gamma_0), \forall \tau \in \mathcal{T}\Gamma, \tag{2.21}$$

  where $\overline{\sigma}$ is a constant in $[0,1)$. In (2.21), $n = (n^1, \dots, n^l)$, $\tau = (\tau^1, \dots, \tau^l)$ and $\mathcal{T}\Gamma$ is the set of tangent vector fields on the boundary Γ. Note that the Stokes boundary condition (2.20) corresponds to the case $\overline{\sigma} = 1$

For the control, one can simply take $u(t,x) :== y(t,x)$ for $(t,x) \in (0,T) \times (\Gamma \setminus \Gamma_0)$. Of course, due to the smoothing effects of the Navier-Stokes equations, one cannot expect to move from a given state y^0 to another given state y^1 unless severe restrictions on the smoothness of y^1, restrictions which are moreover not very explicit. In that case the good notion of controllability is the following: Given a state y^0 and a solution $(\hat{y}, \hat{p}) : [0,T] \times \Omega \to \mathbb{R}^l \times \mathbb{R}$ of the control system, does there exists a control $u : [0,T] \times \Gamma_0 \to \mathbb{R}$ such that the solution $(y,p) : [0,T] \times \Omega \to \mathbb{R}^l \times \mathbb{R}$ of the Navier-Stokes control system such that $y(0,\cdot) = y^0$ satisfies $y(T,\cdot) = \hat{y}(T,\cdot)$?

3 A general framework for control systems modeled by linear PDE's

3.1 The framework

For two normed linear spaces H_1 and H_2, we denote by $\mathcal{L}(H_1; H_2)$ the set of continuous linear maps from H_1 into H_2 and denote by $\| \cdot \|_{\mathcal{L}(H_1;H_2)}$ the usual norm in this space.

Let H and U be two Hilbert spaces. Just to simplify the notations, these Hilbert spaces are assumed, in this section, to be real Hilbert spaces (the case of complex Hilbert spaces follows directly from the case of real Hilbert spaces). The space H is the state space and the space U is the control space. We denote by $(\cdot,\cdot)_H$ the scalar product in H, by $(\cdot,\cdot)_U$ the scalar product in U, by $\|\cdot\|_H$ the norm in H and by $\|\cdot\|_U$ the norm in U.

Let $S(t)$, $t\in[0,+\infty)$, be a strongly continuous semigroup of continuous linear operators on H. Let A be the infinitesimal generator of the semigroup $S(t)$, $t\in[0,+\infty)$. As usual, we denote by $S(t)^*$ the adjoint of $S(t)$. Then $S(t)^*$, $t\in[0,+\infty)$, is a strongly continuous semigroup of continuous linear operators and the infinitesimal generator of this semigroup is the adjoint A^* of A. The domain $D(A^*)$ is equipped with the usual graph norm $\|\cdot\|_{D(A^*)}$ of the unbounded operator A^*:

$$\|z\|_{D(A^*)} := \|z\|_H + \|A^*z\|_H,\ \forall z\in D(A^*).$$

This norm is associated to the scalar product in $D(A^*)$ defined by

$$(z_1,z_2)_{D(A^*)} := (z_1,z_2)_H + (A^*z_1,A^*z_2)_H,\ \forall (z_1,z_2)\in D(A^*)^2.$$

With this scalar product, $D(A^*)$ is a Hilbert space. Let $D(A^*)'$ be the dual of $D(A^*)$ with the pivot space H. In particular,

$$D(A^*)\subset H\subset D(A^*)'.$$

Let

$$B\in\mathcal{L}(U,D(A^*)'). \tag{3.1}$$

In other words, B is a linear map from U into the set of linear functions from $D(A^*)$ into $\mathbb{R}$ such that, for some $C>0$,

$$|(Bu)z|\leqslant C\|u\|_U\|z\|_{D(A^*)},\ \forall u\in U,\ \forall z\in D(A^*).$$

We also assume the following regularity property (also called admissibility condition):

$$\forall T>0, \exists C_T>0 \text{ such that } \int_0^T \|B^*S(t)^*z\|_U^2 dt\leqslant C_T\|z\|_H^2,\ \forall z\in D(A^*). \tag{3.2}$$

In (3.2) and in the following, $B^*\in\mathcal{L}(D(A^*);U)$ is the adjoint of B. It follows from (3.2) that the operators

$$\big(z\in D(A^*)\big)\mapsto\big((t\mapsto B^*S(t)^*z)\in C^0([0,T];U)\big),$$
$$\big(z\in D(A^*)\big)\mapsto\big((t\mapsto B^*S(T-t)^*z)\in C^0([0,T];U)\big)$$

can be extended in a unique way as continuous linear maps from H into $L^2((0,T);U)$. We use the same symbols to denote these extensions.

Note that, using the fact that $S(t)^*$, $t\in[0,+\infty)$, is a strongly continuous semigroup of continuous linear operators on H, it is not hard to check that (3.2) is equivalent to

$$\exists T>0, \exists C_T>0 \text{ such that } \int_0^T \|B^*S(t)^*z\|_U^2 dt\leqslant C_T\|z\|_H^2,\ \forall z\in D(A^*).$$

The control system we consider here is

$$\dot y = Ay+Bu,\ t\in(0,T), \tag{3.3}$$

where, at time t, the control is $u(t) \in U$ and the state is $y(t) \in H$.

Let $T > 0$, $y^0 \in H$ and $u \in L^2((0,T);U)$. We are interested in the Cauchy problem

$$\dot{y} = Ay + Bu(t), \; t \in (0,T), \tag{3.4}$$
$$y(0) = y^0. \tag{3.5}$$

We first give the definition of a solution to (3.4)-(3.5). Let us first motivate our definition Let $\tau \in [0,T]$ and $\varphi : [0,\tau] \to H$. We take the scalar product in H of (3.4) with φ and integrate on $[0,\tau]$. At least formally, we get, using an integration by parts together with (3.5),

$$(y(\tau),\varphi(\tau))_H - (y^0,\varphi(0))_H - \int_0^\tau (y(t),\dot{\varphi}(t) + A^*\varphi(t))_H dt = \int_0^\tau (u(t),B^*\varphi(t))_U dt.$$

Taking $\varphi(t) = S(\tau - t)^* z^\tau$, for every given $z^\tau \in H$, we have formally $\dot{\varphi}(t) + A^*\phi(t) = 0$, which leads to the following definition.

Definition 2 *Let $T > 0$, $y^0 \in H$ and $u \in L^2((0,T);U)$. A solution of the Cauchy problem* (3.4)-(3.5) *is a function $y \in C^0([0,T];H)$ such that*

$$(y(\tau),z^\tau)_H - (y^0,S(\tau)^* z^\tau)_H = \int_0^\tau (u(t),B^*S(\tau-t)^*z^\tau)_U dt, \; \forall \tau \in [0,T], \; \forall z^\tau \in H. \tag{3.6}$$

Note that, by the regularity property (3.2), the right hand side of (3.6) is well defined (see page 7).

With this definition one has the following theorem.

Theorem 3 *Let $T > 0$. Then, for every $y^0 \in H$ and for every $u \in L^2((0,T);U)$, the Cauchy problem* (3.4)-(3.5) *has a unique solution y. Moreover, there exists $C = C(T) > 0$, independent of $y^0 \in H$ and $u \in L^2((0,T);U)$, such that*

$$\|y(\tau)\|_H \leqslant C(\|y^0\|_H + \|u\|_{L^2((0,T);U)}), \; \forall \tau \in [0,T]. \tag{3.7}$$

For a proof of this theorem, see, for example [38, pages 53–54].

3.2 Examples

In this section, we show how to put the linear control systems of Section 2 in the above general framework.

3.2.1 A transport equation

We return to the control system (2.1)-(2.2). Let $L > 0$. The linear control system we study is

$$y_t + y_x = 0, \; t \in (0,T), \; x \in (0,L), \tag{3.8}$$
$$y(t,0) = u(t), \; t \in (0,T), \tag{3.9}$$

where, at time t, the control is $u(t) \in \mathbb{R}$ and the state is $y(t,\cdot) : (0,L) \to \mathbb{R}$.

For the Hilbert space H, we take $H := L^2(0,L)$. For the operator $A : D(A) \to H$ we take

$$D(A) := \{f \in H^1(0,L); \; f(0) = 0\},$$
$$Af := -f_x, \; \forall f \in D(A).$$

Then $D(A)$ is dense in $L^2(0,L)$, A is closed. Moreover

$$(Af,f)_{L^2(0,L)} = -\frac{1}{2} f_x(L)^2, \ \forall f \in D(A),$$

showing that A is dissipative. The adjoint A^* of A is defined by

$$D(A^*) := \{f \in H^1(0,L);\ f(L) = 0\},$$
$$A^* f := f_x, \ \forall f \in D(A^*).$$

As the operator A, the operator A^* is also dissipative. Hence, by the Lumer-Phillips theorem, the operator A is the infinitesimal generator of a strongly continuous semigroup $S(t)$, $t \in [0,+\infty)$, of continuous linear operators on H.

For the Hilbert space U, we take $U := \mathbb{R}$. The operator $B : \mathbb{R} \to D(A^*)'$ is defined by

$$(Bu)z = uz(0), \ \forall u \in \mathbb{R}, \forall z \in D(A^*). \tag{3.10}$$

Note that $B^* : D(A^*) \to \mathbb{R}$ is defined by

$$B^* z = z(0), \ \forall z \in D(A^*).$$

Let us deal with the regularity property (3.2). Let $z^0 \in D(A^*)$. Let

$$z \in C^0([0,T]; D(A^*)) \cap C^1([0,T]; L^2(0,L))$$

be defined by $z(t,\cdot) = S(t)^* z^0$. Inequality (3.2) is equivalent to

$$\int_0^T z(t,0)^2 dt \leqslant C_T \int_0^L z^0(x)^2 dx. \tag{3.11}$$

Let us prove this inequality for $C_T := 1$. We have

$$z_t = z_x, \ t \in (0,T), \ x \in (0,L), \tag{3.12}$$
$$z(t,L) = 0, \ t \in (0,T), \tag{3.13}$$
$$z(0,x) = z^0(x), \ x \in (0,L). \tag{3.14}$$

We multiply (3.12) by z and integrate on $[0,T] \times [0,L]$. Using (3.13), (3.14) and integrations by parts, we get

$$\int_0^T z(t,0)^2 dt = \int_0^L z^0(x)^2 dx - \int_0^L z(T,x)^2 dx \leqslant \int_0^L z^0(x)^2 dx, \tag{3.15}$$

which shows that (3.11) holds for $C_T := 1$.

In fact, as one can easily check, the solution to the following Cauchy problem (in the senses of Definition 2)

$$y_t + y_x = 0, \ t \in (0,T), \ x \in (0,L), \tag{3.16}$$
$$y(t,0) = u(t), \ t \in (0,T), \tag{3.17}$$
$$y(0,x) = y^0(x), \ x \in (0,L), \tag{3.18}$$

where $T > 0$, $y^0 \in L^2(0,L)$ and $u \in L^2(0,T)$ are given data, is

$$y(t,x) = y^0(x-t), \ \forall (t,x) \in [0,T] \times (0,L) \text{ such that } t \leqslant x, \tag{3.19}$$
$$y(t,x) = u(t-x), \ \forall (t,x) \in [0,T] \times (0,L) \text{ such that } t > x. \tag{3.20}$$

3.2.2 A linear Korteweg-de Vries equation

We return to the linear Korteweg-de Vries equation already mentioned in Section 2. Let $L > 0$. The linear control system we study is

$$y_t + y_x + y_{xxx} = 0,\ t \in (0,T),\ x \in (0,L), \tag{3.21}$$

$$y(t,0) = y(t,L) = 0,\ y_x(t,L) = u(t),\ t \in (0,T), \tag{3.22}$$

where, at time t, the control is $u(t) \in \mathbb{R}$ and the state is $y(t,\cdot) : (0,L) \to \mathbb{R}$.

For the Hilbert space H, we take $H = L^2(0,L)$. For the operator $A : D(A) \to H$, we take

$$D(A) := \{f \in H^3(0,L);\ f(0) = f(L) = f_x(L) = 0\},$$
$$Af := -f_x - f_{xxx},\ \forall f \in D(A).$$

Then $D(A)$ is dense in $L^2(0,L)$, A is closed. Simple integrations by parts give

$$(Af,f)_{L^2(0,L)} = -\frac{1}{2} f_x(0)^2,\ \forall f \in L^2(0,L),$$

which shows that A is dissipative. The adjoint A^* of A is defined by

$$D(A^*) := \{f \in H^3(0,L);\ f(0) = f(L) = f_x(0) = 0\},$$
$$A^* f := f_x + f_{xxx},\ \forall f \in D(A^*).$$

As A, the operator A^* is also dissipative. Hence, by the Lumer-Phillips theorem, the operator A is the infinitesimal generator of a strongly continuous semigroup $S(t)$, $t \in [0,+\infty)$, of continuous linear operators on $L^2(0,L)$.

For the Hilbert space U, we take $U := \mathbb{R}$. The operator $B : \mathbb{R} \to D(A^*)'$ is defined by

$$(Bu)z = u z_x(L),\ \forall u \in \mathbb{R}, \forall z \in D(A^*). \tag{3.23}$$

Note that $B^* : D(A^*) \to \mathbb{R}$ is defined by

$$B^* z = z_x(L),\ \forall z \in D(A^*).$$

Let us check the regularity property (3.2). Let $z^0 \in D(A^*)$. Let

$$z \in C^0([0,+\infty); D(A^*)) \cap C^1([0,+\infty); L^2(0,L))$$

be defined by

$$z(t,\cdot) = S(t)^* z^0. \tag{3.24}$$

The regularity property (3.2) is equivalent to

$$\int_0^T |z_x(t,L)|^2 dt \leqslant C_T \int_0^L |z^0(x)|^2 dx. \tag{3.25}$$

From (3.24), one has

$$z_t - z_x - z_{xxx} = 0 \text{ in } C^0([0,+\infty); L^2(0,L)), \tag{3.26}$$

$$z(t,0) = z_x(t,0) = z(t,L) = 0,\ t \in [0,+\infty), \tag{3.27}$$

$$z(0,x) = z^0(x),\ x \in [0,L]. \tag{3.28}$$

We multiply (3.26) by z and integrate on $(0,T) \times (0,L)$. Using (3.27), (3.28) and simple integrations by parts one gets

$$\int_0^T |z_x(t,L)|^2 dt = \int_0^L |z^0(x)|^2 dx - \int_0^L |z(T,x)|^2 dx \leqslant \int_0^L |z^0(x)|^2 dx, \tag{3.29}$$

which shows that (3.25) holds with $C_T := 1$.

3.2.3 A heat equation

We return to the linear heat equation already considered in Section 2. Let Ω be a non empty open subset of $\mathbb{R}^l$ and let ω be a non empty open subset of Ω. The linear heat equation considered in this section is

$$y_t - \Delta y = u(t,x),\ t \in (0,T),\ x \in \Omega, \tag{3.30}$$
$$y = 0 \text{ on } (0,T) \times \partial\Omega, \tag{3.31}$$

where, at time $t \in [0,T]$, the state is $y(t,\cdot) \in L^2(\Omega)$ and the control is $u(t,\cdot) \in L^2(\Omega)$. We require that

$$u(\cdot,x) = 0,\ x \in \Omega \setminus \omega. \tag{3.32}$$

One can put this linear control system in the general framework detailed in Section 3 in the following way. One chooses

$$H := L^2(\Omega),$$

equipped with the usual scalar product. Let $A : D(A) \subset H \to H$ be the linear operator defined by

$$D(A) := \left\{ y \in H_0^1(\Omega);\ \Delta y \in L^2(\Omega) \right\},$$
$$Ay := \Delta y \in H.$$

Note that, if Ω is smooth enough (for example of class C^2), then

$$D(A) = H_0^1(\Omega) \cap H^2(\Omega). \tag{3.33}$$

However, without any regularity assumption on Ω, (3.33) is wrong in general (see in particular [66, Theorem 2.4.3, page 57] by Pierre Grisvard). One easily checks that

$$D(A) \text{ is dense in } L^2(\Omega), \tag{3.34}$$
$$A \text{ is closed.} \tag{3.35}$$

Moreover,

$$(Ay, y)_H = -\int_\Omega |\nabla y|^2 dx,\ \forall y \in D(A). \tag{3.36}$$

Let A^* be the adjoint of A. One easily checks that

$$A^* = A. \tag{3.37}$$

From the Lumer-Phillips theorem, (3.34), (3.35), (3.36) and (3.37), A is the infinitesimal generator of a strongly continuous semigroup of linear contractions $S(t)$, $t \in [0,+\infty)$, on H. For the Hilbert space U we take $L^2(\omega)$. The linear map $B \in \mathcal{L}(U; D(A^*)')$ is the map which is defined by

$$(Bu)\varphi = \int_\omega u\varphi dx.$$

Note that $B \in \mathcal{L}(U; H)$. Hence the regularity property (3.2) is automatically satisfied.

4 Controllability of linear control systems

4.1 Different types of controllability

In this section we are interested in the controllability of the control system (3.3). In contrast to the case of linear finite-dimensional control systems, many types of controllability are possible and interesting. We define here three types of controllability.

Definition 4 *Let $T > 0$. The control system* (3.3) *is exactly controllable in time T if, for every $y^0 \in H$ and for every $y^1 \in H$, there exists $u \in L^2((0,T);U)$ such that the solution y of the Cauchy problem*

$$\dot{y} = Ay + Bu(t),\ y(0) = y^0, \tag{4.1}$$

satisfies $y(T) = y^1$.

Definition 5 *Let $T > 0$. The control system* (3.3) *is null controllable in time T if, for every $y^0 \in H$ and for every $\tilde{y}^0 \in H$, there exists $u \in L^2((0,T);U)$ such that the solution of the Cauchy problem* (4.1) *satisfies $y(T) = S(T)\tilde{y}^0$.*

Let us point out that, by linearity, we get an equivalent definition of "null controllable in time T" if, in Definition 5, one assumes that $\tilde{y}^0 = 0$. This explains the usual terminology "null controllability".

Definition 6 *Let $T > 0$. The control system* (3.3) *is approximately controllable in time T if, for every $y^0 \in H$, for every $y^1 \in H$, and for every $\varepsilon > 0$, there exists $u \in L^2((0,T);U)$ such that the solution y of the Cauchy problem* (4.1) *satisfies $\|y(T) - y^1\|_H \leqslant \varepsilon$.*

Clearly

$$\text{(exact controllability)} \Rightarrow \text{(null controllability and approximate controllability)}.$$

The converse is false in general (see, for example, the control system (3.30)-(3.31) below). However, the converse holds if S is a strongly continuous group of linear operators. More precisely, one has the following theorem.

Theorem 7 *Assume that $S(t)$, $t \in \mathbb{R}$, is a strongly continuous group of linear operators. Let $T > 0$. Assume that the control system* (3.3) *is null controllable in time T. Then the control system* (3.3) *is exactly controllable in time T.*

Proof of Theorem 7. Let $y^0 \in H$ and $y^1 \in H$. From the null controllability assumption applied to the initial data $y^0 - S(-T)y^1$, there exists $u \in L^2((0,T);U)$ such that the solution $\tilde{y}$ of the Cauchy problem

$$\dot{\tilde{y}} = A\tilde{y} + Bu(t),\ \tilde{y}(0) = y^0 - S(-T)y^1,$$

satisfies

$$\tilde{y}(T) = 0. \tag{4.2}$$

One easily sees that the solution y of the Cauchy problem

$$\dot{y} = Ay + Bu(t),\ y(0) = y^0,$$

is given by

$$y(t) = \tilde{y}(t) + S(t-T)y^1, \ \forall t \in [0,T]. \tag{4.3}$$

In particular, from (4.2) and (4.3),

$$y(T) = y^1.$$

This concludes the proof of Theorem 7. ■

4.2 Methods to study controllability

Rouhgly speaking there are essentially two types of methods to study the controllability of linear PDE, namely *direct methods* and *duality methods*.

4.2.1 Direct methods

Among these methods, let us mention in particular

- The extension method. See, for example, [121] and [122, Proof of Theorem 5.3, pages 688–690] by David Russell, [105] by Walter Littman and [38, Section 2.1.2.2, pages 30-34].
- Moment theory. See, for example, [89] by Werner Krabs, [88] by Vilmos Komornik and Paola Loreti, [7] by Sergei Avdonin and Sergei Ivanov, and [38, Section 2.6, pages 95-99]. We give an example of an application of the moment theory for a Schrödinger equation in Section 4.3.2.
- Flatness. This approach has been initiated in the framework of control theory in finite dimension by Michel Fliess, Jean Lévine, Pierre Rouchon and Philippe Martin in [56]. For applications of this method for the control of linear PDE, see, in particular, [108] by Hugues Mounier, Joachim Rudolph, Michel Fliess and Pierre Rouchon,, [91] by Béatrice Laroche, Philippe Martin and Pierre Rouchon, [110] by Nicolas Petit and Pierre Rouchon, as well as the article by Pierre Rouchon in Scholarpedia.

4.2.2 Duality methods

Let us now introduce some "optimal control maps". Let us first deal with the case where the control system (3.3) is exactly controllable in time T. Then, for every y^1, the set $U^T(y^1)$ of $u \in L^2((0,T);U)$ such that

$$(\dot{y} = Ay + Bu(t),\ y(0) = 0) \Rightarrow (y(T) = y^1)$$

is nonempty. Clearly the set $U^T(y^1)$ is a closed affine subspace of $L^2((0,T);U)$. Let us denote by $\mathcal{U}^T(y^1)$ the projection of 0 on this closed affine subspace, i.e., the element of $U^T(y^1)$ of the smallest $L^2((0,T);U)$-norm. Then it is not hard to see that the map

$$\begin{array}{rccl} \mathcal{U}^T: & H & \rightarrow & L^2((0,T);U) \\ & y^1 & \mapsto & \mathcal{U}^T(y^1) \end{array}$$

is a linear map. Moreover, using the closed graph theorem (see, for example, [119, Theorem 2.15, page 50]) one readily checks that this linear map is continuous.

Let us now deal with the case where the control system (3.3) is null controllable in time T. Then, for every y^0, the set $U_T(y^0)$ of $u \in L^2((0,T);U)$ such that

$$(\dot{y} = Ay + Bu(t),\, y(0) = y^0) \Rightarrow (y(T) = 0)$$

is nonempty. Clearly the set $U_T(y^0)$ is a closed affine subspace of $L^2((0,T);U)$. Let us denote by $\mathcal{U}_T(y^0)$ the projection of 0 on this closed affine subspace, i.e., the element of $U_T(y^0)$ of the smallest $L^2((0,T);U)$-norm. Then, again, it is not hard to see that the map

$$\begin{array}{rccc} \mathcal{U}_T: & H & \to & L^2((0,T);U) \\ & y^0 & \mapsto & \mathcal{U}_T(y^0) \end{array}$$

is a continuous linear map.

The main results of this section are the following ones.

Theorem 8 *Let $T > 0$. The control system* (3.3) *is exactly controllable in time T if and only if there exists $c > 0$ such that*

$$\int_0^T \|B^* S(t)^* z\|_U^2 dt \geqslant c\|z\|_H^2,\ \forall z \in D(A^*). \tag{4.4}$$

Moreover, if such a $c > 0$ exists and if c^T is the maximum of the set of $c > 0$ such that (4.4) *holds, one has*

$$\|\mathcal{U}^T\|_{\mathcal{L}(H;L^2((0,T);U))} = \frac{1}{\sqrt{c^T}}. \tag{4.5}$$

Theorem 9 *The control system* (3.3) *is approximately controllable in time T if and only if, for every $z \in H$,*

$$(B^* S(\cdot)^* z = 0 \textit{ in } L^2((0,T);U)) \Rightarrow (z = 0). \tag{4.6}$$

Theorem 10 *Let $T > 0$. The control system* (3.3) *is null controllable in time T if and only if there exists $c > 0$ such that*

$$\int_0^T \|B^* S(t)^* z\|_U^2 dt \geqslant c\|S(T)^* z\|_H^2,\ \forall z \in D(A^*). \tag{4.7}$$

Moreover, if such a $c > 0$ exists and if c_T is the maximum of the set of $c > 0$ such that (4.7) *holds, then*

$$\|\mathcal{U}_T\|_{\mathcal{L}(H;L^2((0,T);U))} = \frac{1}{\sqrt{c_T}}. \tag{4.8}$$

Theorem 11 *Assume that, for every $T > 0$, the control system* (3.3) *is null controllable in time T. Then, for every $T > 0$, the control system* (3.3) *is approximately controllable in time T.*

For a proof of these theorems, see, for example [38, Section 2.3.2]. Inequalities (4.4) and (4.7) are usually called observability inequalities for the abstract linear control system $\dot{y} = Ay + Bu$. The difficulty is to prove them! For this purpose, there are many methods available (but still many open problems). Among these methods, let us mention in particular

- Multiplier methods. See in particular, [103] by Jacques-Louis Lions, [86] by Vilmos Komornik, [139] by Enrique Zuazua. For a simple example of this method, see Section 4.3.1.
- Microlocal analysis. See in particular [12] by Claude Bardos, Gilles Lebeau and Jeffrey Rauch and the appendix 2 of the book [103].

Remark 12 *In contrast to Theorem 11, note that, for a given $T > 0$, the null controllability in time T does not imply the approximate controllability in time T. For example, let $L > 0$ and let us take $H := L^2(0, L)$ and $U := \{0\}$. We consider the linear control system*

$$y_t + y_x = 0,\ t \in (0, T),\ x \in (0, L), \tag{4.9}$$

$$y(t, 0) = u(t) = 0,\ t \in (0, T). \tag{4.10}$$

In Section 4.3.1, we shall see how to put this control system in the abstract framework $\dot{y} = Ay + Bu$. It follows from (3.20)*, that, whatever $y^0 \in L^2(0, L)$ is, the solution to the Cauchy problem (see Definition 2)*

$$y_t + y_x = 0,\ t \in (0, T),\ x \in (0, L),$$
$$y(t, 0) = u(t) = 0,\ t \in (0, T),$$
$$y(0, x) = y^0(x),\ x \in (0, L),$$

satisfies

$$y(T, \cdot) = 0,\ \text{if } T \geqslant L.$$

In particular, if $T \geqslant L$, the linear control system (4.9)-(4.10) *is null controllable but is not approximately controllable.*

4.3 Examples

4.3.1 A transport equation

We return to the transport control system (3.8)-(3.9). This example is pedagogically interesting since one can give explicitly the solution to the Cauchy problem (3.16)-(3.17)-(3.18) where $T > 0$, $y^0 \in L^2(0, L)$ and $u \in L^2(0, T)$ are given. This solution is given by (3.19)-(3.20). From this explicit solution one readily gets

Proposition 13 *The control system (3.8)-(3.9) is*

- *exactly controllable in time T if and only if $T \geqslant L$,*
- *null controllable in time T if and only if $T \geqslant L$,*
- *approximately controllable in time T if and only if $T \geqslant L$.*

Let us show how to use the multiplier method in order to prove that if

$$T > L \tag{4.11}$$

then the control system (3.8)-(3.9) is exactly controllable. By Theorem 8, the exact controllability in time T is equivalent to the existence of $c > 0$ such that

$$\int_0^T z(t, 0)^2 dt \geqslant c \int_0^L z^0(x)^2 dx, \tag{4.12}$$

where $z : [0,T] \times [0,L] \to \mathbb{R}$ is the solution of the Cauchy problem

$$z_t - z_x = 0, \tag{4.13}$$

$$z(t,L) = 0,\ t \in (0,T), \tag{4.14}$$

$$z(0,\cdot) = z^0. \tag{4.15}$$

Let us prove prove (4.12). With simple density arguments, we may assume that z is of class C^1. Let us multiply (4.13) by the *mutiplier* z and integrate the obtained equality on $[0,L]$. Using (4.14), one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^L |z(t,x)|^2 dx\right) = -|z(t,0)|^2. \tag{4.16}$$

Let us now multiply (4.13) by the *mutiplier* xz and integrate the obtained equality on $[0,L]$. Using (4.14), one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^L x|z(t,x)|^2 dx\right) = -\int_0^L |z(t,x)|^2 dx. \tag{4.17}$$

For $t \in [0,T]$, let $e(t) := \int_0^L |z(t,x)|^2 dx$. From (4.17), we have

$$\begin{aligned}\int_0^T e(t)dt &= -\int_0^L x|z(T,x)|^2 dx + \int_0^L x|z(0,x)|^2 dx \\ &\leqslant L\int_0^L |z(0,x)|^2 dx = Le(0).\end{aligned} \tag{4.18}$$

From (4.16), we get

$$e(t) = e(0) - \int_0^t |z(\tau,0)|^2 d\tau \geqslant e(0) - \int_0^T |z(\tau,0)|^2 d\tau. \tag{4.19}$$

From (4.15), (4.18) and (4.19), we get

$$(T-L)\|z^0\|^2_{L^2(0,L)} \leqslant T\int_0^T |z(\tau,0)|^2 d\tau, \tag{4.20}$$

which proves the observability inequality (4.12) with c given by

$$c := \sqrt{\frac{T-L}{T}}. \tag{4.21}$$

4.3.2 A linear Schrödinger equation

We return to the nonlinear Schrödinger control system consider in Section 2.5. For simplicity, we forget the variables S and D: the control system is simply (2.12)-(2.13). In this section how the moment methods can be used in order to prove the controllability of linearized control system around important trajectories of the control system (2.12)-(2.13).

Let I be the open interval $(-1,1)$. For $\gamma \in \mathbb{R}$, let $A_\gamma : D(A_\gamma) \subset L^2(I;\mathbb{C}) \to L^2(I;\mathbb{C})$ be the operator defined on

$$D(A_\gamma) := H^2(I;\mathbb{C}) \cap H_0^1(I;\mathbb{C}) \tag{4.22}$$

by

$$A_\gamma\varphi := -\varphi_{xx} - \gamma x\varphi. \tag{4.23}$$

In (4.22), as usual,

$$H^1_0(I;\mathbb{C}) := \{\varphi \in H^1((0,L);\mathbb{C});\ \varphi(0) = \varphi(L) = 0\}.$$

We denote by $\langle \cdot, \cdot \rangle$ the usual Hermitian scalar product in the Hilbert space $L^2(I;\mathbb{C})$:

$$\langle \varphi, \psi \rangle := \int_I \varphi(x)\overline{\psi(x)}dx, \tag{4.24}$$

where $\overline{z}$ denotes the complex conjugate of the complex number z. Note that

$$D(A_\gamma) \text{ is dense in } L^2(I;\mathbb{C}), \tag{4.25}$$

$$A_\gamma \text{ is closed}, \tag{4.26}$$

$$A^*_\gamma = A_\gamma,\ \text{(i.e., } A_\gamma \text{ is self-adjoint)}, \tag{4.27}$$

$$A_\gamma \text{ has compact resolvent}. \tag{4.28}$$

Let us recall that (4.28) means that there exists a real α in the resolvent set of A_γ such that the operator $(\alpha\mathrm{Id} - A_\gamma)^{-1}$ is compact from $L^2(I;\mathbb{C})$ into $L^2(I;\mathbb{C})$, where Id denotes the identity map on H (see, for example, [84, pages 36 and 187]). Then (see, for example, [84, page 277]), the Hilbert space $L^2(I;\mathbb{C})$ has a complete orthonormal system $(\varphi_{k,\gamma})_{k\in\mathbb{N}\setminus\{0\}}$ of eigenfunctions for the operator A_γ:

$$A_\gamma\varphi_{k,\gamma} = \lambda_{k,\gamma}\varphi_{k,\gamma},$$

where $(\lambda_{k,\gamma})_{k\in\mathbb{N}\setminus\{0\}}$ is an increasing sequence of positive real numbers. Let $\mathbb{S}$ be the unit sphere of $L^2(I;\mathbb{C})$:

$$\mathbb{S} := \{\phi \in L^2(I;\mathbb{C});\ \int_I |\phi(x)|^2dx = 1\} \tag{4.29}$$

and, for $\phi \in \mathbb{S}$, let $T_{\mathbb{S}}\phi$ be the tangent space to $\mathbb{S}$ at ϕ:

$$T_{\mathbb{S}}\phi := \{\Phi \in L^2(I;\mathbb{C});\ \Re\langle \Phi, \phi\rangle = 0\}, \tag{4.30}$$

where, as usual, $\Re z$ denotes the real part of the complex number z. Let

$$\psi_{1,\gamma}(t,x) := e^{-i\lambda_{1,\gamma}t}\varphi_{1,\gamma}(x),\ (t,x) \in (0,T) \times I. \tag{4.31}$$

Note that

$$\psi_{1,\gamma t} = i\psi_{1,\gamma xx} + i\gamma x\psi_{1,\gamma},\ t \in (0,T),\ x \in I,$$
$$\psi_{1,\gamma}(t,-1) = \psi_{1,\gamma}(t,1) = 0,\ t \in (0,T),$$
$$\int_I |\psi_{1,\gamma}(t,x)|^2dx = 1,\ t \in (0,T).$$

Hence $(\psi, u) = (\psi_{1,\gamma}, \gamma)$ is a trajectory of the control system (2.12)-(2.13). The linearized control system around this trajectory is the following linear control system:

$$\Psi_t = i\Psi_{xx} + i\gamma x\Psi + iux\psi_{1,\gamma},\ (t,x) \in (0,T) \times I, \tag{4.32}$$

$$\Psi(t,-1) = \Psi(t,1) = 0,\ t \in (0,T). \tag{4.33}$$

This is a control system where, at time $t \in [0,T]$,

- The state is $\Psi(t,\cdot) \in L^2(I;\mathbb{C})$ with $\Psi(t,\cdot) \in T_{\mathbb{S}}(\psi_{1,\gamma}(t,\cdot))$.
- The control is $u(t) \in \mathbb{R}$.

Let us first deal with the Cauchy problem

$$\Psi_t = i\Psi_{xx} + i\gamma x\Psi + iux\psi_{1,\gamma},\ (t,x) \in (0,T) \times I, \tag{4.34}$$

$$\Psi(t,-1) = \Psi(t,1) = 0,\ t \in (0,T), \tag{4.35}$$

$$\Psi(0,x) = \Psi^0(x), \tag{4.36}$$

where $T > 0$, $u \in L^1(0,T)$ and $\Psi^0 \in L^2(I;\mathbb{C})$ are given. By (4.27),

$$(-iA_\gamma)^* = -(-iA_\gamma).$$

Therefore, by the Lumer-Phillips theorem $-iA_\gamma$ is the infinitesimal generator of a strongly continuous group of linear isometries on $L^2(I;\mathbb{C})$. We denote by $S_\gamma(t)$, $t \in \mathbb{R}$, this group.

Note that, since $\psi_{1,\gamma}$ depends on time, Section 3.1 cannot be applied. Our notion of solution to the Cauchy problem (4.34)-(4.35)-(4.36) is given in the following definition.

Definition 14 *Let $T > 0$, $u \in L^1(0,T)$ and $\Psi^0 \in L^2(I;\mathbb{C})$. A solution $\Psi : [0,T] \times I \to \mathbb{C}$ to the Cauchy problem* (4.34)-(4.35)-(4.36) *is the function $\Psi \in C^0([0,T];L^2(I;\mathbb{C}))$ defined by*

$$\Psi(t) = S_\gamma(t)\Psi^0 + \int_0^t S_\gamma(t-\tau)iu(\tau)x\psi_{1,\gamma}(\tau,\cdot)d\tau. \tag{4.37}$$

With this definition and standard arguments, one can show that the Cauchy problem (4.34)-(4.35)-(4.36) is well posed (see e.g. [38, Theorem A.7, page 375]).

Let us now study the controllability of the linear Schrödinger control system (4.32)-(4.33). Let

$$H^3_{(0)}(I;\mathbb{C}) := \{\psi \in H^3(I;\mathbb{C});\ \psi(-1) = \psi_{xx}(-1) = \psi(1) = \psi_{xx}(1) = 0\}. \tag{4.38}$$

The goal of this section is to prove the following controllability result due to Karine Beauchard [13, Theorem 5, page 862].

Theorem 15 *There exists $\gamma_0 > 0$ such that, for every $T > 0$, for every $\gamma \in (0,\gamma_0]$, for every $\Psi^0 \in T_{\mathbb{S}}\psi_{1,\gamma}(0,\cdot) \cap H^3_{(0)}(I;\mathbb{C})$ and for every $\Psi^1 \in T_{\mathbb{S}}\psi_{1,\gamma}(T,\cdot) \cap H^3_{(0)}(I;\mathbb{C})$, there exists $u \in L^2(0,T)$ such that the solution of the Cauchy problem*

$$\Psi_t = i\Psi_{xx} + i\gamma x\Psi + iu(t)x\psi_{1,\gamma},\ t \in (0,T),\ x \in I, \tag{4.39}$$

$$\Psi(t,-1) = \Psi(t,1) = 0,\ t \in (0,T), \tag{4.40}$$

$$\Psi(0,x) = \Psi^0(x),\ x \in I, \tag{4.41}$$

satisfies

$$\Psi(T,x) = \Psi^1(x),\ x \in I. \tag{4.42}$$

We are also going to see that the conclusion of Theorem 15 does not hold for $\gamma = 0$ (as already noted by Pierre Rouchon in [118]).

Proof of Theorem 15. Let $T > 0$,

$$\Psi^0 \in T_{\mathbb{S}}(\psi_{1,\gamma}(0,\cdot)) \text{ and } \Psi^1 \in T_{\mathbb{S}}(\psi_{1,\gamma}(T,\cdot)).$$

Let $u \in L^2(0,T)$. Let Ψ be the solution of the Cauchy problem (4.39)-(4.40)-(4.41). Let us decompose $\Psi(t,\cdot)$ in the complete orthonormal system $(\varphi_{k,\gamma})_{k\in\mathbb{N}\setminus\{0\}}$ of eigenfunctions for the operator A_γ:

$$\Psi(t,\cdot) = \sum_{k=1}^{\infty} y_k(t)\varphi_{k,\gamma}.$$

Taking the Hermitian product of (4.39) with $\varphi_{k,\gamma}$, one readily gets, using (4.40) and integrations by parts,

$$\dot{y}_k = -i\lambda_{k,\gamma} y_k + i b_{k,\gamma} u(t) e^{-i\lambda_{1,\gamma} t}, \tag{4.43}$$

with

$$b_{k,\gamma} := \langle \varphi_{k,\gamma}, x\varphi_{1,\gamma}\rangle \in \mathbb{R}. \tag{4.44}$$

Note that (4.41) is equivalent to

$$y_k(0) = \langle \Psi^0, \varphi_{k,\gamma}\rangle,\ \forall k \in \mathbb{N}\setminus\{0\}. \tag{4.45}$$

From (4.44) and (4.45) one gets

$$y_k(T) = e^{-i\lambda_{k,\gamma}T}(\langle \Psi^0, \varphi_{k,\gamma}\rangle + i b_{k,\gamma}\int_0^T u(t) e^{i(\lambda_{k,\gamma}-\lambda_{1,\gamma})t} dt). \tag{4.46}$$

By (4.46), (4.42) is equivalent to the following so-called moment problem on u:

$$b_{k,\gamma}\int_0^T u(t) e^{i(\lambda_{k,\gamma}-\lambda_{1,\gamma})t} dt = i\left(\langle \Psi^0, \varphi_{k,\gamma}\rangle - \langle \Psi^1, \varphi_{k,\gamma}\rangle e^{i\lambda_{k,\gamma}T}\right), \forall k \in \mathbb{N}\setminus\{0\}. \tag{4.47}$$

Let us now explain why for $\gamma = 0$ the conclusion of Theorem 15 does not hold. Indeed, one has

$$\varphi_{n,0}(x) := \sin(n\pi x/2),\ n \in \mathbb{N}\setminus\{0\},\ \text{if } n \text{ is even}, \tag{4.48}$$
$$\varphi_{n,0}(x) := \cos(n\pi x/2),\ n \in \mathbb{N}\setminus\{0\},\ \text{if } n \text{ is odd}. \tag{4.49}$$

In particular, $x\varphi_{1,0}\varphi_{k,0}$ is an odd function if k is odd. Therefore

$$b_{k,0} = 0 \text{ if } k \text{ is odd}.$$

Hence, by (4.47), if there exists k odd such that

$$\langle \Psi^0, \varphi_{k,0}\rangle - \langle \Psi^1, \varphi_{k,0}\rangle e^{i\lambda_{k,0}T} \neq 0,$$

there is no control $u \in L^2(0,T)$ such that the solution of the Cauchy problem (4.39)-(4.40)-(4.41) (with $\gamma = 0$) satisfies (4.42).

Let us now turn to the case where γ is small but not 0. Since Ψ^0 is in $T_{\mathbb{S}}(\psi_{1,\gamma}(0,\cdot))$,

$$\Re\langle \Psi^0, \varphi_{1,\gamma}\rangle = 0. \tag{4.50}$$

Similarly, the fact that Ψ^1 is in $T_{\mathbb{S}}(\psi_{1,\gamma}(T,\cdot))$ tells us that

$$\Re(\langle \Psi^1, \varphi_{1,\gamma}\rangle e^{i\lambda_{1,\gamma}T}) = 0. \tag{4.51}$$

The key ingredient to prove Theorem 15 is the following theorem.

Theorem 16 *Let* $(\mu_i)_{i\in\mathbb{N}\setminus\{0\}}$ *be a sequence of real numbers such that*

$$\mu_1 = 0, \tag{4.52}$$

$$\text{there exists } \rho > 0 \text{ such that } \mu_{i+1} - \mu_i \geqslant \rho, \forall i \in \mathbb{N} \setminus \{0\}. \tag{4.53}$$

Let $T > 0$ *be such that*

$$\lim_{x\to+\infty} \frac{N(x)}{x} < \frac{T}{2\pi}, \tag{4.54}$$

where, for every $x > 0$*,* $N(x)$ *is the largest number of* μ_j*'s contained in an interval of length* x*. Then there exists* $C > 0$ *such that, for every sequence* $(c_k)_{k\in\mathbb{N}\setminus\{0\}}$ *of complex numbers such that*

$$c_1 \in \mathbb{R}, \tag{4.55}$$

$$\sum_{k=1}^{\infty} |c_k|^2 < \infty, \tag{4.56}$$

there exists a (real-valued) function $u \in L^2(0,T)$ *such that*

$$\int_0^T u(t)e^{i\mu_k t}dt = c_k, \forall k \in \mathbb{N} \setminus \{0\}, \tag{4.57}$$

$$\int_0^T u(t)^2 dt \leqslant C \sum_{k=1}^{\infty} |c_k|^2. \tag{4.58}$$

Remark 17 *Theorem 16 is due do Jean-Pierre Kahane [83, Theorem III.6.1, page 114]; see also [20, pages 341–365] by Arne Beurling. See also, in the context of control theory, [120, Section 3] by David Russell who uses prior works [77] by Albert Ingham, [115] by Ray Redheffer and [124] by Laurent Schwartz. For a proof of Theorem 16, see, for example, [89, Section 1.2.2] by Werner Krabs, [88, Chapter 9] by Vilmos Komornik and Paola Loreti or [7, Chapter II, Section 4] by Sergei Avdonin and Sergei Ivanov. Improvements of Theorem 16 have been obtained by Stéphane Jaffard, Marius Tucsnak and Enrique Zuazua in [81, 82], by Stéphane Jaffard and Sorin Micu in [80], by Claudio Baiocchi, Vilmos Komornik and Paola Loreti in [8] and by Vilmos Komornik and Paola Loreti in [87] and in [88, Theorem 9.4, page 177].*

Note that, by (4.50) and (4.51),

$$i(\langle \Psi^0, \varphi_{1,\gamma}\rangle - \langle \Psi^1, \varphi_{1,\gamma}\rangle e^{i\lambda_{1,\gamma}T}) \in \mathbb{R}. \tag{4.59}$$

Hence, in order to apply Theorem 15 to our moment problem, it remains to estimate $\lambda_{k,\gamma}$ and $b_{k,\gamma}$. This is done in the following propositions, due to Karine Beauchard.

Proposition 18 ([13, Proposition 41, pages 937–938]) *There exist* $\gamma_0 > 0$ *and* $C_0 > 0$ *such that, for every* $\gamma \in [-\gamma_0, \gamma_0]$ *and for every* $k \in \mathbb{N} \setminus \{0\}$*,*

$$\left|\lambda_{k,\gamma} - \frac{\pi^2 k^2}{4}\right| \leqslant C_0 \frac{\gamma^2}{k}.$$

Proposition 19 ([13, Proposition 1, page 860]) *There exist $\gamma_1 > 0$ and $C > 0$ such that, for every $\gamma \in (0, \gamma_1]$ and for every even integer $k \geqslant 2$,*

$$\left| b_{k,\gamma} - \frac{(-1)^{\frac{k}{2}+1} 8k}{\pi^2 (k^2-1)^2} \right| < \frac{C\gamma}{k^3},$$

and for every odd integer $k \geqslant 3$,

$$\left| b_{k,\gamma} - \gamma \frac{2(-1)^{\frac{k-1}{2}}(k^2+1)}{\pi^4 k (k^2-1)^2} \right| < \frac{C\gamma^2}{k^3}.$$

It is a classical result that

$$\varphi := \sum_{k=1}^{+\infty} d_k \varphi_{k,\gamma} \in H^3_{(0)}(I;\mathbb{C})$$

if and only if

$$\sum_{k=1}^{+\infty} k^6 |d_k|^2 < +\infty.$$

Hence, Theorem 15 readily follows from Theorem 16 applied to the moment problem (4.47) with the help of Proposition 18 and Proposition 19. ∎

Remark 20 *The moment method does not work well for the Schrödinger in dimension larger than 1 (see, however, [79] by Stéphane Jaffard in dimension 2). For these dimensions controllability results have been obtained by means of other methods. Let us mention, in particular,*

- *The use of the multipliers method. See, in particular, [93] by Irena Lasiecka and Roberto Triggiani, [48] by Caroline Fabre, [106] by Elaine Machtyngier, [107] by Elaine Machtyngier and Enrique Zuazua.*
- *The use of microlocal analysis. See, in particular, [97] by Gilles Lebeau and [111] by Kim-Dang Phung, which rely use of the exact controllability result [12] by Claude Bardos, Gilles Lebeau and Jeffrey Rauch for the wave equation. See also [25] by Nicolas Burq.*

For a survey on these results, see, in particular, [136] by Enrique Zuazua.

4.3.3 A linear Korteweg-de Vries equation

We go back to the linear Korteweg-de Vries equation control system (3.21)-(3.22) already considered in Section 3.2.2. Let

$$\mathcal{N} := \left\{ 2\pi \sqrt{\frac{j^2 + l^2 + jl}{3}};\ j, l \in \mathbb{N} \setminus \{0\} \right\}. \tag{4.60}$$

Then one has the following theorem, due to Lionel Rosier [116],

Theorem 21 *Let $T > 0$. The control system (3.21)-(3.22) is exactly controllable in time T if and only if $L \notin \mathcal{N}$. Moreover, if $L \in \mathcal{N}$, then the control system (3.21)-(3.22) is neither approximately controllable nor null controllable in time T.*

For a proof of Theorem 21, see [116] or [38, Section 2.2.2, pages 42-48]. Let us only point out that $2\pi \in \mathcal{N}$ (take $j = l = 1$ in (4.60) and that for $L = 2\pi$ and for every $T > 0$ the control system (3.21)-(3.22) is neither approximately controllable nor null controllable in time T. This readily follows from the following observation. Let $T > 0$, let $y^0 \in L^2(0,L)$, and let $u \in L^2(0,L)$. Let $y \in C^0([0,T]; L^2(0,L))$ be the solution to the Cauchy problem (see Definition 2)

$$y_t + y_x + y_{xxx} = 0,\ t \in (0,T),\ x \in (0,2\pi), \tag{4.61}$$

$$y(t,0) = y(t,L) = 0,\ y_x(t,L) = u(t),\ t \in (0,T), \tag{4.62}$$

$$y(0,x) = y^0(x),\ x \in (0,2\pi). \tag{4.63}$$

We multiply (4.61) by $(1-\cos(x))$ and integrate on $(0,2\pi)$. Using (4.62) together with integrations by parts, one gets (first when y is smooth enough and by density for the general case)

$$\frac{d}{dt}\int_0^{2\pi} (1-\cos(x))y dx = 0,$$

which shows that the control system (3.21)-(3.22) is neither approximately controllable nor null controllable in time T.

4.3.4 A heat equation

We go back to the control system (3.30)-(3.31)-(3.32) and show how Carleman allows to prove the following theorem, due to Hector Fattorini and David Russell [53, Theorem 3.3] if $n = 1$, to Oleg Imanuvilov [73, 74] (see also the book [59] by Andrei Fursikov and Oleg Imanuvilov) and to Gilles Lebeau and Luc Robbiano [98] for $n > 1$.

Theorem 22 *Let us assume that Ω is of class C^2 and connected. Then, for every $T > 0$ the control system* (3.30)-(3.31)-(3.32) *is null controllable and approximately controllable in time T.*

Sketch of the proof of Theorem 22. Let $T > 0$. With the notations of Section 3.2.3, for $y^0 \in L^2(\Omega)$,

$$(B^*S^*(t)y^0)(x) = y(t,x),\ t \in (0,T),\ x \in \omega, \tag{4.64}$$

where $y : (0,+\infty) \times \Omega \to \mathbb{R}$ is defined by

$$y_t - \Delta y = 0,\ (t,x) \in (0,T) \times \Omega, \tag{4.65}$$

$$y = 0 \text{ on } (0,T) \times \partial\Omega, \tag{4.66}$$

$$y(0,x) = y^0(x),\ x \in \Omega, \tag{4.67}$$

The first step is the following lemma, due to Oleg Imanuvilov [74, Lemma 1.2] (see also [59, Lemma 1.1 page 4] and [38, Lemma 2.68 page 80]) and whose proof is omitted.

Lemma 23 *There exists $\psi \in C^2(\overline{\Omega})$ such that*

$$\psi > 0 \text{ in } \Omega,\ \psi = 0 \text{ on } \partial\Omega, \tag{4.68}$$

$$|\nabla\psi(x)| > 0,\ \forall x \in \overline{\Omega} \setminus \omega_0. \tag{4.69}$$

Remark 24 *In the case $n = 1$, $\Omega = (a,b)$ for some real numbers $a < b$. Let us take $c \in \omega$. Then $\psi : \bar{\Omega} \to \mathbb{R}$ defined by*

$$\psi(x) := (b-c)^3 - (x-c)^3,\ \text{if } x \in [c,b],\ \psi(x) := (c-a)^3 - (c-x)^3,\ \text{if } x \in [a,c],$$

satisfies (4.68)-(4.69).

Let us fix ψ as in Lemma 23. Let $\alpha : (0,T)\times\overline{\Omega} \to (0,+\infty)$ and $\phi : (0,T)\times\overline{\Omega} \to (0,+\infty)$ be defined by

$$\alpha(t,x) = \frac{e^{2\lambda\|\psi\|_{C^0(\overline{\Omega})}} - e^{\lambda\psi(x)}}{t(T-t)},\ \forall (t,x) \in (0,T)\times\overline{\Omega}, \tag{4.70}$$

$$\phi(t,x) = \frac{e^{\lambda\psi(x)}}{t(T-t)},\ \forall (t,x) \in (0,T)\times\overline{\Omega}, \tag{4.71}$$

where $\lambda \in [1,+\infty)$ will be chosen later on. Let $z : [0,T]\times\overline{\Omega} \to \mathbb{R}$ be defined by

$$z(t,x) := e^{-s\alpha(t,x)}y(t,x),\ (t,x) \in (0,T)\times\overline{\Omega}, \tag{4.72}$$

$$z(0,x) = z(T,x) = 0,\ x \in \overline{\Omega}, \tag{4.73}$$

where $s \in [1,+\infty)$ will be chosen later on. From (4.65), (4.70), (4.71) and (4.72), we have

$$P_1 + P_2 = P_3 \tag{4.74}$$

with

$$P_1 := -\Delta z - s^2\lambda^2\phi^2|\nabla\psi|^2 z + s\alpha_t z, \tag{4.75}$$

$$P_2 := z_t + 2s\lambda\phi\nabla\psi\nabla z + 2s\lambda^2\phi|\nabla\psi|^2 z, \tag{4.76}$$

$$P_3 := -s\lambda\phi(\Delta\psi)z + s\lambda^2\phi|\nabla\psi|^2 z. \tag{4.77}$$

Let $Q := (0,T)\times\Omega$. From (4.74), we have

$$2\iint_Q P_1P_2\,dxdt \leqslant \iint_Q P_3^2\,dxdt. \tag{4.78}$$

Let n denote the outward unit normal vector field on $\partial\Omega$. Note that z vanishes on $[0,T]\times\partial\Omega$ (see (4.66) and (4.72)) and on $\{0,T\}\times\overline{\Omega}$ (see (4.73)). Then straightforward computations using integrations by parts lead to

$$2\iint_Q P_1P_2\,dxdt = I_1 + I_2 \tag{4.79}$$

with

$$I_1 := \iint_Q (2s^3\lambda^4\phi^3|\nabla\psi|^4|z|^2 + 4s\lambda^2\phi|\nabla\psi|^2|\nabla z|^2)dxdt - \int_0^T\int_{\partial\Omega} 2s\lambda\phi\frac{\partial\psi}{\partial n}\left(\frac{\partial z}{\partial n}\right)^2 d\sigma dt, \tag{4.80}$$

$$\begin{aligned} I_2 := \iint_Q \Big(&4s\lambda(\phi\psi_i)_j z_i z_j - 2s\lambda(\phi\psi_i)_i|\nabla z|^2 \\ &+ 2s^3\lambda^3\phi^3(|\nabla\psi|^2\psi_i)_i z^2 - 2s\lambda^2(\phi|\nabla\psi|^2)_{ii}z^2 \\ &- s\alpha_{tt}z^2 - 2s^2\lambda(\phi\psi_i\alpha_t)_i z^2 + 4s^2\lambda^2\phi\alpha_t|\nabla\psi|^2z^2 + 2s^2\lambda^2\phi\phi_t|\nabla\psi|^2z^2\Big)dxdt. \end{aligned} \tag{4.81}$$

In (4.81) and until the end of the proof of Theorem 22, we use the usual repeated-index sum convention. By (4.69) and (4.71), there exists Λ such that, for every $\lambda \geqslant \Lambda$, we have, on $(0,T)\times(\Omega\setminus\omega_0)$,

$$-4\lambda^2\phi|\nabla\psi|^2|a|^2 \leqslant 4\lambda(\phi\psi_i)_j a_i a_j - 2\lambda(\phi\psi_i)_i|a|^2,\ \forall a = (a_1,\dots,a_l)^{\mathrm{tr}} \in \mathbb{R}^l, \tag{4.82}$$

$$-\lambda^4\phi^3|\nabla\psi|^4 \leqslant 2\lambda^3\phi^3(|\nabla\psi|^2\psi_i)_i. \tag{4.83}$$

We take $\lambda := \Lambda$. Note that, by (4.68),

$$\frac{\partial \psi}{\partial n} \leqslant 0 \text{ on } \partial\Omega. \tag{4.84}$$

Moreover, using (4.70) and (4.71), one gets, the existence of $C > 0$, such that, for every $(t, x) \in (0, T) \times \Omega$,

$$|\alpha_{tt}| + |(\phi\psi_i\alpha_t)_i| + |\phi\alpha_t|\nabla\psi|^2| + |\phi\phi_t|\nabla\psi|^2| + |\phi^3(|\nabla\psi|^2\psi_i)_i| + |\phi^3|\nabla\psi|^4| \leqslant \frac{C}{t^3(T-t)^3}, \tag{4.85}$$

$$|\phi(\Delta\psi)| + |\phi|\nabla\psi|^2| + |(\phi\psi_i)_i| + |(\phi\psi_i)_i| + |(\phi|\nabla\psi|^2)_{ii}| \leqslant \frac{C}{t(T-t)}, \tag{4.86}$$

$$|(\phi\psi_i)_j| \leqslant \frac{C}{t(T-t)}, \, \forall (i,j) \in \{1, \dots, l\}^2. \tag{4.87}$$

From (4.69) and (4.71), one gets the existence of $C > 0$ such that

$$\frac{1}{t^3(T-t)^3} \leqslant C\phi^3|\nabla\psi|^4(t,x), \, \forall (t,x) \in (0,T) \times (\Omega \setminus \omega_0). \tag{4.88}$$

Using (4.78) to (4.88), we get the existence of $C > 0$ such that, for every $s \geqslant 1$ and for every y^0,

$$s^3 \int_{(0,T)} \int_{\Omega\setminus\omega_0} \frac{|z|^2}{t^3(T-t)^3} dxdt \leqslant Cs^2 \iint_Q \frac{|z|^2}{t^3(T-t)^3} dxdt + Cs^3 \int_{(0,T)} \int_{\omega_0} \frac{|\nabla z|^2 + |z|^2}{t^3(T-t)^3} dxdt. \tag{4.89}$$

Taking for $s \geqslant 1$ large enough in (4.89), one gets

$$s^3 \int_{(0,T)} \int_{\Omega\setminus\omega_0} \frac{|z|^2}{t^3(T-t)^3} dxdt \leqslant Cs^3 \int_{(0,T)} \int_{\omega_0} \frac{|\nabla z|^2 + |z|^2}{t^3(T-t)^3} dxdt. \tag{4.90}$$

From Theorem 9, (4.72), and (4.90), one gets the approximate controllability in time T (with $H := L^2(\Omega)$ and $U := L^2(\omega)$; see Definition 6) of the control system (3.30)-(3.31)-(3.32).

Let us now deal with the null controllability. Taking $s \geqslant 1$ large enough in (4.89), one gets the existence of $c_0 > 0$ independent of y^0 such that

$$\int_{T/3}^{2T/3} \int_\Omega |z|^2 dxdt \leqslant c_0 \int_0^T \int_{\omega_0} \frac{|\nabla z|^2 + |z|^2}{t^3(T-t)^3} dxdt. \tag{4.91}$$

We choose such an s and such a c_0. Coming back to y using (4.70) and (4.72), we deduce from (4.91) the existence of $c_1 > 0$ independent of y^0 such that

$$\int_{T/3}^{2T/3} \int_\Omega |y|^2 dxdt \leqslant c_1 \int_0^T \int_{\omega_0} t(T-t)(|\nabla y|^2 + |y|^2) dxdt. \tag{4.92}$$

Let $\rho \in C^\infty(\overline{\Omega})$ be such that

$$\rho = 1 \text{ in } \omega_0,$$
$$\rho = 0 \text{ in } \overline{\Omega} \setminus \omega.$$

We multiply (4.65) by $t(T-t)\rho y$ and integrate on Q. Using (4.66) and integrations by parts, we get the existence of $c_2>0$ independent of y^0 such that

$$\int_0^T \int_{\omega_0} t(T-t)(|\nabla y|^2+|y|^2)dxdt \leqslant c_2 \int_0^T \int_\omega |y|^2 dxdt. \tag{4.93}$$

From (4.92) and (4.93), we get

$$\int_{T/3}^{2T/3} \int_\Omega |y|^2 dxdt \leqslant c_1 c_2 \int_0^T \int_\omega |y|^2 dxdt. \tag{4.94}$$

Let us now multiply (4.65) by y and integrate on Ω. Using integrations by parts together with (4.66), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |y(t,x)|^2 dx \leqslant 0. \tag{4.95}$$

From (4.94) and (4.95), one gets that (4.7) holds with

$$M := \frac{T}{3c_1c_2}.$$

With Theorem 10 and Theorem 11, this concludes the proof of Theorem 22.

4.4 Numerical methods

Again, there are two possibilities to study numerically the controllability of a linear control systems: direct methods, duality methods. The most popular ones use duality methods and in particular the Hilbert Uniqueness Method (HUM) due to Jacques Louis Lions [103, 104]. For the numerical approximation, one uses often discretization by finite difference methods. However a new problem appear: the control for the discretized model does not necessarily lead to a good approximation to the control for the original continuous problem. In particular, the classical convergence requirements, namely stability and consistency, of the numerical scheme used does not suffice to guarantee good approximations to the controls that one wants to compute. Observability/controllability may be lost under numerical discretization as the mesh size tends to zero. To overcome this problem, several remedies have been used, in particular, filtering, Tychonoff regularization, multigrid methods, and mixed finite element methods. For precise informations and references, we refer to the survey papers [137, 138] by Enrique Zuazua.

4.5 Complements and further references

In this section on the controllability of linear PDE, we have already given references to books and papers. But there are of course many other references which must also be mentioned. If one restricts to books or surveys we would like to add in particular (but this is a very incomplete list):

- The survey [5] by Fatiha Alabau-Boussouira and Piermarco Cannarsa. It deals, in particular, with abstract evolution equations, wave equations, heat equations and quadratic optimal control for linear PDE.

- The book [17] by Alain Bensoussan on stochastic control.

- The books [18, 19] by Alain Bensoussan, Giuseppe Da Prato, Michel Delfour and Sanjoy Mitter, which deal, in particular, with differential control systems with delays and partial differential control systems with specific emphasis on controllability, stabilizability and the Riccati equations.

- The book [43] by Ruth Curtain and Hans Zwart which deals with general infinite-dimensional linear control systems theory. It includes the usual classical topics in linear control theory such as controllability, observability, stabilizability, and the linear-quadratic optimal problem. For a more advanced level on this general approach, one can look at the book [127] by Olof Staffans.

- The book [44] by René Dáger and Enrique Zuazua on partial differential equations on planar graphs modeling networked flexible mechanical structures (with extensions to the heat, beam and Schrödinger equations on planar graphs).

- The book [47] by Abdelhaq El Jaï and Anthony Pritchard on the input-output map and the importance of the location of the actuators/sensors for a better controllability/observability.

- The books [51, 52] by Hector Fattorini on optimal control for infinite-dimensional control problems (linear or nonlinear, including partial differential equations).

- The book [57] by Andrei Fursikov on study of optimal control problems for infinite-dimensional control systems with many examples coming from physical systems governed by partial differential equations (including the Navier-Stokes equations).

- The book [88] by Vilmos Komornik and Paola Loreti on harmonic (and nonharmonic) analysis methods with many applications to the controllability of various time-reversible systems.

- The book [90] by John Lagnese and Günter Leugering on optimal control on networked domains for elliptic and hyperbolic equations, with a special emphasis on domain decomposition methods.

- The books [94, 95] by Irena Lasiecka and Roberto Triggiani which deal with finite horizon quadratic regulator problems and related differential Riccati equations for general parabolic and hyperbolic equations with numerous important specific examples.

- The survey [122] by David Russell, which deals with the hyperbolic and parabolic equations, quadratic optimal control for linear PDE, moments and duality methods, controllability and stabilizability.

- The book [131] by Marius Tucsnak and George Weiss on passive and conservative linear systems, with a detailed chapter on the controllability of these systems.

- The survey [139] by Enrique Zuazua on recent results on the controllability of linear partial differential equations. It includes the study of the controllability of wave equations, heat equations, in particular with low regularity coefficients, which is important to treat semi-linear equations, fluid-structure interaction models.

5 Controllability of nonlinear control systems

In this section, we consider the controllability of nonlinear control system modeled by nonlinear PDE's. There are no general methods to deal with this difficult problem. It is natural to first consider the problem of the controllability around an equilibrium of a nonlinear partial differential equation. Then the natural next step is to look if the linearized control system around this equilibrium is controllable. This is what we call the linear test in the following. We give an example of application below and mention methods and tools to deal with the cases where there is a problem of loss of derivatives.

Of course when the linearized control system is not controllable, one cannot say that the same noncontrollabilty holds for the nonlinear system. Various methods can be used to deal with this case. Let us mention , in particular,

- Iterated Lie brackets. This is the most popular and powerful method for finite dimensional control system (see e.g. [38, Section 3.2], [78, Chapters 1 and 2], [109, Section 3.1] as well as the papers [1, 2, 21, 23, 22, 69, 70, 85, 128, 129, 130]. It is also useful for control of PDE; see in particular the papers on incompressible fluids [3, 4] by Andrei Agrachev and Andrei Sarychev, [125] by Armen Shirikyan. However, for many control PDE the iterated Lie brackets are not well defined; see, e.g. [38, Chapter 5].
- The return method: See Section 5.3.
- Quasi-static deformations: See Section 5.4.
- Power series expansion: See Section 5.5.

5.1 The linear test: the regular case

When the linearized control systems around an equilibrium in controllable, one can try to use the inverse mapping theorem to get a local controllability result for the nonlinear control system. We illustrate this method on the Korteweg-de Vries nonlinear control system (2.3)-(2.4). (Of course this is just an example: this method can be applied too many nonlinear PDE). The linearized control system around $(y, u) = (0, 0)$ is the control system

$$y_t + y_x + y_{xxx} = 0,\ x \in (0, L),\ t \in (0, T), \tag{5.1}$$

$$y(t, 0) = y(t, L) = 0,\ y_x(t, L) = u(t),\ t \in (0, T), \tag{5.2}$$

where, at time t, the control is $u(t) \in \mathbb{R}$ and the state is $y(t, \cdot) : (0, L) \to \mathbb{R}$. We have previously seen (see Theorem 21) that if

$$L \notin \mathcal{N} := \left\{ 2\pi \sqrt{\frac{j^2 + l^2 + jl}{3}};\ j, l \in \mathbb{N} \setminus \{0\} \right\}, \tag{5.3}$$

then, for every time $T > 0$, the control system (5.1)-(5.2) is controllable in time T. Hence one may expect that the nonlinear control system (2.3)-(2.4) is at least locally controllable if (5.3) holds. The goal of this section is to prove that this is indeed true, a result due to Lionel Rosier [116, Theorem 1.3].

5.1.1 Well-posedness of the Cauchy problem

Let us first define the notion of solutions for the Cauchy problem associated to (2.3)-(2.4). Multiplying (2.3) by $\phi : [0,\tau] \times [0,L] \to \mathbb{R}$, using (2.4) and performing integrations by parts lead to the following definition.

Definition 25 *Let $T > 0$, $y^0 \in L^2(0,L)$ and $u \in L^2(0,T)$ be given. A solution of the Cauchy problem*

$$y_t + y_x + y_{xxx} + yy_x = 0,\ x \in [0,L],\ t \in [0,T], \tag{5.4}$$

$$y(t,0) = y(t,L) = 0,\ y_x(t,L) = u(t),\ t \in [0,T], \tag{5.5}$$

$$y(0,x) = y^0(x),\ x \in [0,L], \tag{5.6}$$

is a function $y \in C^0([0,T]; L^2(0,L)) \cap L^2((0,T); H^1(0,L))$ such that, for every $\tau \in [0,T]$ and for every $\phi \in C^3([0,\tau] \times [0,L])$ such that

$$\phi(t,0) = \phi(t,L) = \phi_x(t,0) = 0,\ \forall t \in [0,\tau], \tag{5.7}$$

one has

$$-\int_0^\tau \int_0^L (\phi_t + \phi_x + \phi_{xxx}) y dx dt - \int_0^\tau u(t)\phi_x(t,L)dt + \int_0^L y(\tau,x)\phi(\tau,x)dx - \int_0^L y^0(x)\phi(0,x)dx = 0. \tag{5.8}$$

Then one has the following theorem which is proved in [39, Appendix A].

Theorem 26 *Let $T > 0$. Then there exists $\varepsilon > 0$ such that, for every $y^0 \in L^2(0,L)$ and $u \in L^2(0,T)$ satisfying*

$$\|y^0\|_{L^2(0,L)} + \|u\|_{L^2(0,T)} \leqslant \varepsilon,$$

the Cauchy problem (5.4)-(5.5)-(5.6) *has a unique solution.*

The proof is rather lengthy and technical. We omit it.

5.1.2 Local controllability

The goal of this section is to prove the following local controllability result due to Lionel Rosier [116, Theorem 1.3].

Theorem 27 *Let $T > 0$, and let us assume that*

$$L \notin \mathcal{N}, \tag{5.9}$$

with

$$\mathcal{N} := \left\{ 2\pi\sqrt{\frac{j^2 + l^2 + jl}{3}};\ j, l \in \mathbb{N} \setminus \{0\} \right\}. \tag{5.10}$$

Then there exist $C > 0$ and $r_0 > 0$ such that for every $y^0, y^1 \in L^2(0,L)$, with $\|y^0\|_{L^2(0,L)} < r_0$ and $\|y^1\|_{L^2(0,L)} < r_0$, there exist

$$y \in C^0([0,T], L^2(0,L)) \cap L^2((0,T); H^1(0,L))$$

and $u \in L^2(0,T)$ *satisfying* (2.3)-(2.4) *such that*

$$y(0,\cdot) = y^0, \tag{5.11}$$

$$y(T,\cdot) = y^1, \tag{5.12}$$

$$\|u\|_{L^2(0,T)} \leqslant C(\|y^0\|_{L^2(0,L)} + \|y^1\|_{L^2(0,L)}). \tag{5.13}$$

Proof of Theorem 27. Let $\mathcal{F} : L^2(0,L) \times L^2(0,T) \to L^2(0,L)^2$, $(y^0, u) \mapsto (y^0, y(T,\cdot))$ where $y \in C^0([0,T]; L^2(0,L))$ is the solution to the Cauchy problem (5.4)-(5.5)-(5.6). It follows from 26 that $\mathcal{F}$ is well defined on a neighborhood of $(0,0) \in L^2(0,L) \times L^2(0,T)$. With some lengthy bur straightforward estimates (see in particular [39, Appendix A]), one can check that $\mathcal{F}$ is of class C^1 on a neighborhood of $(0,0) \in L^2(0,L) \times L^2(0,T)$ and that, as expected $\mathcal{F}'(0,0) : L^2(0,L) \times L^2(0,T) \to L^2(0,L)^2$, associates to $(y^0,u) \in L^2(0,L) \times L^2(0,T)$ $(y^0, y(T,\cdot)) \in L^2(0,L)^2$ where $y \in C^0([0,T]; L^2(0,L))$ is the solution to the Cauchy problem

$$y_t + y_x + y_{xxx} = 0,\ t \in (0,T),\ x \in (0,L),$$
$$y(t,0) = y(t,L) = 0,\ y_x(t,L) = u(t),\ t \in (0,T),$$
$$y(0,x) = y^0(x),\ x \in (0,L).$$

From Theorem 21, one gets that $\mathcal{F}'(0,0)$ is onto, which the inverse mapping theorem implies Theorem 27. ■

5.2 The linear test: the problem of loss of derivatives

In fact, in many situations one cannot applied directly the usual inverse mapping theorem to deduce from the controllability of the linearized control system at an equilibrium the local controllability of the nonlinear system at this equilibrium. This is due to some problem of loss of derivatives. Let us give a simple example where this problem appear. We consider the following simple nonlinear transport equation:

$$y_t + a(y)y_x = 0,\ x \in [0,L],\ t \in [0,T], \tag{5.14}$$

$$y(t,0) = u(t),\ t \in [0,T], \tag{5.15}$$

where $a \in C^2(\mathbb{R})$ satisfies

$$a(0) > 0. \tag{5.16}$$

For this control system, at time $t \in [0,T]$, the state is $y(t,\cdot) \in C^1([0,L])$ and the control is $u(t) \in \mathbb{R}$. If one wants to have a Hilbert space as a state space, one can also work with suitable Sobolev spaces (for example $y(t,\cdot) \in H^2(0,L)$ is a suitable space). We are interested in the local controllability of the control system (5.14)-(5.15) at the equilibrium $(\bar{y}, \bar{u}) = (0,0)$. Hence we first look at the linearized control system at the equilibrium $(\bar{y}, \bar{u}) = (0,0)$. This linear control system is the following one:

$$y_t + a(0)y_x = 0,\ t \in [0,T],\ x \in [0,L], \tag{5.17}$$

$$y(t,0) = u(t),\ t \in [0,T]. \tag{5.18}$$

For this linear control system, at time $t \in [0,T]$, the state is $y(t,\cdot) \in C^1([0,L])$ and the control is $u(t) \in \mathbb{R}$. Concerning the well-posedness of the Cauchy problem of this linear control system, one easily get the following proposition.

Proposition 28 *Let $T > 0$. Let $y^0 \in C^1([0,L])$ and $u \in C^1([0,T])$ be such that the following compatibility conditions hold*

$$u(0) = y^0(0), \tag{5.19}$$
$$\dot{u}(0) + a(0)y_x^0(0) = 0. \tag{5.20}$$

Then the Cauchy problem

$$y_t + a(0)y_x = 0,\ t \in [0,T],\ x \in [0,L], \tag{5.21}$$
$$y(t,0) = u(t),\ t \in [0,T], \tag{5.22}$$
$$y(0,x) = y^0(x),\ x \in [0,L], \tag{5.23}$$

has a unique solution $y \in C^1([0,T] \times [0,L])$.

Of course (5.19) is a consequence of (5.22) and (5.23): it is a necessary condition for the existence of a solution $y \in C^0([0,T] \times [0,L])$ to the Cauchy problem (5.21)-(5.22)-(5.23). Similarly (5.20) is a direct consequence of (5.21), (5.22) and (5.23): it is a necessary condition for the existence of a solution $y \in C^1([0,T] \times [0,L])$ to the Cauchy problem (5.21)-(5.22)-(5.23).

One has the following (easy) proposition.

Proposition 29 *Let $T > L/a(0)$. The linear control system* (5.17)-(5.18) *is controllable in time T. In other words, for every $y^0 \in C^1([0,L])$ and for every $y^1 \in C^1([0,L])$, there exists $u \in C^1([0,T])$ such that the solution y of the Cauchy problem* (5.21)-(5.22)-(5.23) *satisfies*

$$y(T,x) = y^1(x),\ x \in [0,L]. \tag{5.24}$$

In fact one can give an explicit example of such a u. Let $u \in C^1([0,T])$ be such that

$$u(t) = y^1(a(0)(T-t)),\ \text{for every } t \in [T - (L/a(0)), T], \tag{5.25}$$
$$u(0) = y^0(0), \tag{5.26}$$
$$\dot{u}(0) = -a(0)y_x^0(0). \tag{5.27}$$

Such a u exists since $T > L/a(0)$. Then the solution y of the Cauchy problem (5.21)-(5.22)-(5.23) is given by

$$y(t,x) = y^0(x - a(0)t),\ \forall (t,x) \in [0,T] \times [0,L] \text{ such that } a(0)t \leqslant x, \tag{5.28}$$
$$y(t,x) = u(t - (x/a(0))),\ \forall (t,x) \in [0,T] \times [0,L] \text{ such that } a(0)t > x. \tag{5.29}$$

(The fact that such a y is in $C^1([0,T] \times [0,L])$ follows from the fact that y and u are of class C^1 and from the compatibility conditions (5.26)-(5.27).) From (5.28), one has (5.23). From $T > L/a(0)$, (5.25) and (5.28), one gets (5.24). With this method, one can easily construct a continuous linear map

$$\begin{array}{rccc} \Gamma : & C^1([0,L]) \times C^1([0,L]) & \to & C^1([0,T]) \\ & (y^0, y^1) & \mapsto & u \end{array}$$

such that

- The compatibility conditions $u(0) = y^0(0)$ and $\dot{u}(0) + a(y^0(0))y_x^0(0) = 0$ hold.

- The solution $y \in C^1([0,T]\times[0,L])$ of the Cauchy problem

$$\begin{gathered} y_t + a(0)y_x = 0,\ (t,x)\in[0,T]\times[0,L],\\ y(t,0) = u(t),\ t\in[0,T],\\ y(0,x) = y^0(x),\ x\in[0,L], \end{gathered}$$

  satisfies

$$y(T,x) = y^1(x),\ x\in[0,L].$$

In order to prove the local controllability of the control system (5.14)-(5.15) at the equilibrium $(\bar{y},\bar{u}) = (0,0)$, let us try to mimic what we have done for the nonlinear Korteweg-de Vries equation (2.3)-(2.4). Now the map $\mathcal{F}$ is the following one. Let

$$\begin{gathered} F := \{(y^0,u)\in C^1([0,L])\times C^1([0,T]);\ u(0) = y^0(0),\ \dot{u}(0) + a(y^0(0))y^0_x(0) = 0\},\\ G := C^1([0,L])^2. \end{gathered}$$

$$\begin{array}{cccc} \mathcal{F}: & F & \to & G \\ & (y^0,u) & \mapsto & (y^0, y(T,\cdot)) \end{array}$$

where $y \in C^1([0,T]\times[0,L]))$ is the solution to the Cauchy problem

$$y_t + a(y)y_x = 0,\ x\in[0,L],\ t\in[0,T], \tag{5.30}$$

$$y(t,0) = u(t),\ t\in[0,T], \tag{5.31}$$

$$y(0,x) = y^0(x), \forall x\in[0,L]. \tag{5.32}$$

One can prove that this map $\mathcal{F}$ is well defined and continuous in a neighborhood of $(0,0)$ (see [101] for much more general result). Note that, formally, the linearized control system at $(y,u) := (0,0)$ is the control system (5.17)-(5.18), which by Proposition 29 is controllable (at least if $T > L/a(0)$). Unfortunately the map $\mathcal{F}$ is not of class C^1. One could thing to avoid this problem by replacing G by $G := C^1([0,L])\times C^0([0,T])$. Then the map $\mathcal{F}$ is now of class C^1, but $\mathcal{F}'(0,0): F\to G$ is no longer onto: there are controls $u\in C^0([0,T])$ allowing to go from $y^0\in C^1([0,L]$ to $y^1\in C^0([0,L])$ but these controls are not of class C^1 if y^0 is not of class C^1. We have lost one derivative.

There is a general tool, namely the Nash-Moser method, which allows us to deal with this problem of loss of derivatives. There are many forms of this method. Let us mention, in particular, the ones given by Mikhael Gromov in [67, Section 2.3.2], Lars Hörmander in [71], Richard Hamilton in [68]; see also the book by Serge Alinhac and Patrick Gérard [6]. This approach can also be used in the context of the control system (5.14)-(5.15). See the papers [13, 14, 15] by Karine Beauchard, and [16], which show the power and the flexibility of the Nash-Moser method in the context of control theory. However the Nash-Moser method has two major drawbacks

1. It does not give the optimal functional spaces for the state and the control.

2. It is more complicated to apply than the method we want to present here.

There is a more standard fixed point method which works for many control systems, in particular the control system (5.17)-(5.18) which allows to prove for this control system the following controllability result.

Theorem 30 *Let us assume that*

$$T > \frac{L}{a(0)}. \tag{5.33}$$

Then there exist $\varepsilon > 0$ *and* $C > 0$ *such that, for every* $y^0 \in C^1([0, L])$ *and for every* $y^1 \in C^1([0, L])$ *such that*

$$\|y^0\|_{C^1([0,L])} \leqslant \varepsilon \text{ and } \|y^1\|_{C^1([0,L])} \leqslant \varepsilon,$$

there exists $u \in C^1([0, T])$ *such that*

$$\|u\|_{C^1([0,T])} \leqslant C(\|y^0\|_{C^1([0,L])} + \|y^1\|_{C^1([0,L])}) \tag{5.34}$$

and such that the solution of the Cauchy problem (5.30)-(5.31)-(5.32) *exists, is of class* C^1 *on* $[0, T] \times [0, L]$ *and satisfies*

$$y(T, x) = y^1(x),\ x \in [0, L].$$

The fixed point method is the following one. Let $z \in C^1([0, L] \times [0, L]$. One consider the following linear control system

$$y_t + a(z)y_x = 0,\ y(t, 0) = u(t),\ t \in [0, T],\ x \in [0, L]. \tag{5.35}$$

If the C^1-norm of z is small enough this system is controllable in time $T > L/a(0)$ and there exist $u \in C^1([0, T])$ such that the Cauchy problem

$$y_t + a(z)y_x = 0,\ y(t, 0) = u(t),\ y(0, x) = y^0(x),\ t \in [0, T],\ x \in [0, L],$$

has a (unique) solution $y \in C^1([0, T] \times [0, L])$ and this solution satisfies $y(T, x) = y^1(x)$, $\forall x \in [0, L]$. Of course this u is not unique. However, if one chooses it well on can prove, using the Brouwer fixed point theorem, that, at least if the C^1-norm of y^0 and y^1 are small enough, the map $z \mapsto y$ has a fixed point, which shows that the control $u := y(\cdot, 0)$ steer the control system (5.14)-(5.15) from y^0 to y^1. See [38, Section 4.2] for more details.

Remark 31 *One can find similar controllability results for much more general hyperbolic systems and with a different proof in the papers [30] by Marco Cirinà, [102] by Ta Tsien Li and Bing-Yu Zhang, and [100] by Ta Tsien Li and Bo-Peng Rao. See also [38, Section 4.2.1] as well as the book [99] by Ta-tsien Li.*

Remark 32 *As for the Nash-Moser method, the controllability of the linearized control system* (5.21)-(5.22)-(5.23) *at the equilibrium* $(y, u) := (0, 0)$ *is not sufficient for our proof of Theorem 30: one needs a controllability result for linear control systems which are close to the linear control system* (5.21)-(5.22)-(5.23).

Remark 33 *Sometimes these fixed point methods used with careful estimates can lead to global controllability results if the nonlinearity is not too strong at infinity. See in particular*

- *For semilinear heat equations: [50] by Caroline Fabre, Jean-Pierre Puel and Enrique Zuazua, [59, Chapter I, Section 3] by Andrei Fursikov and Oleg Imanuvilov, and [55] by Enrique Fernández-Cara and Enrique Zuazua.*
- *For wave equations: [134, 135] by Enrique Zuazua, [92] by Irena Lasiecka and Roberto Triggiani.*
- *For a truncated Navier-Stokes equation: [49] by Caroline Fabre.*

5.3 The return method

In order to explain this method, let us first consider the problem of local controllability of the following control system in finite dimension

$$\dot{y} = f(y, u),$$

where $y \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control; we assume that f is of class C^∞ and satisfies

$$f(0,0) = 0.$$

The return method consists in reducing the local controllability of a nonlinear control system to the existence of suitable trajectories and to the controllability of *linear* systems. The idea is the following one: Assume that, for every positive real number T and every positive real number ε, there exists a measurable bounded function $\bar{u} : [0,T] \to \mathbb{R}^m$ with $\|\bar{u}\|_{L^\infty(0,T)} \leqslant \varepsilon$ such that, if we denote by $\bar{y}$ the solution of $\dot{\bar{y}} = f(\bar{y}, \bar{u}(t))$, $\bar{y}(0) = 0$, then

$$\bar{y}(T) = 0, \tag{5.36}$$

$$\text{the linearized control system around } (\bar{y}, \bar{u}) \text{ is controllable on } [0,T]. \tag{5.37}$$

Then, from the inverse function theorem, one gets the existence of $\eta > 0$ such that, for every $y^0 \in \mathbb{R}^n$ and for every $y^1 \in \mathbb{R}^n$ satisfying

$$|y^0| < \eta, \ |y^1| < \eta,$$

there exists $u \in L^\infty((0,T); \mathbb{R}^m)$ such that

$$|u(t) - \bar{u}(t)| \leqslant \varepsilon, \ t \in [0,T],$$

and such that, if $y : [0,T] \to \mathbb{R}^n$ is the solution of the Cauchy problem

$$\dot{y} = f(y, u(t)), \ y(0) = y^0,$$

then

$$y(T) = y^1.$$

Since $T > 0$ and $\varepsilon > 0$ are arbitrary, one gets that $\dot{y} = f(y,u)$ is small-time locally controllable at the equilibrium $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$.

Example 34 *Let us consider the nonholonomic integrator, i.e. the following control system*

$$\dot{y}_1 = u_1, \ \dot{y}_2 = u_2, \ \dot{y}_3 = y_1 u_2 - y_2 u_1, \tag{5.38}$$

where the state is $y = (y_1, y_2, y_3)^{\mathrm{tr}} \in \mathbb{R}^3$ and the control is $u = (u_1, u_2)^{\mathrm{tr}} \in \mathbb{R}^2$ Let us recall that this system is small-time locally controllable at $(0,0) \in \mathbb{R}^3 \times \mathbb{R}^2$. The classical proof of this property relies on Lie brackets and on the the Rashevski-Chow theorem [112, 29]. Let us show how the return method also gives this controllability property. Take any $T > 0$ and any $\bar{u} : [0,T] \to \mathbb{R}^2$ such that $\bar{u}(t-t) = -\bar{u}(t)$. Let $\bar{y} : [0,T] \to \mathbb{R}^3$ be the solution of the Cauchy problem

$$\dot{\bar{y}} = f(\bar{y}, \bar{u}(t)), \ \bar{y}(0) = 0.$$

One easily checks that $\bar{y}(T) = 0$ and that the linearized control system around $(\bar{y}, \bar{u})$ is controllable if (and only if) $\bar{u} \not\equiv 0$. Hence we recover small-time locally controllable at $(0,0) \in \mathbb{R}^3 \times \mathbb{R}^2$. Let us point out that this new proof does not use Lie bracket and uses only controllability of linear control systems. This is exactly what we want in order to deal with nonlinear control system modeled by partial differential equations: For these systems Lie brackets often do not work and one knows a lot of tools to study the controllability of linear control system modeled by partial differential equations.

With this return method one has got in [32, 34, 61, 62] the following global controllability result of the Euler equations of incompressible fluids (where, for simplicity we do not specify the regularity of the functions and of the domain Ω):

Theorem 35 *Let $l \in \{2,3\}$. Let Ω be a nonempty bounded open subset of $\mathbb{R}^l$. Let Γ_0 be a nonempty open subset of the boundary $\Gamma := \partial\Omega$ of Ω. We assume that Γ_0 meets every connected components of Γ Let $y^0 : \overline{\Omega} \to \mathbb{R}^2$ and $y^1 : \overline{\Omega} \to \mathbb{R}^2$ be such that*

$$\text{div } y^0 = \text{div } y^1 = 0, y^0(x) \cdot n(x) = y^1(x) \cdot n(x) = 0, \forall x \in \Gamma \backslash \Gamma_0,$$

where $n : \partial\Omega \to \mathbb{R}^2$ denotes the outward normal. Then, for every $T > 0$, there exist $(y, p) : \overline{\Omega} \times [0, T] \to \mathbb{R}^2$ such that

$$\begin{gathered} y_t + (y \cdot \nabla)y + \nabla p = 0 \text{ in } [0,T] \times \overline{\Omega}, \\ \text{div } y = 0 \text{ in } [0,T] \times \overline{\Omega}, \\ y(t,\cdot) \cdot n(x) = 0, \forall t \in [0,T], \forall x \in \Gamma \setminus \Gamma_0, \\ y(0,\cdot) = y_0, \; y(T,\cdot) = 0. \end{gathered}$$

Main ingredients for the proof of Theorem 35. Note that the linearized control system of the Euler control system is far from being controllable (for this linear system the vorticity cannot be modified). The proof relies on the return method. In order to use this method, one needs to construct a (good) trajectory $\bar{y}$ going from 0 to 0. Such trajectory is constructed using a potential flow (i.e. a flow of the form $\bar{y}(t,x) = \nabla\varphi(t,x)$). If the potential flow is well chosen the linearized control system around this trajectory is controllable. Using this controllability and a suitable fixed point argument one gets the local controllability of the Euler control system. The global controllability follows from this local result by a suitable scaling argument.

Remark 36 *The return method has been introduced in [31] for a stabilization problem. It has been used for the first time in [32, 34] for the controllability of a partial differential equation, namely the Euler equations of incompressible fluids. The return method has been used to study the controllability of the following partial differential equations.*

1. *Navier-Stokes equations of incompressible fluids in [33] (with the Navier boundary condition: see Section 2.7), in [40] for $\Gamma_0 = \Gamma$, and by Andrei Fursikov et Oleg Imanuvilov in [60].*

2. *Boussinesq equations, by Andrei Fursikov and Oleg Imanuvilov in [60],*

3. *Burgers equation, by Thierry Horsin in [72] and Marianne Chapouly in [28].*

4. *Shallow water equations in [36], a paper motivated by the prior paper [46] by François Dubois, Nicolas Petit and Pierre Rouchon (see also Section 5.4).*

5. *Vlasov-Poisson equations, by Olivier Glass in [63].*

6. *1-D Euler isentropic equations by Olivier Glass in [65].*

7. *Schrödinger equations, by Karine Beauchard in [13], and in [16]. (These two papers are motivated by the prior paper [117] by Pierre Rouchon).*

Remark 37 *In fact, as already mentioned in Section 5.2, for many nonlinear partial differential equations, the fact that the linearized control system along the trajectory $(\bar{y}, \bar{u})$ is controllable is not sufficient to get the local controllability along $(\bar{y}, \bar{u})$. This is due to some loss of derivatives problems. To take care of this problem, one use some suitable fixed point methods. Note that these fixed point methods rely often on the controllability of some (and many) linear control systems which are not the linearized control system along the trajectory $(\bar{y}, \bar{u})$. (It can also rely on some specific methods which do not use the controllability of any linear control system. This last case appears in the papers [72] by Thierry Horsin and [65] by Olivier Glass.)*

Remark 38 *For the Navier-Stokes control system, the linearized control system around of* 0 *is controllable (this result is due to Oleg Imanuvilov [75, 76]). (See also [54] by Enrique Fernńandez-Cara, Sergio Guerrero, Oleg Imanuvilov and Jean-Pierre Puel.) However it is not clear how to deduce from this controllability a global controllability result for the Navier-Stokes control system. Roughly speaking, the global controllability results for the Navier-Stokes equations in [33, 40] are deduced from the controllability of the Euler equations. This is possible because the Euler equations are quadratic and the Navier-Stokes equations are the Euler equations plus a linear "perturbation" (of course some technical problems appear due to the fact that this linear perturbation involves more derivatives than the Euler equations: One faces a problem of singular perturbations).*

5.4 Quasi-static deformations

Let us explain how this method can be used on a specific example. We consider the water-tank control system (2.7)-(2.8)-(2.9)-(2.10)-(2.11) (see Figure 1). The state space is
This is a control system, denoted Σ, where, at time $t \in [0, T]$,

- the state is $Y(t) = (H(t, \cdot), v(t, \cdot), s(t), D(t))$,
- the control is $u(t) \in \mathbb{R}$.

Of course, the total mass of the fluid is conserved so that, for every solution of (2.7) to (2.9),

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^L H(t, x)\, \mathrm{d}x = 0. \tag{5.39}$$

(One gets (5.39) by integrating (2.7) on $[0, L]$ and by using (2.9) together with an integration by parts.) Moreover, if H and v are of class C^1, it follows from (2.8) and (2.9) that

$$H_x(t, 0) = H_x(t, L), \tag{5.40}$$

which is also $-u(t)/g$. Therefore we introduce the vector space E of functions $Y = (H, v, s, D) \in C^1([0, L]) \times C^1([0, L]) \times \mathbb{R} \times \mathbb{R}$ such that

$$H_x(0) = H_x(L), \tag{5.41}$$

$$v(0) = v(L) = 0, \tag{5.42}$$

and we consider the affine subspace $\mathcal{Y} \subset E$ consisting of elements $Y = (H, v, s, D) \in E$ satisfying

$$\int_0^L H(x)\mathrm{d}x = LH_e. \tag{5.43}$$

The vector space E is equipped with the natural norm

$$|Y| := \|H\|_{C^1([0,L])} + \|v\|_{C^1([0,L])} + |s| + |D|.$$

One has the following local controllability theorem, proved in [36].

Theorem 39 *There exist $T > 0$, $C > 0$ and $\eta > 0$ such that, for every $Y^0 = \left(H^0, v^0, s^0, D^0\right) \in \mathcal{Y}$, and for every $Y^1 = \left(H^1, v^1, s^1, D^1\right) \in \mathcal{Y}$ such that*

$$\left\|H^0 - H_e\right\|_{C^1([0,L])} + \left\|v^0\right\|_{C^1([0,L])} < \eta, \; \left\|H^1 - H_e\right\|_{C^1([0,L])} + \left\|v^1\right\|_{C^1([0,L])} < \eta,$$
$$\left|s^1 - s^0\right| + \left|D^1 - s^0 T - D^0\right| < \eta,$$

there exists $u \in C^0([0,T])$ satisfying the compatibility condition $u(0) = gH_x(0) = gH_x(L)$ such that the solution $(H, v, s, D) \in C^1([0,T] \times [0,L]) \times C^1([0,T] \times [0,L]) \times C^1([0,T]) \times C^1([0,T])$ to the Cauchy problem (2.7) to (2.11) with the initial condition

$$(H(0,\cdot), v(0,\cdot), s(0), D(0)) = (H^0, v^0, s^0, D^0),$$

satisfies

$$(H(T,\cdot), v(T,\cdot), s(T), D(T)) = (H^1, v^1, s^1, D^1)$$

and, for every $t \in [0,T]$,

$$\begin{aligned} \|H(t) - H_e\|_{C^1([0,L])} + \|v(t)\|_{C^1([0,L])} + |u(t)| \leqslant \\ C\left(\sqrt{\|H^0 - H_e\|_{C^1([0,L])} + \|v^0\|_{C^1([0,L])} + \|H^1 - H_e\|_{C^1([0,L])} + \|v^1\|_{C^1([0,L])}}\right) \\ + C\left(\left|s^1 - s^0\right| + \left|D^1 - s^0 T - D^0\right|\right). \end{aligned} \tag{5.44}$$

As a corollary of this theorem, any steady state $Y^1 = (H_e, 0, 0, D^1)$ can be reached from any other steady state $Y^0 = (H_e, 0, 0, D^0)$.

Main ideas of the proof of Theorem 39. For simplicity, let us forget around the variables s and D. Without loss of generality, we may assume that $H_e = g = L = 1$. Again the linearized control system around $(H, v, u) := (1, 0, 0)$ is not controllable. Indeed, this linearized control system is

$$\Sigma_{\text{lin}} \qquad h_t + v_x = 0,\; v_t + h_x = -u(t),\; v(t,0) = v(t,1) = 0,$$

and, if we let

$$v_{\text{o}}(t,x) := \frac{1}{2}(v(t,x) - v(t,1-x)),\; h_{\text{e}}(t,x) := \frac{1}{2}(h(t,x) + h(t,1-x)),$$

one gets

$$h_{\text{e}t} + v_{\text{o}x} = 0,\; v_{\text{o}t} + h_{\text{e}x} = 0, v_{\text{o}}(t,0) = v_{\text{o}}(t,1) = 0.$$

Hence the control has no effect on $(h_{\text{e}}, v_{\text{o}})$, which shows that Σ_{lin} is far from being controllable (it misses an infinite dimensional space).

Again, one tries to use the return method (see Section 5.3). For this method one needs to find trajectories such that the the linearized control system around this trajectory is controllable. One can check that this is the case for the trajectory given by the following equilibrium point

$$H_\gamma(x) := 1 + \gamma(1/2 - x),\; v_\gamma(x) := 0,\; u_\gamma := \gamma.$$

where $\gamma \in \mathbb{R}$ is small but not 0. However this trajectory does not go from 0 to 0: from the controllability of the linearized control system around $(H_\gamma, v_\gamma, u_\gamma)$ one gets only the local controllability around (H_γ, v_γ), i.e., there exists an open neighborhood $\mathcal{N}$ of (H_γ, v_γ) such that, given two states in $\mathcal{N}$, there exists a trajectory of the control system Σ going from the first state to the secund one. This does not imply the local controllability around $(1, 0)$. However, let us assume that

(i) There exists a trajectory of the control system Σ going from $(1, 0)$ to $\mathcal{N}$,

(ii) There exists a trajectory of the control system Σ going from some point in $\mathcal{N}$ to $(1, 0)$.

Then it is not hard to then prove the desired local controllability around $(1, 0)$. In order to get the trajectory mentioned in (i) and (ii), one uses quasi-static deformations: For example, for (i), one fixes some functions $g : [0, 1] \to \mathbb{R}$ such that $g(0) = 0$, $g(1) = 1$ and consider for $\varepsilon \in (0, +\infty)$ the control $u_\varepsilon : [0, 1/\varepsilon] \to \mathbb{R}$ defined by

$$u_\varepsilon(t) := g(\varepsilon t).$$

Let us now start from $(1, 0)$ and uses the control u_ε. Then one can check that, uniformly in $t \in [0, 1/\varepsilon]$,

$$H(t, \cdot) - H_{g(\varepsilon t)} \to 0 \text{ and } v(t, \cdot) \to 0 \text{ as } \varepsilon \to 0.$$

In particular $(H(1/\varepsilon, \cdot), v(t, \cdot)$ in $\mathcal{N}$ for $\varepsilon \in (0, +\infty)$ small enough. This proves (i). The proof of (ii) is similar.

Remark 40 *There is in fact a problem of loss of derivatives (see Section 5.2) and it is not easy to deduce the local controllability around $(H_\gamma, v_\gamma, u_\gamma)$ from the controllability of the linearized control system around $(H_\gamma, v_\gamma, u_\gamma)$. To avoid the Nash-Moser method, a suitable fixed point method is used. This method requires the controllability "many" (essentially a family of codimension 4)) linear control systems which are close to the linear control system Σ_{lin}.*

Remark 41 *The quasi-static deformations works easily here since $(H_\gamma, v_\gamma, u_\gamma)$ are stable equilibriums. When the equilibriums are not stable, one can first stabilize them by using suitable feedback laws (see, in particular, [41] for semilinear heat equations and in [42] for semilinear wave equations.*

Remark 42 *Note that, due to the finite speed of propagation, it is natural that only large-time local controllability hods. However, for a Schrödinger analog of control system Σ, it is proved in [37] (see also [38, Remark 9.20, pages 269-270] that the small-time local controllability also does not hold, even if the Schrödinger equation has an infinite speed of propagation. (The proof of the large-time local controllability for this Schrödinger control system is due to Karine Beauchard [13]; see also [16] if one deals also with the variables S and D.)*

5.5 Power series expansion

Again, we present this method on an example. Let $L > 0$. Let us consider the following Korteweg-de Vries control system

$$y_t + y_x + y_{xxx} + yy_x = 0,\ t \in (0, T),\ x \in (0, L), \tag{5.45}$$
$$y(t, 0) = y(t, L) = 0,\ y_x(t, L) = u(t),\ t \in (0, T), \tag{5.46}$$

where, at time $t \in [0, T]$, the control is $u(t) \in \mathbb{R}$ and the state is $y(t, \cdot) : (0, L) \to \mathbb{R}$.

We are interested in the local controllability of the control system (5.45)-(5.46) around the equilibrium $(y_e, u_e) := (0, 0)$. Lionel Rosier has proved in [116] that this local controllability holds if

$$L \notin \mathcal{N} := \left\{ 2\pi\sqrt{\frac{j^2 + l^2 + jl}{3}};\, j \in \mathbb{N} \setminus \{0\},\, l \in \mathbb{N} \setminus \{0\} \right\}. \tag{5.47}$$

Lionel Rosier has got his result by first proving that, if $L \notin \mathcal{N}$, then the linearized control system around $(y_e, u_e) := (0, 0)$ is controllable. Note that $2\pi \in \mathcal{N}$ (take $j = l = 1$) and, as shown also by Lionel Rosier, if $L \in \mathcal{N}$, then the linearized control system around $(y_e, u_e) := (0, 0)$ is not controllable. However the nonlinear yy_x helps to recover the local controllability: One has the following theorem

Theorem 43 ([39, Theorem 2]) *Let $T > 0$ and let $L = 2\pi$. (Thus, in particular, $L \in \mathcal{N}$.) Then there exist $C > 0$ and $r_1 > 0$ such that for any y^0, $y^1 \in L^2(0, L)$, with $\|y^0\|_{L^2(0,L)} < r_1$ and $\|y^1\|_{L^2(0,L)} < r_1$, there exist*

$$y \in C^0([0, T]; L^2(0, L)) \cap L^2((0, T); H^1(0, L))$$

and $u \in L^2(0, T)$ satisfying (5.45)-(5.46)*, such that*

$$y(0, \cdot) = y^0, \tag{5.48}$$

$$y(T, \cdot) = y^1, \tag{5.49}$$

$$\|u\|_{L^2(0,T)} \leqslant C(\|y^0\|_{L^2(0,L)} + \|y^1\|_{L^2(0,L)})^{1/3}. \tag{5.50}$$

Main ideas of the proof of Theorem 43. The proof relies on some kind of power series expansion. Let us just explain the method on the control system of finite dimension

$$\dot{y} = f(y, u), \tag{5.51}$$

where the state is $y \in \mathbb{R}^n$ and the control is $u \in \mathbb{R}^m$. Here f is a function of class C^∞ on a neighborhood of $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ and we assume that $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ is an equilibrium of the control system (5.51), i.e $f(0, 0) = 0$. Let

$$H := \operatorname{Span}\{A^i B u;\, u \in \mathbb{R}^m,\, i \in \{0, \ldots, n-1\}\}$$

with

$$A := \frac{\partial f}{\partial y}(0, 0),\ B := \frac{\partial f}{\partial u}(0, 0).$$

If $H = \mathbb{R}^n$, the linearized control system around $(0, 0)$ is controllable and therefore the nonlinear control system (5.51) is small-time locally controllable at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$. Let us look at the case where the dimension of H is $n - 1$. Let us make a (formal) power series expansion of the control system (5.51) in (y, u) around the constant trajectory $t \mapsto (0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$. We write

$$y = y^1 + y^2 + \ldots,\ u = u^1 + u^2 + \ldots.$$

The order 1 is given by (y^1, u^1); the order 2 is given by (y^2, u^2) and so on. The dynamics of these different orders are given by

$$\dot{y}^1 = \frac{\partial f}{\partial y}(0, 0)y^1 + \frac{\partial f}{\partial u}(0, 0)u^1, \tag{5.52}$$

$$\dot{y}^2 = \frac{\partial f}{\partial y}(0,0)y^2 + \frac{\partial f}{\partial u}(0,0)u^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,0)(y^1,y^1) + \frac{\partial^2 f}{\partial y \partial u}(0,0)(y^1,u^1) + \frac{1}{2}\frac{\partial^2 f}{\partial u^2}(0,0)(u^1,u^1), \quad (5.53)$$

and so on. Let $e_1 \in H^\perp$. Let $T > 0$. Let us assume that there are controls $u^1_\pm$ and $u^2_\pm$, both in $L^\infty((0,T);\mathbb{R}^m)$, such that, if $y^1_\pm$ and $y^2_\pm$ are solutions of

$$\dot{y}^1_\pm = \frac{\partial f}{\partial y}(0,0)y^1_\pm + \frac{\partial f}{\partial u}(0,0)u^1_\pm,$$
$$y^1_\pm(0) = 0,$$

$$\dot{y}^2_\pm = \frac{\partial f}{\partial y}(0,0)y^2_\pm + \frac{\partial f}{\partial u}(0,0)u^2_\pm + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,0)(y^1_\pm,y^1_\pm) + \frac{\partial^2 f}{\partial y \partial u}(0,0)(y^1_\pm,u^1_\pm) + \frac{1}{2}\frac{\partial^2 f}{\partial u^2}(0,0)(u^1_\pm,u^1_\pm),$$

$$y^2_\pm(0) = 0,$$

then

$$y^1_\pm(T) = 0,$$
$$y^2_\pm(T) = \pm e_1.$$

Let $(e_i)_{i\in\{2,\dots n\}}$ be a basis of H. By the definition of H and a classical result about the controllable part of a linear system (see e.g. [126, Section 3.3]), there are $(u_i)_{i=2,\dots,n}$, all in $L^\infty(0,T)^m$, such that, if $(y_i)_{i=2,\dots,n}$ are the solutions of

$$\dot{y}_i = \frac{\partial f}{\partial y}(0,0)y_i + \frac{\partial f}{\partial u}(0,0)u_i,$$
$$y_i(0) = 0,$$

then, for every $i \in \{2,\dots,n\}$,

$$y_i(T) = e_i.$$

Now let

$$b = \sum_{i=1}^{n} b_i e_i$$

be a point in $\mathbb{R}^n$. Let $u^1 \in L^\infty((0,T);\mathbb{R}^m)$ be defined by the following

- If $b_1 \geqslant 0$, then $u^1 := u^1_+$ and $u^2 := u^2_+$.
- If $b_1 < 0$, then $u^1 := u^1_-$ and $u^2 := u^2_-$.

Then let $u : (0,T) \to \mathbb{R}^m$ be defined by

$$u(t) := |b_1|^{1/2}u^1(t) + |b_1|u^2(t) + \sum_{i=2}^{n} b_i u_i(t).$$

Let $y : [0,T] \to \mathbb{R}^n$ be the solution of

$$\dot{y} = f(y, u(t)),\ y(0) = 0.$$

Then one has, as $b \to 0$,

$$y(T) = b + o(b). \tag{5.54}$$

Hence, using the Brouwer fixed-point theorem and standard estimates on ordinary differential equations, one gets the local controllability of $\dot{y} = f(y,u)$ (around $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$) in time T, that is, for every $\varepsilon > 0$, there exists $\eta > 0$ such that, for every $(a,b) \in \mathbb{R}^n \times \mathbb{R}^n$ with $|a| < \eta$ and $|b| < \eta$, there exists a trajectory $(y,u) : [0,T] \to \mathbb{R}^n \times \mathbb{R}^m$ of the control system (5.51) such that

$$y(0) = a,\ y(T) = b,$$
$$|u(t)| \leqslant \varepsilon,\ t \in (0,T).$$

We use this power series expansion method to get Theorem 43. In fact, for this theorem, an expansion to order 2 is not sufficient: we obtain the local controllability by means of an expansion up to order 3 (which makes the computations very lengthy).

Remark 44 *The power series expansion method has been used in the context of partial differential equations for the first time in [39]. It has then been used*

- *For the above Korteweg-de Vries control system and for the other values of $L \in \mathcal{N}$ by Eduardo Cerpa in [26] and by Eduardo Cerpa and Emmanuelle Crépeau in [27].*
- *For a Schrödinger equation in [16].*

6 Complements and further references

There are many importants problems which are not discussed in this paper. Perhaps the more fundamental ones are optimal control theory and the stabilization problem. For the optimal control theory, see references already given in Section 4.5. The stabilization problem is the following one. We have an equilibrium which is unstable (or not enough stable) without the use of the control. Let us give a concrete example. One has a stick that is placed vertically on one of his fingers. In principle, if the stick is exactly vertical with a speed exactly equal to 0, it should remain vertical. But, due to various small errors (the stick is not exactly vertical, for example), in practice, the stick falls down. In order to avoid this, one moves the finger in a suitable way, depending on the position and speed of the stick; one uses a "*feedback law*" (or "*closed-loop control*") which stabilizes the equilibrium. The problem of the stabilization is the existence and construction of such stabilizing feedback laws for a given control system. More precisely, let us consider the control system (1.1) and let us assume that $f(0,0) = 0$. The stabilization problem is to find a feedback law $y \to u(y)$ such that 0 is asymptotically stable for the closed loop system $\dot{y} = f(y, u(y))$.

Again, as for the controllability, the first step to study the stabilization problem is to look at the linearized control system at the equilibrium. Roughly speaking one expects that a linear feedback which stabilizes (exponentially) the linearized control systems stabilizes (locally) the nonlinear control system. This is indeed the case in many important situations. For example, for the Navier control system mentioned in Section, see in particular [9] by Viorel Barbu, [11] by Viorel Barbu and Roberto Triggiani, [10] by Viorel Barbu, Irena Lasiecka and Roberto Triggiani, [58] by Andrei Fursikov, [113, 114] by Jean-Pierre Raymond, and [132].

When the linearized control system cannot be stabilized it still may happen that the nonlinearity helps. This for example the case for the Euler control system (2.16-(2.17)-(2.17): see [35], and [64] by Olivier Glass.

The most popular approach to construct stabilizing feedbacks relies on Lyapunov functions. See [38, Chapter 12] for various methods to design Lyapunov functions.

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