Homework II: Solutions

1. Repeatedly differentiating the Riesz map we get

$$f^{(k)}(s) = \frac{(-1)^k k!}{2\pi j} \int_{-\infty}^{\infty} \frac{f(j\omega)}{(s - j\omega)^{k+1}}\, d\omega, \qquad s \in \Pi.$$

Evaluating these equations at $s = a$,

$$f^{(k)}(a) = \frac{(-1)^k k!}{j} (f, \widetilde{B}_k), \qquad k = 1, \cdots$$

where $\widetilde{B}_k(s) = (s + a)^{-k}$, $k \geq 1$. If the Laguerre functions were not complete in $\mathcal{H}_2(\Pi)$, then there would be a non-trivial $f \in \mathcal{H}_2(\Pi)$ satisfying $(f, B_k) = 0$ for all k. Since the linear span of the functions $\widetilde{B}_n$ are equal to the linear span of the Laguerre functions, this implies that f is orthogonal to $\widetilde{B}_n$ for all n. But, this means that all the derivatives of $f(s)$ vanish at $a \in \Pi$ forcing $f \equiv 0$. Hence, we reach a contradiction.

2. Under the bilinear transformation $s = \psi(z) = \frac{z-1}{z+1}$, the closed right-half plane is mapped onto the exterior of the open unit disk and the imaginary axis is mapped onto the unit circle. The bilinear map also preserves the minimality of the transformed realization. Thus, if $g(z) = f(\psi(z))$, then the corresponding discrete-time realization of $g(z)$ is stable and minimal. Now, the results in Akçay and Türkay (SIMAX,2007) can be applied. Expressions for the derivatives of $g(z)$ at the points $z_k = \psi(s_k)$ are obtained by applying the chain rule of differentiation to $g = f \circ \psi$.

3. From the constraint and the substitutions, note that

$$\begin{pmatrix} P - APA^T & G - APC^T \\ G^T - CPA^T & \Lambda_0 - CPC^T \end{pmatrix} \geq 0.$$

Now, the result follows from Theorem 7.1 in Katayama.

4. (a) Fix k and let $K_k = (x_1, x_2, \cdots)$. Put $f = e_n$ in $f(k) = (f, K_k)$. Then,

$$\delta_{nk} = e_n(k) = K_k(n) = x_n.$$

As n is varied, we have $x_n = 1$ if $n = k$ and 0 otherwise. It follows that $K_k = e_k$. Now, from $K(m,n) = K_n(m)$ we get $K(m,n) = e_n(m) = \delta_{mn}$ for all m, n.

(b) Let $f = \sum_{n=0}^{\infty} f_n z^n \in \mathcal{H}^2$. Note that for each $\beta \in \mathbf{C}$ with modulus less than one, $g(z) = \sum_{k=0}^{\infty} \overline{\beta}^n z^n \in \mathcal{H}^2$. Thus, for all $f \in \mathcal{H}^2$

$$(f, g) = \sum_{n=0}^{\infty} f_n \beta^n = f(\beta)$$

which shows $K_\beta(z) = g(z)$. Let α, β be inside the unit circle. Then,

$$K(\alpha, \beta) = K_\beta(\alpha) = \sum_{k=0}^{\infty} \bar{\beta}^n \alpha^n = \frac{1}{1 - \alpha\overline{\beta}}.$$