Homework I: Solutions

1. Since $LT = I_r$, we get $P^2 = TLTL = TL = P$. Also, T and L are of full rank, so that $Im(P) = Im(TL) = Im(T)$ and $Ker(P) = Ker(TL) = Ker(L)$. This implies that P is the oblique projection onto $Im(T)$ along $Ker(L)$. Similarly, we can prove that Q is a projection.

2. Define $L = [L_1 \;\; L_2]$ and $V = [V_1 \;\; V_2]$. Since $\begin{bmatrix} L \\ V \end{bmatrix} [T \;\; U] = \begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix}$, we have

$$\begin{bmatrix} L_1 & L_2 \\ V_1 & V_2 \end{bmatrix} \begin{bmatrix} I_r & -X \\ 0 & I_{n-r} \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix}.$$

This implies that $L_1 = I_r$, $L_2 = X$, $V_1 = 0$, $V_2 = I_{n-r}$, and hence

$$P = \begin{bmatrix} I_r & X \\ 0 & 0 \end{bmatrix}.$$

3. Since the orthogonal projection is expressed as $\hat{E}\{A|B\} = KB$, $K \in \mathbf{R}^{p\times q}$, the optimality condition is reduced to $A - KB \perp B$. Hence, we have

$$(A - KB)B^T = 0 \Rightarrow K = (AB^T)(BB^T)^\dagger$$

showing that $\hat{E}\{A|B\} = (AB^T)(BB^T)^\dagger B$.

4. Since $Q_1^T Q_2 = 0$, two terms in the right-hand side of $A = L_{21}Q_1^T + L_{22}Q_2^T$ are orthogonal. From $B = L_{11}Q_1^T$ with B full row rank, we see that L_{11} is nonsingular and Q_1^T forms a basis of the space spanned by the row vectors of B. It therefore follows that $\hat{E}\{A|B\} = L_{21}Q_1^T = L_{21}Q_1^T = L_{21}L_{11}^{-1}B$. Also, from $AQ_1 = L_{21}$, we get $\hat{E}\{A|B\} = A(Q_1Q_1^T)$. since $L_{22}Q_2^T$ is orthogonal to the row space of B, it follows that $\hat{E}\{A|B^\perp\} = L_{22}Q_2^T$.