Parallel generation of pseudo-random sequences

Who?

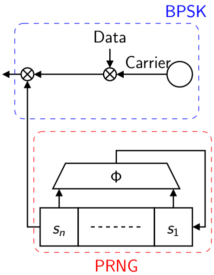
$$\begin{aligned}&1100001010100110001001100100111010010110110001100000\\&0100001100101000011010101110010011101000011000100110\\&1111011010101110000011110\cdots =\end{aligned}$$

$$-\frac{(\text{Cedric Lauradoux})_{10}}{19935245913182750153280416113442150364601400879 63}$$

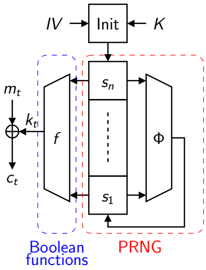
When?

14/10/2008 (simply today)

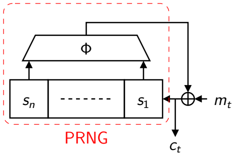
Applications of sequences



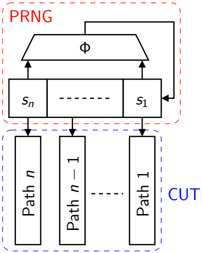
Spread spectrum



Stream ciphers



Data scrambler



Build In Self Test

Outline

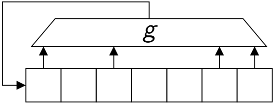
- Is it interesting to study shift register theory ?
- History of the parallel generation of m-sequences
  - *m-sequences*
  - *Decimation*
  - *Shift register transformations*
  - *The windmill generator*
- The extended windmill generator
  - *PFB transformation and windmill generator*
  - *NLFSRs*
- Wind to water: the case of ℓ-sequences
  - *ℓ-sequences*
  - *the watermill*
- Conclusions

Introduction

Is it interesting to study shift register theory ?

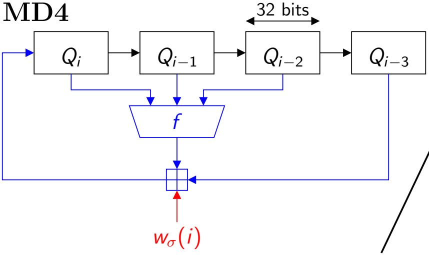
Sequences the backbone of symmetric cryptography: more precisely Non-Linear Feedback Shift Registers.

NLFSRs

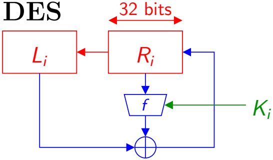


Problems:
- Period
- alphabet
- speed

MD4



DES



Introduction

Is it interesting to study shift register theory ?

Remenbering some discussion:

[Student] How to choose the parameters for a PRNG ?

[Advisor] Well, there exist security parameters like a proven period, the size or the number of taps in the feedback...

[Student] Okay, but there is still many candidates that meet the criteria. So what is the next step ?

[Advisor] Do you know how to roll a dice ?

Example

- $\frac{1+x}{1+x+x^2} = 11011011011\cdots$
- $\frac{1+x+x^2}{1+x+x^4} = 01111010110\cdots$
- $\frac{1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8}{1+x^6+x^8} = 11111111000000110000\cdots$

If we have $a(x) = \sum_{i=0}^{\infty} a_i X^i = \frac{p(x)}{q^\star(x)}$:

$$a_i = Tr(p(x)\alpha^i).$$

History of the parallel generation of *m*-sequences

Definitions

Theorem *Let $S = (s_i)$ an infinite sequence. S is periodic iff $\exists\ p$ and $q, q^\star(0) \neq 0$, $deg(p) \leq deg(q^\star)$ such that $s(x) = p(x)/q^\star(x)$.*

Theorem *If p and $q^\star$ are relatively prime, the period T of $s(X) = p(x)/q^\star(x)$ is the order of $q(x)$.*

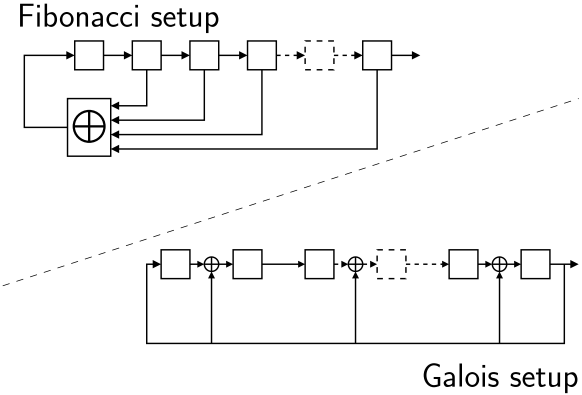
Result *If $q^\star(x)$ is* **primitive**, *i.e.* **irreductible** *and $ord(q(x)) = 2^m - 1$, then $T = 2^m - 1$ with $m = deg(q^\star(x))$.*

Comment *$q^\star(x)$ is the* **characteristic polynomial** *of S defined as the reciprocical of the connection/feedback polynomial $q(x)$:*

$$q^\star(x) = x^n q(\frac{1}{x}).$$

History of the parallel generation of *m*-sequences

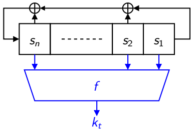
Linear Feedback Shift Registers (LFSRs)



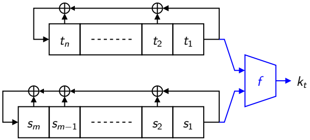
History of the parallel generation of *m*-sequences

The stream ciphers of our grandfathers

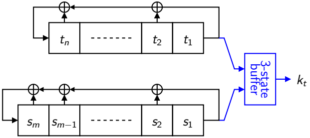
The filter generator



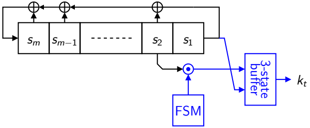
The combiner generator



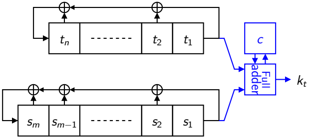
The shrinking generator



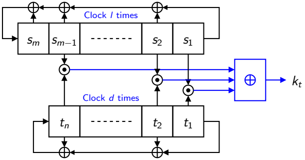
The self shrinking generator



The summation generator



The Multispeed inner product generator



Let S be an infinite sequence over an alphabet $\mathcal{A}$:

$$S = s_0, s_1, s_2 \cdots$$

For an integer v, a *v–decimation* of S is the set of sub-sequences defined by:

$$\begin{array}{ccc} S_v^0 & = & (s_0, s_v, \cdots) \\ S_v^1 & = & (s_1, s_{1+v}, \cdots) \\ \vdots & \vdots & \vdots \\ S_v^{v-2} & = & (s_{v-2}, s_{2v-2}, \cdots) \\ S_v^{v-1} & = & (s_{v-1}, s_{2v-1}, \cdots)\,. \end{array}$$

History of the parallel generation of m-sequences

4 solutions

- Strict decimation
- Parallel feedforward transformation (PFF)
- Parallel feedback transformation (PFB)
- Windmill generator

History of the parallel generation of m-sequences

Strict decimation

Theorem **[Zierler1959,Rueppel1986]**. *Let S be a sequence produced by an LFSR whose feedback polynomial $q(x)$ is irreducible in $\mathbf{F}_2$ of degree n. Let α be a root of $q(x)$ and let T be the period of $q(x)$. Let S_v^i be a sub-sequence resulting from the v-decimation of S. Then, S_v^i can be generated by an LFSR with the following properties:*

- *The minimum polynomial of α^v in $\mathbf{F}_{2^m}$ is the connection polynomial $q'(x)$ of the resulting LFSR.*
- *The period T' of $q'(x)$ is equal to $\frac{T}{gcd(v,T)}$.*
- *The degree n' of $q'(x)$ is equal to the multiplicative order of $q(x)$ in $\mathbf{Z}_{T'}$.*

History of the parallel generation of m-sequences

PFB transformation

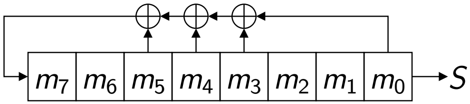
Notation

	Memory cell	*Content*
One register	m_i	$(m_i)_t$
Many registers	m_i^k *of* R_k	$(m_i^k)_t$

Example

Let consider the LFSR defined by the following relations:

$$
\begin{aligned}
(m_7)_{t+1} &= (m_3)_t \oplus (m_4)_t \oplus (m_5)_t \oplus (m_0)_t \\
(m_i)_{t+1} &= (m_{i+1}) \ \textit{if} \ i \neq 7.
\end{aligned}
$$



History of the parallel generation of m-sequences

PFB transformation

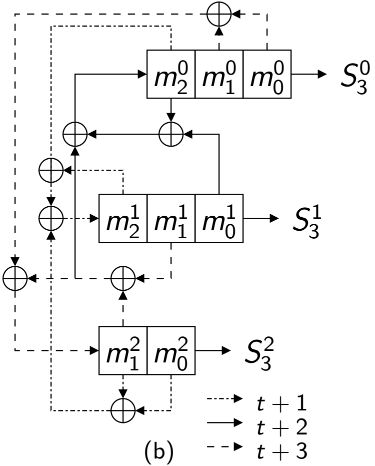
The PFB transformation virtually clocks an LFSR v-times.

Thus, we need to implements the previous equations for the successive states $(m_7)_{t+j}$ for $1 \leq j \leq v$ ($v = 3$):

$$\begin{aligned}
(m_7)_{t+1} &= (m_3)_t \oplus (m_4)_t \oplus (m_5)_t \oplus (m_0)_t \\
(m_7)_{t+2} &= (m_4)_t \oplus (m_5)_t \oplus (m_6)_t \oplus (m_1)_t \\
(m_7)_{t+3} &= (m_5)_t \oplus (m_6)_t \oplus (m_7)_t \oplus (m_2)_t \\
(m_i)_{t+3} &= (m_{i+3})_t \text{ if } i < 5.
\end{aligned}$$

History of the parallel generation of m-sequences

PFB transformation



(b)

Well, it is a bloody mess !

History of the parallel generation of m-sequences

The windmill generator

Theorem

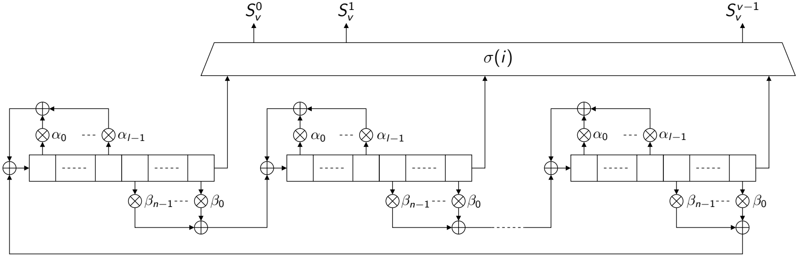
[Smeets1988] *Let n and v be integers such that $1 \leq v < n$. Let $\alpha(x) = \sum \alpha_i x^i$ and $\beta(x^{-1}) = \sum \beta_i x^{-i}$ be two polynomials over $\mathbf{F}_k$ such that $\alpha(0) = 1$ and $\beta(0) \neq 1$. There exist a permutation σ of $1, 2 \cdots v-1$ and a length parameters $\ell(i)$ such that the polynomial defined by:*

$$q(x) = \alpha(x^v) - \beta(x^{-v} x^n)$$

is the primitive feedback polynomial of the sequence S associated to the generator shown on the next slide!

History of the parallel generation of m-sequences

The windmill generator

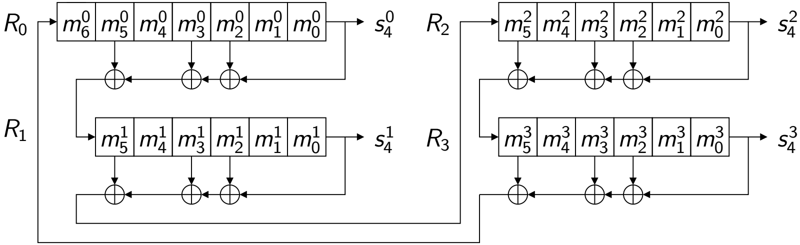


History of the parallel generation of m-sequences

The windmill generator

The windmill generator has been used in the E0 stream cipher (Bluetooth):

Four LFSRs ⇒ Four 4-vane windmills



$$q(x) = x^{25} + x^{20} + x^{12} + x^8 + 1$$

History of the parallel generation of m-sequences

The windmill generator

v	4		8		16	
n	$\#_{pri}$	$\#_{irr}$	$\#_{pri}$	$\#_{irr}$	$\#_{pri}$	$\#_{irr}$
9	1	1				
15	2	4				
17	28	28	0	0		
23	82	86	1	1	0	0
25	314	318	6	6	0	0
31	1063	1063	3	3	0	0
33	3285	4092	15	18	0	0
39	11482	13566	10	12	0	0
41	51144	51148	54	54	0	0
47	178253	178368	40	40	1	1
49	678916	684122	170	172	0	0
55	2229834	2439982	137	161	1	3

How to compute this table ?

Irreducibility test

Definition A polynomial $q \in \mathbf{F}_k[X]$ is irreducible, if $deg(q) > 0$ and if all the divisor of q is a constant or a multiple of q by a constant.

Algorithm	Worst case
Ben-Or	$nM(n) \log kn$
Rabin	$nM(n) \log k \log n$

- $M(n) = n \log n \log \log n$ (assuming FFT-based multiplication)

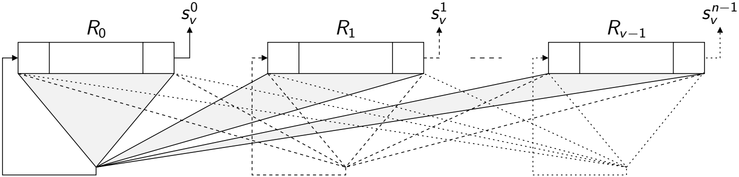
Comment *However, in practice we can expect to have $\log n \; M(n) \log kn$ with Ben-Or because a random polynomial is expected to have a factor of small degree.*

The extended windmill generator

PFB transformation and windmill generator

The feedback function F_i in the PFB transformation can be decomposed as the sum modulo two of v sub-functions $f_{i,j}$ which depends only of a given register R_j:

$$F_i = \bigoplus_{j=0}^{v-1} f_{i,j}.$$



The extended windmill generator

PFB transformation and windmill generator

Prop. *A v-vane windmill polynomial of degree n corresponds to a shift-registers network issue from a PFB transformation with at most 2 functions $f_{i,j}$ associated to the feedback function F_i, $0 \leq i < v$.*

Proof *The feedback function can be written:*

$$(m^k_{n-1})_{t+1} = \bigoplus_{i=0}^{\lfloor n/v \rfloor} \alpha_{vi+j-1}(m^{\sigma_1(k)}_{vi+j-1})_t \oplus \bigoplus_{j=0}^{\lfloor n/v \rfloor} \beta_{m-iv+j-1}(m^{\sigma_2(k)}_{m-vi+j-1})_t$$

with $k > n - v$ and σ_1 and σ_2 are two permutation of $1, 2 \cdots v - 1$ defined by:

$$\begin{aligned} \sigma_1(k) &= \lfloor \tfrac{n}{v} \rfloor + k - 1 \bmod v \\ \sigma_2(k) &= n + k \bmod v. \end{aligned}$$

The extended windmill generator

PFB transformation and windmill generator

Result *The windmill generator is only a subset of the PFB transformation with only 2 $f_{i,j}$ per F_i.*

How to find the others ?

- modify σ_1 ? not possible because $\alpha(0) \neq 0$.
- so modify σ_2:

$$\sigma_2'(k) = n + k - \phi \bmod v.$$

- if $\phi = 0 \leftarrow$ the orginal windmill setup
- if $n + k - \phi = 0 \bmod v \leftarrow$ the original setup with $\beta(x) = 1$
- otherwise new setup !

The extended windmill generator

New definition

Definition

The primitive polynomial

$$q(x) = \alpha(x^v) - x^{n-\phi}\beta(x^{-v}) - x^n$$

with $\alpha(0) \neq 0$, $\beta(x) \neq 0$ if $\phi = 0$ and $\beta(0) = 0$ otherwise and $0 \leq \phi < v$ defines the set of all PFB transformation with at most 2 functions $f_{i,j}$ associated to $F_i,\ 0 \leq i < v$ and generating m-sequences.

Is it a good news ? Yes, we can find good polynomials of degree $d = 3 \bmod 8$.

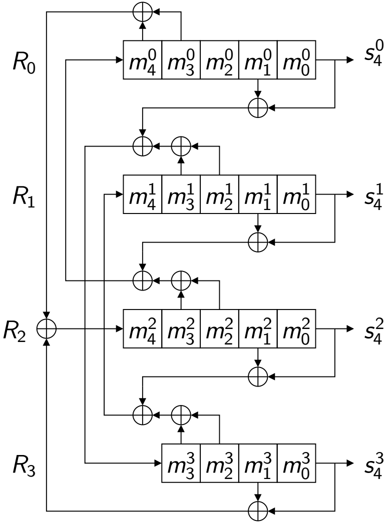
The extended windmill generator

New result

v	4		8		16	
n	$\#_{pri}$	$\#_{irr}$	$\#_{pri}$	$\#_{irr}$	$\#_{pri}$	$\#_{irr}$
9	1	1	0	0	0	0
11	1	1	0	0	0	0
13	6	6	0	0	0	0
15	9	12	0	0	0	0
17	38	38	2	2	0	0
19	31	31	3	3	0	0
21	39	41	2	2	0	0
23	172	179	4	4	0	0
25	479	491	19	19	0	0
27	238	281	4	5	0	0
29	571	573	2	2	0	0
31	2133	2133	16	16	0	0
33	4901	6100	34	46	3	3
35	3473	3702	18	18	4	4

The extended windmill generator

New result



$$q(x) = x^{19} + x^{13} + x^9 + x^4 + 1$$

Definition

The feedback functions of a non-linear non-singular extended windmill generator are defined by:

$$F_k = m_0^{\sigma_1(k)} \oplus g(m_{\alpha_{i_1}}^{\sigma_1(k)}, m_{\alpha_{i_2}}^{\sigma_1(k)}, \cdots, m_{\beta_{j_1}}^{\sigma_2(k)}, m_{\beta_{j_2}}^{\sigma_2(k)}, \cdots)$$

with g a Boolean function and:

$$\begin{aligned} \sigma_1(k) &= \lfloor \tfrac{n}{v} \rfloor + k - 1 \bmod v \\ \sigma_2(k) &= n + k - \phi \bmod v. \end{aligned}$$

Is it a good news ? Tt is an empty definition (choice for g: 2^{2^m})
but at least it is a research direction...

Example

- $\frac{1}{5} = \cdots 110011001101$
- $\frac{1}{7} = \cdots 010101010111$
- $-\frac{1}{7} = \cdots 1001001001001$

Definition

The canonical Hensel form of a 2-adic integer a is defined by:

$$a = \sum_{i=0}^{\infty} a_i 2^i.$$

If we have $\sum_{i=0}^{\infty} a_i 2^i = \frac{A}{q}$:

$$a_i = 2^{-i} A \mod q \mod 2.$$

Wind to water: the case of ℓ-sequences

Definitions

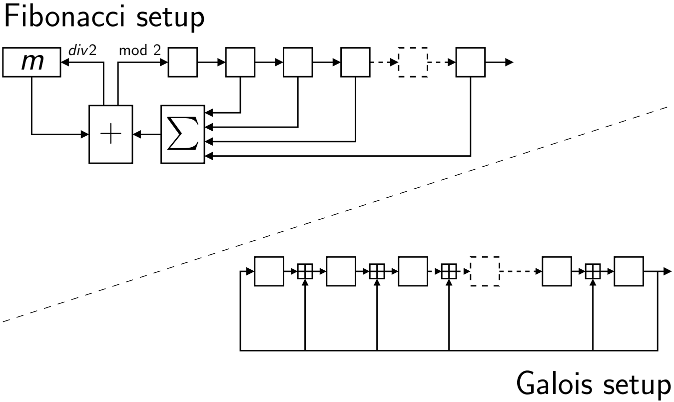
Theorem *$S = (s_i)$ an infinite sequence. S is periodic iff $\exists$ p and q,* **relatively prime**, *q odd such that $p/q = \sum_{i=0}^{\infty} s_i 2^i$ with $q < 0 \leq p$, $p \leq -q$.*

Theorem *If p and q are relatively prime, q odd, the period T of p/q is the order of 2 modulo q.*

Result *If q is well chosen, then $T = q - 1$.*

Wind to water: the case of ℓ-sequences

Feedback with Carry Shift Registers (FCSRs)



Wind to water: the case of ℓ-sequences

What about the 4 solutions ?

- Strict decimation: very bad **[Lauradoux2008]**
- Parallel feedforward transformation (PFF): not known. . .
- Parallel feedback transformation (PFB) **[Lauradoux2008]**
- Watermill generator **[Lauradoux2009 ?]**

Wind to water: the case of ℓ-sequences

The watermill generator

Let q be a prime number of maximal order such that:

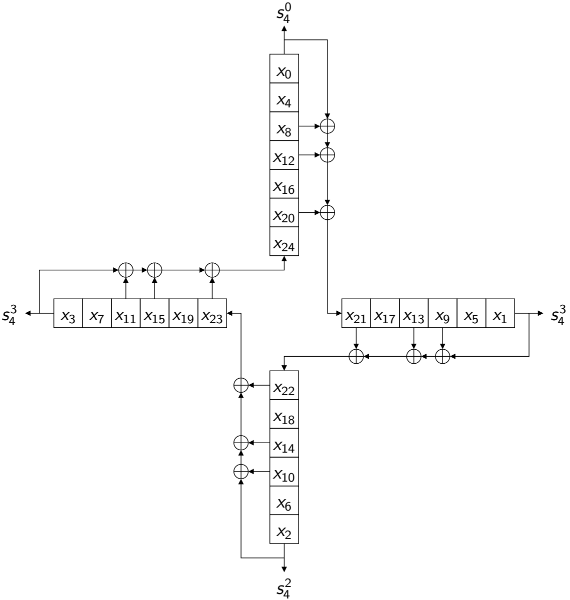
$$q = \alpha + 2^{n-\phi}\beta + 2^n$$

with $\alpha = \sum \alpha_i 2^{iv}$, $\alpha_0 = 1$ and $\beta = \sum \beta_i 2^{-iv} \cdots$

Conclusion

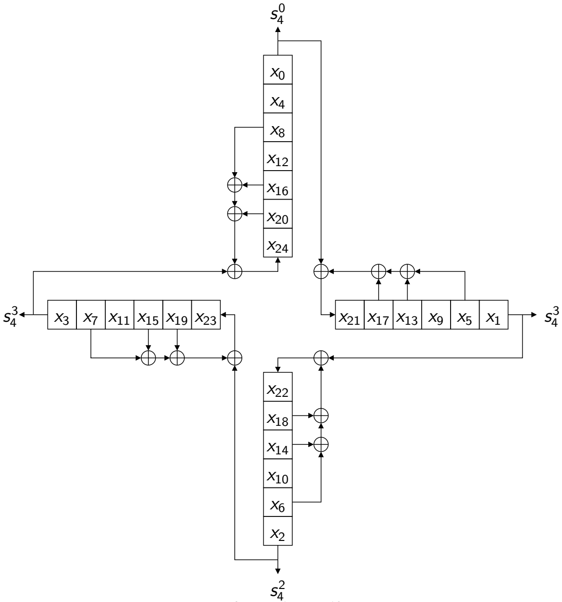
Why is it called the windmill generator ?

Conclusion



A. $f(x) = x^{25} + x^{20} + x^{12} + x^8 + 1$

Conclusion



B. $f(x) = x^{25} + x^{17} + x^{13} + x^5 + 1$