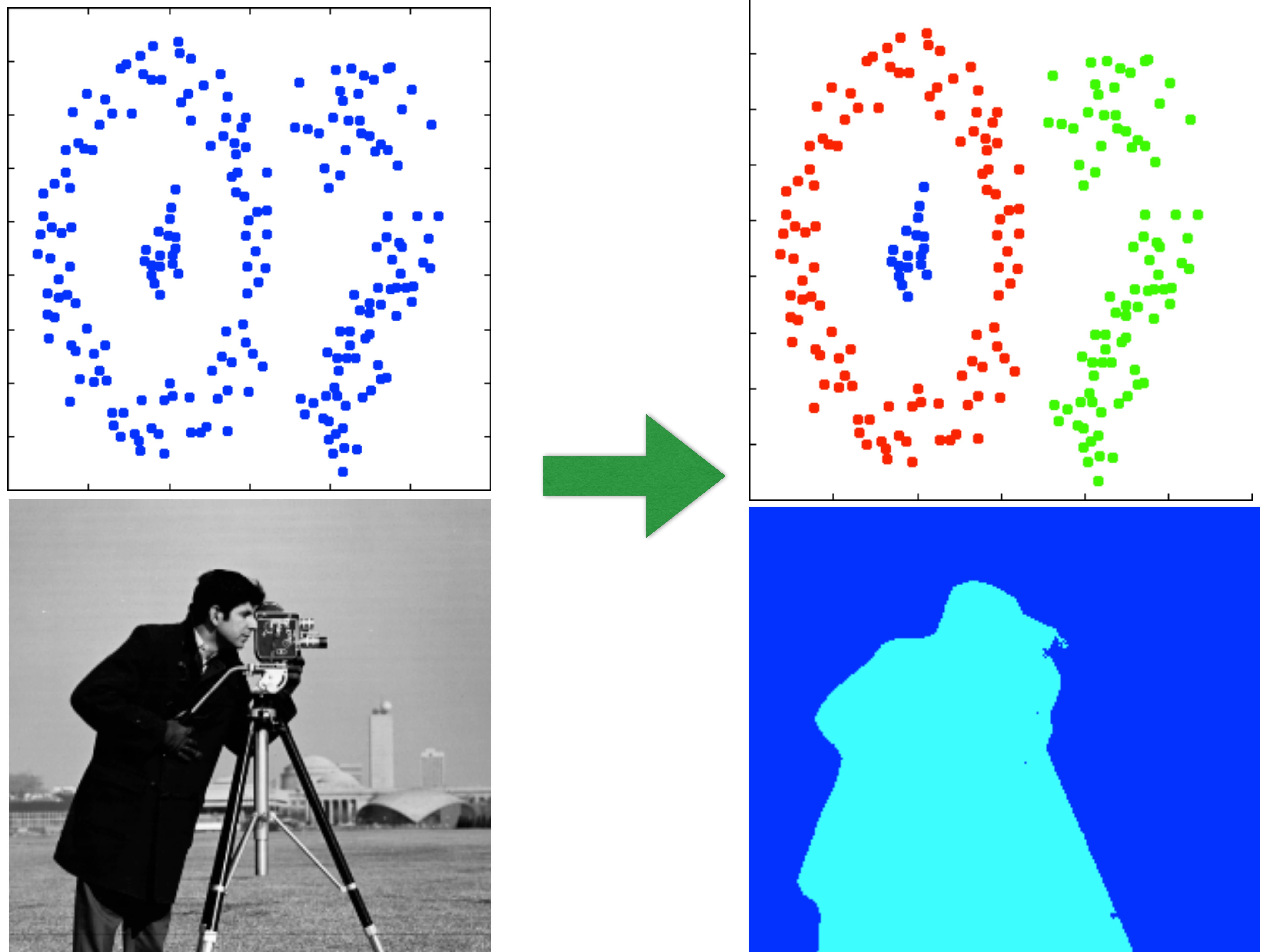
Normalized Cuts

Presented by:
Amit Kumar K.C.

Jean Gallier, “Spectral theory of Unsigned and Signed Graphs”



Normalized cuts and image segmentation

J Shi, J Malik - Pattern Analysis and Machine Intelligence, IEEE …, 2000 - ieeexplore.ieee.org

NEARLY 75 years ago, Wertheimer [24] pointed out the importance of perceptual grouping and organization in vision and listed several key factors, such as similarity, proximity, and good continuation, which lead to visual grouping. However, even to this day, many of the **...**

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Organization

- Introduction
- Graph clustering
  - Normalized cuts
- Signed graphs

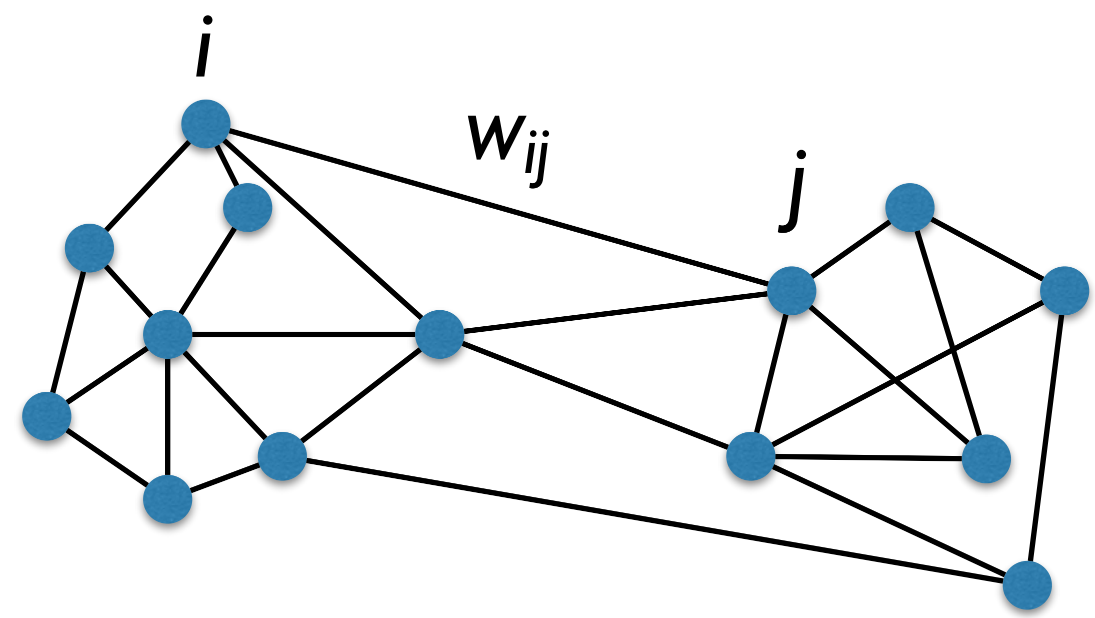
Few terminologies

- Adjacency matrix
- Degree
- Volume
- Graph cuts

Few terminologies

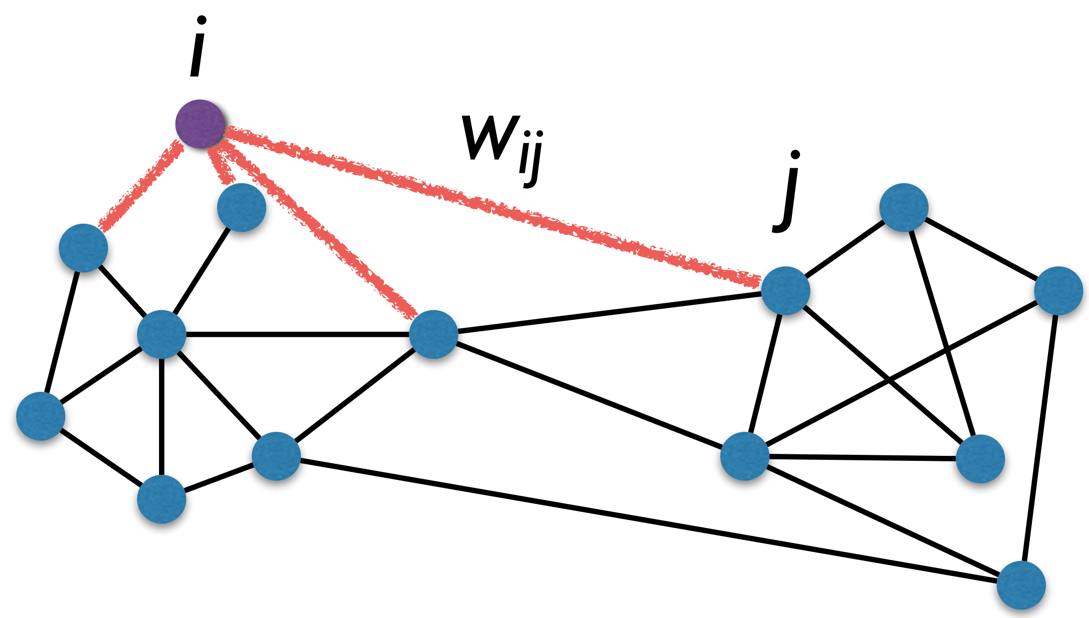
$G = (V, E, W)$

Adjacency matrix



Degree

$$d_i := \sum_{j=1}^{n} w_{ij}$$

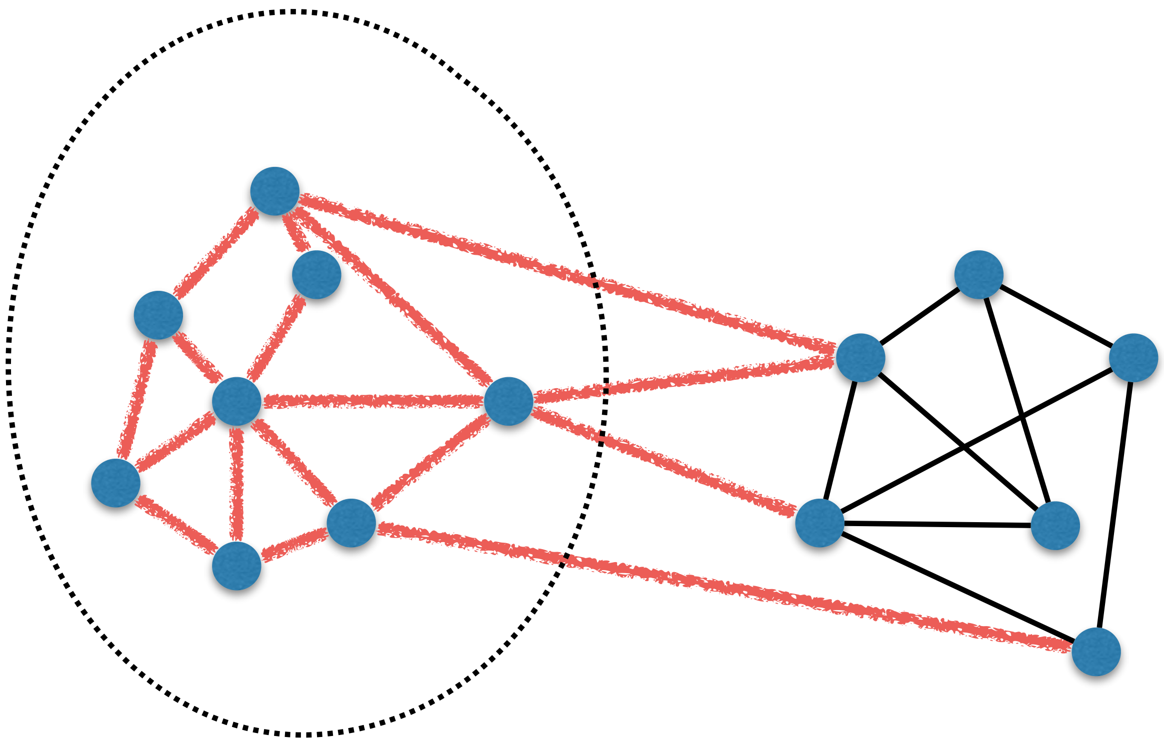


Degree matrix

$$D := \mathrm{diag}(d_1, \cdots, d_n)$$

Volume

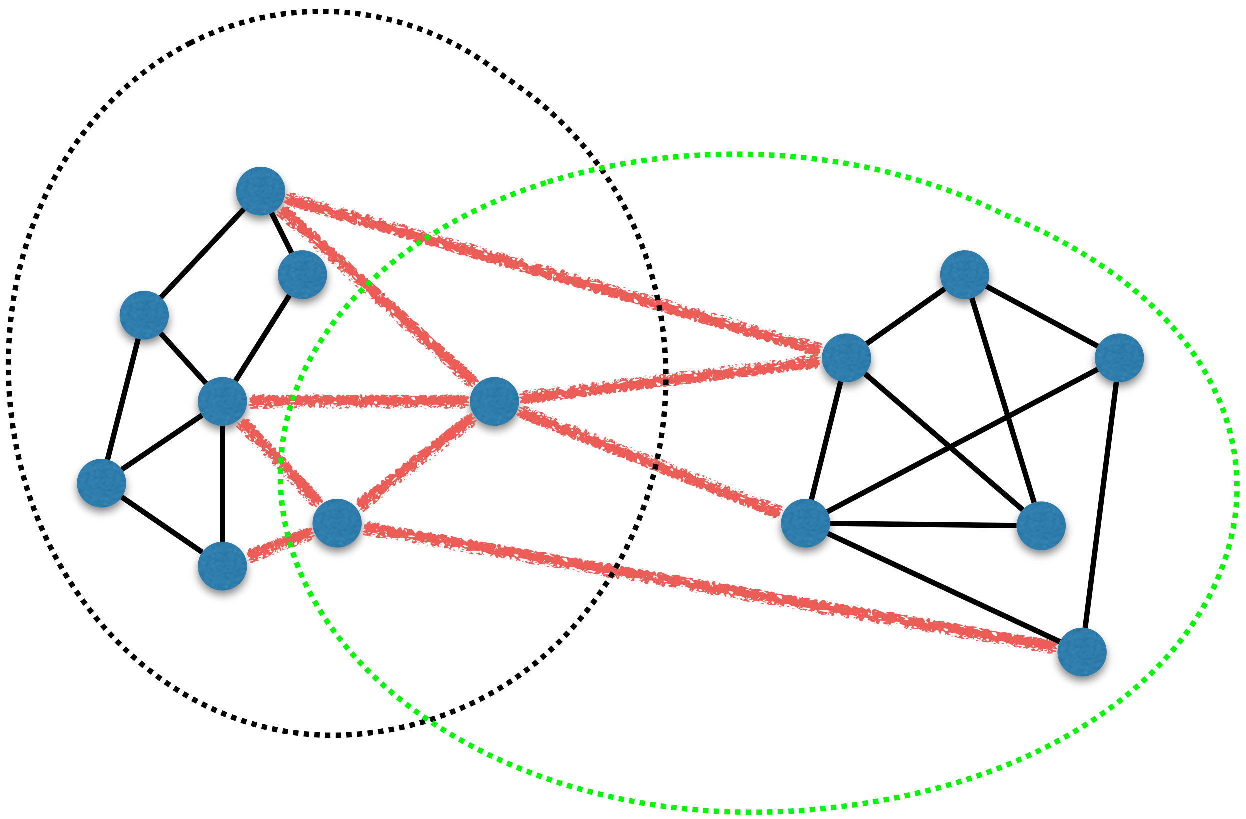
$$\mathrm{vol}(A) := \sum_{i \in A} d_i$$



$A \subseteq V$

Links

$$\text{links}(A, B) := \sum_{i \in A, j \in B} w_{ij}$$

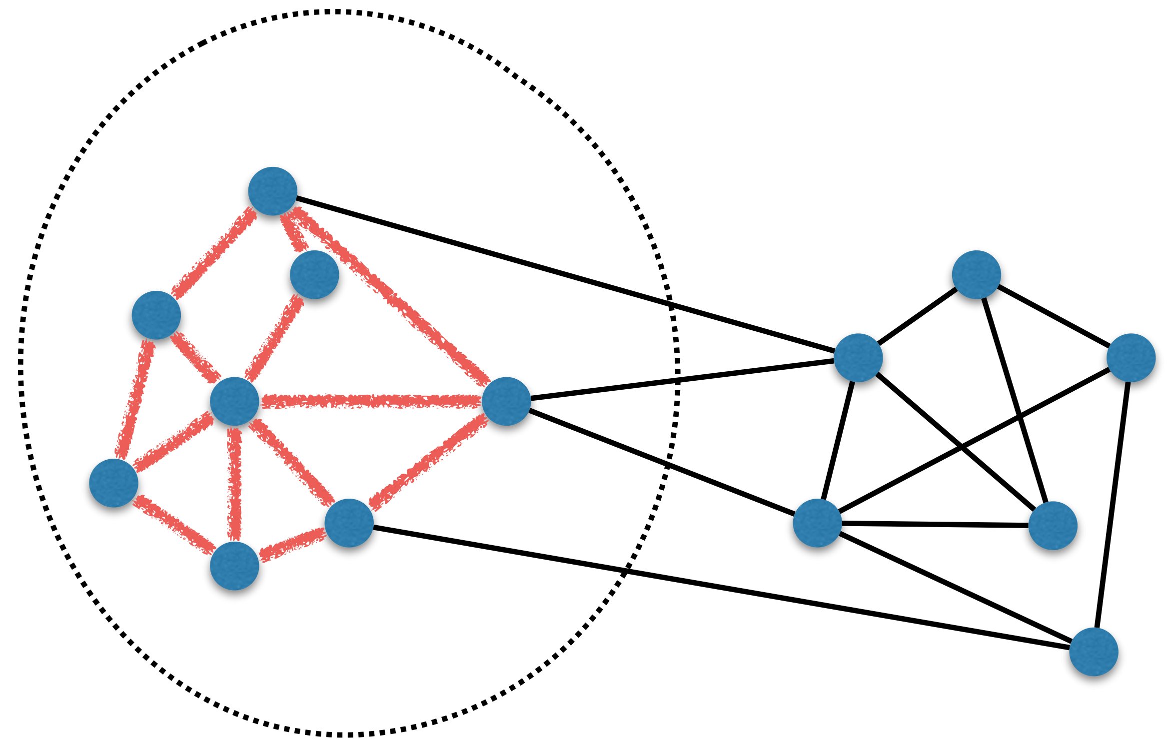


$A \subseteq V$ $B \subseteq V$

Association

$$\text{assoc}(A) = \text{links}(A, A)$$

$$= \sum_{i \in A, j \in A} w_{ij}$$



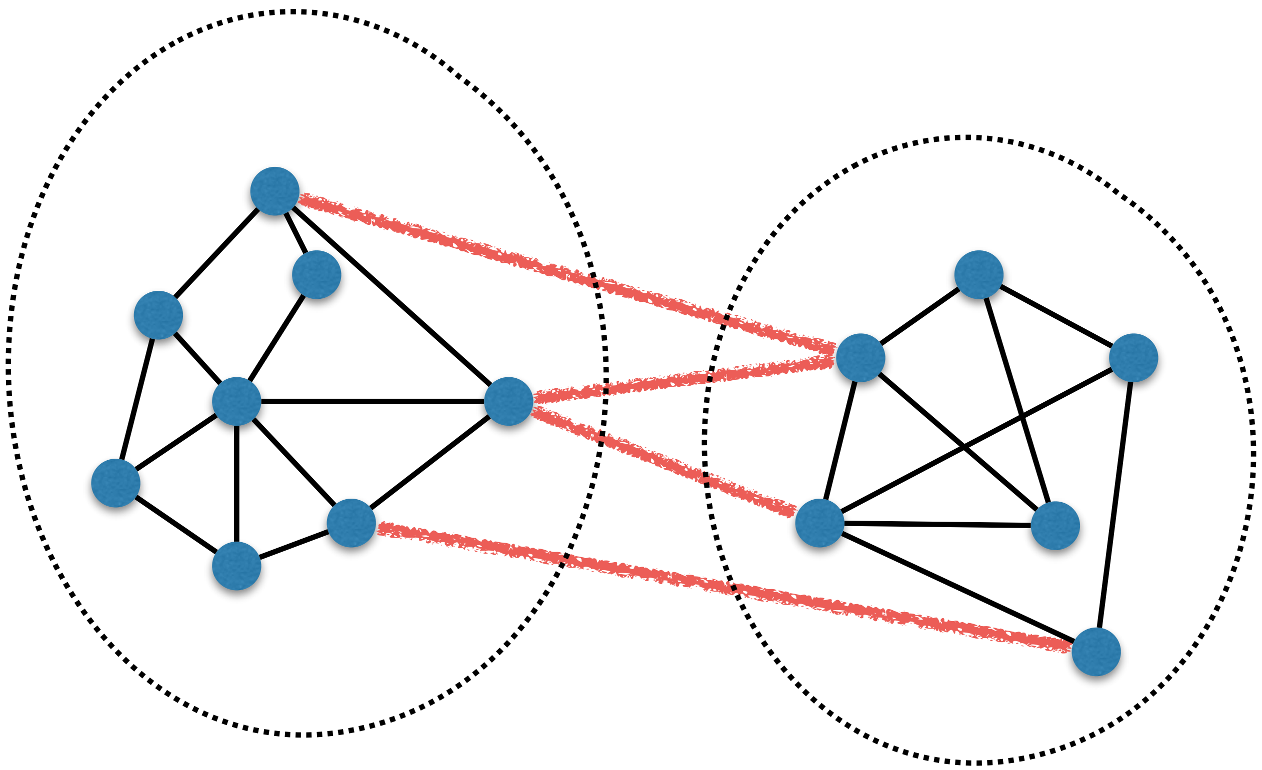
$A \subseteq V$

Prob. that a random walker (RW) remains within A

Cut

$$\mathbf{cut}(A) = \mathbf{links}(A, \bar{A})$$

$$= \sum_{i \in A, j \in \bar{A}} w_{ij}$$



$A \subseteq V$ $\qquad$ $\bar{A} := V \setminus A$

Prob. that a random walker (RW) jumps from A to $\bar{A}$

Quick question?

$$\text{cut}(A) \quad = \quad \text{vol}(A) \quad ? \quad \text{assoc}(A)$$

Quick question?

$$\mathrm{cut}(A) = \mathrm{vol}(A) - \mathrm{assoc}(A)$$

Graph Laplacian

$$L := D - W$$

Degree matrix

For every $x \in \mathbb{R}^{|V|}$ $\quad x^\top L x = \sum_{(i,j)\in E} w_{ij}(x_i - x_j)^2 \geq 0$

Graph clustering

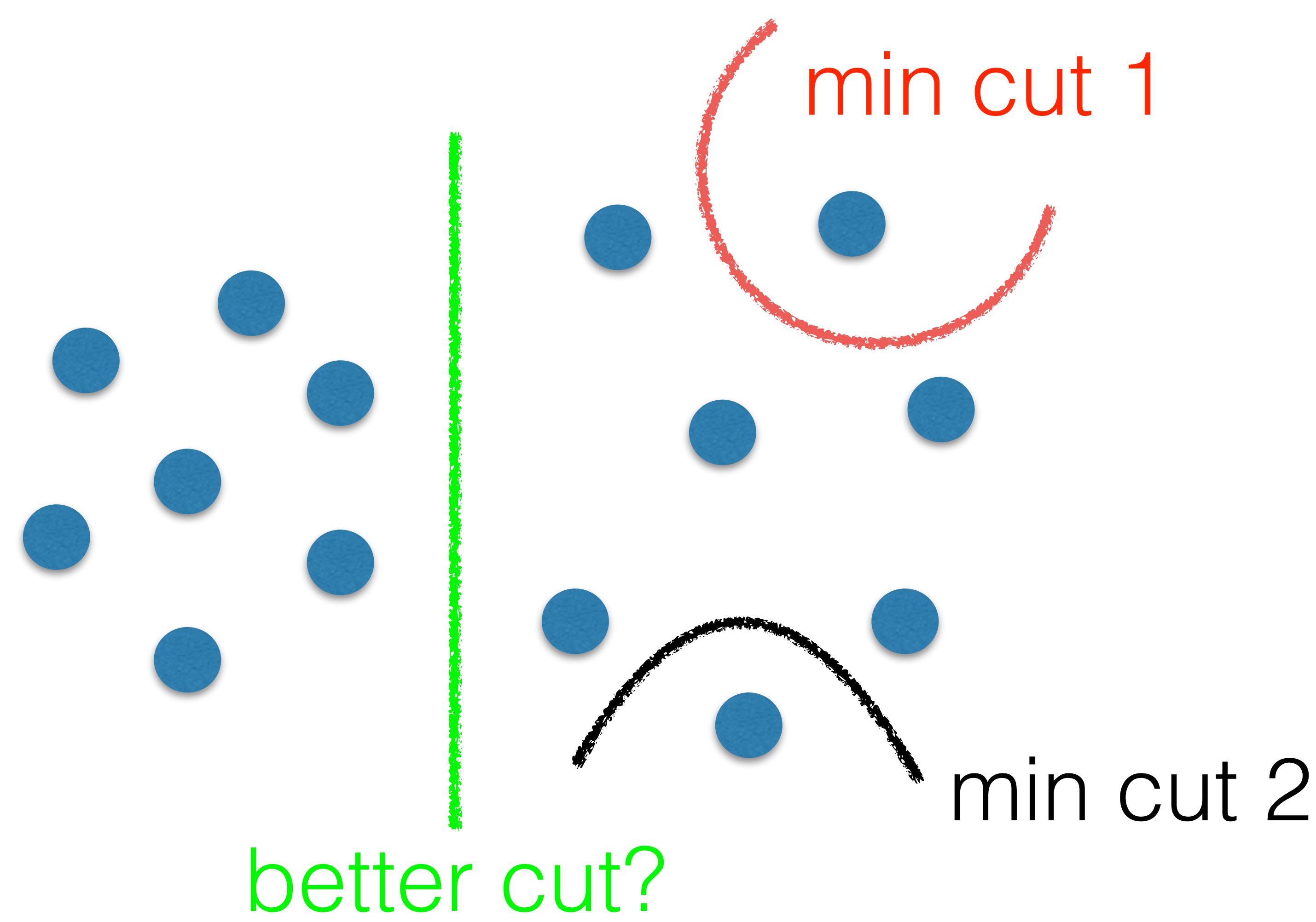
Find K partitions $(A_1, ..., A_K)$ such that

$$\mathbf{cut}(A_1, \cdots, A_K) = \sum_{j=1}^{K} \mathbf{cut}(A_j, \bar{A}_j)$$

is minimized.

Min-cut problem

Min-cut favours ‘smaller’ partitions



$$\mathbf{cut}(A, \bar{A}) = \sum_{i \in A, j \in \bar{A}} w_{ij}$$

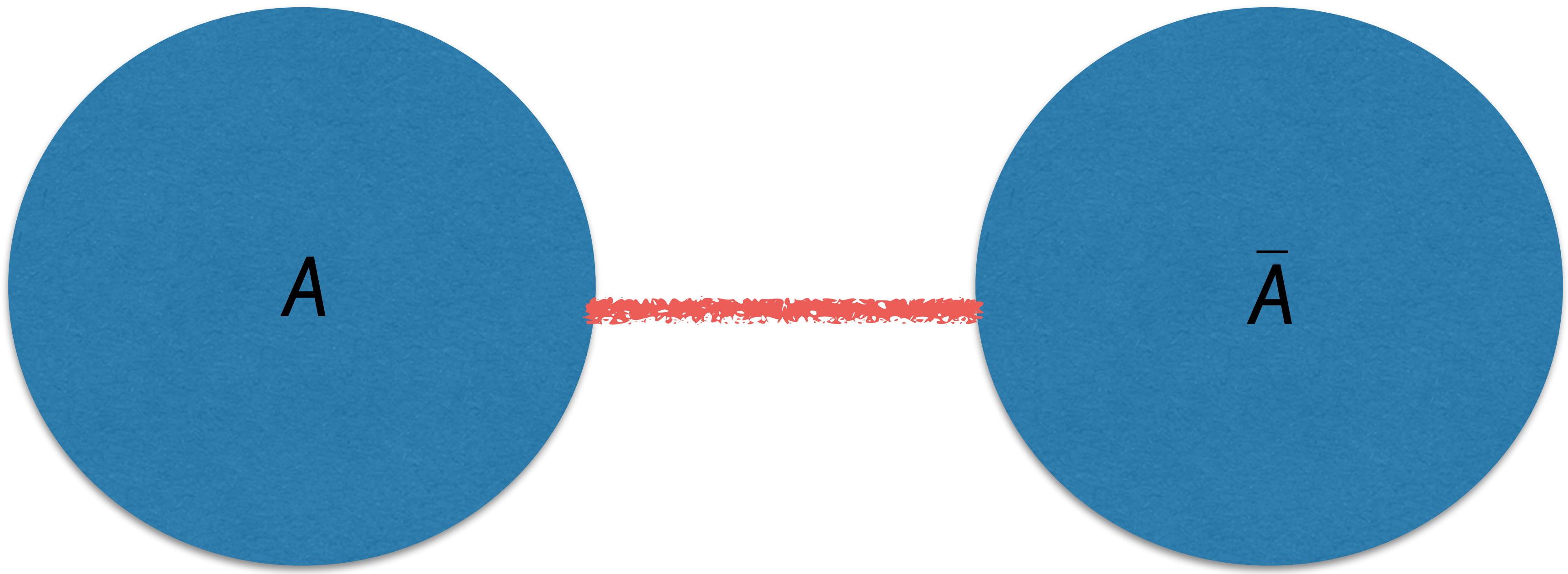
We need to 'normalize' the cut

Minimize $\dfrac{\mathbf{cut}(A, \bar{A})}{\mathbf{size}(A)}$

When $\mathbf{size}(A) = |A|$ Ratio Cut

$\mathbf{size}(A) = \mathbf{vol}(A)$ Normalized Cut

Interpretation



$\mathbf{cut}(A, \bar{A})$ prob. that a random walker will escape from A to $\bar{A}$

$\mathbf{vol}(A)$ prob. that a random walker will remain inside A

Formally,

$$\text{Ncut}(A, \bar{A}) := \frac{\text{cut}(A, \bar{A})}{\text{vol}(A)} + \frac{\text{cut}(\bar{A}, A)}{\text{vol}(\bar{A})}$$

$$= \text{cut}(A, \bar{A}) \left[\frac{1}{\text{vol}(A)} + \frac{1}{\text{vol}(\bar{A})} \right]$$

Undirected graphs only!

Binary partitioning

$$x_i = \begin{cases} a & \text{if } i \in A, \\ b & \text{if } i \in \bar{A} \end{cases}$$

$$x^\top L x = \frac{1}{2} \sum_{i,j=1}^{n} w_{ij} (x_i - x_j)^2 \quad = (b-a)^2 \text{cut}(A, \bar{A})$$

Binary partitioning

minimize $\dfrac{x^\top L x}{x^\top D x}$ subject to $x \in \{a, b\}^n$, $x^\top D \mathbf{1} = \mathbf{0}$

NP-complete!!

Binary partitioning

$$\text{minimize} \quad \frac{x^\top L x}{x^\top D x} \quad \text{subject to} \quad x \in \{a, b\}^n \;, \quad x^\top D \mathbf{1} = 0$$

Relax

$$\text{minimize} \quad \frac{x^\top L x}{x^\top D x} \quad \text{subject to} \quad \cancel{x \in \{a, b\}^n} \;, \quad x^\top D \mathbf{1} = 0$$

Binary partitioning

minimize $\dfrac{x^\top L x}{x^\top D x}$ subject to $x \in \{a, b\}^n$, $x^\top D \mathbf{1} = 0$

Relax

minimize $\dfrac{x^\top L x}{x^\top D x}$ subject to $\cancel{x \in \{a, b\}^n}$, $x^\top D \mathbf{1} = 0$

$y = D^{1/2} x$

minimize $\dfrac{y^\top D^{-1/2} L D^{-1/2} y}{y^\top y}$ subject to $y^\top D^{1/2} \mathbf{1} = 0$

Binary partitioning

$$\text{minimize} \quad \frac{y^\top D^{-1/2} L D^{-1/2} y}{y^\top y} \quad \text{subject to} \quad y^\top D^{1/2} \mathbf{1} = \mathbf{0}$$

$$y^\star = \quad \text{Second smallest eigenvector of} \quad D^{-1/2} L D^{-1/2}$$

$$x^\star = D^{-1/2} y^\star$$

Extending to K cases

minimize $\quad \mathbf{Ncut}(A_1, \cdots, A_K) = \sum_{j=1}^{K} \frac{\mathbf{cut}(A_j, \bar{A}_j)}{\mathbf{vol}(A_j)}$

subject to $\quad A_i \cap A_j = \emptyset$

$A_i \cup A_j = V$

$1 \leq i, j \leq K, i \neq j$

Extending to K cases

Define $$X_i^j = \begin{cases} a_j & \text{if } i \in A_j, \\ 0 & \text{if } i \notin A_j \end{cases}$$

Extending to K cases

Define $$X_i^j = \begin{cases} a_j & \text{if } i \in A_j, \\ 0 & \text{if } i \notin A_j \end{cases}$$

$$X = \begin{bmatrix} a_1 & 0 & 0 \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

$$X^1$$

Extending to K cases

Define $X_i^j = \begin{cases} a_j & \text{if } i \in A_j, \\ 0 & \text{if } i \notin A_j \end{cases}$

$$X = \begin{bmatrix} a_1 & 0 & 0 \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

X^1

$$\text{Ncut}(A_1, \cdots, A_K) = \sum_{j=1}^{K} \frac{\text{cut}(A_j, \bar{A}_j)}{\text{vol}(A_j)} = \sum_{j=1}^{K} \frac{(X^j)^\top L X^j}{(X^j)^\top D X^j}$$

Ensure $A_i \cap A_j = \emptyset$

Columns of X should be orthogonal

$$(X^i)^\top X^j = \mathbf{0} \quad \forall i \neq j$$

D is a diagonal matrix with positive entries

$$(X^i)^\top D X^j = \mathbf{0} \quad \forall i \neq j$$

Ensure $A_i \cup A_j = V$

Each vertex should appear in some partition.

Each row of X must have some non-zero entry.

Each diagonal element of $XX^\top$ must be non-zero

$$X(X^\top X)^{-1}X^\top 1_N = 1_N$$

Putting all together

minimize $$\mathbf{Ncut}(X) = \sum_{j=1}^{K} \frac{(X^j)^\top L X^j}{(X^j)^\top D X^j}$$

subject to $$(X^i)^\top D X^j = 0 \quad 1 \leq i, j \leq K, i \neq j$$

$$X(X^\top X)^{-1} X^\top \mathbf{1}_N = \mathbf{1}_N$$

$$X \in \mathcal{X}$$

Putting all together

minimize $\sum_{j=1}^{K} (X^j)^\top L X^j$

subject to $(X^i)^\top D X^j = \mathbf{0} \quad 1 \leq i, j \leq K, i \neq j$

$X(X^\top X)^{-1} X^\top 1_N = 1_N$

$X \in \mathcal{X}$

$(X^j)^\top D X^j = 1 \quad 1 \leq j \leq K$

Putting all together

minimize $\mathbf{tr}(X^\top L X)$

subject to $X^\top D X = I$

$$X(X^\top X)^{-1} X^\top \mathbf{1}_N = \mathbf{1}_N$$

$$X \in \mathcal{X}$$

Putting all together

minimize $\mathrm{tr}(Y^\top D^{-1/2} L D^{-1/2} Y)$

subject to $Y^\top Y = I$

$YY^\top D^{1/2}\mathbf{1} = D^{1/2}\mathbf{1}$

From continuous to discrete

Continuous solution X might not be
aligned with actual discrete solution Z

Find an orthogonal transform R to align X with Z

From continuous to discrete

$$\begin{aligned} \text{minimize} \quad & \phi(Z, R) = \|Z - XR\|_F^2 \\ \text{subject to} \quad & Z \in \{0, 1\}^{N \times K} \\ & R^\top R = I \end{aligned}$$

Alternatively solve for Z and R

Demo

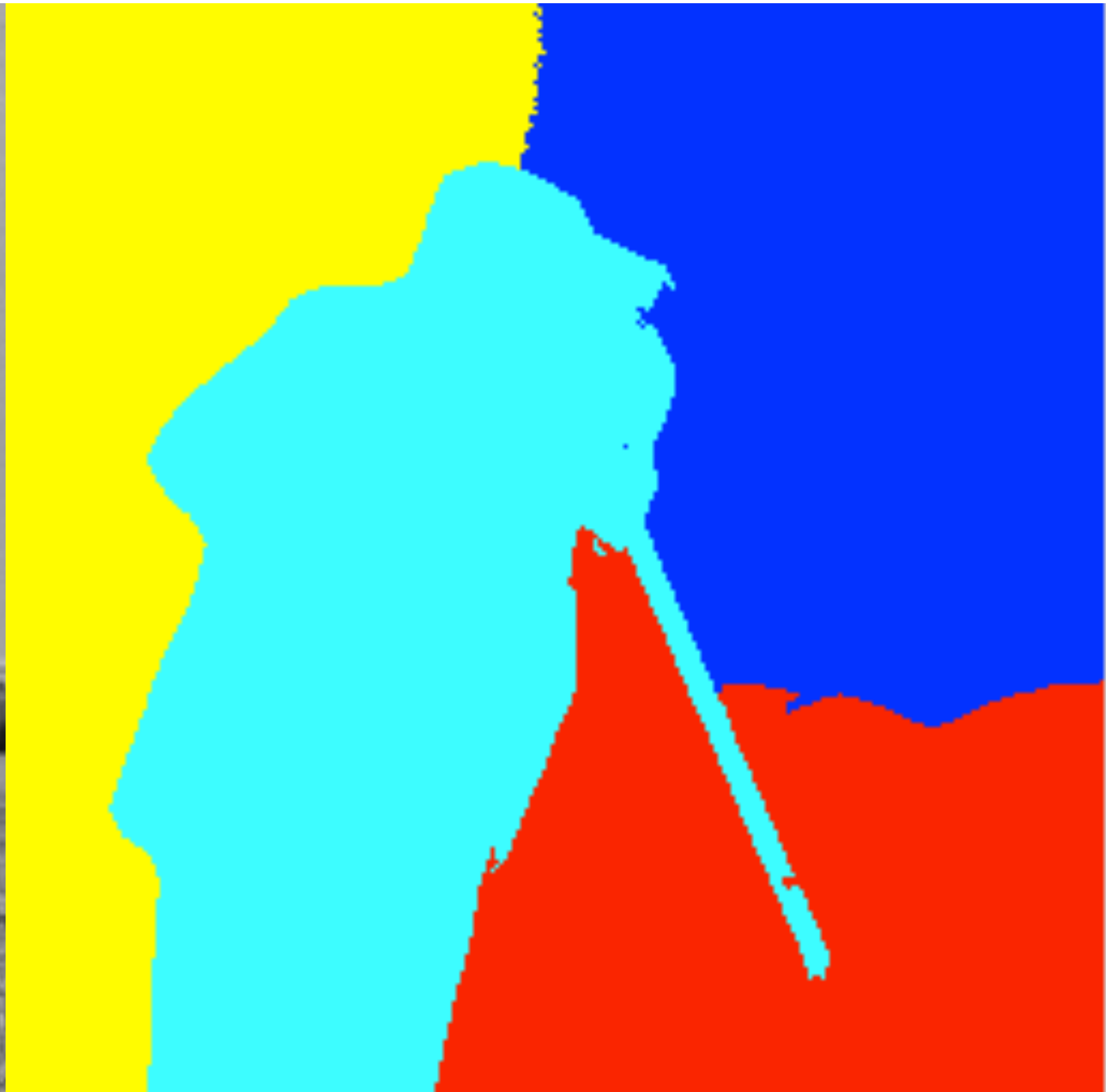
Input Image

Ncut with K=2

Input Image

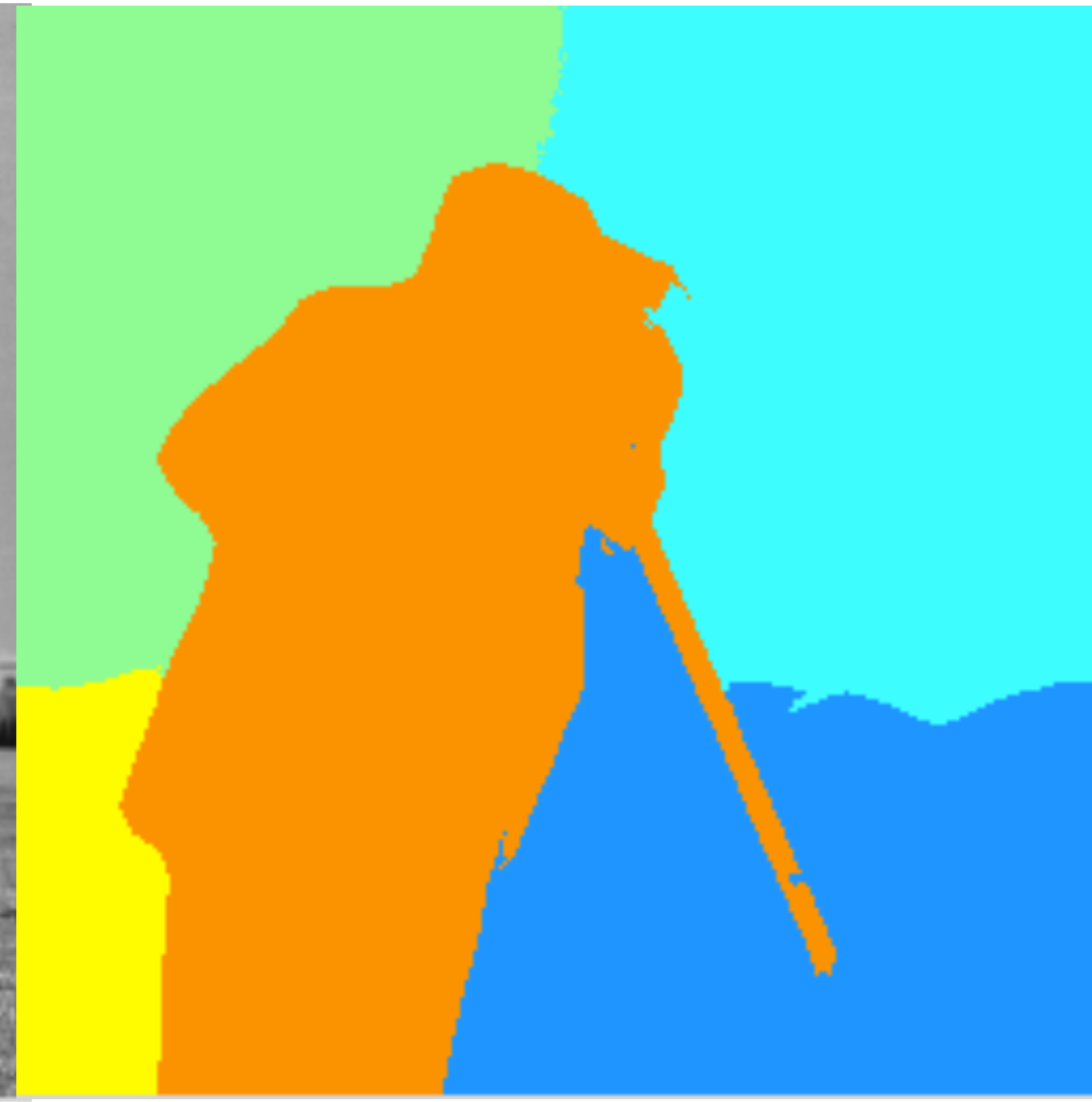
Ncut with K=3

Input Image



Ncut with K=4

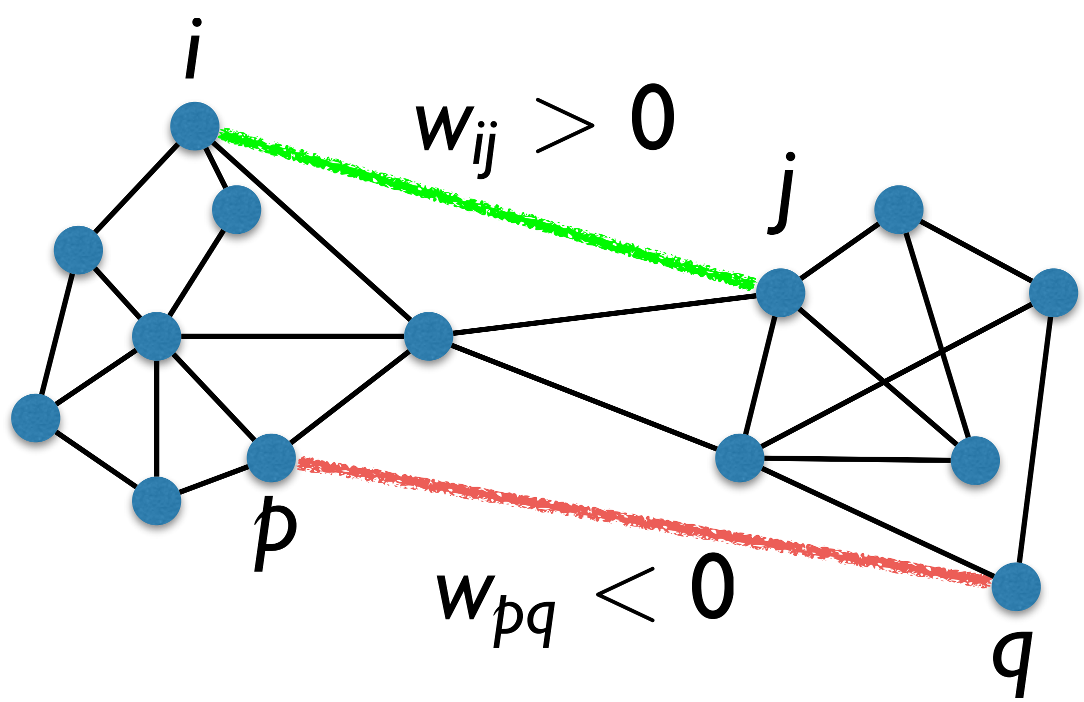
Input Image



Ncut with K=5

Top 5 eigenvectors

Ncut with signed graphs



positive: similarity

negative: dissimilarity

Let's adapt our definitions

	Unsigned graph	Signed graph
Degree	$d_i := \sum_{j=1}^{n} w_{ij}$	$\bar{d}_i = \sum_{j=1}^{n} \lvert w_{ij} \rvert$

Let's adapt our definitions

	Unsigned graph	Signed graph
Degree	$d_i := \sum_{j=1}^{n} w_{ij}$	$\bar{d}_i = \sum_{j=1}^{n} \lvert w_{ij} \rvert$
Volume	$\mathrm{vol}(A) := \sum_{i \in A} d_i$	$\mathrm{vol}(A) := \sum_{i \in A} \sum_{j=1}^{n} \lvert w_{ij} \rvert$

Let's adapt our definitions

	Unsigned graph	Signed graph
Degree	$d_i := \sum_{j=1}^{n} w_{ij}$	$\bar{d}_i = \sum_{j=1}^{n} \lvert w_{ij} \rvert$
Volume	$\mathbf{vol}(A) := \sum_{i \in A} d_i$	$\mathbf{vol}(A) := \sum_{i \in A} \sum_{j=1}^{n} \lvert w_{ij} \rvert$
Links	$\mathbf{links}(A, B) := \sum_{i \in A, j \in B} w_{ij}$	$\mathbf{links}^{+}(A, B) := \sum_{\substack{i \in A, j \in B \\ w_{ij} > 0}} w_{ij}$ $\mathbf{links}^{-}(A, B) := \sum_{\substack{i \in A, j \in B \\ w_{ij} < 0}} -w_{ij}$

Let's adapt our definitions

	Unsigned graph	Signed graph
Cut	$\mathbf{cut}(A,\bar{A}) := \sum_{i\in A, j\in\bar{A}} w_{ij}$	$\mathbf{cut}(A,\bar{A}) := \sum_{\substack{i\in A, j\in\bar{A} \\ w_{ij}\neq 0}} \lvert w_{ij}\rvert$

Let’s adapt our definitions

	Unsigned graph	Signed graph
Cut	$\mathbf{cut}(A,\bar{A}) := \sum_{i\in A, j\in\bar{A}} w_{ij}$	$\mathbf{cut}(A,\bar{A}) := \sum_{\substack{i\in A, j\in\bar{A} \\ w_{ij}\neq 0}} \lvert w_{ij}\rvert$

$$\mathbf{cut}(A,\bar{A}) \quad = \quad \mathbf{links}^{+}(A,\bar{A}) \quad ? \quad \mathbf{links}^{-}(A,\bar{A})$$

Let's adapt our definitions

Unsigned graph

Signed graph

Cut

$$\mathbf{cut}(A, \bar{A}) := \sum_{i \in A, j \in \bar{A}} w_{ij}$$

$$\mathbf{cut}(A, \bar{A}) := \sum_{\substack{i \in A, j \in \bar{A} \\ w_{ij} \neq 0}} |w_{ij}|$$

$$\mathbf{cut}(A, \bar{A}) \quad = \quad \mathbf{links}^{+}(A, \bar{A}) \quad + \quad \mathbf{links}^{-}(A, \bar{A})$$

Let’s adapt our definitions

	Unsigned graph	Signed graph
Cut	$\mathbf{cut}(A,\bar{A}) := \sum_{i\in A, j\in \bar{A}} w_{ij}$	$\mathbf{cut}(A,\bar{A}) := \sum_{\substack{i\in A, j\in \bar{A} \\ w_{ij}\neq 0}} \lvert w_{ij}\rvert$
Laplacian	$L = D - W$	$\bar{L} = \bar{D} - W$

Why all these?

Define $\quad \bar{d}_i = \sum_{j=1}^{n} |w_{ij}|$

$$\bar{D} = \mathbf{diag}(\bar{d}_1, \cdots, \bar{d}_n)$$

$$\bar{L} = \bar{D} - W$$

Then, $\quad x^\top \bar{L} x = \frac{1}{2} \sum_{i,j=1}^{n} |w_{ij}| (x_i - \mathbf{sign}(w_{ij}) x_j)^2 \geq 0$

$\bar{L}$ is 'positive semi-definite'

Define

$$X_i^j = \begin{cases} a_j & \text{if } i \in A_j, \\ 0 & \text{if } i \notin A_j \end{cases} \qquad X = \begin{bmatrix} a_1 & 0 & 0 \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

Then

$$\frac{(X^j)^\top \bar{L} X^j}{(X^j)^\top \bar{D} X^j} = \frac{\mathrm{cut}(A_j, \bar{A}_j) + 2\mathrm{links}^-(A_j, A_j)}{\mathrm{vol}(A_j)}$$

$$\mathrm{sNcut}(A_1, \cdots, A_K) = \sum_{j=1}^{K} \frac{\mathrm{cut}(A_j, \bar{A}_j)}{\mathrm{vol}(A_j)} + 2 \sum_{j=1}^{K} \frac{\mathrm{links}^-(A_j, A_j)}{\mathrm{vol}(A_j)}$$

Define

$$X_i^j = \begin{cases} a_j & \text{if } i \in A_j, \\ 0 & \text{if } i \notin A_j \end{cases} \qquad X = \begin{bmatrix} a_1 & 0 & 0 \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

Then

$$\frac{(X^j)^\top \bar{L} X^j}{(X^j)^\top \bar{D} X^j} = \frac{\text{cut}(A_j, \bar{A}_j) + 2\text{links}^-(A_j, A_j)}{\text{vol}(A_j)}$$

$$\text{sNcut}(A_1, \cdots, A_K) = \sum_{j=1}^{K} \frac{\text{cut}(A_j, \bar{A}_j)}{\text{vol}(A_j)} + 2\sum_{j=1}^{K} \frac{\text{links}^-(A_j, A_j)}{\text{vol}(A_j)}$$

Minimize positive and negative edges between clusters

minimize negative edges within clusters

How to proceed?

Replace L by $\bar{L}$ and D by $\bar{D}$

Use existing pipeline

However,…

How to proceed?

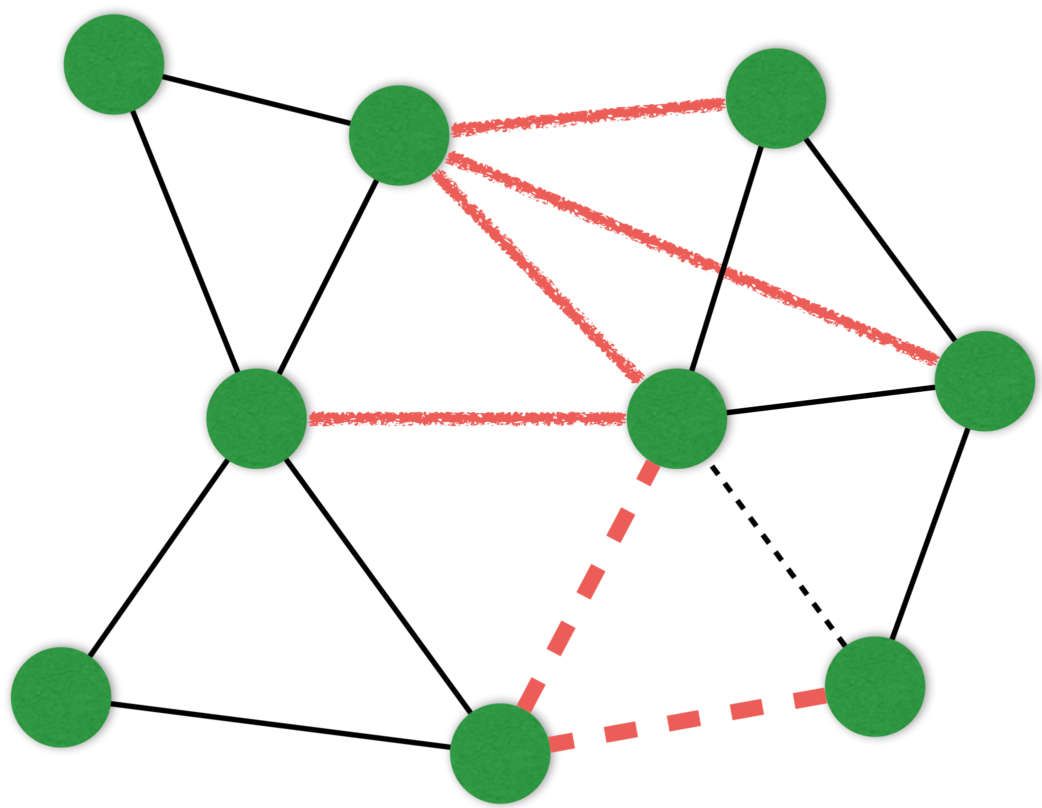
Replace L by $\bar{L}$ and D by $\bar{D}$

Use existing pipeline

However,…

Unlike L, $\bar{L}$ can be positive definite.

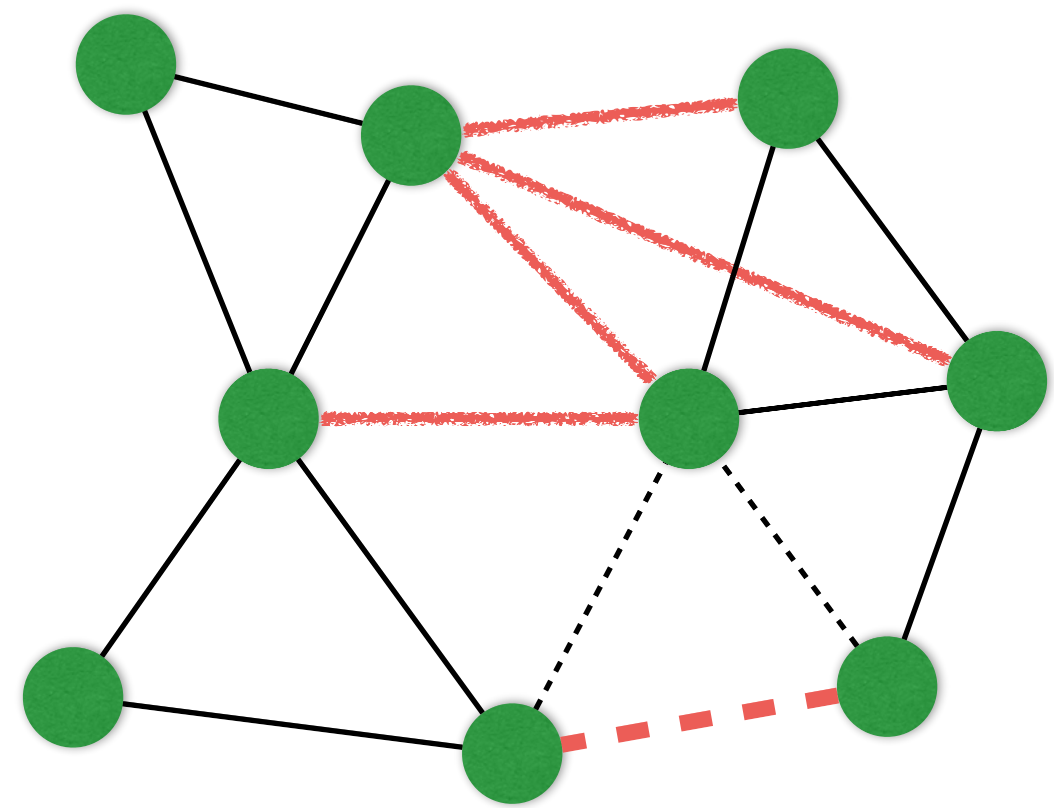
Balanced and unbalanced graphs



Balanced graph

=

contains cycles
with only even number
of negative edges



Unbalanced graph

=

contains some cycle
with odd number of
negative edges