# A very short introduction to Digital Topology

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# Definition

Analysis and filtering of 2D / 2.5D / 3D / *n*-D digital images

Applications:

- connectedness and connected components;
- classification of points in a digital set;
- thinning;
- skeletization;
- ...

# Key authors

| | |
|---|---|
| A. Rosenfeld | 1970 |
| S. Yokoi | 1975 |
| T. Pavlidis | 1979 |
| T.Y. Kong | 1985 |
| G. Bertrand | 1994 |

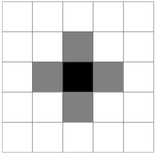
# Adjacency

Two points $(i_1, j_1)$ and $(i_2, j_2)$ are said 4-adjacents iff

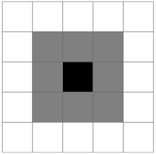
$$(i_1 - i_2)^2 + (j_1 - j_2)^2 = 1$$

Two points $(i_1, j_1)$ and $(i_2, j_2)$ are said 8-adjacents iff

$$1 \leqslant (i_1 - i_2)^2 + (j_1 - j_2)^2 \leqslant 2$$



4-neighbors



8-neighbors

# Connexity

Let $\Pi$ be a digital picture, $S$ a non-empty subset of $\Pi$, $P$ and $Q$ points of $\Pi$.

- $P$ and $Q$ are connected in $S$ if there exists a path from $P$ to $Q$ consisting entirely of points of $S$.
- "Connectedness in $S$" is an equivalence relation.
- The equivalence classes defined by this relation are called the *connected components* of $S$.
  If $S$ has only one component, it is called *connected*.
- The unique component of $\overline{S}$ that contains the border of $\Pi$ is called the *background* of $S$; all other components, if any, are called *holes* in $S$.
  If $S$ has no holes, it is called *simply connected*.

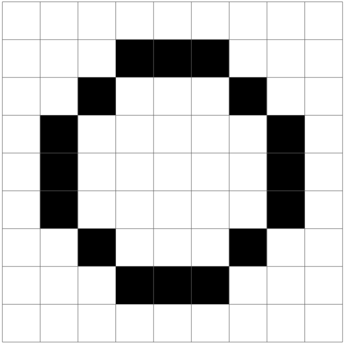
# Jordan's Curve Theorem

Given a simple closed polygonal curve $\mathcal{C}$ in the plane $\mathbb{R}^2$.

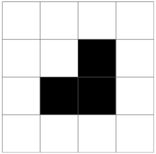
Then $\mathbb{R}^2 \backslash \mathcal{C}$ consists of exactly two open connected sets (in the sense of the ordinary $\mathbb{R}^2$-topology).

Exactly one of these sets is bounded and is called the *interior* with respect to $\mathcal{C}$ and the other one is unbounded and is called the *exterior* with respect to $\mathcal{C}$.

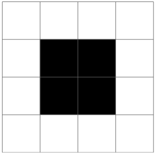
# Connectivity paradoxa



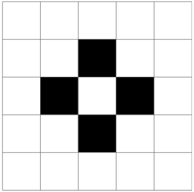
# Connectivity paradoxa

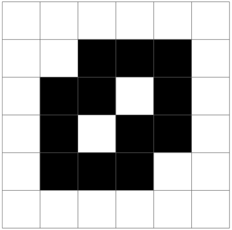


8-connectivity



4-connectivity





# Digital Jordan's Curve Theorem

Given a closed simple $\kappa$-curve $\mathcal{P}$ in the digital plane $\mathbb{Z}^2$.

Then $\mathbb{Z}^2 \backslash \mathcal{P}$ consists of exactly two $\overline{\kappa}$-connected sets.

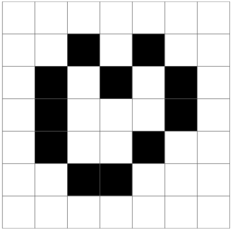
Exactly one of these sets is bounded and is called the *interior* with respect to $\mathcal{P}$ and the other is unbounded and is called the *exterior* with respect to $\mathcal{P}$.

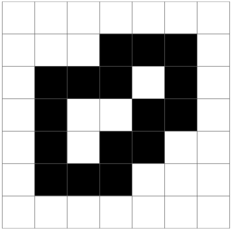
# Discrete analogs of Jordan curves and arcs

A component $\mathcal{C}$ is called a *curve* if it is connected, and each of its point has exactly two neighbors in $\mathcal{C}$.

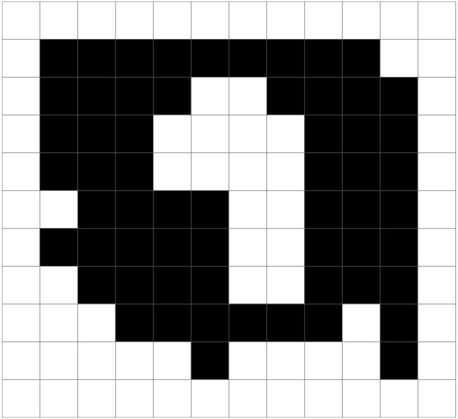
A curve has at most one hole.

Every point of a curve $\mathcal{C}$ is adjacent (in the sense of $\overline{\mathcal{C}}$'s connectedness) to both components of $\overline{\mathcal{C}}$.

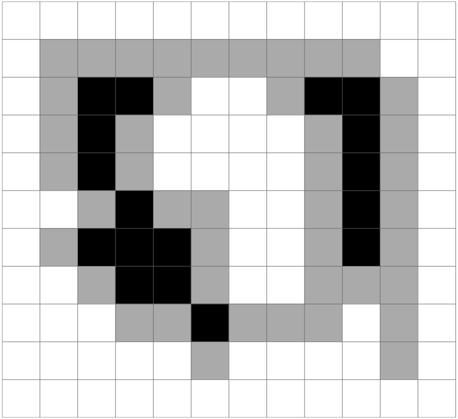




# Borders



# Borders



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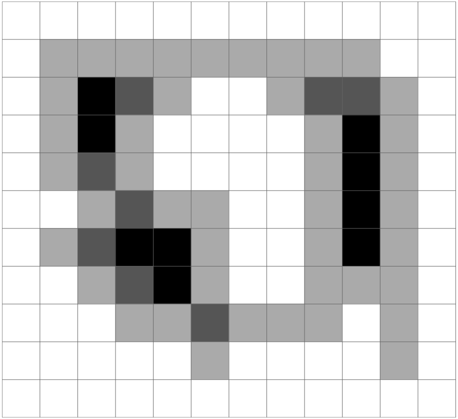

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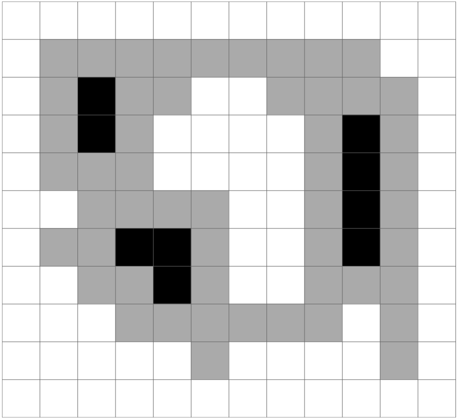

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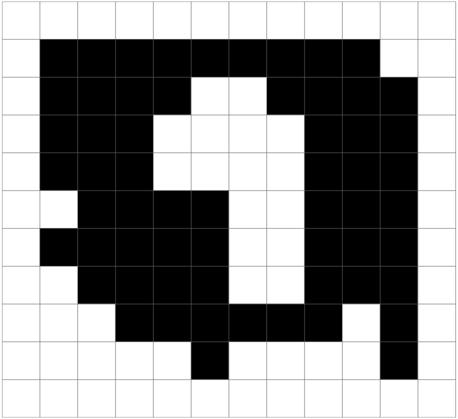
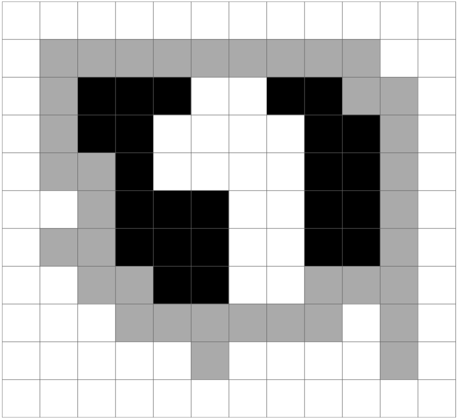


# Borders



# Thinning

Goal: removing points without changing the connectedness properties of either $S$ or $\overline{S}$

# Thinning (simple point)

The following properties of the point P of S are all equivalent ($N(P)$ denotes the set of 8-neighbors of $P$):

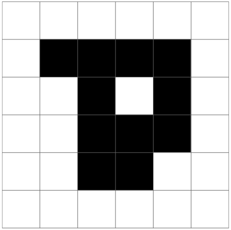
- $S \cap N(P)$ has the same number of components (in the $S$ sense) as $S \cap [N(P) \cup \{P\}]$
- $\bar{S} \cap N(P)$ has the same number of components (in the $\bar{S}$ sense) as $[\bar{S} \cap N(P)] \cup \{P\}$
- $S \cap N(P)$ has just one component adjacent to $P$ ("component" and "adjacent" in the $S$ sense)
- $\bar{S} \cap N(P)$ has just one component adjacent to $P$ ("component" and "adjacent" in the $\bar{S}$ sense)
- $S - \{P\}$ has the same number of components (in the $S$ sense) as $S$, and $\bar{S} \cup \{P\}$ has the same number of components (in the $\bar{S}$ sense) as $\bar{S}$.

# Thinning (simple point)

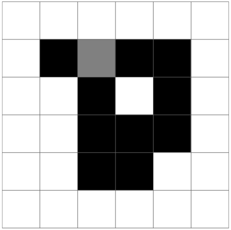
S is an arc iff it is simply connected and has exactly two simple points

S is a curve iff it is connected, has exactly one hole, and has no simple points

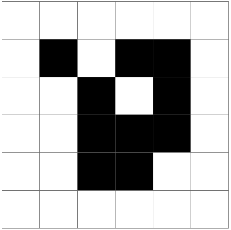
# Simple points



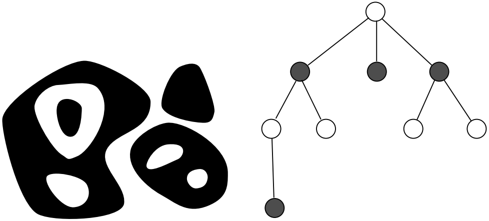
# Simple points



# Simple points



# Adjacency tree



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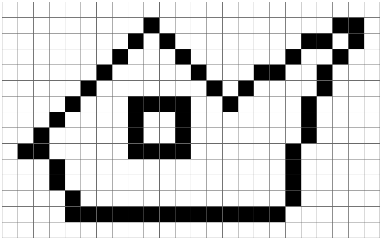
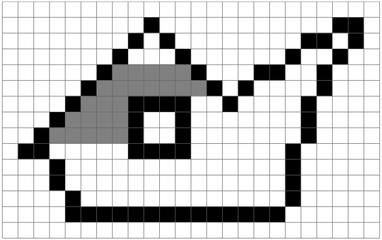
# Shape filling





# 3D Digital Topology