Constructive LMI Approaches for Anti-Windup Compensator Design for Systems Subject to Saturation

Sophie Tarbouriech
LAAS-CNRS, Toulouse, France

Part III - Approximating the saturation term

Outline of part III

1. Introduction
2. Representation by regions of saturation
3. Representation through differential inclusion (DI)
4. Representation by sector nonlinearities
5. Discussion

1. Introduction (1)

- In order to derive solutions to the problems of stability analysis or control design, we will use the Lyapunov theory.
  - ▷ Then, we need to express adequate mathematical representations of the saturation.
  - ▷ Three main mathematical representations can be considered.
- In the sequel, we will present:
  - ▷ First, a representation by saturation regions, which consists in dividing the state-space in subregions.
  - ▷ Secondly, a polytopic representation, based on the use of differential inclusions
  - ▷ Finally, a sector nonlinearity representation, which consists in including the saturation function in a sector.
- It is important to note that the first representation is exact whereas the two others are approximations.

☞ Throughout this chapter, we consider the following closed-loop system:

$$\dot{x}(t) = Ax(t) + Bu(t) \text{ with } u = sat(Kx)$$

where the function *sat* is given by:

$$sat(u_{(i)}) = \begin{cases} u_{max(i)} & \text{if} \quad u_{(i)} > u_{max(i)} \\ u_{(i)} & \text{if} \quad -u_{min(i)} \leq u_{(i)} \leq u_{max(i)} \\ -u_{min(i)} & \text{if} \quad u_{(i)} < -u_{min(i)} \end{cases}$$

2. Representation by saturation regions (1)

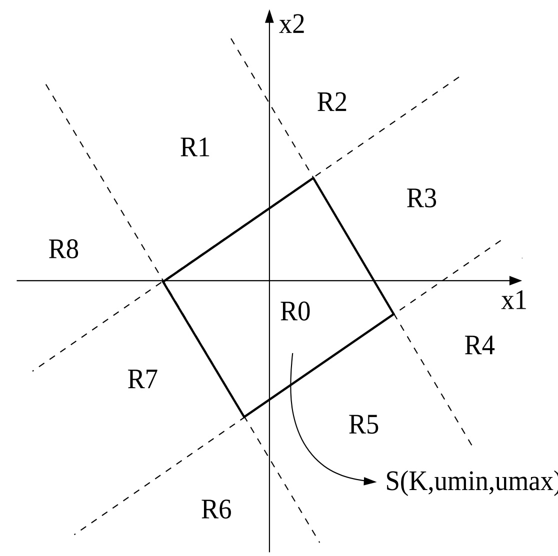
☞ Division of the state-space in regions called saturation regions.

☞ $\exists\ 3^m$ saturation regions defined by: $S(R_j, d_j) = \{x \in \Re^n; R_j x \leq d_j\}$ where $d_j \in \Re^{l_j}$ is defined from the components of $\pm u_{max}$, $\pm u_{min}$, and $R_j \in \Re^{l_j \times n}$ is defined from the lines of K and $-K$.

- ▷ In each region, one can define a vector $\xi_j \in \Re^m$ which components are:
  - ⋆ 1 when $(sat(K_{(i)}x) = u_{max(i)}$,
  - ⋆ 0 when $sat(K_{(i)}x) = K_{(i)}x$,
  - ⋆ -1 when $sat(K_{(i)}x) = -u_{min(i)}$.
- ▷ $\xi_j = 0$ corresponds to the region of linearity $S(K, u_{min}, u_{max})$.
- ▷ Each ξ_j corresponds to a combination between saturated and non-saturated inputs.

2. Representation by saturation regions (2)

☞ Each ξ_j , $j = 1, \ldots, 3^m$, is associated to a polyhedral region



$$S(R_j, d_j) = \{x \in \Re^n \; ; \; R_j x \leq d_j\}$$

☞ In each region $S(R_j, d_j)$, the saturated system is described by an affine system:

$$\dot{x}(t) = (A + Bdiag(I_m - |\xi_j|)K)x(t) + Bu(\xi_j) = \bar{A}_j x(t) + v_j \qquad (1)$$

with $u_{(i)}(\xi_j) = \begin{cases} u_{max(i)} & \text{if} \quad \xi_{j(i)} = 1 \\ 0 & \text{if} \quad \xi_{j(i)} = 0 \\ -u_{min(i)} & \text{if} \quad \xi_{j(i)} = -1 \end{cases}$.

Differential inclusions *[Boyd et al]*

Definition 1 *A differential inclusion (DI) is described by:*

$$\dot{x} \in F(x(t), t) \; ; \quad x(0) = x_0 \tag{2}$$

where F is a set-valued function on $\Re^n \times \Re_+$.

- ▷ Any $x : \Re_+ \to \Re^n$ that satisfies (2) is called solution or trajectory of the DI (2).
- ▷ If we assume that $F(x(t), t)$ is a convex set for every x and t, the DI (2) writes:

$$\dot{x} \in co\{F(x(t), t)\} \; ; \quad x(0) = x_0 \tag{3}$$

  and (3) is called a relaxed version of (2).

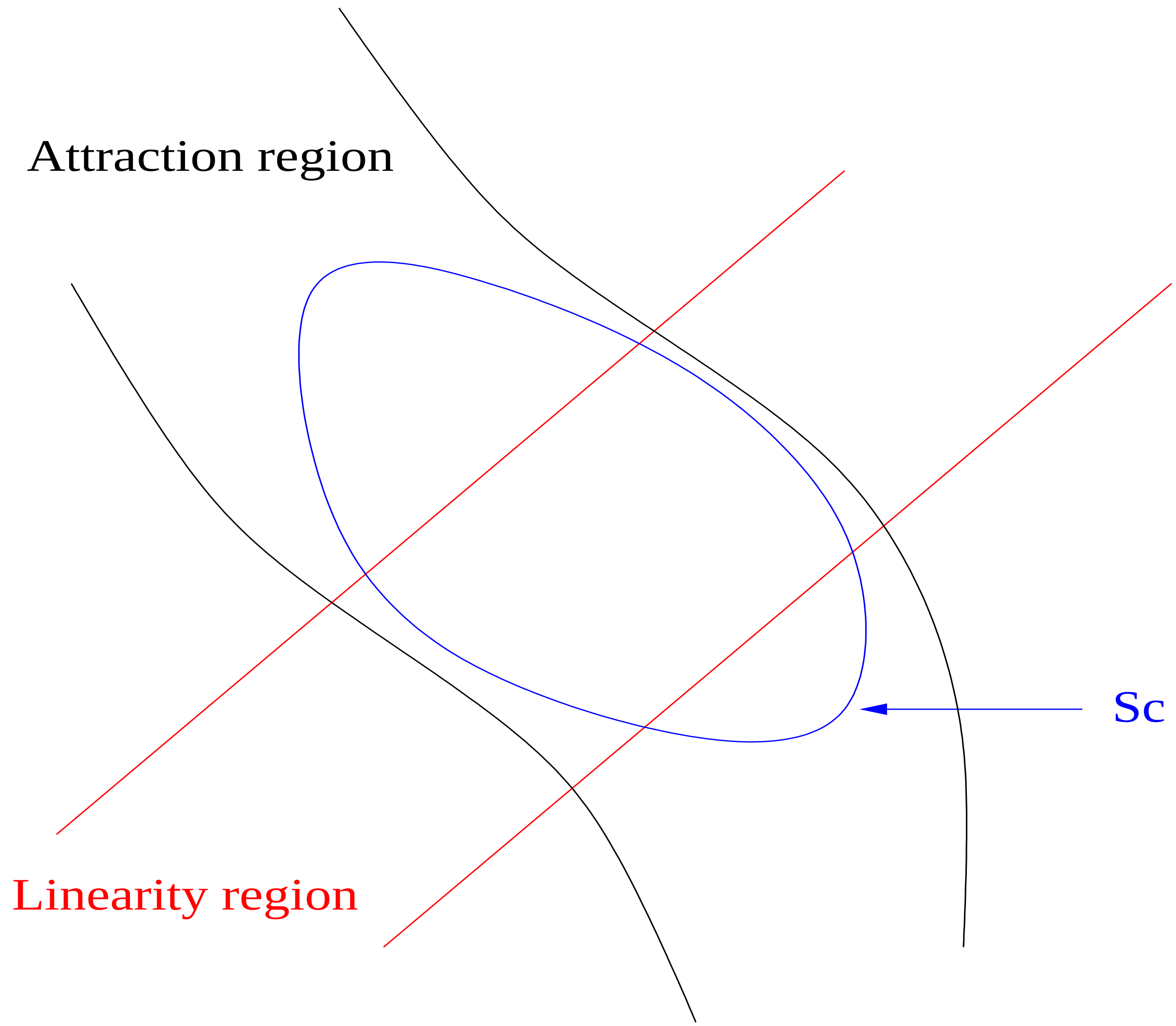
- ▷ Note that since $F(x(t), t) \in co\{F(x(t), t)\}$ every trajectory of the DI (2) is also a trajectory of (3).

Definition 2 *A linear differential inclusion (LDI) is given by:*

$$\dot{x} \in \Omega_x \; ; \quad x(0) = x_0 \tag{4}$$

where Ω_x is a subset of $\Re^n$.

3. Representation through DI (2)



☞ Use differential inclusion representation for $x \in \mathcal{S}_c$

3. Representation through DI (3)

☞ The saturated term may be written as: $sat(Kx(t)) = \Gamma(\alpha(x))Kx(t)$

▷ where the matrix $\Gamma(\alpha(x))$ is a diagonal matrix whose diagonal elements are defined for $i = 1, ..., m$ as

$$\alpha_{(i)}(x) = \begin{cases} \frac{u_{max(i)}}{K_{(i)}x(t)} & \text{if} \quad K_{(i)}x(t) > u_{max(i)} \\ 1 & \text{if} \quad -u_{min(i)} \leq K_{(i)}x(t) \leq u_{max(i)} \\ -\frac{u_{min(i)}}{K_{(i)}x(t)} & \text{if} \quad K_{(i)}x(t) < -u_{min(i)} \end{cases} \quad , \quad i = 1, \ldots, m$$

▷ Then $0 < \alpha_{(i)}(x) \leq 1$, $i = 1, \ldots, m$, $\forall x \in \Re^n$.

▷ The closed-loop system becomes:

$$\dot{x}(t) = (A + B\Gamma(\alpha(x))K)x(t)$$

☞ References: (Gomes da Silva Jr. et al., 1997),(Tarbouriech and Henrion, 1999], (Gomes da Silva Jr. and Tarbouriech, 2001), (Gomes da Silva Jr. et al., 2003)

3. Representation through DI (4)

☞ Define the polyhedral set:

$$S(H, u_{min}, u_{max}) = \{x \in \Re^n; -u_{min(i)} \leq H_{(i)}x \leq u_{max(i)},\ i = 1, ..., m\}$$

Lemma 1 *(Hu et al, 2002) If* $x(t) \in S(H, u_{min}, u_{max})$ *then*

$$sat(Kx) \in Co\{(G_jK + (I_m - G_j)H)x, i = 1, ..., 2^m\}$$

where G_j, $j = 1, \ldots, 2^m$, *are diagonal matrices whose diagonal elements take the value* 1 *or* 0*. Then it follows:*

$$\dot{x} \in Co\{(A + B(G_jK + (I_m - G_j)H)x, i = 1, ..., 2^m\} \quad (5)$$

Proof:

- If $x \in S(H, u_{min}, u_{max})$ then $sat(Kx) = \Gamma(x)Kx$ is a vector composed by some elements from Kx and the rest from Hx.
- Hence, for $x \in S(H, u_{min}, u_{max})$, one can compute $A + B\Gamma(\alpha(x))K$ as a linear combination of matrices $A + B(G_jK + (I_m - G_j)H)$, $j = 1, \cdots, 2^m$.
  - ▷ Therefore, condition (5) follows.

☞ Relation (5) can be considered as a parameterized LDI due to the degree of freedom injected through matrix H.

▷ We study the behavior of the following polytopic model to deduce properties for the saturated system:

$$\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_j (A + B(G_j K + (I_m - G_j)H))x(t)$$

with $\sum_{j=1}^{2^m} \lambda_j = 1,\ \lambda_j \geq 0.$

☞ References: (Hu et al., 2003, 2005)

☞ Particular cases

▷ $H_{(i)} = \alpha_{l(i)} K_{(i)} \Leftrightarrow$ Approach 1 due to (Gomes da Silva Jr and Tarbouriech, 2001 and 2003)

▷ $H = K$: saturation avoidance. The linear behavior of the system is studied

3. Representation through DI (6)

☞ Extension to NLDI: approach developed by Alamo and his co-workers. The authors describes a model:

- ▷ by describing $sat(Kx) = (sat(K_{(i)}x))_{i \in S} + (K_{(i)}x)_{i \in S^c}$, where S^c is the complementary of set S.
- ▷ Example: $m = 2$. $S =, 1, 2, \{1, 2\}$ and $S^c = \{1, 2\}, 2, 1, .$
- ▷ by using some adequate upper-bounding of quadratic product involving the saturation.

☞ Moreover, the deadzone $\phi(Kx) = sat(Kx) - Kx$ can also be approximated through LDI (see, Castelan and al., 2008).

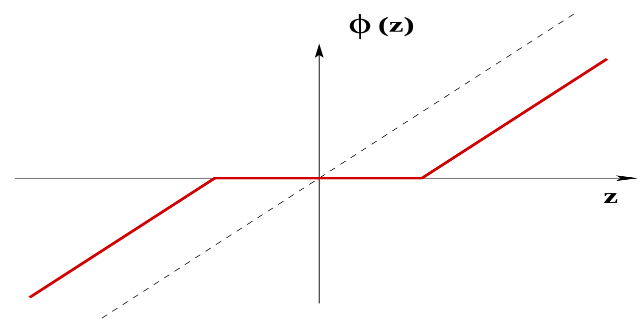
☞ For simplicity, we consider a symmetrical saturation function:
$u_{max} = u_{min} = u_0$.

☞ Furthermore, we consider the deadzone nonlineatity $\phi(Kx) = sat(Kx) - Kx$.

Classical conditions

☞ Let us consider a generic nonlinearity $\phi(z) = z - sat(z)$.

☞ The deadzone nonlinearity ϕ belongs to the sector [0 1] as depicted in the following figure



☞ ϕ verify the following condition

$$\phi(z)'\lambda(\phi(z) - z) \leq 0, \forall z \in \Re^m$$

with $\lambda = diag(\lambda_i) \geq 0$

▷ Condition extremely conservative: it does not allow to discriminate a saturation from a deadzone.

▷ For $\phi(z) = sat(z) - z$, condition writes $\phi(z)'\lambda(\phi(z) + z) \leq 0, \forall z \in \Re^m$

Lemma 2 *The nonlinearity $\phi(Kx)$ satisfies the sector condition:*

$$\phi(Kx)'T[\phi(Kx)+\Lambda Kx] \leq 0, \quad \forall x \in S(K,u_0^\lambda)$$

where T is a positive diagonal matrix, $0 \leq \Lambda < I_m$ is a non-negative diagonal matrix and the set $S(\mathbb{K},u_0^\lambda)$ is a polyhedral set defined as follows:

$$S(K,u_0^\lambda) = \{x \in \Re^n; -\frac{u_{0(i)}}{1-\Lambda_{(i,i)}} \leq K_{(i)}x \leq \frac{u_{0(i)}}{1-\Lambda_{(i,i)}}, \ i = 1,...,m\}$$

☞ References: Bernstein, Boyd, Gomes da Silva Jr.

- ▷ Condition less conservative than the previous one.
- ▷ Main drawback: product between the potential decision variables T, Λ and K.

Modified sector condition

Lemma 3 *The nonlinearity* $\phi(Kx)$ *satisfies the sector condition:*

$$\phi(Kx)'T[\phi(Kx) + Gx] \leq 0, \quad \forall x \in S(K - G, u_0)$$

where T *is a positive diagonal matrix and the set* $S(K - G, u_0)$ *is a polyhedral set defined as follows:*

$$S(K - G, u_0) = \{x \in \Re^n; -u_{0(i)} \leq (K_{(i)} - G_{(i)})x \leq u_{0(i)}, \ i = 1, ..., m\}$$

▷ OBS: Particular case $G = \Lambda K \Leftrightarrow$ previous classical approach

▷ The global case is treated by considering $G = K$.

☞ References: (Gomes da Silva Jr. and Tarbouriech, 2003, 2005)

☞ Elements of proof.

▷ Recall that $\phi(Kx)$ is defined as follows:

$$\phi(K_{(i)}x) = \begin{cases} u_{0(i)} - K_{(i)}x < 0 & \text{if} \quad K_{(i)}x > u_{0(i)} \\ 0 & \text{if} \quad -u_{0(i)} \leq K_{(i)}x \leq u_{0(i)} \\ -u_{0(i)} - K_{(i)}x > 0 & \text{if} \quad K_{(i)}x < -u_{0(i)} \end{cases} , \quad \forall i = 1, \cdots, m$$

▷ From the definition of the nonlinearity $\phi(Kx)$, we can consider three cases:

– Case 1. $\phi(K_{(i)}x) < 0$. Then, it follows that:

$$\phi(K_{(i)}x)T_{(i,i)}(\phi(K_{(i)}x) + G_{(i)}x) = \phi(K_{(i)}x)T_{(i,i)}(u_{0(i)} - K_{(i)}x + G_{(i)}x) \leq 0$$

provided that $T_{(i,i)} > 0$ and $u_{0(i)} - K_{(i)}x + G_{(i)}x \geq 0$.

– Case 2. $\phi(K_{(i)}x) = 0$. Then, it follows that:

$$\phi(K_{(i)}x)T_{(i,i)}(\phi(K_{(i)}x) + G_{(i)}x) = 0$$

– Case 3. $\phi(K_{(i)}x) > 0$. Then, it follows that:

$$\phi(K_{(i)}x)T_{(i,i)}(\phi(K_{(i)}x) + G_{(i)}x) = \phi(K_{(i)}x)T_{(i,i)}(-u_{0(i)} - K_{(i)}x + G_{(i)}x) \leq 0$$

provided that $T_{(i,i)} > 0$ and $-u_{0(i)} - K_{(i)}x + G_{(i)}x \leq 0$.

☞ Generalization. Let us consider a generic nonlinearity $\phi(v) = sat(v) - v \in \Re^m$ and the following set:

$$S(u_0) = \{v, w \in \Re^m; -u_{0(i)} \leq v_{(i)} - w_{(i)} \leq u_{0(i)},\ i = 1, ..., m\}$$

Lemma 4 *(Tarbouriech et al., IEEE-TAC 2006) If the vectors v and w belong to the set $S(u_0)$ then the nonlinearity $\phi(v)$ satisfies the following inequality:*

$$\phi(v)'T[\phi(v) + w] \leq 0$$

for any diagonal positive definite matrix $T \in \Re^{m \times m}$.

▷ Vectors v and w can be complex functions.

▷ This lemma allows to take into account simply nested saturations and implicit functions.

▷ The global case is treated by considering $w = v$.

5. Discussion

☞ To summarize:

① $V(x) = x'Px,\ P = P' > 0$

② $V(x) = x'Px - 2\sum_{i=1}^{m} \int_0^{K_{(i)}x} (sat(\sigma_{(i)}) - \sigma_{(i)}) N_{(i,i)} d\sigma_{(i)}$

③ $V(x) = \max_i \left\{ \frac{G_{(i)}x}{\omega_{(i)}} \right\}$