II.4. POLYNOMIAL DESIGN

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Fixed-order robust design: a difficult problem

In this last part of course, we study robust stabilization with a fixed-order controller affected by parametric uncertainty

A difficult problem in general because
- fixed-order controller means non-convexity of the design space
- parametric uncertainty means highly structured uncertainty and exponential (combinatorial) complexity

In the literature: a lot of analysis results, but very few design results..
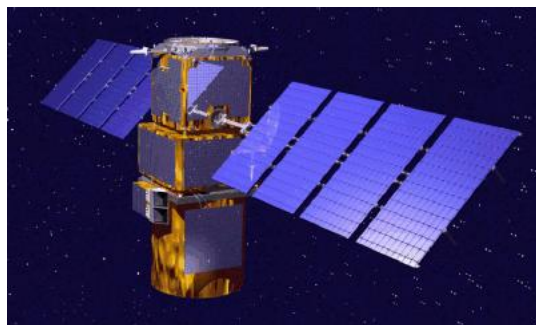
Low-order controllers are required in embedded devices

Existing design results

- H_∞ design methods, state-space techniques

- Critical direction (Nyquist), convex optimization with cutting-plane algorithms

- Infinite-dimensional Youla-Kučera parametrization generally leading to high-order controllers

- Use of linear programming with polytopic sufficient conditions for stability

In this course

- Use of polynomial techniques

- Use of LMI optimization

- Controllers of fixed (hence low) order

Nominal Pole placement

We consider the SISO feedback system
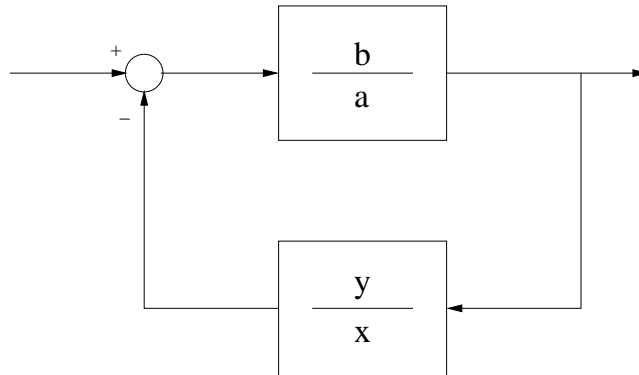
Closed-loop transfer function

$$\frac{bx}{ax + by}$$

In the absence of hidden modes (a and b coprime polynomials), pole placement amounts to finding polynomials x and y solving the Diophantine equation (from Diophantus of Alexandria 200-284)

$$ax + by = c$$

where c is a given closed-loop characteristic polynomial capturing the desired system poles

Pole placement: numerical aspects

The polynomial Diophantine equation

$$ax + by = c$$

is linear in unknowns x and y, and denoting

$$a(s) = a_0 + a_1 s + \cdots + a_{d_a} s^{d_a}$$
$$x(s) = x_0 + x_1 s + \cdots + x_{d_b} x^{d_b}$$
$$\text{etc..}$$

we can identify powers of the indeterminate s to build a linear system of equations

$$\begin{bmatrix} a_0 & & & b_0 & & \\ a_1 & \ddots & & b_1 & \ddots & \\ \vdots & & a_0 & \vdots & & b_0 \\ a_{d_a} & & a_1 & b_{d_b} & & b_1 \\ & \ddots & \vdots & & \ddots & \vdots \\ & & a_{d_a} & & & b_{d_b} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{d_x} \\ \hline y_0 \\ y_1 \\ \vdots \\ y_{d_y} \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d_c} \end{bmatrix}$$

The above matrix is called Sylvester matrix, it has a special Toeplitz banded structure that can be exploited when solving the equation

James J Sylvester
(1814 London - 1897 London)

Otto Toeplitz
(1881 Breslau - 1940 Jerusalem)

Pole placement for MIMO systems

Pole placement can be performed similarly for a plant left MFD

$$A^{-1}(s)B(s)$$

with a controller right MFD

$$Y(s)X^{-1}(s)$$

The Diophantine equation to be solved is now over polynomial matrices

$$A(s)X(s) + B(s)Y(s) = C(s)$$

and right hand-side matrix $C(s)$ captures information on invariant polynomials and eigenstructure

For example $C(s)$ may contain H_2 or H_∞ optimal dynamics (obtained with spectral factorization)

Robust pole placement

Now assume that the plant transfer function

$$\frac{b(q)}{a(q)}$$

contains some uncertain parameter q

The problem of robust pole placement will then consists in finding a controller

$$\frac{y}{x}$$

such that the uncertain closed-loop charact. polynomial

$$a(q)x + b(q)y = c(q)$$

is robustly stable

How can we find x, y to ensure robust stability of $c(q)$ for all admissible uncertainty q ?

Coefficients of c are linear in x and y, but we saw that stability conditions are non-linear and highly non-convex in c..

Robust pole placement

One possible remedy is a suitable

Convex approximation of the stability region

Then we can perform design with
- linear programming (polytopes)
- quadratic programming (spheres, ellipsoids)
- semidefinite programming (LMIs)

Complexity of design algorithm increases
Conservatism of control law decreases

Robust design via polytopic approximation

MIMO plant with right MFD

$$B(s)A^{-1}(s) = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & s+1 \end{bmatrix}^{-1}$$

with uncertainty in parameter

$$b \in [0.5,\ 1.5]$$

We seek a proper first order controller

$$X^{-1}(s)Y(s) = \begin{bmatrix} s+x_1 & x_2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} y_1 s + y_2 & y_3 s + y_4 \\ 0 & y_5 \end{bmatrix}$$
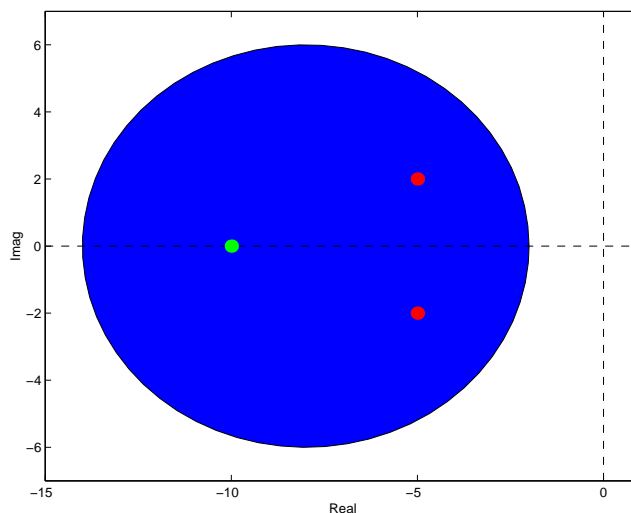
assigning robustly the closed-loop polynomial matrix

$$C(s) = \begin{bmatrix} s^2 + \alpha s + \beta & \delta(s) \\ 0 & s + \gamma \end{bmatrix}$$

whose coefficients live in the polytope

$$\begin{bmatrix} -14 & 1 & 0 \\ 16 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} > \begin{bmatrix} -196 \\ 56 \\ -4 \\ 2 \\ -14 \end{bmatrix}$$

These specifications amounts to assigning the poles within the disk

$$|s+8| < 6$$

Robust design via polytopic approximation (2)

Equating powers of indeterminate s in the polynomial matrix Diophantine equation

$$X(s)A(s) + Y(s)B(s) = C(s)$$

we obtain the design inequalities

$$
\begin{bmatrix}
-13 & -7 & 0.5 \\
14 & 8 & -1 \\
-1 & -1 & 0.5 \\
-13 & -21 & 1.5 \\
14 & 24 & -3 \\
-1 & -3 & 1.5
\end{bmatrix}
\begin{bmatrix}
x_1 \\
y_1 \\
y_2
\end{bmatrix}
>
\begin{bmatrix}
-182 \\
40 \\
-2 \\
-182 \\
40 \\
-2
\end{bmatrix}
$$

characterizing all parameters x_1, y_1 and y_2 of admissible robust controllers
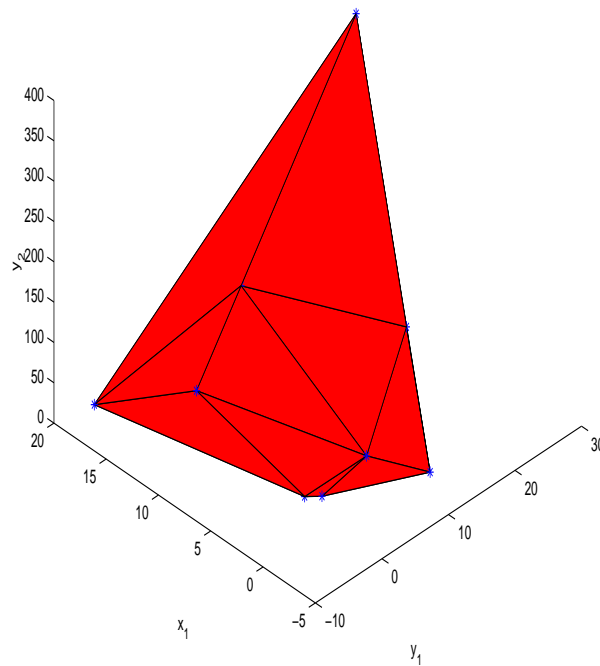
Corresponding polytope with 9 vertices

Robust design via ellipsoidal approximation

Closed-loop characteristic polynomial

$$c(s) = a(s)x(s) + b(s)y(s)$$

$$= c_0 + c_1 s + \cdots + c_{d-1}s^{d-1} + s^d$$

whose coefficients are given by the LSE

$$
\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-1} \end{bmatrix}
=
\begin{bmatrix}
y_0 & & & x_0 & & \\
y_1 & \ddots & & x_1 & \ddots & \\
\vdots & & y_0 & \vdots & & x_0 \\
y_{m-1} & & y_1 & x_{m-1} & & x_1 \\
y_m & & \vdots & 1 & & \vdots \\
& \ddots & y_{m-1} & & \ddots & x_{m-1} \\
& & y_m & & & 1
\end{bmatrix}
\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \\ a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}
+
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ x_0 \\ x_1 \\ \vdots \\ x_{m-1} \end{bmatrix}
$$

$$\boxed{c = S(x,y)p + e(x)}$$

It is assumed that uncertain plant parameters belong to the ellipsoid

$$E_p = \{p \; : \; (p - \bar{p})^\star P(p - \bar{p}) \leq 1\}$$

where $\bar{p}$ is a given nominal plant vector and P is a given positive definite covariance matrix

Robust control problem:

> Find controller coefficients x, y
> robustly stabilizing plant a, b subject to
> ellipsoidal uncertainty $p \in E_p$

Robust design via ellipsoidal approximation

Recall that by solving an LMI we could approximate from the interior the non-convex stability region with an ellipsoid

$$E_q = \{q \ : \ (q - \bar{q})^\star Q(q - \bar{q}) \leq 1\}$$

In other words, $q \in E_q$ implies $q(s)$ stable

Using conditions for inclusion of an ellipsoid into another we can show that finding x and y such that $q \in E_q$ for all $p \in E_p$ amounts to solving another LMI problem (not given here)

Coefficients x, y are such that the controller $y(s)/x(s)$ robustly stabilizes plant $b(s)/a(s)$

Ellipsoidal robust design: example

We consider the two mixing tanks arranged in cascade with recycle stream
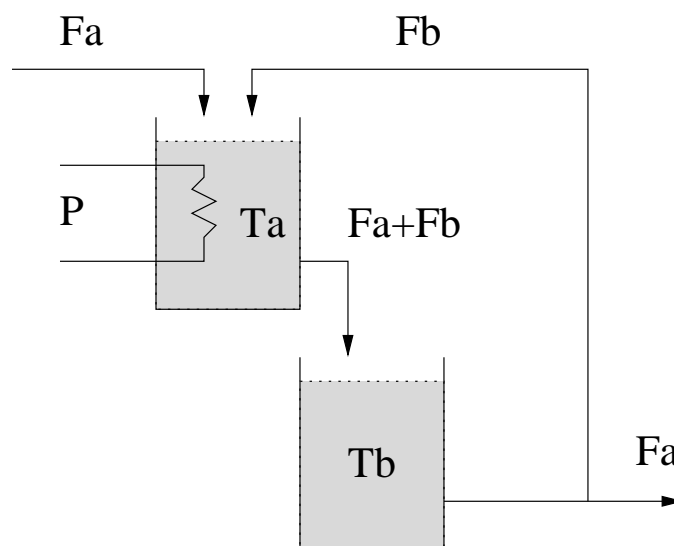
The controller must be designed to maintain the temperature T_b of the second tank at a desired set point by manipulating the power P delivered by the heater located in the first tank

The only available measurement is temperature T_b

Ellipsoidal robust design: example (2)

The identification of the nominal plant model is carried out using a standard least-squares method

A second-order discrete-time model

$$p(z) = \frac{b_0 + b_1 z}{a_0 + a_1 z + z^2}$$

is obtained with nominal plant vector

$$\bar{p} = \begin{bmatrix} 0.0038 & 0.0028 & 0.2087 & -1.1871 \end{bmatrix}^\star$$

The positive definite matrix P characterizing the uncertainty ellipsoid

$$E_p = \{p \ : \ (p - \bar{p})^\star P (p - \bar{p}) \leq 1\}$$

is readily available as a by-product of the identification technique

$$P = 10^5 \begin{bmatrix} 2.4179 & 0.0568 & 0.0069 & 0 \\ 0.0568 & 2.4121 & 0.0045 & 0.0062 \\ 0.0069 & 0.0045 & 0.0015 & 0.0014 \\ 0 & 0.0062 & 0.0014 & 0.0015 \end{bmatrix}$$

Ellipsoidal robust design: example (3)

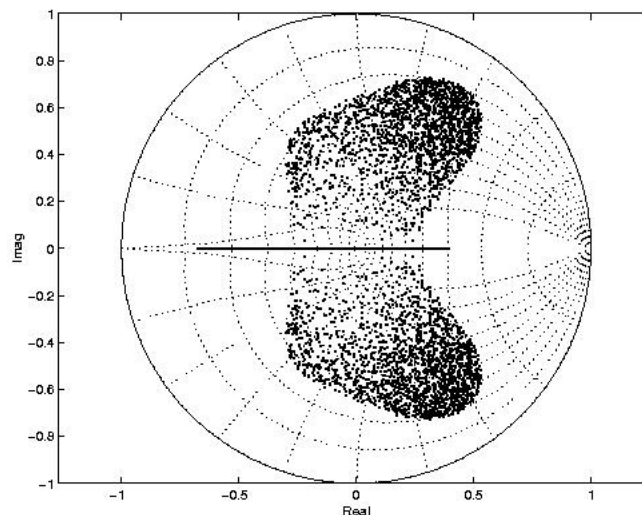Solving the LMI analysis problem we obtain first an inner ellipsoidal approximation

$$E_q = \{q \;:\; (q - \bar{q})^\star Q(q - \bar{q}) \leq 1\}$$

of the non-convex stability region, with

$$Q = \begin{bmatrix} 2.3378 & 0 & 0.5397 \\ 0 & 2.1368 & 0 \\ 0.5397 & 0 & 1.7552 \end{bmatrix} \qquad \bar{q} = \begin{bmatrix} 0 \\ 0.1235 \\ 0 \end{bmatrix}$$

Then we solve the design LMI to obtain the first-order robustly stabilizing controller

$$\frac{y(z)}{x(z)} = \frac{0.3377 + 166.0z}{0.6212 + z}$$

Robust closed-loop root-locus for
random admissible ellipsoidal uncertainty

Strict positive realness

Let

$$\mathcal{D} = \{s \;:\; \begin{bmatrix} 1 \\ s \end{bmatrix}^{\star} \underbrace{\begin{bmatrix} a & b \\ b^{\star} & c \end{bmatrix}}_{H} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0\}$$

be a stability region in the complex plane where Hermitian matrix H has inertia $(1, 0, 1)$

Let $\partial\mathcal{D}$ denote the 1-D boundary of $\mathcal{D}$

Standard choices are

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

for the left half-plane and the unit disk resp.

We say that a rational matrix $G(s)$ is strictly positive real (SPR for short) when

$$\text{Re } G(s) \succ 0 \quad \text{for all} \quad s \in \partial\mathcal{D}$$

Stability and strict positive realness

Consider two square polynomial matrices of size n and degree d

$$
\begin{aligned}
N(s) &= N_0 + N_1 s + \cdots + N_d s^d \\
D(s) &= D_0 + D_1 s + \cdots + D_d s^d
\end{aligned}
$$

Polynomial matrix $N(s)$ is stable iff there is a stable polynomial $D(s)$ such that rational matrix $N(s)D^{-1}(s)$ is strictly positive real

Proof

From the definition of SPRness, $N(s)D^{-1}(s)$ SPR with $D(s)$ stable implies $N(s)$ stable

Conversely, if $N(s)$ is stable then the choice $D(s) = N(s)$ makes rational matrix $N(s)D^{-1}(s) = I$ obviously SPR

It turns out that this condition can be characterized by an LMI..

SPRness as an LMI

Let $N = [N_0 \quad N_1 \cdots N_d]$, $D = [D_0 \quad D_1 \cdots D_d]$ and

$$\Pi = \begin{bmatrix} I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \\ 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \end{bmatrix}$$

Given a stable $D(s)$, $N(s)$ ensures SPRness of $N(s)D^{-1}(s)$ iff there exists a matrix $P = P^\star$ of size dn such that

$$D^\star N + N^\star D - H(P) \succ 0$$

where

$$H(P) = \Pi^\star(S \otimes P)\Pi = \Pi^\star \begin{bmatrix} aP & bP \\ b^\star P & cP \end{bmatrix} \Pi$$

Proof

Similar to the proof on positivity of a polynomial, based on the decomposition as a sum-of-squares with lifting matrix P

LMI condition for design

Given a stable polynomial matrix $D(s)$, polynomial matrix $N(s)$ is stable if there is a matrix P satisfying the LMI

$$D^\star N + N^\star D - H(P) \succ 0$$

(it follows then that $P \succ 0$)

- New convex inner approximation
of stability domain
- Shape described by an LMI
- Depends on the particular choice of $D(s)$
- More general than polytopes and ellipsoids

Useful for design because linear in N

Polynomial $D(s)$ will be referred to as the

central polynomial

LMI condition for analysis

The LMI condition of SPRness can be used also for design if we interchange the respective roles played by $N(s)$ and $D(s)$

If there is a matrix P and a polynomial matrix $D(s)$ satisfying the LMI

$$D^\star N + N^\star D - H(P) \succ 0$$
$$P \succ 0$$

then polynomial matrix $N(s)$ is stable
(it follows that $D(s)$ is stable as well)

Useful for (robust) analysis because linear in D

Note that, in contrast with the design LMI, we have here to enforce $P \succ 0$

Second-degree discrete-time polynomial

Consider the discrete polynomial

$$n(z) = n_0 + n_1 z + z^2$$

We will study the shape of the LMI stability region for the following central polynomial

$$d(z) = z^2$$

We can show that non-strict feasibility of the LMI is equivalent to existence of a matrix P satisfying

$$
\begin{aligned}
p_{00} + p_{11} + p_{22} &= 1 \\
p_{10} + p_{01} + p_{21} + p_{22} &= n_1 \\
p_{20} + p_{02} &= n_0 \\
P &\succeq 0
\end{aligned}
$$

which is an LMI in the primal SDP form

Infeasibility of primal LMI is equivalent to the existence of a vector satisfying the dual LMI

$$y_0 + n_1 y_1 + n_0 y_2 < 0$$

$$Y = \begin{bmatrix} y_0 & y_1 & y_2 \\ y_1 & y_0 & y_1 \\ y_2 & y_1 & y_0 \end{bmatrix} \succeq 0$$

The eigenvalues of Toeplitz matrix Y are

$$y_0 - y_2 \quad \text{and} \quad (2y_0 + y_2 \pm \sqrt{y_2^2 + 8y_1^2})/2$$

so it is positive definite iff y_1 and y_2 belong to the interior of a bounded parabola scaled by y_0

The corresponding values of n_0 and n_1 belong to the interior of the envelope generated by the curve

$$(2\lambda_2 - 1)n_0 + (2\lambda_1 - 1)\sqrt{\lambda_2} n_1 + 1 > 0 \quad 0 \le \lambda_i \le 1$$

Second-degree discrete-time polynomial (3)
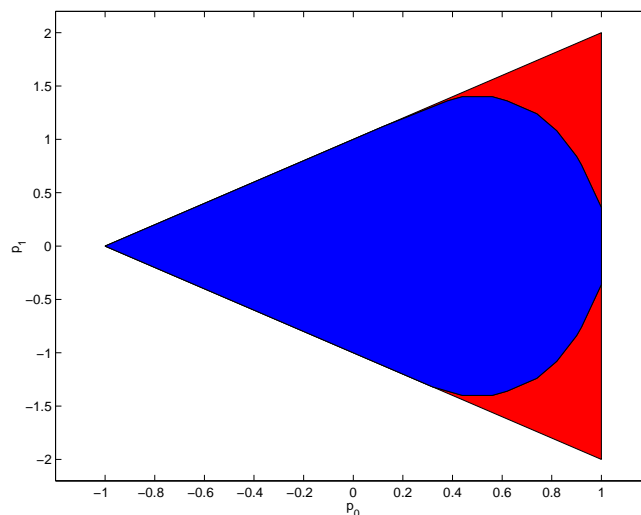
The implicit equation of the envelope is

$$(2n_0 - 1)^2 + (\frac{\sqrt{2}}{2}n_1)^2 = 1$$

a scaled circle

The LMI stability region is then the union of the interior of the circle with the interior of the triangle delimited by the two lines
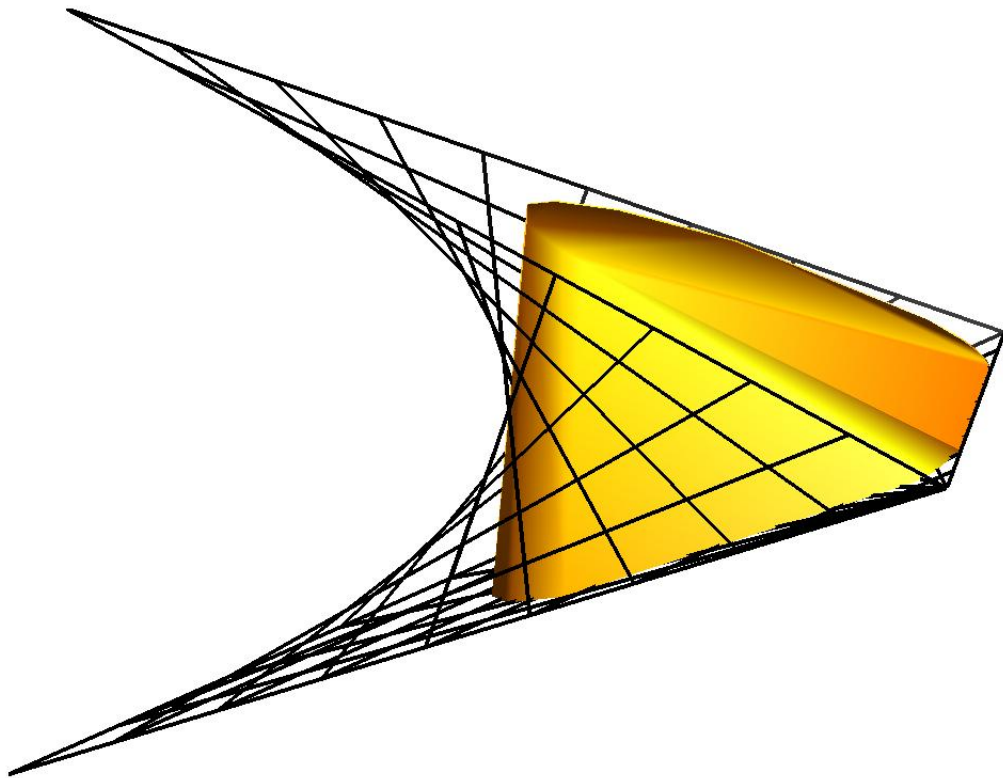
$$n_0 \pm n_1 + 1 = 0$$

tangent to the circle, with vertices $[-1, 0]$, $[1/3, 4/3]$ and $[1/3, -4/3]$

Third-degree discrete-time polynomial

Similarly, we can approximate from the inside the nonconvex third-degree discrete-time polynomial stability region with an LMI

Application to robust stability analysis

Assume that $N(s, \lambda)$ is a polynomial matrix with multi-linear dependence in a parameter vector λ belonging to a polytope Λ

Denote by $N_i(s)$ the vertices obtained by enumerating each vertex in Λ

Polytopic polynomial matrix $N(s, \lambda)$ is robustly stable if there exists a matrix D and matrices P_i satisfying the LMI

$$D^\star N_i + N_i^\star D - H(P_i) \succ 0$$
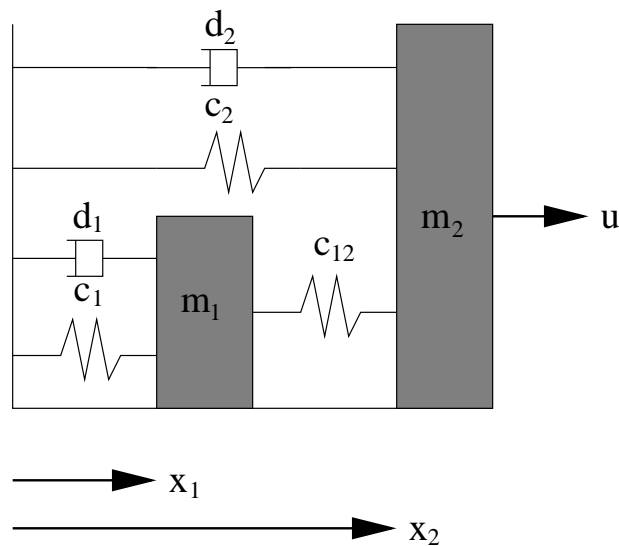$$P_i \succ 0 \quad \forall i$$

Proof

Since the LMI is linear in D - matrix of coefficients of polynomial matrix $D(s)$ - it is enough to check the vertices to prove stability in the whole polytope

Robust stability of polynomial matrices

Example
Consider the following mechanical system

It is described by the polynomial MFD

$$\begin{bmatrix} m_1 s^2 + d_1 s + c_1 + c_{12} & -c_{12} \\ -c_{12} & m_2 s^2 + d_2 s + c_2 + c_{12} \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ u(s) \end{bmatrix}$$

System parameters $\lambda = \begin{bmatrix} m_1 & d_1 & c_1 & m_2 & d_2 & c_2 \end{bmatrix}$
belong to the uncertainty hyper-rectangle
$\Lambda = [1, 3] \times [0.5, 2] \times [1, 2] \times [2, 5] \times [0.5, 2] \times [2, 4]$
and we set $c_{12} = 1$

This mechanical system is passive so it must be open-loop stable (when $u(s) = 0$) independently of the values of the masses, springs, and dampers
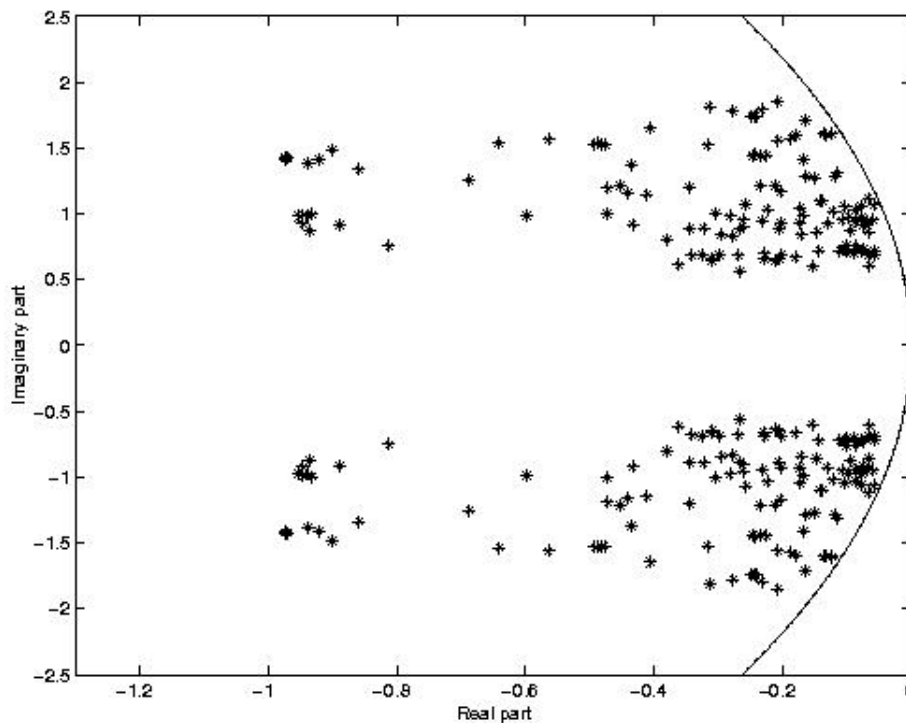
Robust stability of polynomial matrices

However, it is a non-trivial task to know whether the open-loop system is robustly D-stable in some stability region $\mathcal{D}$ ensuring a certain damping. Here we choose the disk of radius 12 centered at -12

$$\mathcal{D} = \{s : (s + 12)^2 < 12^2\}$$

The robust stability analysis problem amounts to assessing whether the second degree polynomial matrix in the MFD has its zeros in $\mathcal{D}$ for all admissible uncertainty in a polytope with $m = 2^6 = 64$ vertices

LMI problem is feasible – vertex zeros shown below

Polytope of polynomials

We can also check robust stability of polytopes of polynomials without using the edge theorem or the graphical value set
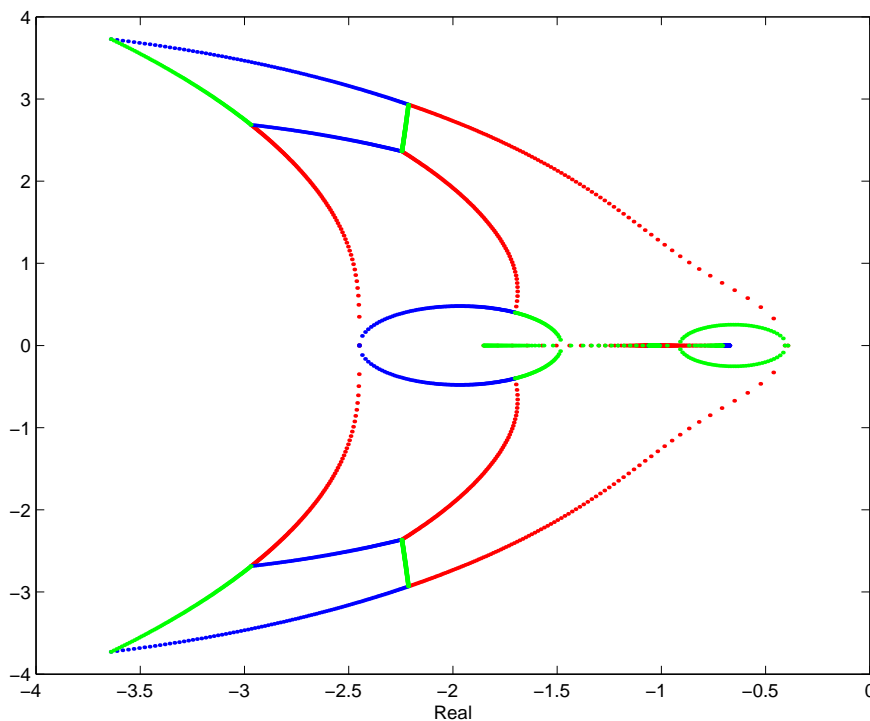
Example
Continuous-time polytope of degree 3 with 3 vertices

$$
\begin{aligned}
n_1(s) &= 28.3820 + 34.7667s + 8.3273s^2 + s^3 \\
n_2(s) &= 0.2985 + 1.6491s + 2.6567s^2 + s^3 \\
n_3(s) &= 4.0421 + 9.3039s + 5.5741s^2 + s^3
\end{aligned}
$$

The LMI problem is feasible, so the polytope is robustly stable − see robust root locus below

Interval polynomial matrices

Similarly, we can assess robust stability of interval polynomial matrices, a difficult problem in general

Example
Continuous-time interval polynomial matrix of degree 2 with $2^3 = 8$ vertices

$$\begin{bmatrix} \begin{aligned} &[7.7 - 2.3s + 4.3s^2, \\ &\quad 3.7 + 2.7s + 4.3s^2] \\ &3.6 + 6.4s + 4.3s^2 \end{aligned} & \begin{aligned} &[-3.1 - 6s - 2.2s^2, \\ &\quad -4.1 - 7s - 2.2s^2] \\ &[3.2 + 11s + 8.2s^2, \\ &\quad 16 + 12s + 8.2s^2] \end{aligned} \end{bmatrix}$$

LMI is feasible so the matrix is robustly stable
See robust root locus below
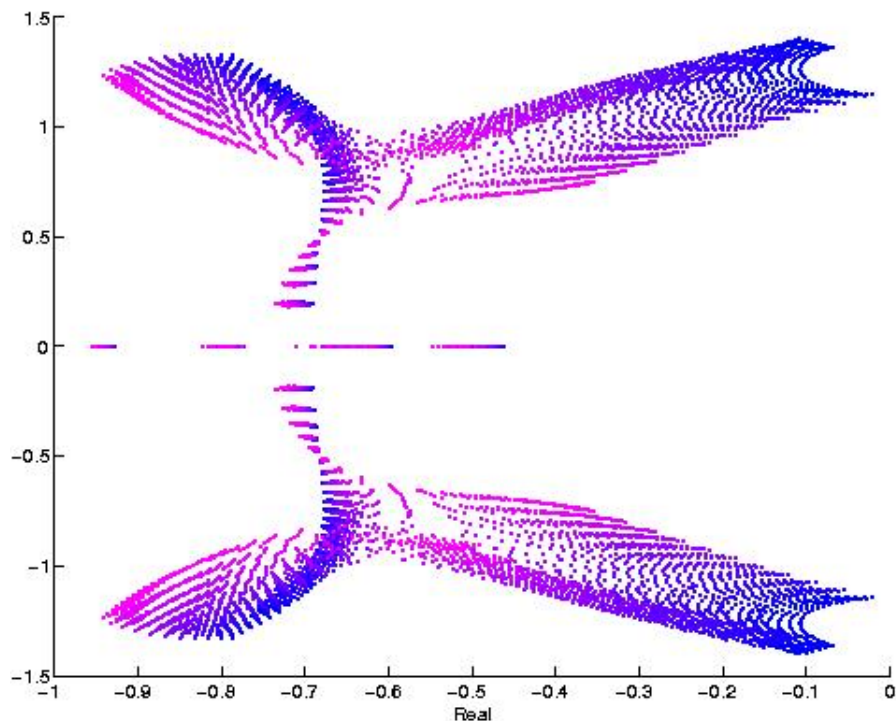
State-space systems

One advantage of our approach is that state-space results can be obtained as simple by-products, since stability of a constant matrix A is equivalent to stability of the pencil matrix

$$N(s) = sI - A$$

Matrix A is stable iff there exists a matrix F and a matrix P solving the LMI

$$\begin{bmatrix} F^\star A + A^\star F - aP & -A^\star - F^\star - bP \\ -A - F - b^\star P & 2I - cP \end{bmatrix} \succ 0$$

$$P \succ 0$$

Proof
Just take $D(s) = sI - F$ and notice that the LMI can be also written more explicitly as

$$\begin{bmatrix} -F^\star \\ I \end{bmatrix} \begin{bmatrix} -A & I \end{bmatrix} + \begin{bmatrix} -A^\star \\ I \end{bmatrix} \begin{bmatrix} -F & I \end{bmatrix} - \begin{bmatrix} aP & bP \\ b^\star P & cP \end{bmatrix} > 0$$

Robust stability of state-space systems

We can recover known LMI stability conditions

Nice decoupling between Lyapunov matrix P and additional variable F allows for construction of parameter-dependent Lyapunov matrix

Assume that uncertain matrix $A(\lambda)$ has multilinear dependence on polytopic uncertain parameter λ and denote by A_i the corresponding vertices

Matrix $A(\lambda)$ is robustly stable if there exists a matrix F and matrices P_i solving the LMI

$$
\begin{bmatrix} F^\star A_i + A_i^\star F - aP_i & -A_i^\star - F^\star - bP_i \\ -A_i - F - b^\star P_i & 2I - cP_i \end{bmatrix} \succ 0
$$
$$
P_i \succ 0 \quad \forall i
$$

Proof
Consider the parameter-dependent Lyapunov matrix $P(\lambda)$ built from vertices P_i

Robust design

Assume now that system matrix $C(s, \lambda)$ comes from a polynomial Diophantine equation

$$C(s, \lambda) = A(s, \lambda)X(s) + B(s, \lambda)Y(s)$$

where system matrices A and B are subject to multi-linear polytopic uncertainty λ

In order to ensure robust SPRness of the rational matrix $D^{-1}(s)C(s, \lambda)$ central polynomial $D(s)$ must be close to the nominal closed-loop matrix, such as

$$D(s) = C(s, \lambda_0)$$

where λ_0 is the nominal parameter vector

A sensible simple choice of $D(s)$ is therefore the nominal closed-loop denominator polynomial matrix, obtained by any standard design method (pole assignment, LQ, H_∞)

Robot example

We consider the problem of designing a robust controller for the approximate ARMAX model of a PUMA robotic disk grinding process

From the results of identification and because of the nonlinearity of the robot, the coefficients of the numerator of the plant transfer function change for different positions of the robot arm. We consider variations of up to 20% around the nominal value of the parameters

The fourth-order discrete-time model is given by

$$\frac{b(z^{-1},q)}{a(z^{-1},q)} = \frac{\left(\begin{array}{c} (0.0257 + q_1) + (-0.0764 + q_2)z^{-1} \\ +(-0.1619 + q_3)z^{-2} + (-0.1688 + q_4)z^{-3} \end{array} \right)}{1 - 1.914z^{-1} + 1.779z^{-2} - 1.0265z^{-3} + 0.2508z^{-4}}$$

where

$$|q_1| \leq 0.00514, |q_2| \leq 0.01528, |q_3| \leq 0.03238, |q_4| \leq 0.03376$$

Robot

Closed-loop polynomial

$$d(z,q) = z^{12}[(1-z^{-1})a(z^{-1},q)x(z^{-1}) + z^{-5}b(z^{-1},q)y(z^{-1})]$$

where the term $1 - z^{-1}$ is introduced in the controller denominator to maintain the steady state error to zero when parameters are changed

With the input central polynomial $d(z) = z^{19}$ the LMI returns the seventh-order robust controller

$$\frac{y(z^{-1})}{x(z^{-1})} = \frac{\left(\begin{array}{c} -0.2863 + 0.2928z^{-1} + 0.0221z^{-2} \\ -0.1558z^{-3} + 0.0809z^{-4} + 0.1420z^{-5} \\ -0.1254z^{-6} + 0.0281z^{-7} \end{array}\right)}{\left(\begin{array}{c} 1 + 1.1590z^{-1} + 0.9428z^{-2} \\ +0.4996z^{-3} + 0.3044z^{-4} + 0.4881z^{-5} \\ +0.4003z^{-6} + 0.3660z^{-7} \end{array}\right)}$$
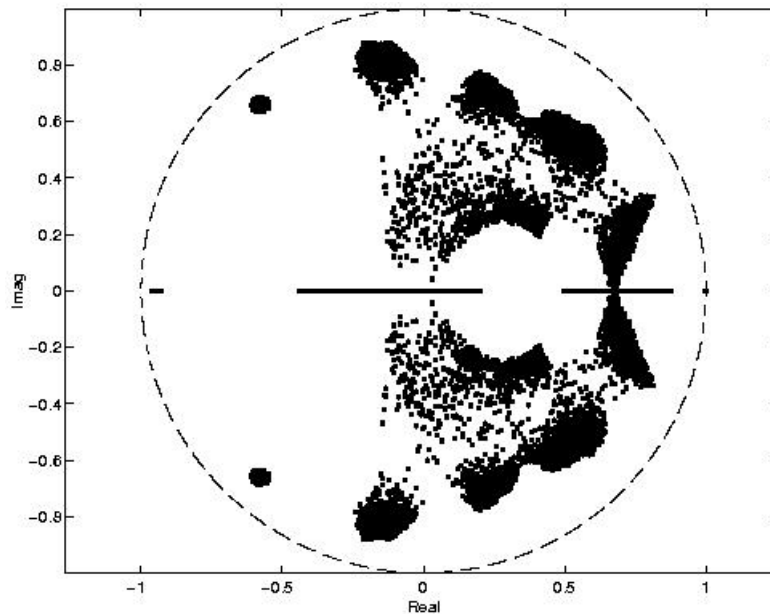
Second-order systems

Second-order linear system

$$\begin{aligned} (A_0 + A_1 s + A_2 s^2)x &= Bu \\ y &= Cx \end{aligned}$$

to be controlled by PD output-feedback controller

$$u = -(F_0 + F_1 s)y$$

Applications: large flexible space structures, earthquake engineering, mechanical multi-body systems, damped gyroscopic systems, robotics control, vibration in structural dynamics, flows in fluid mechanics, electrical circuits

320m long Millenium footbridge over river Thames in London

PD controller

Closed-loop system behavior captured by zeros of quadratic polynomial matrix

$$N(s) = (A_0 + BF_0C) + (A_1 + BF_1C)s + A_2s^2$$

Zeros of $N(s)$ must be located in some stability region $\mathcal{D}$ characterized as before by matrix H

Uncertainty can affect A_0 (stiffness) A_1 (damping) and A_2 (mass)

Given A_0, A_1, A_2, B, C
find F_0, F_1
ensuring robust pole placement

Robust LMI stability condition

- **Norm-bounded** (unstructured) uncertainty

$$N(s) + \Delta M(s) \quad \sigma_{\mathsf{max}}(\Delta) \leq \delta$$

LMI robust stability condition on $N(s)$

$$\begin{bmatrix} D^\star N + N^\star D - H(P) - \gamma D^\star D & \delta M^\star \\ \delta M & \gamma I \end{bmatrix} \succ 0$$

- **Polytopic** (structured) uncertainty

$$N(s) = \sum_i \lambda^i N^i(s) \quad \sum_i \lambda^i = 1 \quad \lambda^i \geq 0$$

Vertex LMI robust stability conditions:

$$DN^i + (N^i)^\star D - H(P^i) \succ 0, \quad i = 1, 2, \ldots$$

Parameter-dependent Lyapunov matrix
$P(\lambda) = \sum_i \lambda^i P^i$

Robust design

Once central polynomial matrix $D(s)$ is fixed, robust stability condition is LMI in $N(s)$, so extension to design is straightforward

Easy incorporation of structural constraints on controller coefficient matrices F_0, F_1:
- minimization of 2-norm (SOCP)
- enforcing some entries to zero (LP)
- maximization of uncertainty radius (SDP)

Key point is choice of central polynomial matrix

Good policy: set $D(s)$ to some nominal system matrix obtained by some standard design method (pole placement, LQ, H_2 or H_∞), then try to optimize around $D(s)$

Example: five masses

Five masses linked by elastic springs
controlled by two external forces
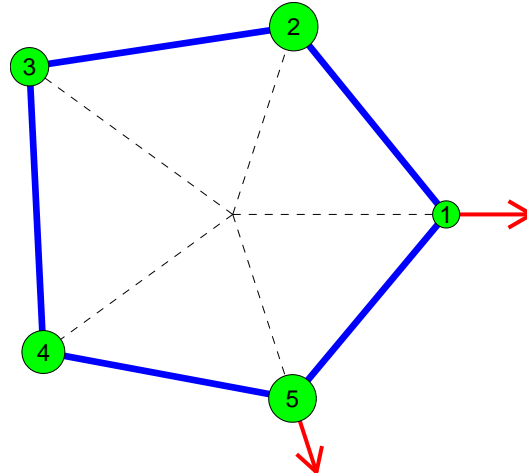
$$A_0 = \begin{bmatrix} 2.565 & 1.080 & 0 & 0 & 1.089 \\ 0.6038 & 0.8206 & 0.4766 & 0 & 0 \\ 0 & 0.6009 & 1.504 & 0.4808 & 0 \\ 0 & 0 & 0.4300 & 1.114 & 0.5131 \\ 0.6190 & 0 & 0 & 0.4626 & 0.8352 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1.964 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1.116 & 0 \end{bmatrix} \qquad \begin{array}{c} A_1 = 0_5 \\ A_2 = I_5 \\ C = I_5 \end{array}$$

Purely imaginary open-loop poles $\pm i1.783$, $\pm i1.380$, $\pm i1.145$,
$\pm i0.5675$ and $\pm i0.3507$

Nominal PD controller F_0^0, F_1^0 obtained with LQ design
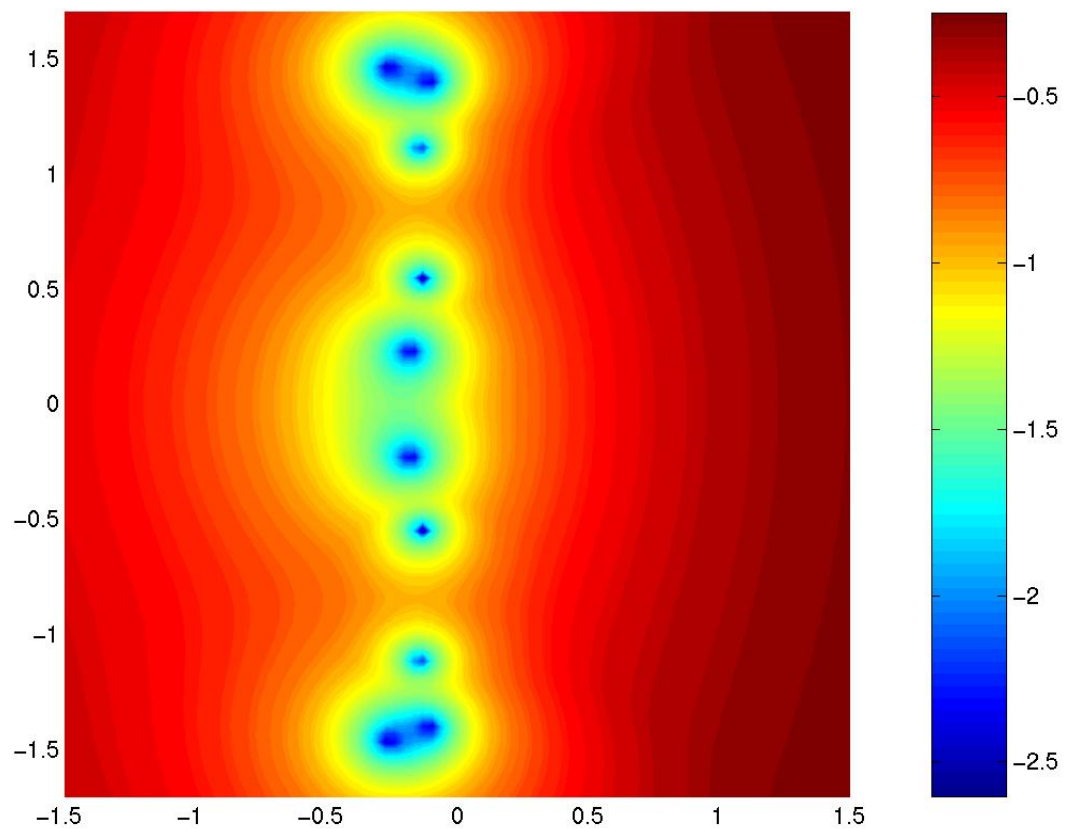Resulting central polynomial matrix

$$D(s) = (A_0 + BF_0^0 C) + (A_1 + BF_1^0 C)s + A_2 s^2$$

Stability region $\mathcal{D} = \{s : \operatorname{Re} s < -0.1\}$

Five mass example (2)

Minimizing the norm of feedback matrices F_0, F_1 over the design LMI yields

$$\|[F_0 \quad F_1]\| = 0.7537 < \|[F_0^0 \quad F_1^0]\| = 1.462$$

Closed-loop pseudospectrum of the five masses example