II.2. STATE-SPACE DESIGN

Didier HENRION
henrion@laas.fr

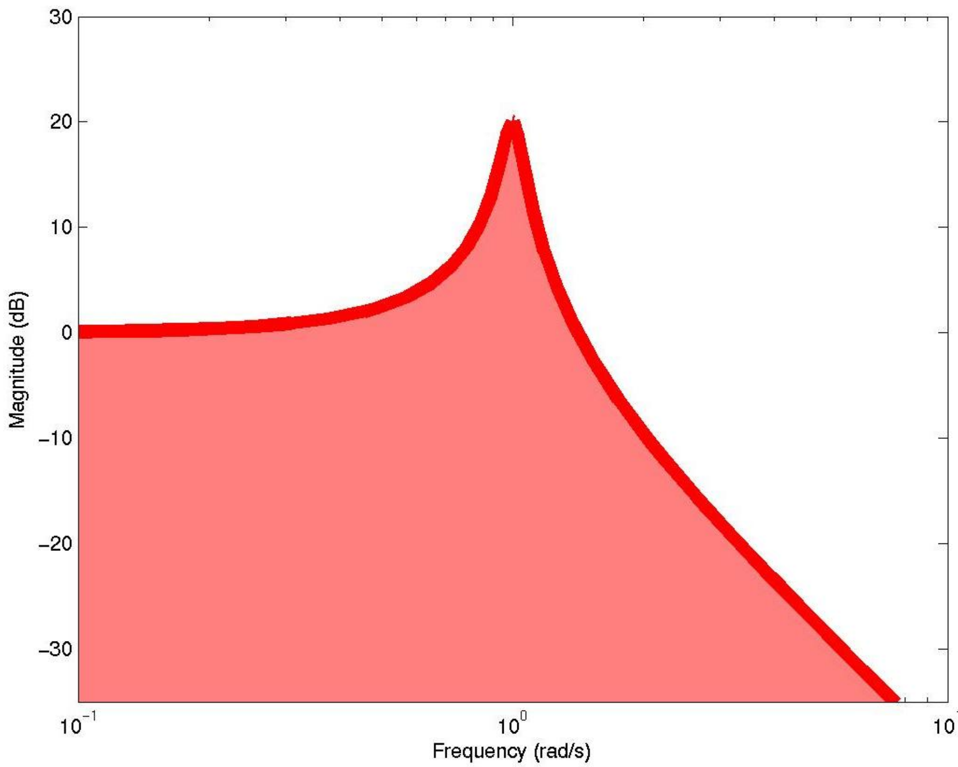
Belgian Graduate School on
Systems, Control, Optimization and Networks

Leuven - April and May 2010

H_2 norm

The H_2 norm is the energy (l_2 norm) of the impulse response $h(t)$ of a system G

$$\|G\|_2 = \left(\int_0^{\infty} h^\star(t)h(t)dt\right)^{1/2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)H^\star(j\omega)d\omega\right)^{1/2}$$



For a continuous-time system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

with transfer function $G(s) = C(sI - A)^{-1}B + D$ we must assume $D = 0$ to have $\|G\|_2$ finite

Computing the H_2 norm

Let $h_i(t) = Ce^{At}B_i$ denote the i-th column of the impulse response of G, then

$$\begin{aligned} \|G\|_2^2 &= \sum_i \|h_i\|_2^2 \\ &= \sum_i \int_0^\infty B_i^\star e^{A^\star t} C^\star C e^{At} B_i dt \\ &= \text{trace}\, B^\star \left(\int_0^\infty e^{A^\star t} C^\star C e^{At} dt \right) B \end{aligned}$$

Matrix

$$P_o = \int_0^\infty e^{A^\star t} C^\star C e^{At} dt$$

is the observability Grammian solution to the Lyapunov equation

$$A^\star P_o + P_o A + C^\star C = 0$$

and hence

$$\|G\|_2^2 = \text{trace}\, B^\star P_o B$$

If (A, C) observable then $P_o \succ 0$

Dual and LMI computation of the H_2 norm

Defining the controllability Grammian

$$P_c = \int_0^\infty e^{At} B B^\star e^{A^\star t} dt$$

solution to the Lyapunov equation

$$A P_c + P_c A^\star + B B^\star = 0$$

we have a dual expression

$$\|G\|_2^2 = \text{trace}\, C P_c C^\star$$

Dual lyapunov equations formulated as LMIs

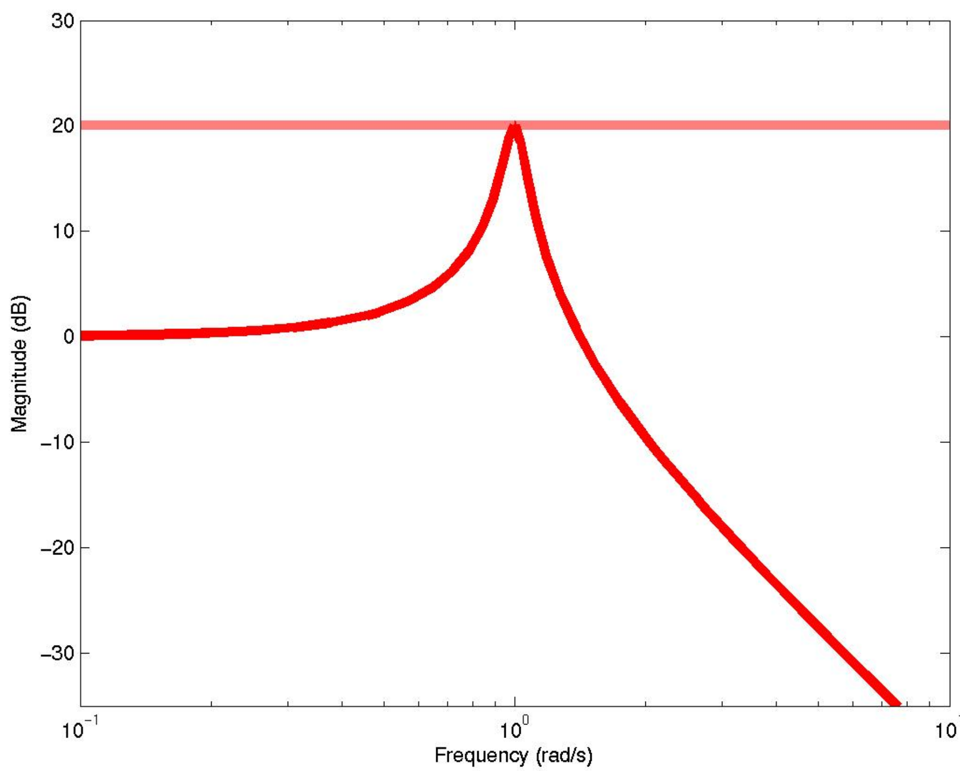
$$\begin{array}{rcll} \|G\|_2^2 & = & \min & \text{trace}\, B^\star P B \\ & & \text{s.t.} & A^\star P + P A + C^\star C \preceq 0 \\ & & & P \succeq 0 \end{array}$$

$$\begin{array}{rcll} \|G\|_2^2 & = & \min & \text{trace}\, C Q C^\star \\ & & \text{s.t.} & A Q + Q A^\star + B B^\star \preceq 0 \\ & & & Q \succeq 0 \end{array}$$

H_∞ norm

The H_∞ norm is the induced energy gain (l_2 to l_2)

$$\|G\|_\infty = \sup_{\|x\|_2=1} \|Gx\|_2 = \sup_\omega \|G(j\omega)\|$$



It is the worst case gain

Computing the H_∞ norm

In contrast with the H_2 norm, computation of the H_∞ norm is iterative

For the continuous-time linear system

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx + Du \end{array}$$

with transfer function $G(s) = C(sI - A)^{-1}B + D$ we have $\|G(s)\|_\infty < \gamma$ iff $R = \gamma^2 I - D^\star D \succ 0$ and the Hamiltonian matrix

$$\left[\begin{array}{cc} A + BR^{-1}D^\star C & BR^{-1}B^\star \\ -C^\star(I + DR^{-1}D^\star)C & -(A + BR^{-1}D^\star C)^\star \end{array}\right]$$

has no eigenvalues on the imaginary axis

We can then design a bisection algorithm with guaranteed quadratic convergence to find the minimum value of γ such that the Hamiltonian has no imaginary eigenvalues

LMI computation of the H_∞ norm

Refer to the part of the course on norm-bounded uncertainty

$$\sup_{\|z\|_2=1} \|w\| = \|\Delta\| < \gamma^{-1}$$

to prove that for the continuous-time system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

with transfer function $G(s) = C(sI-A)^{-1}B+D$ we have $\|G(s)\|_\infty < \gamma$ iff there exists a matrix P solving the LMI

$$\begin{bmatrix} A^\star P + PA + C^\star C & PB + C^\star D \\ B^\star P + D^\star C & D^\star D - \gamma^2 I \end{bmatrix} \prec 0 \quad P \succ 0$$

Using the Schur complement and a change of variables this can be expanded to

$$\begin{bmatrix} A^\star P + PA & PB & C^\star \\ B^\star P & -\gamma I & D^\star \\ C & D & -\gamma I \end{bmatrix} \prec 0 \quad P \succ 0$$

Linear systems design

Open-loop continuous-time LTI system

$$\dot{x} = Ax + Bu$$

with state-feedback controller

$$u = Kx$$

produces closed-loop system

$$\dot{x} = (A + BK)x$$

Applying Lyapunov LMI stability condition

$$(A + BK)^\star P + P(A + BK) \prec 0 \qquad P \succ 0$$

we get bilinear terms..

Bilinear Matrix Inequalities (BMIs) are non-convex in general !

State-feedback design:
linearizing change of variables

Project BMI onto $P^{-1} \succ 0$

$$(A+BK)^\star P + P(A+BK) \prec 0$$
$$\Longleftrightarrow$$
$$P^{-1}\left[(A+BK)^\star P + P(A+BK)\right]P^{-1} \prec 0$$
$$\Longleftrightarrow$$
$$P^{-1}A^\star + P^{-1}K^\star B^\star + AP^{-1} + BKP^{-1} \prec 0$$

Denoting

$$Q = P^{-1} \quad Y = KP^{-1}$$

we derive a state-feedback design LMI

$$AQ + QA^\star + BY + Y^\star B^\star \prec 0 \quad Q \succ 0$$

We obtained an LMI thanks to a one-to-one linearizing change of variables

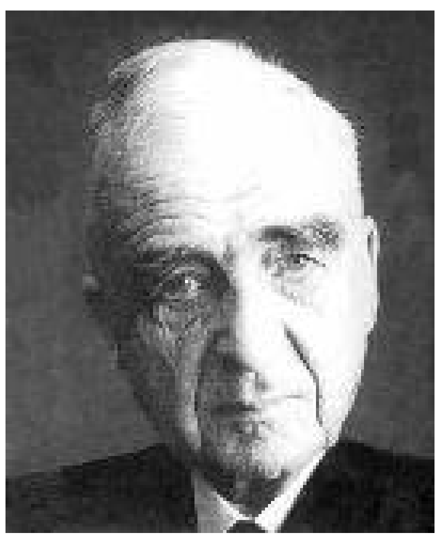
Simple but very useful trick..

Finsler's lemma

A very useful trick in robust control..

The following statements are equivalent

1. $x^\star A x > 0$ for all $x \neq 0$ s.t. $Bx = 0$
2. $\tilde{B}^\star A \tilde{B} > 0$ where $B\tilde{B} = 0$
3. $A + \lambda B^\star B > 0$ for some scalar λ
4. $A + XB + B^\star X^\star > 0$ for some matrix X



Paul Finsler
(1894 Heilbronn - 1970 Zurich)

State-feedback design: null-space projection

We can also use item 2 of Finsler's lemma, projecting onto the (full column rank) null-space $\tilde{B}$ of $B^\star$

$$B^\star \tilde{B} = 0$$

so that BMI

$$A^\star P + PA + K^\star B^\star P + PBK \prec 0$$

is equivalent to the projected LMI

$$\tilde{B}^\star (AQ + QA^\star)\tilde{B} \prec 0 \quad Q \succ 0$$

Feedback K can be recovered from Lyapunov matrix Q as

$$K = -\lambda B^\star Q^{-1}$$

where λ is a suitably large scalar

Here we obtained an LMI thanks to a projection onto a null-space

State-feedback design: Riccati inequality

We can also use item 3 of Finsler's lemma to convert BMI

$$A^\star P + PA + K^\star B^\star P + PBK \prec 0$$

into

$$A^\star P + PA - \lambda PBB^\star P \prec 0$$

where $\lambda \geq 0$ is an unknown scalar

Now replacing P with λP we get

$$A^\star P + PA - PBB^\star P \prec 0$$

which is equivalent to the Riccati equation

$$A^\star P + PA - PBB^\star P + Q = 0$$

for some matrix $Q \succ 0$

Shows equivalence between state-feedback LMI stabilizability and the linear quadratic regulator (LQR) problem

Robust state-feedback design for polytopic uncertainty

Open-loop system $\dot{x} = Ax + Bu$ with polytopic uncertainty

$$(A, B) \in \text{conv}\,\{(A_1, B_1), \ldots, (A_N, B_N)\}$$

and robust state-feedback controller $u = Kx$

In order to derive synthesis condition,
we start with analysis conditions

$$(A_i + B_i K)^\star P + P(A_i + B_i K) \prec 0 \ \forall i \quad Q \succ 0$$

and we obtain the quadratic stabilizability LMI

$$A_i Q + Q A_i^\star + B_i Y + Y^\star B_i^\star \prec 0 \ \forall i \quad Q \succ 0$$

with the linearizing change of variables

$$Q = P^{-1} \quad Y = KP^{-1}$$

State-feedback H_2 control

Continuous-time LTI open-loop system

$$\begin{aligned} \dot{x} &= Ax + B_w w + B_u u \\ z &= C_z x + D_{zw} w + D_{zu} u \end{aligned}$$

with state-feedback controller

$$u = Kx$$

yields closed-loop system

$$\begin{aligned} \dot{x} &= (A + B_u K)x + B_w w \\ z &= (C_z + D_{zu} K)x + D_{zw} w \end{aligned}$$

with transfer function

$$G(s) = D_{zw} + (C_z + D_{zu}K)(sI - A - B_u K)^{-1} B_w$$

between performance signals w and z

H_2 performance specification

$$\|G(s)\|_2 < \gamma$$

We must have $D_{zw} = 0$ (finite gain)

H_2 design LMIs

As usual, start with analysis condition:
there exists K such that $\|G(s)\|_2 < \gamma$ iff

$$\begin{array}{c} \text{trace}\,(C_z + D_{zu}K)Q(C_z + D_{zu}K)^\star < \gamma \\ (A + B_uK)Q + Q(A + B_uK)^\star + BB^\star \prec 0 \end{array}$$

The trace inequality can be written as

$$\text{trace}(C_z + D_{zu}K)Q(C_z + D_{zu}K)^\star < \text{trace}\,W < \gamma$$

for some matrix W such that

$$\begin{bmatrix} W & (C_z + D_{zu}K)Q \\ Q(C_z + D_{zu}K)^\star & Q \end{bmatrix} \succ 0$$

We obtain the overall LMI formulation

$$\begin{array}{c} \text{trace}\,W < \gamma \\ \begin{bmatrix} W & C_zQ + D_{zu}Y \\ QC_z^\star + Y^\star D_{zu}^\star & Q \end{bmatrix} \succ 0 \\ AQ + QA^\star + B_uY + Y^\star B_u^\star + B_wB_w^\star \prec 0 \end{array}$$

with resulting H_2 suboptimal state-feedback

$$K = YQ^{-1}$$

State-feedback H_∞ control

Similarly, with closed-loop system

$$\begin{aligned} \dot{x} &= (A + B_u K)x + B_w w \\ z &= (C_z + D_{zu} K)x + D_{zw} w \end{aligned}$$

and H_∞ performance specification

$$\|G(s)\|_\infty < \gamma$$

on transfer function between w and z we obtain the design LMI

$$\left[\begin{array}{cc} AQ + QA^\star + B_u Y + Y^\star B_u^\star + B_w B_w^\star & \star \\ C_z Q + D_{zu} Y + D_{zw} B_w^\star & D_{zw} D_{zw}^\star - \gamma^2 I \end{array} \right] \prec 0$$
$$Q \succ 0$$

with resulting H_∞ suboptimal state-feedback

$$K = YQ^{-1}$$

Optimal H_∞ control: minimize γ

Mixed H_2/H_∞ control

State-feedback controller system
with two performance channels

$$\begin{array}{rcl} \dot{x} & = & (A + B_u K)x + B_w w \\ z_\infty & = & (C_\infty + D_{\infty u} K)x + D_{\infty w} w \\ z_2 & = & (C_2 + D_{2u} K)x \end{array}$$

and mixed performance specifications

$$\|G_\infty(s)\|_\infty < \gamma_\infty \quad \|G_2(s)\|_2 < \gamma_2$$

on transfer functions from w to z_∞ and z_2 respectively

Formulation of H_∞ constraint

$$\left[\begin{array}{cc} AQ_\infty + B_u K Q_\infty + (\star) + B_w B_w^\star & \star \\ C_\infty Q_\infty + D_{\infty u} K Q_\infty + D_{\infty w} B_w^\star & D_{\infty w} D_{\infty w}^\star - \gamma_\infty^2 I \end{array}\right] \prec 0$$
$$Q_\infty \succ 0$$

BMI formulation of H_2 constraint

$$\text{trace}\, W < \gamma_2$$
$$\left[\begin{array}{cc} W & C_2 Q_2 + D_{2u} K Q_2 \\ \star & Q_2 \end{array}\right] \succ 0$$
$$AQ_2 + B_u K Q_2 + (\star) + B_w B_w^\star \prec 0$$

Problem:

We cannot linearize simultaneously
terms KQ_∞ and KQ_2 !

Mixed H_2/H_∞ control design LMI

Remedy:

Enforce $Q_2 = Q_\infty = Q$!

Conservative but useful.. Always trade-off between conservatism and tractability

Resulting mixed H_2/H_∞ design LMI

$$\begin{bmatrix} AQ + B_u Y + (\star) + B_w B_w^\star & \star \\ C_\infty Q + D_{\infty u} Y + D_{\infty w} B_w^\star & D_{\infty w} D_{\infty w}^\star - \gamma_\infty^2 I \end{bmatrix} \prec 0$$

$$\text{trace}\, W < \gamma_2$$

$$\begin{bmatrix} W & C_2 Q + D_{2u} Y \\ \star & Q \end{bmatrix} \succ 0$$

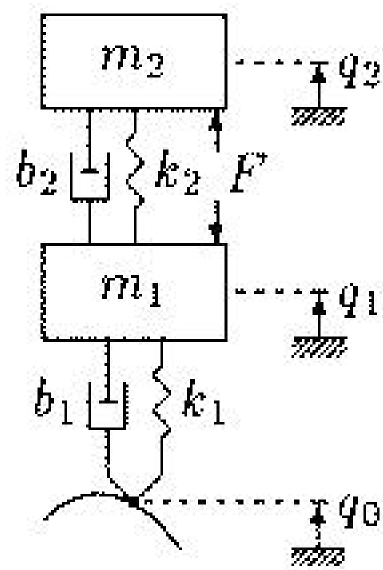
$$AQ + B_u Y + (\star) + B_w B_w^\star \prec 0$$

Guaranteed cost mixed H_2/H_∞:
given γ_∞ minimize γ_2

Can be used for H_2 design of uncertain systems with norm-bounded uncertainty

Mixed H_2/H_∞ control: example

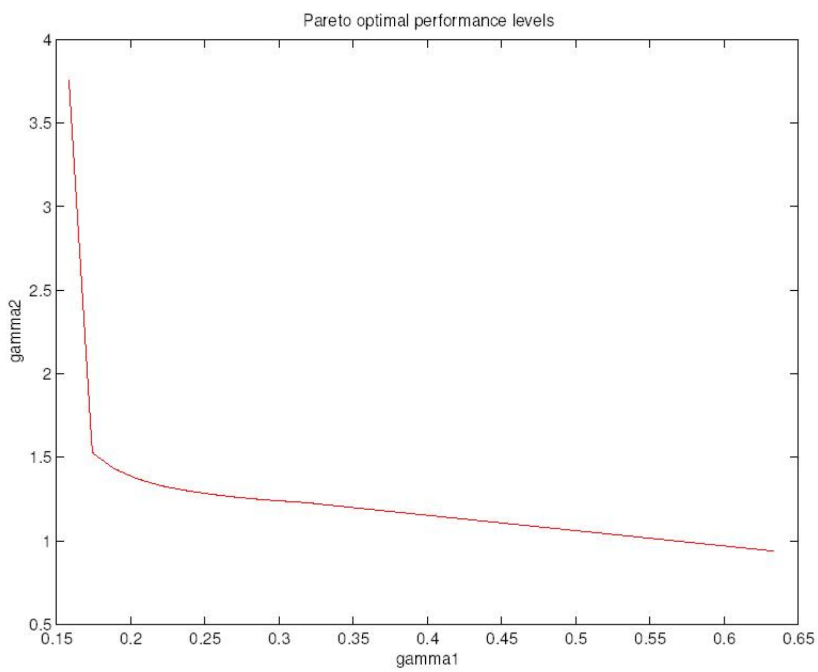
Active suspension system (Weiland)



$$m_2\ddot{q}_2 + b_2(\dot{q}_2 - \dot{q}_1) + k_2(q_2 - q_1) + F = 0$$
$$m_1\ddot{q}_1 + b_2(\dot{q}_1 - \dot{q}_2) + k_2(q_1 - q_2)$$
$$+k_1(q_1 - q_0) + b_1(\dot{q}_1 - \dot{q}_0) + F = 0$$

$$z = \begin{bmatrix} q_1 - q_0 \\ F \\ \ddot{q}_2 \\ q_2 - q_1 \end{bmatrix} \; y = \begin{bmatrix} \ddot{q}_2 \\ q_2 - q_1 \end{bmatrix} \; w = q_0 \;\; u = F$$

$G_\infty(s)$ from q_0 to $[q_1 - q_0 \quad F]$
$G_2(s)$ from q_0 to $[\ddot{q}_2 \quad q_2 - q_1]$



Trade-off between $\|G_\infty\|_\infty \leq \gamma_1$ and $\|G_2\|_2 \leq \gamma_2$

Dynamic output-feedback

Continuous-time LTI open-loop system

$$\begin{aligned} \dot{x} &= Ax + B_w w + B_u u \\ z &= C_z x + D_{zw} w + D_{zu} u \\ y &= C_y x + D_{yw} w \end{aligned}$$

with dynamic output-feedback controller

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y \end{aligned}$$

Denote closed-loop system as

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}w \\ z &= \tilde{C}\tilde{x} + \tilde{D}w \end{aligned} \quad \text{with } \tilde{x} = \begin{bmatrix} x \\ x_c \end{bmatrix} \text{ and}$$

$$\tilde{A} = \begin{bmatrix} A + B_u D_c C_y & B_u C_c \\ B_c C_y & A_c \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} B_w + B_u D_c D_{yw} \\ B_c D_{yw} \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} C_z + D_{zu} D_c C_y & D_{zu} C_c \end{bmatrix} \qquad \tilde{D} = D_{zw} + D_{zu} D_c D_{yw}$$

Affine expressions on controller matrices

H_2 output feedback design

H_2 design conditions

$$\begin{gathered}\text{trace}\, W < \gamma \\ \begin{bmatrix} W & \tilde{C}\tilde{Q} \\ \star & \tilde{Q} \end{bmatrix} \succ 0 \\ \begin{bmatrix} \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^\star & \tilde{B} \\ \tilde{B}^\star & -I \end{bmatrix} \prec 0 \end{gathered}$$

can be linearized with a specific change of variables

Denote

$$\tilde{Q} = \begin{bmatrix} Q & \bar{Q}^\star \\ \bar{Q} & \times \end{bmatrix} \qquad \tilde{P} = \tilde{Q}^{-1} = \begin{bmatrix} P & \bar{P} \\ \bar{P}^\star & \times \end{bmatrix}$$

so that $\bar{P}$ and $\bar{Q}$ can be obtained from P and Q via relation

$$PQ + \bar{P}\bar{Q} = I$$

Always possible when controller has same order than the open-loop plant

Linearizing change of variables for H_2 output-feedback design

Then define

$$\begin{bmatrix} X & U \\ Y & V \end{bmatrix} = \begin{bmatrix} \bar{P} & PB_u \\ 0 & I \end{bmatrix} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} \bar{Q} & 0 \\ C_y Q & I \end{bmatrix} + \begin{bmatrix} P \\ 0 \end{bmatrix} A \begin{bmatrix} Q & 0 \end{bmatrix}$$

which is a one-to-one affine relation with converse

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} \bar{P}^{-1} & -\bar{P}^{-1}PB_u \\ 0 & I \end{bmatrix} \begin{bmatrix} X - PAQ & U \\ Y & V \end{bmatrix} \begin{bmatrix} \bar{Q}^{-1} & 0 \\ -C_y Q \bar{Q}^{-1} & I \end{bmatrix}$$

We derive the following H_2 design LMI

$$\begin{gathered} \text{trace } W < \gamma \\ D_{zw} + D_{zu} V D_{yw} = 0 \\ \begin{bmatrix} W & C_z Q + D_{zu} Y & C_z + D_{zu} V C_y \\ \star & Q & I \\ \star & \star & P \end{bmatrix} \succ 0 \\ \begin{bmatrix} AQ + B_u Y + (\star) & A + B_u V C_y + X^\star & B_w + B_u V D_{yw} \\ \star & PA + UC_y + (\star) & PB_w + UD_{yw} \\ \star & \star & -I \end{bmatrix} \prec 0 \end{gathered}$$

in decision variables Q, P, W (Lyapunov)
and X, Y, U, V (controller)

Controller matrices are obtained via the relation

$$PQ + \bar{P}\bar{Q} = I$$

(tedious but straightforward linear algebra)

H_∞ output-feedback design

Similarly two-step procedure for full-order H_∞ output-feedback design:

- solve LMI for Lyapunov variables Q, P, W and controller variables X, Y, U, V
- retrieve controller matrices via linear algebra

For reduced-order controller of order $n_c < n$ there exists a solution $\bar{P}$,$\bar{Q}$ to the equation

$$PQ + \bar{P}\bar{Q} = I$$

iff

$$\text{rank}\,(PQ - I) = n_c$$
$$\Longleftrightarrow$$
$$\text{rank}\begin{bmatrix} Q & I \\ I & P \end{bmatrix} = n + n_c$$

Static output feedback iff $PQ = I$

Difficult rank constrained LMI !