II.1. STATE-SPACE ANALYSIS

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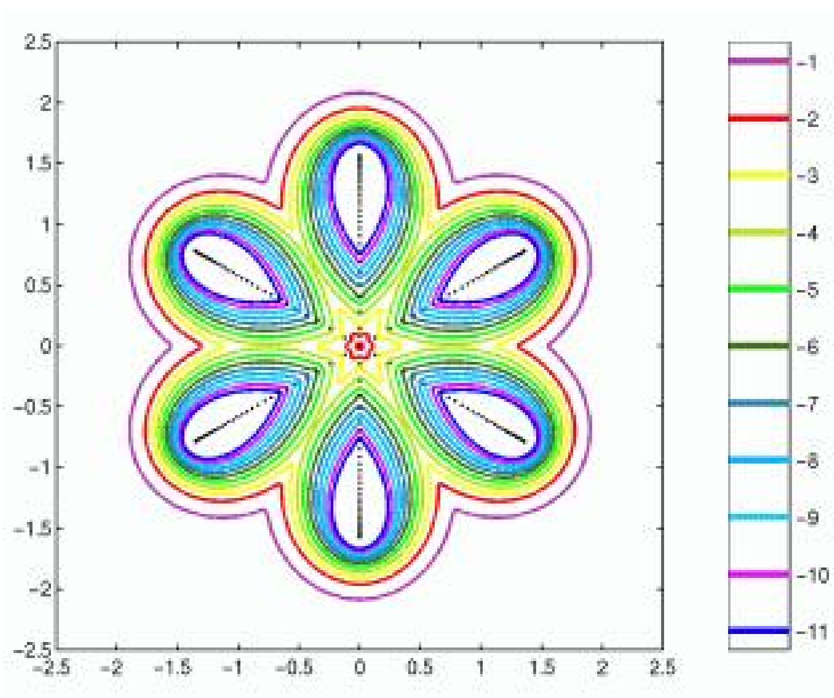
State-space methods

Developed by Kalman and colleagues in the 1960s as an alternative to frequency-domain techniques (Bode, Nichols..)

Starting in the 1980s, numerical analysts developed powerful linear algebra routines for matrix equations: numerical stability, low computational complexity, large-scale problems

Matlab launched by Cleve Moler (1977-1984) heavily relies on LINPACK, EISPACK & LAPACK packages



Pseudospectrum of a Toeplitz matrix

Linear systems stability

The continuous-time linear time invariant (LTI) system

$$\dot{x}(t) = Ax(t) \quad x(0) = x_0$$

where $x(t) \in \mathbb{R}^n$ is asymptotically stable, meaning

$$\lim_{t \to \infty} x(t) = 0 \quad \forall x_0$$

if and only if

- there exists a quadratic Lyapunov function $V(x) = x^\star P x$ such that

$$\begin{array}{rcl} V(x(t)) & > & 0 \\ \dot{V}(x(t)) & < & 0 \end{array}$$

along system trajectories

- equivalently, matrix A satisfies

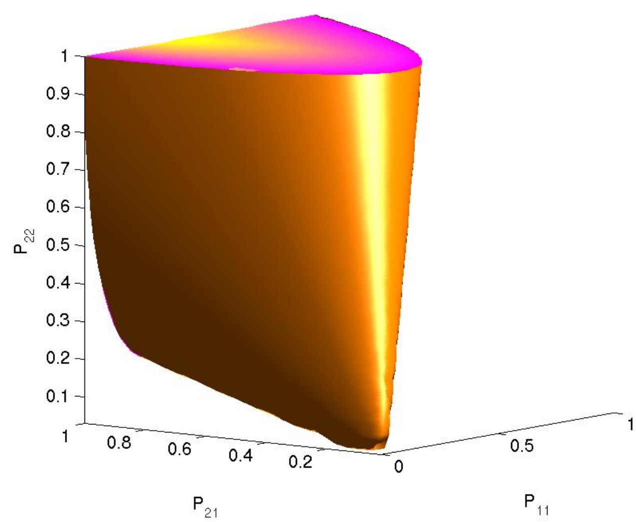
$$\max_i \text{real}\, \lambda_i(A) < 0$$

Lyapunov stability

Note that $V(x) = x^\star P x = x^\star (P + P^\star) x/2$
so that Lyapunov matrix P can be chosen
symmetric without loss of generality

Since $\dot{V}(x) = \dot{x}^\star P x + x^\star P \dot{x} = x^\star A^\star P x + x^\star P A x$ positivity of $V(x)$ and negativity of $\dot{V}(x)$ along system trajectories can be expressed as an LMI

$$A^\star P + P A \prec 0 \qquad P \succ 0$$



Matrices P satisfying Lyapunov's LMI with $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

Lyapunov equation

The Lyapunov LMI can be written equivalently as the Lyapunov equation

$$A^\star P + PA + Q = 0$$

where $Q \succ 0$

The following statements are equivalent

- the system $\dot{x} = Ax$ is asymptotically stable
- for some matrix $Q \succ 0$ the matrix P solving the Lyapunov equation satisfies $P \succ 0$
- for all matrices $Q \succ 0$ the matrix P solving the Lyapunov equation satisfies $P \succ 0$

The Lyapunov LMI can be solved numerically without IP methods since solving the above equation amounts to solving a linear system of $n(n+1)/2$ equations in $n(n+1)/2$ unknowns

Alternative to Lyapunov LMI

Recall the theorem of alternatives for LMI

$$F(x) = F_0 + \sum_i x_i F_i$$

Exactly one statement is true

- there exists x s.t. $F(x) \succ 0$
- there exists a nonzero $Z \succeq 0$ s.t. trace $F_0 Z \le 0$ and trace $F_i Z = 0$ for $i > 0$

Alternative to Lyapunov LMI

$$F(x) = \begin{bmatrix} -A^\star P - PA & 0 \\ 0 & P \end{bmatrix} \succ 0$$

is the existence of a nonzero matrix

$$Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \succeq 0$$

such that

$$Z_1 A^\star + A Z_1 - Z_2 = 0$$

Alternative to Lyapunov LMI (proof)

Suppose that there exists such a matrix $Z \neq 0$ and extract Cholesky factor

$$Z_1 = UU^\star$$

Since $Z_1 A^\star + A Z_1 \succeq 0$ we must have

$$AUU^\star = USU^\star$$

where $S = S_1 + S_2$ and $S_1 = -S_1^\star$, $S_2 \succeq 0$

It follows from

$$AU = US$$

that U spans an invariant subspace of A associated with eigenvalues of S, which all satisfy real $\lambda_i(S) \geq 0$

Conversely, suppose $\lambda_i(A) = \sigma + j\omega$ with $\sigma \geq 0$ for some i with eigenvector v

Then rank-one matrices

$$Z_1 = vv^\star \quad Z_2 = 2\sigma vv^\star$$

solve the alternative LMI

Discrete-time Lyapunov LMI

Similarly, the discrete-time LTI system

$$x_{k+1} = Ax_k$$

is asymptotically stable iff

- there exists a quadratic Lyapunov function $V(x) = x^\star P x$ such that

$$V(x_k) > 0$$
$$V(x_{k+1}) - V(x_k) < 0$$

along system trajectories

- equivalently, matrix A satisfies

$$\max_i |\lambda_i(A)| < 1$$

Here too this can be expressed as an LMI

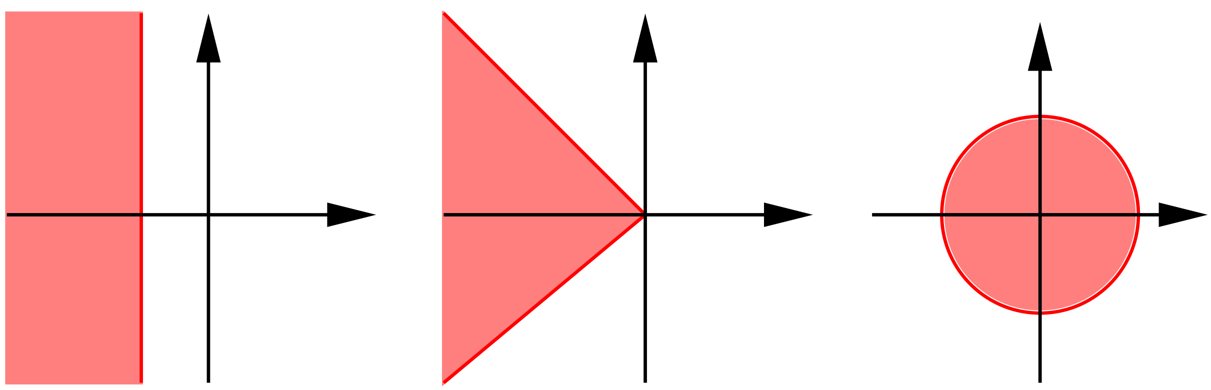
$$A^\star P A - P \prec 0 \quad P \succ 0$$

More general stability regions

Let

$$\mathcal{D} = \{s \in \mathbb{C} \; : \; \begin{bmatrix} 1 \\ s \end{bmatrix}^{\star} \begin{bmatrix} d_0 & d_1 \\ d_1^{\star} & d_2 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0\}$$

with $d_0, d_1, d_2 \in \mathbb{C}^3$ be a region of the complex plane (half-plane or disk)



Matrix A is said $\mathcal{D}$-stable when its spectrum $\sigma(A) = \{\lambda_i(A)\}$ belongs to region $\mathcal{D}$

Equivalent to generalized Lyapunov LMI

$$\begin{bmatrix} I \\ A \end{bmatrix}^{\star} \begin{bmatrix} d_0 P & d_1 P \\ d_1^{\star} P & d_2 P \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} \prec 0 \quad P \succ 0$$

LMI stability regions

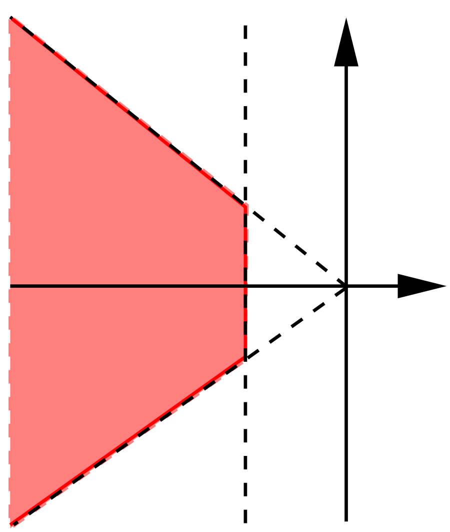
We can consider $\mathcal{D}$-stability in LMI regions

$$\mathcal{D} = \{s \in \mathbb{C} \,:\, D(s) = D_0 + D_1 s + D_1^\star s^\star \prec 0\}$$

such as

$\mathcal{D}$	dynamics
real$(s) < -\alpha$	dominant behavior
real$(s) < -\alpha,\ \lvert s\rvert < r$	oscillations
$\alpha_1 <$ real$(s) < \alpha_2$	bandwidth
$\lvert$imag$(s)\rvert < \alpha$	horizontal strip
real$(s)\tan\theta < -\lvert$imag$(s)\rvert$	damping cone

or intersections thereof



Example for the cone

$$D_0 = 0 \quad D_1 = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

Lyapunov LMI for LMI stability regions

Matrix A has all its eigenvalues in the region

$$\mathcal{D} = \{s \in \mathbb{C} \,:\, D_0 + D_1 s + D_1^\star s^\star \prec 0\}$$

if and only if the following LMI is feasible

$$D_0 \otimes P + D_1 \otimes AP + D_1^\star \otimes PA^\star \prec 0 \quad P \succ 0$$

where $\otimes$ denotes the Kronecker product

Litterally replace s with A !

Can be extended readily to quadratic matrix inequality stability regions

$$\mathcal{D} = \{s \in \mathbb{C} \,:\, D_0 + D_1 s + D_1^\star s^\star + D_2 s^\star s \prec 0\}$$

parabolae, hyperbolae, ellipses etc
convex ($D_2 \succeq 0$) or not

Uncertain systems and robustness

When modeling systems we face several sources of uncertainty, including

- non-parametric (unstructured) uncertainty
  - unmodeled dynamics
  - truncated high frequency modes
  - non-linearities
  - effects of linearization, time-variation..
- parametric (structured) uncertainty
  - physical parameters vary within given bounds
  - interval uncertainty (l_∞)
  - ellipsoidal uncertainty (l_2)
  - diamond uncertainty (l_1)

How can we overcome uncertainty ?

- model predictive control
- adaptive control
- robust control

A control law is robust if it is valid over the whole range of admissible uncertainty (can be designed off-line, usually cheap)

Uncertainty modeling

Consider the continuous-time LTI system

$$\dot{x}(t) = Ax(t) \quad A \in \mathcal{A}$$

where matrix A belongs to an uncertainty set $\mathcal{A}$

For unstructured uncertainties we consider norm-bounded matrices

$$\mathcal{A} = \{A + B\Delta C \;:\; \|\Delta\|_2 \leq \mu\}$$

For structured uncertainties we consider polytopic matrices

$$\mathcal{A} = \text{conv}\,\{A_1, \ldots, A_N\}$$

There are other more sophisticated uncertainty models not covered here

Uncertainty modeling is an important and difficult step in control system design !

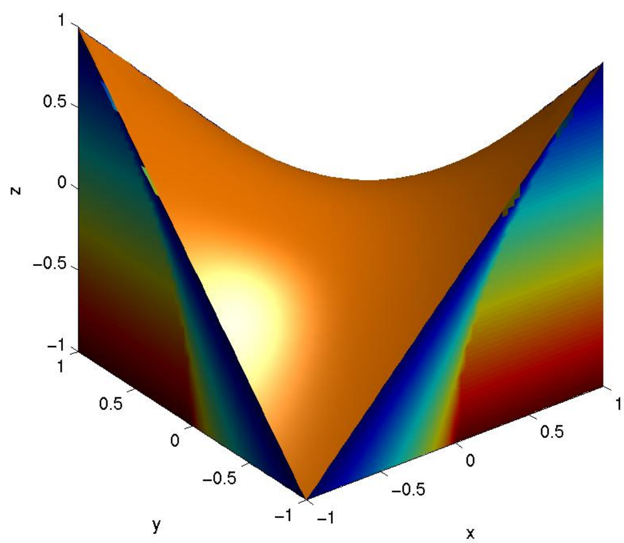
Robust stability

The continuous-time LTI system

$$\dot{x}(t) = Ax(t) \quad A \in \mathcal{A}$$

is robustly stable when it is asymptotically stable for all $A \in \mathcal{A}$

If $\mathcal{S}$ denotes the set of stable matrices, then robust stability is ensured as soon as $\mathcal{A} \subset \mathcal{S}$
Unfortunately $\mathcal{S}$ is a non-convex cone !



Non-convex set of continuous-time stable matrices

$$\begin{bmatrix} -1 & x \\ y & z \end{bmatrix}$$

Symmetry

If dynamic systems were symmetric, i.e

$$A = A^\star$$

continuous-time stability $\max_i \text{real}\, \lambda_i(A) < 0$ would be equivalent to

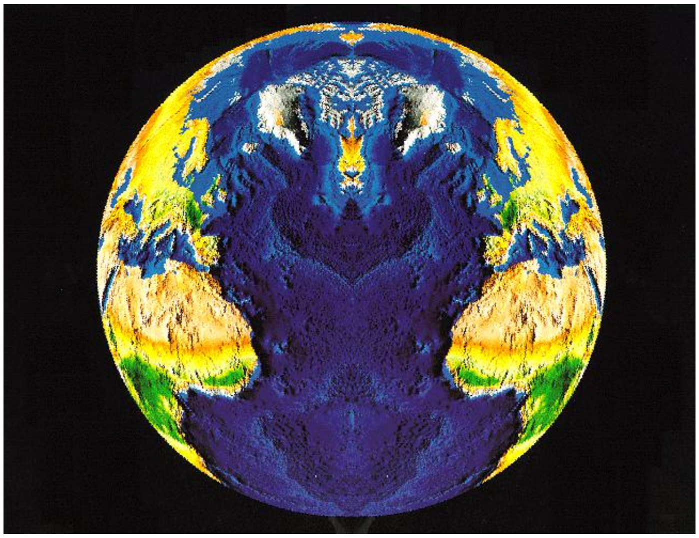
$$A + A^\star \prec 0$$

and discrete-time stability $\max_i |\lambda_i(A)| < 1$ to

$$A^\star A \prec I \Longleftrightarrow \begin{bmatrix} -I & A \\ A^\star & -I \end{bmatrix} \prec 0$$

which are both LMIs !

We can show that stability of a symmetric linear system can be always proven with the Lyapunov matrix $P = I$



Fortunately, the world is not symmetric !

Robust and quadratic stability

Because of non-convexity of the cone of stable matrices, robust stability is sometimes difficult to check numerically, meaning that

> computational cost is an exponential function of the number of system parameters

Remedy:
The continuous-time LTI system $\dot{x}(t) = Ax(t)$ is quadratically stable if its robust stability can be guaranteed with the same quadratic Lyapunov function for all $A \in \mathcal{A}$

Obviously, quadratic stability is more pessimistic, or more conservative than robust stability:

> Quadratic stability $\Longrightarrow$ Robust stability

but the converse is not always true

Quadratic stability for polytopic uncertainty

The system with polytopic uncertainty

$$\dot{x}(t) = Ax(t) \quad A \in \text{conv}\,\{A_1, \ldots, A_N\}$$

is quadratically stable iff there exists a matrix P solving the LMI

$$A_i^T P + P A_i \prec 0 \quad P \succ 0$$

Proof by convexity

$$\sum_i \lambda_i (A_i^T P + P A_i) = A^T(\lambda) P + P A(\lambda) \prec 0$$

for all $\lambda_i \geq 0$ such that $\sum_i \lambda_i = 1$

This is a vertex result: stability of a whole family of matrices is ensured by stability of the vertices of the family

Usually vertex results ensure computational tractability

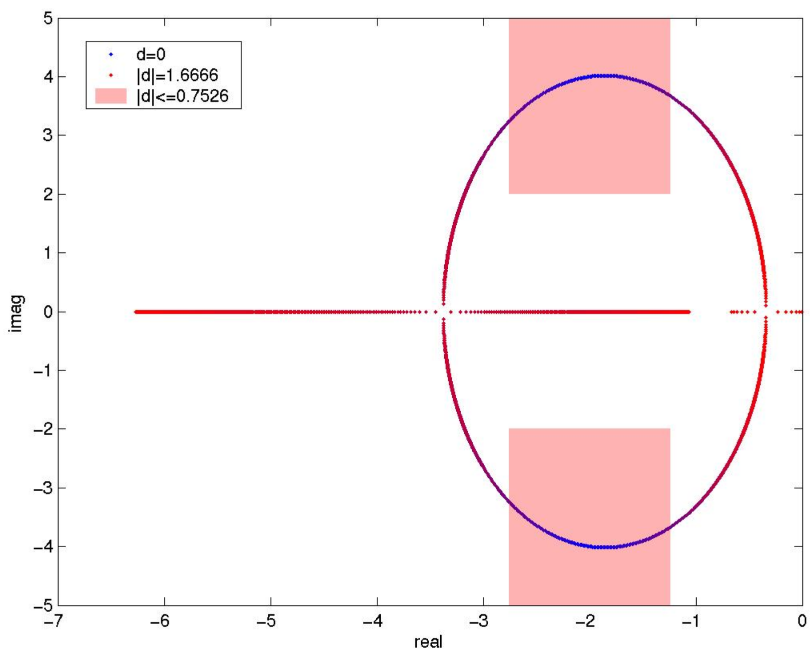
Quadratic and robust stability: example

Consider the uncertain system matrix

$$A(\delta) = \begin{bmatrix} -4 & 4 \\ -5 & 0 \end{bmatrix} + \delta \begin{bmatrix} -2 & 2 \\ -1 & 4 \end{bmatrix}$$

with real parameter δ such that $|\delta| \le \mu$
= polytope with vertices $A(-\mu)$ and $A(\mu)$

stability	max μ
quadratic	0.7526
robust	1.6666



Quadratic stability for norm-bounded uncertainty

The system with norm-bounded uncertainty

$$\dot{x}(t) = (A + B\Delta C)x(t) \quad \|\Delta\|_2 \leq \mu$$

is quadratically stable iff there exists a matrix P solving the LMI

$$\left[\begin{array}{cc} A^\star P + PA + C^\star C & PB \\ B^\star P & -\gamma^2 I \end{array}\right] \prec 0 \quad P \succ 0$$

with $\gamma^{-1} = \mu$

This is called the bounded-real lemma
proved next with the S-procedure

We can maximize the level of
allowed uncertainty by minimizing scalar γ

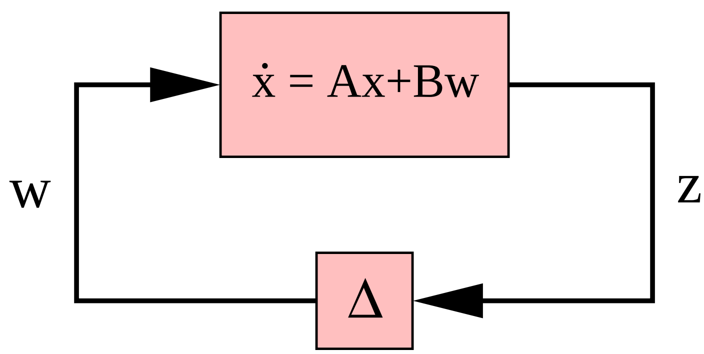
Norm-bounded uncertainty as feedback

Uncertain system

$$\dot{x} = (A + B\Delta C)x$$

can be written as the feedback system

$$\begin{array}{rcl} \dot{x} & = & Ax + Bw \\ z & = & Cx \\ w & = & \Delta z \end{array}$$



so that for the Lyapunov function $V(x) = x^\star P x$ we have

$$\begin{array}{rcl} \dot{V}(x) & = & 2x^\star P \dot{x} \\ & = & 2x^\star P(Ax + Bw) \\ & = & x^\star(A^\star P + PA)x + 2x^\star PBw \\ & = & \left[\begin{array}{c} x \\ w \end{array}\right]^\star \left[\begin{array}{cc} A^\star P + PA & PB \\ B^\star P & 0 \end{array}\right] \left[\begin{array}{c} x \\ w \end{array}\right] \end{array}$$

Norm-bounded uncertainty as feedback (2)

Since $\Delta^\star\Delta \preceq \mu^2 I$ it follows that

$$w^\star w = z^\star \Delta^\star \Delta z \preceq \mu^2 z^\star z$$
$$\Longleftrightarrow$$
$$w^\star w - \mu^2 z^\star z = \begin{bmatrix} x \\ w \end{bmatrix}^\star \begin{bmatrix} -C^\star C & 0 \\ 0 & \gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \leq 0$$

Combining with the quadratic inequality

$$\dot{V}(x) = \begin{bmatrix} x \\ w \end{bmatrix}^\star \begin{bmatrix} A^\star P + PA & PB \\ B^\star P & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0$$

and using the S-procedure we obtain

$$\begin{bmatrix} A^\star P + PA & PB \\ B^\star P & 0 \end{bmatrix} \prec \begin{bmatrix} -C^\star C & 0 \\ 0 & \gamma^2 I \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} A^\star P + PA + C^\star C & PB \\ B^\star P & -\gamma^2 I \end{bmatrix} \prec 0 \quad P \succ 0$$

Norm-bounded uncertainty: generalization

Now consider the feedback system

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx + Dw \\ w &= \Delta z \end{aligned}$$

with additional feedthrough term Dw

We assume that matrix $I - \Delta D$ is non-singular
$=$ well-posedness of feedback interconnection
so that we can write

$$w = \Delta z = \Delta(Cx + Dw)$$
$$(I - \Delta D)w = \Delta Cx$$
$$w = (I - \Delta D)^{-1}\Delta Cx$$

and derive the linear fractional transformation (LFT) uncertainty description

$$\dot{x} = Ax + Bw = (A + B(I - \Delta D)^{-1}\Delta C)x$$

Norm-bounded LFT uncertainty

The system with norm-bounded LFT uncertainty

$$\dot{x} = \left(A + B(I - \Delta D)^{-1}\Delta C\right)x \quad \|\Delta\|_2 \leq \mu$$

is quadratically stable iff there exists a matrix P solving the LMI

$$\begin{bmatrix} A^\star P + PA + C^\star C & PB + C^\star D \\ B^\star P + D^\star C & D^\star D - \gamma^2 I \end{bmatrix} \prec 0 \quad P \succ 0$$

Notice the lower right block $D^\star D - \gamma^2 I \prec 0$ which ensures non-singularity of $I - \Delta D$ hence well-posedness

LFT modeling can be used more generally to cope with rational functions of uncertain parameters, but this is not covered in this course..

Sector-bounded uncertainty

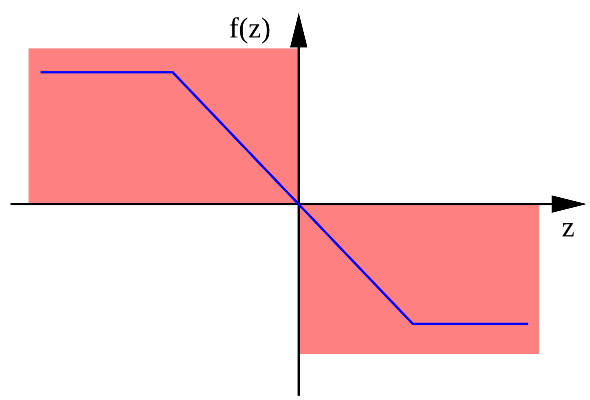
Consider the feedback system

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx + Dw \\ w &= f(z) \end{aligned}$$

where vector function $f(z)$ satisfies

$$z^\star f(z) \leq 0 \quad f(0) = 0$$

which is a sector condition



$f(z)$ can also be considered as an uncertainty but also as a non-linearity

Quadratic stability for sector-bounded uncertainty

We want to establish quadratic stability with the quadratic Lyapunov matrix $V(x) = x^\star P x$ whose derivative

$$\begin{aligned} \dot{V}(x) &= 2x^\star P(Ax + Bf(z)) \\ &= \begin{bmatrix} x \\ f(z) \end{bmatrix}^\star \begin{bmatrix} A^\star P + PA & PB \\ B^\star P & 0 \end{bmatrix} \begin{bmatrix} x \\ f(z) \end{bmatrix} \end{aligned}$$

must be negative when

$$\begin{aligned} 2z^\star f(z) &= 2(Cx + Df(z))^\star f(z) \\ &= \begin{bmatrix} x \\ f(z) \end{bmatrix}^\star \begin{bmatrix} 0 & C^\star \\ C & D + D^\star \end{bmatrix} \begin{bmatrix} x \\ f(z) \end{bmatrix} \end{aligned}$$

is non-positive, so we invoke the S-procedure to derive the LMI

$$\begin{bmatrix} A^\star P + PA & PB - C^\star \\ B^\star P - C & -D - D^\star \end{bmatrix} \prec 0 \qquad P \succ 0$$

This is called the positive-real lemma

Parameter-dependent Lyapunov functions

Quadratic stability:

- fast variation of parameters
- computationally tractable
- conservative, or pessimistic (worst-case)

Robust stability:

- no variation of parameters
- computationally difficult (in general)
- exact (is it really relevant ?)

Is there something in between ?

For example, given an LTI system affected by box, or interval uncertainty

$$\dot{x}(t) = A(\lambda(t))x(t) = (A_0 + \sum_{i=1}^{N} \lambda_i(t)A_i)x(t)$$

where

$$\lambda \in \Lambda = \{\lambda_i \in [\underline{\lambda_i},\ \overline{\lambda_i}]\}$$

we may consider parameter-dependent Lyapunov matrices, such as

$$P(\lambda(t)) = P_0 + \sum_i \lambda_i(t)P_i$$

Polytopic Lyapunov certificate

Quadratic Lyapunov function $V(x) = x^\star P(\lambda) x$ must be positive with negative derivative along system trajectory hence

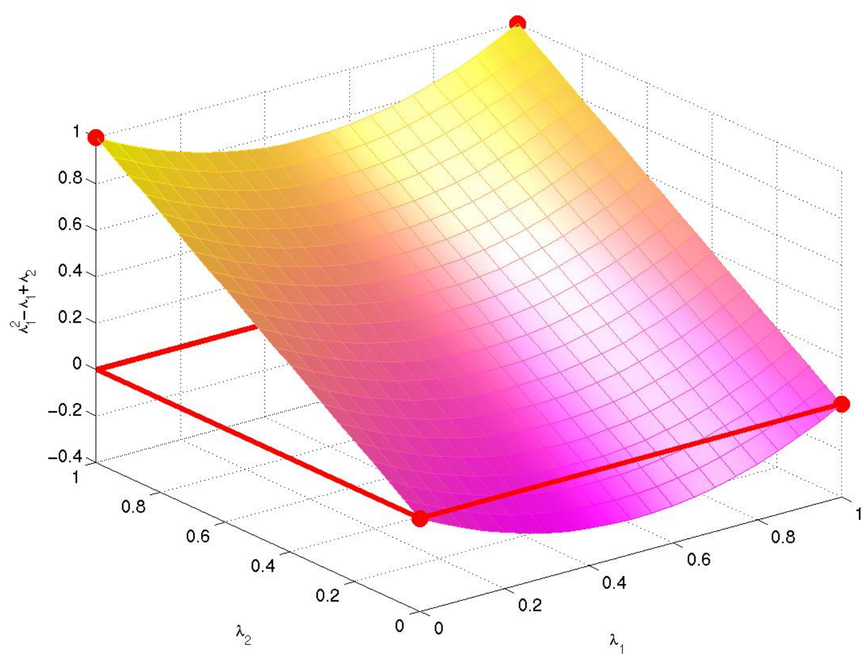
$$P(\lambda) \succ 0 \quad \forall \lambda \in \Lambda$$

and

$$A^\star(\lambda)P(\lambda) + P(\lambda)A^\star(\lambda) + \dot{P}(\lambda) \prec 0 \quad \forall \lambda \in \Lambda$$

We have to solve parametrized LMIs

Parametrized LMIs feature non-linear terms in λ so it is not enough to check vertices of Λ, denoted by vertΛ



$\lambda_1^2 - \lambda_1 + \lambda_2 \geq 0$ on vert Δ
but not everywhere on $\Delta = [0,\ 1] \times [0,\ 1]$

Parametrized LMIs

Central problem in robustness analysis:
find x such that

$$F(x,\lambda) = \sum_\alpha \lambda^\alpha F_\alpha(x) \succ 0, \quad \forall \lambda \in \Lambda$$

where Λ is a compact set, typically the unit simplex or the unit ball

Convex but infinite-dimensional problem which is difficult in general

Matrix extensions of polynomial positivity conditions, for which various hierarchies of LMIs are available:

- Pólya's theorem
- Schmüdgen's representation
- Putinar representation

See EJC 2006 survey by Carsten Scherer