I.4. CONVEX LMI MODELLING

Didier HENRION
henrion@laas.fr

Belgian Graduate School on
Systems, Control, Optimization and Networks

Leuven - April and May 2010

Minors

A minor of a matrix F is the determinant of a submatrix of F, say with row index I and column index J

If $I = J$ this is a principal minor

If $I = J = 1 \dots k$ this is a leading principal minor

A symmetric matrix F is positive definite iff all its leading principal minors are positive

A symmetric matrix F is positive semidefinite iff all its principal minors are nonnegative

Positive semidefiniteness

If $F \in \mathbb{R}^{m \times m}$ we have $2^m - 1$ principal minors

A simpler and equivalent criterion follows from the fact that a univariate polynomial $t \mapsto f(t) = \sum_k f_{m-k} t^k = \prod_k (t - t_k)$ which has only real roots satisfies $t_k \leq 0$ iff $f_k \geq 0$

Apply to characteristic polynomial

$$t \mapsto f(t) = \det(tI_m + F) = \sum_{k=0}^{m} f_{m-k}(F) t^k$$

A symmetric matrix F is positive semidefinite iff $f_i(F) \geq 0$, $\forall i$

Only m polynomials to be checked, they are (signed) sums of principal minors

Geometry of LMI sets

Given symmetric matrices F_i we want to characterize the shape in $\mathbb{R}^n$ of the LMI set

$$\mathcal{F} = \{x \in \mathbb{R}^n \ : \ F(x) = F_0 + \textstyle\sum_{i=1}^n x_i F_i \succeq 0\}$$

Build characteristic polynomial

$$t \mapsto f(t,x) = \det(tI_m + F(x)) = \sum_{k=0}^{m} f_{m-k}(x) t^k$$

which is monic, i.e. $f_0(x) = 1$

Matrix $F(x)$ is PSD iff $f_i(x) \geq 0$ for all $i = 1, \ldots, m$

Semialgebraic description

Diagonal minors are multivariate polynomials of the x_i

So the LMI set can be described as

$$\mathcal{F} = \{x \in \mathbb{R}^n \ : \ f_i(x) \geq 0,\ i = 1, 2, \ldots\}$$

which is a basic semialgebraic set
(basic = intersection of polynomial level-sets)

Moreover, it is a convex set

Example of 2D LMI feasible set

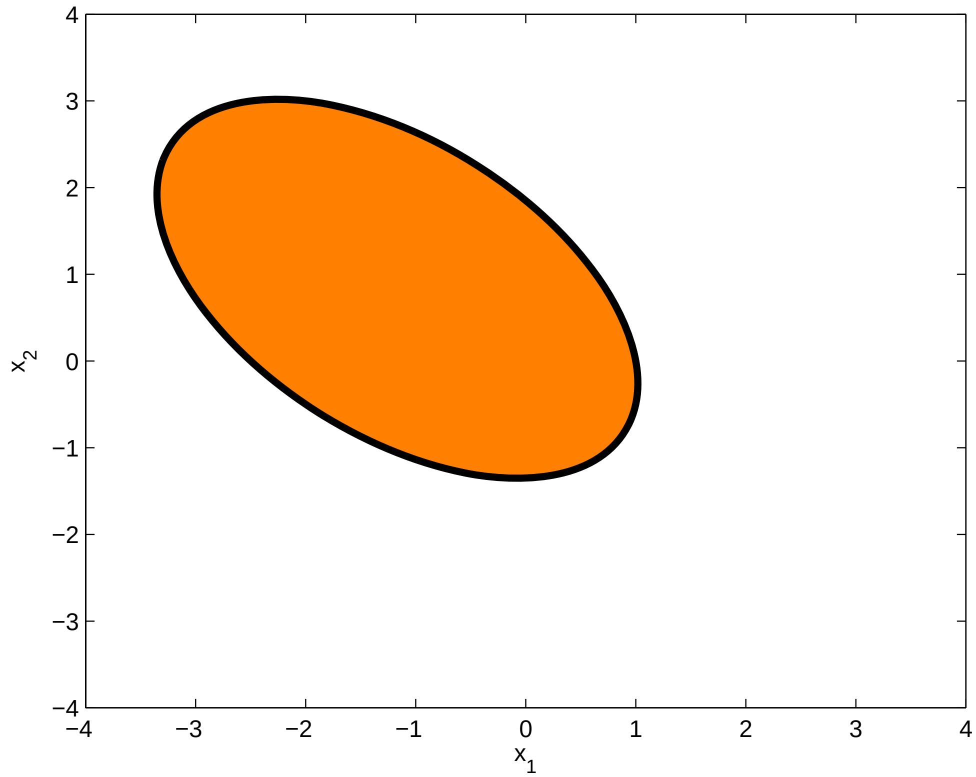
$$F(x) = \begin{bmatrix} 1 - x_1 & x_1 + x_2 & x_1 \\ x_1 + x_2 & 2 - x_2 & 0 \\ x_1 & 0 & 1 + x_2 \end{bmatrix} \succeq 0$$

System of 3 polynomial inequalities $f_i(x) \geq 0$

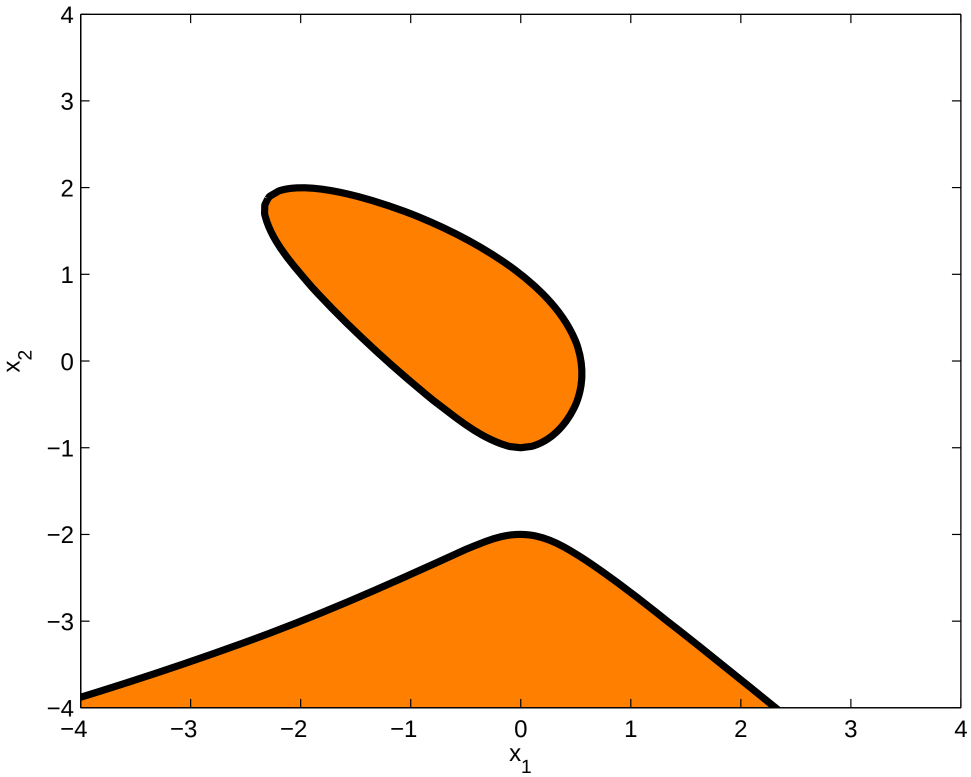
1st order minors: $f_1(x) = 4 - x_1 \geq 0$

LMI set = intersection of an infinite number of halfspaces
$\{x \,:\, y^T F(x) y \geq 0\}$ for all $y \in \mathbb{R}^3$

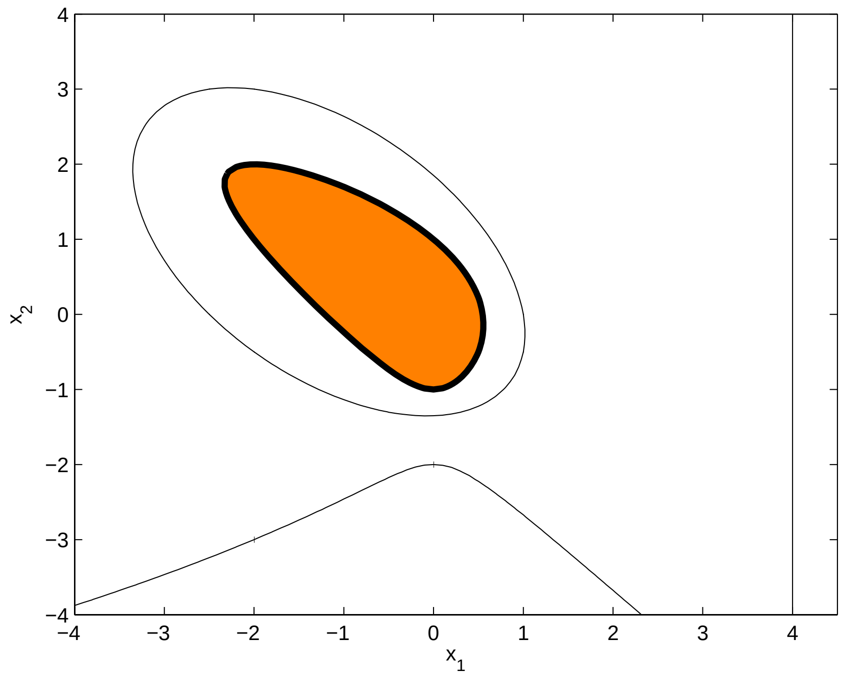
2nd order: $f_2(x) = 5 - 3x_1 + x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2 \geq 0$



3rd order: $f_3(x) = 2 - 2x_1 + x_2 - 3x_1^2 - 3x_1x_2 - 2x_2^2 - x_1x_2^2 - x_2^3 \geq 0$



LMI feasible set = intersection of sets $\{x : f_i(x) \geq 0\}$, $i = 1, 2, 3$

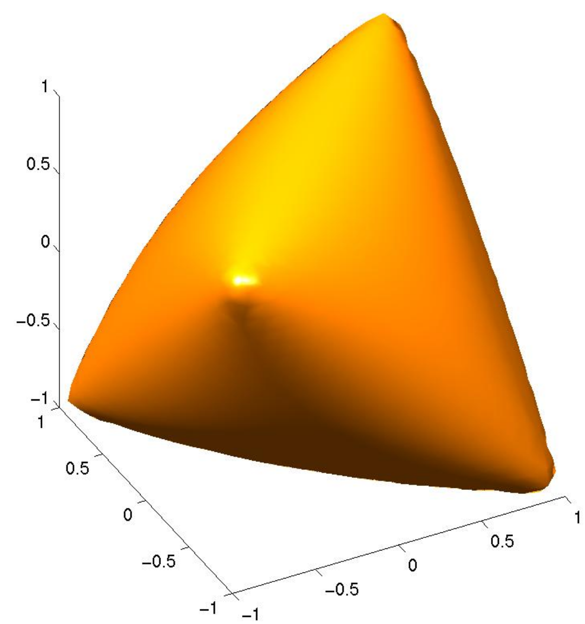


Boundary of LMI region shaped by determinant

Other polynomials only isolate convex connected component

Example of 3D LMI feasible set

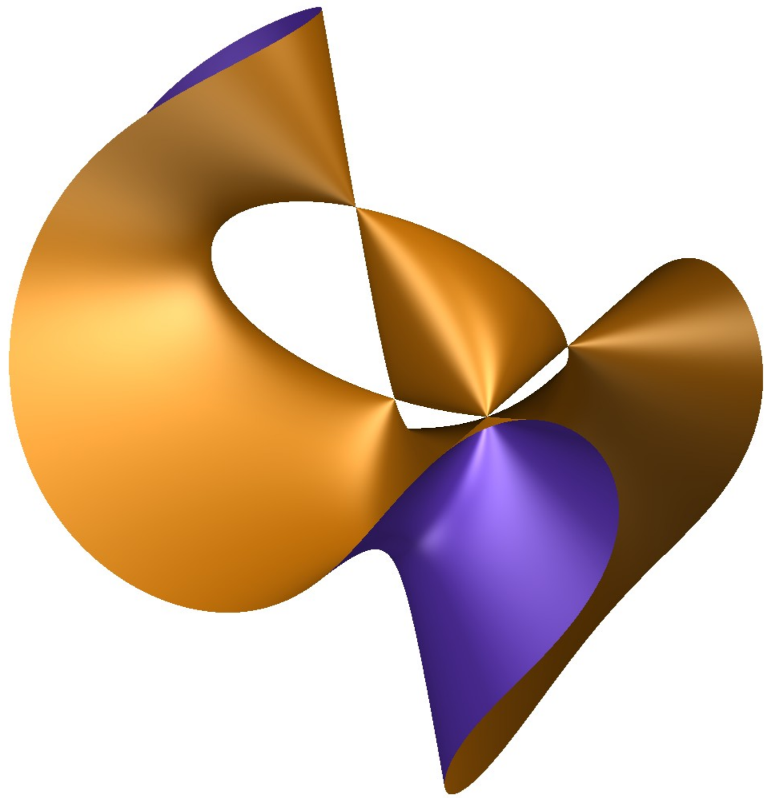
$$\mathcal{F} = \{x \in \mathbb{R}^3 : \underbrace{\begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{bmatrix}}_{F(x)} \succeq 0\}$$



A smoothened tetrahedron..
vertices correspond to points x for which rank $F(x) = 1$

Semialgebraic formulation

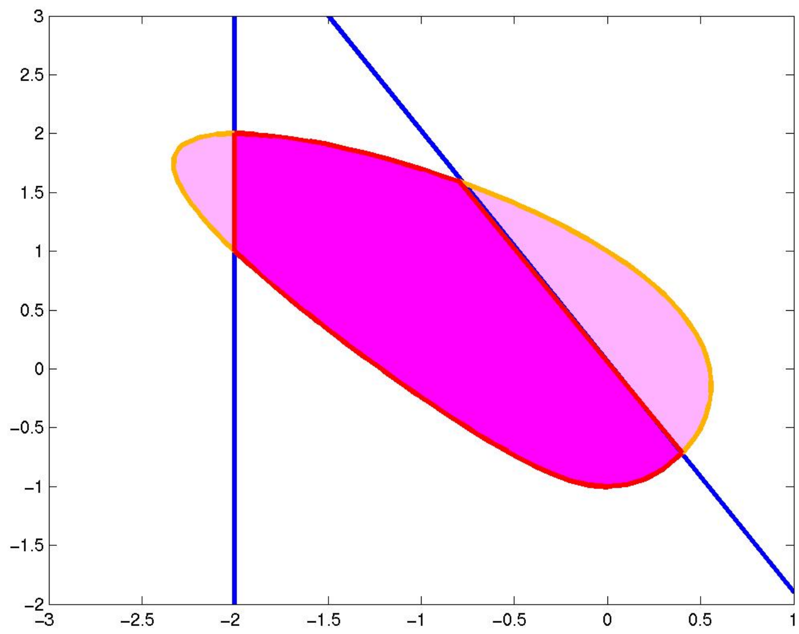
$$\mathcal{F} = \{x \in \mathbb{R}^3 : 1+2x_1x_2x_3-(x_1^2+x_2^2+x_3^2) \geq 0,\ 3-x_1^2-x_2^2-x_3^2 \geq 0\}$$



Intersection of LMI sets

Intersection of LMI feasible sets is also LMI

$F(x) \succeq 0 \quad x_1 \geq -2 \quad 2x_1 + x_2 \leq 0$



LMI sets

LMI sets are convex basic semialgebraic sets..
but are all convex basic semialgebraic sets LMI ?

Let us make a fundamental distinction between

- LMI representable sets
- lifted-LMI representable sets

We say that a convex set $X \subset \mathbb{R}^n$ is LMI representable if there exists an affine mapping $F(x)$ such that

$$x \in X \iff F(x) \succeq 0$$

LMI and lifted-LMI representability

We say that a convex set $X \subset \mathbb{R}^n$ is lifted-LMI representable if there exists an affine mapping $F(x, u)$ such that

$$x \in X \iff \exists u \in \mathbb{R}^m \,:\, F(x, u) \succeq 0$$

A set X is lifted-LMI representable when

$$x \in X \iff \exists u \,:\, F(x, u) \succeq 0$$

i.e. when it is the projection of the solution set of the LMI $F(x, u) \succeq 0$ onto the x-space and u are additional, or lifting variables

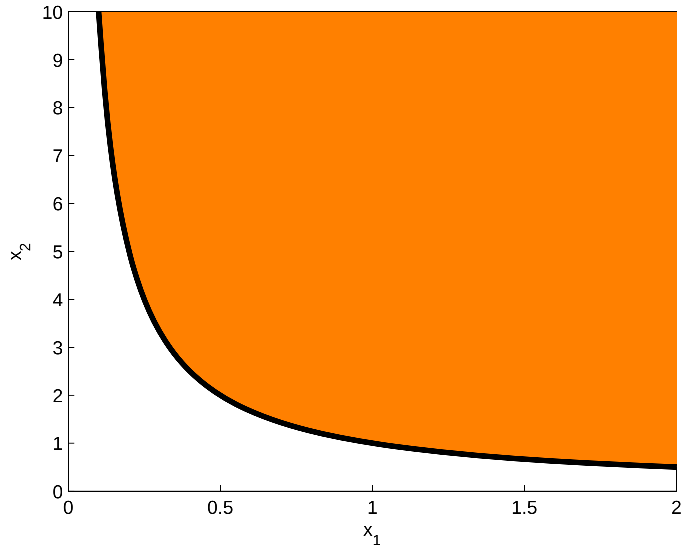
In other words, lifting variables u are not allowed
in LMI representations

LMI and lifted-LMI functions

Similarly, a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is LMI (or lifted-LMI) representable if its epigraph

$$\{x, t : f(x) \leq t\}$$

is an LMI (or lifted-LMI) representable set



Conic quadratic forms

The Lorentz, or ice-cream cone

$$\{x, t \in \mathbb{R}^n \times \mathbb{R} \;:\; \|x\|_2 \leq t\}$$

is LMI representable as

$$\left\{ x, t \in \mathbb{R}^n \times \mathbb{R} \;:\; \begin{bmatrix} tI_n & x \\ x^T & t \end{bmatrix} \succeq 0 \right\}$$

As a result, all second-order conic sets are LMI representable

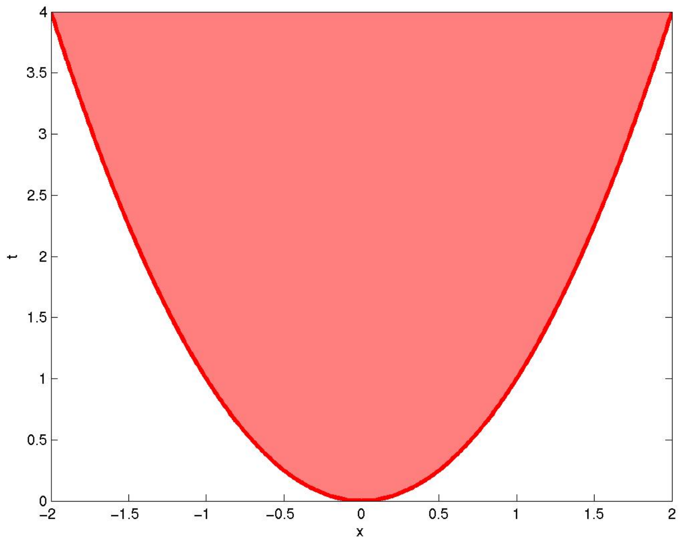
In the sequel we give a list of LMI and lifted-LMI representable sets (following Ben-Tal, Nemirovski, Nesterov)

Quadratic forms

The Euclidean norm $\{x, t \in \mathbb{R}^n \times \mathbb{R} \; : \; \|x\|_2 \leq t\}$
is LMI representable (see previous slide)

The squared Euclidean norm $\{x, t \in \mathbb{R}^n \times \mathbb{R} \; : \; x^T x \leq t\}$
is also LMI representable as

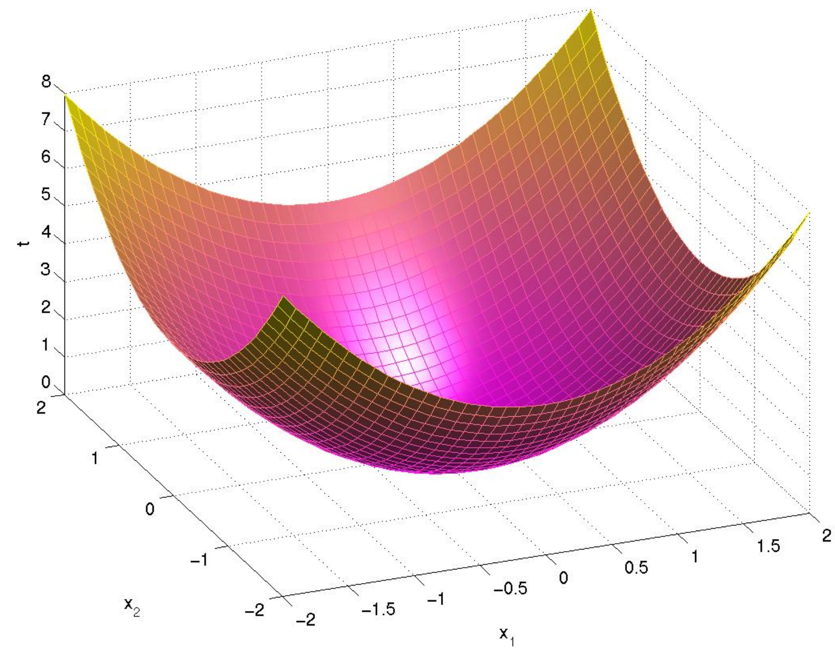
$$\begin{bmatrix} t & x^T \\ x & I_n \end{bmatrix} \succeq 0$$



Quadratic forms (2)

The convex quadratic set $\{x \in \mathbb{R}^n \; : \; x^T A x + b^T x + c \leq 0\}$ with $A = A^T \succeq 0$ is LMI representable as

$$\left[\begin{array}{cc} -b^T x - c & x^T D^T \\ Dx & I_n \end{array}\right] \succeq 0$$

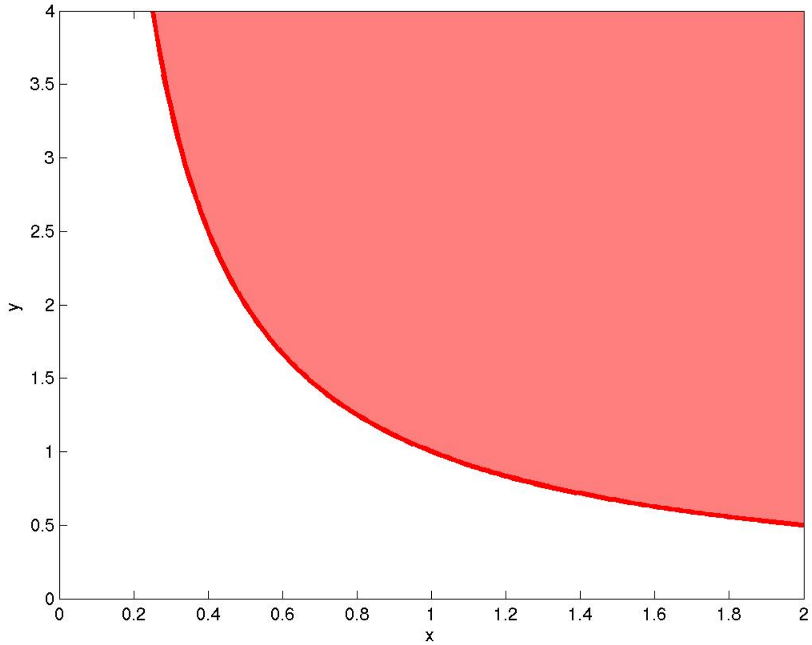


where D is the Cholesky factor of $A = D^T D$

Hyperbola

The branch of hyperbola $\{x, y \in \mathbb{R}^2 \ : \ x \geq 0, \ \ xy \geq 1\}$ is LMI representable as

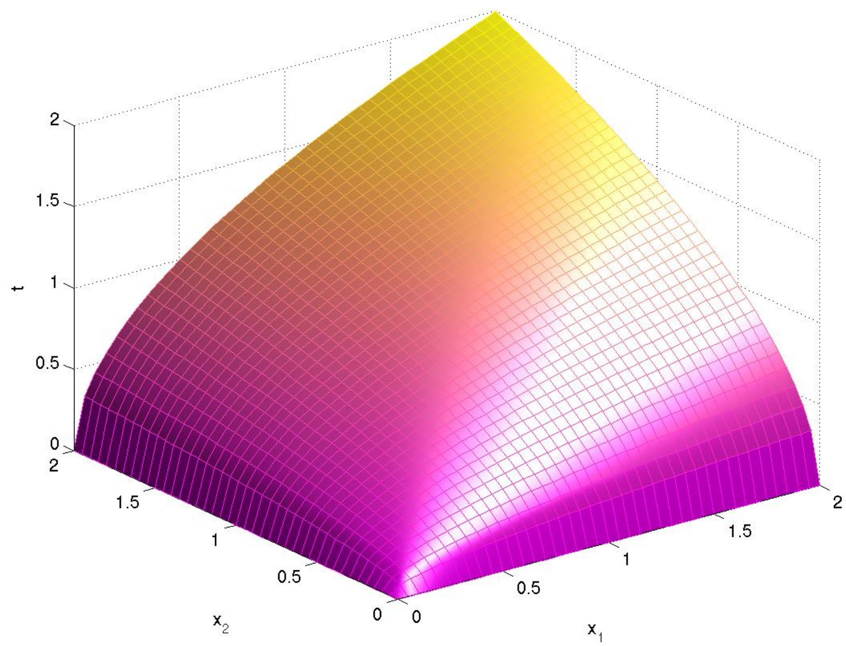
$$\begin{bmatrix} x & 1 \\ 1 & y \end{bmatrix} \succeq 0$$



Geometric mean of two variables

The hypograph of the geometric mean of 2 variables
$\{x_1, x_2, t \in \mathbb{R}^3 \;:\; x_1, x_2 \geq 0,\ \sqrt{x_1 x_2} \geq t\}$ is LMI representable as

$$\begin{bmatrix} x_1 & t \\ t & x_2 \end{bmatrix} \succeq 0$$



Geometric mean of several variables

The hypograph of the geometric mean of 2^k variables
$\{x_1, \ldots, x_{2^k}, t \in \mathbb{R}^{2^k+1} \; : \; x_i \geq 0, \; (x_1 \cdots x_{2^k})^{1/2^k} \geq t\}$
is lifted-LMI representable

Proof: iterate the previous construction by introducing lifting variables

Example with $k = 3$:

$$(x_1 x_2 \cdots x_8)^{1/8} \geq t$$

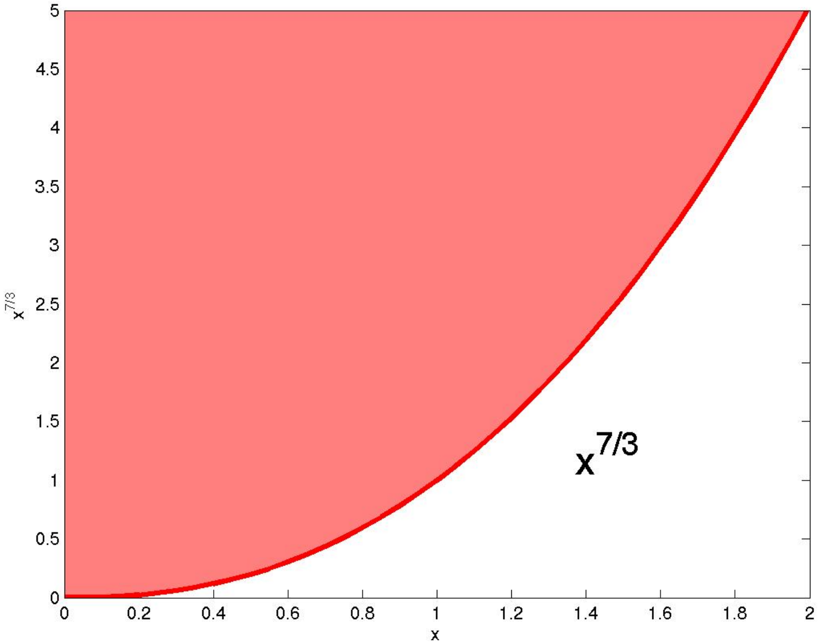
$$\left.\begin{array}{rcl} \sqrt{x_1 x_2} & \geq & x_{11} \\ \sqrt{x_3 x_4} & \geq & x_{12} \\ \sqrt{x_5 x_6} & \geq & x_{13} \\ \sqrt{x_7 x_8} & \geq & x_{14} \end{array}\right\} \left.\begin{array}{rcl} \sqrt{x_{11} x_{12}} & \geq & x_{21} \\ \sqrt{x_{13} x_{14}} & \geq & x_{22} \end{array}\right\} \sqrt{x_{21} x_{22}} \geq t$$

Useful idea in other LMI representability problems

Rational power functions

Following the same ideas, the increasing rational power functions

$$f(x) = x^{p/q}, \quad x \geq 0$$

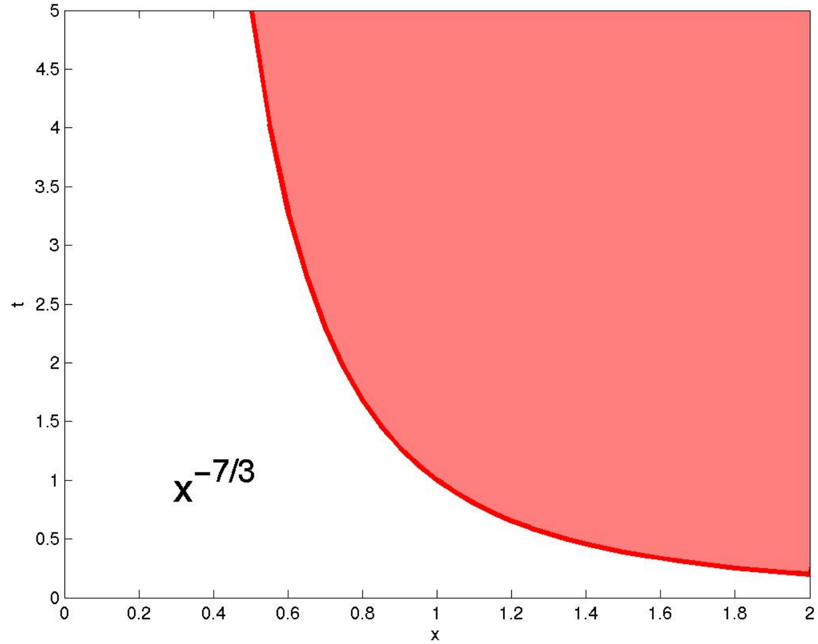


with rational $p/q \geq 1$, are lifted-LMI representable

Rational power functions

Similarly, the decreasing rational power functions

$$g(x) = x^{-p/q}, \quad x \geq 0$$



with rational $p/q \geq 0$, are lifted-LMI representable

Rational power functions

Example: $\{x, t \,:\, x \geq 0,\ x^{7/3} \leq t\}$
Start from lifted-LMI representable $\hat{t} \leq (\hat{x}_1 \cdots \hat{x}_8)^{1/8}$
and replace

$$\begin{aligned} &\hat{t} = \hat{x}_1 = x \geq 0 \\ &\hat{x}_2 = \hat{x}_3 = \hat{x}_4 = t \geq 0 \\ &\hat{x}_5 = \hat{x}_6 = \hat{x}_7 = \hat{x}_8 = 1 \end{aligned}$$

to get

$$\begin{aligned} x &\leq x^{1/8} t^{3/8} \\ x^{7/8} &\leq t^{3/8} \\ x^{7/3} &\leq t \end{aligned}$$

Same idea works for any rational $p/q \geq 1$

- lift = use additional variables, and
- project in the space of original variables

Even power monomial

The epigraph of even power monomial $\mathcal{F} = \{x, t \,:\, x^{2p} \leq t\}$ where p is a positive integer is lifted-LMI representable

Indeed $\{x, t \,:\, x^{2p} \leq t\} \Longleftrightarrow \{x, y, t \,:\, x^2 \leq y\}$ and $\{x, y, t \,:\, y \geq 0,\ y^p \leq t\}$, both lifted-LMI representable

Use lifting y and project back onto x, t

Similarly, even power polynomials are lifted-LMI representable (several monomials)

Quartic level set

Model quartic level set

$$\mathcal{F} = \{x, t \;:\; x^4 \leq t\}$$

as

$$\mathcal{F} = \{x, t \;:\; \exists y \;:\; y \geq x^2,\;\; t \geq y^2,\;\; y \geq 0\}$$

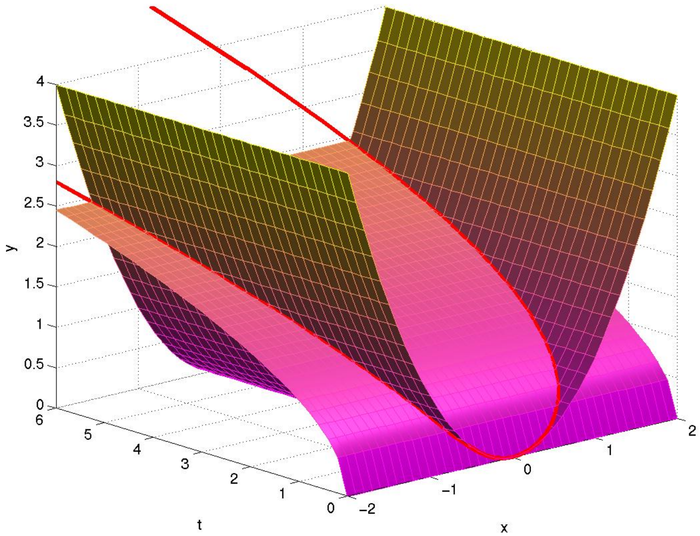
LMI in x, t and y

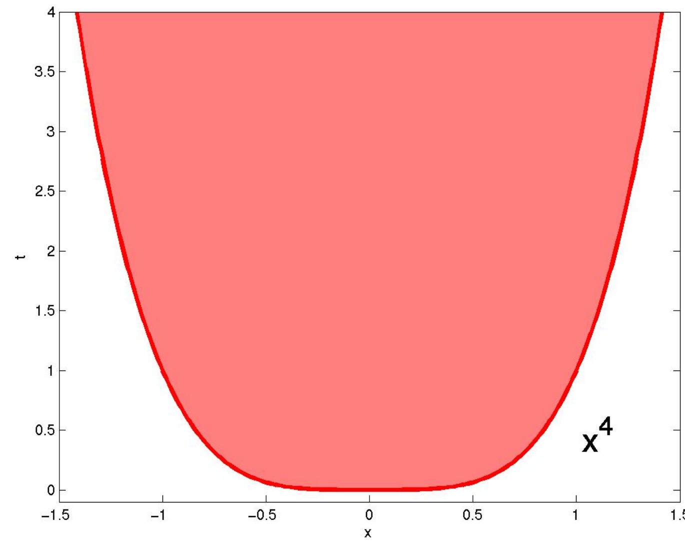
$$\begin{bmatrix} 1 & x \\ x & y \end{bmatrix} \succeq 0 \quad \begin{bmatrix} 1 & y \\ y & t \end{bmatrix} \succeq 0$$

It can be shown that it is impossible to remove the lifting variable y while keeping a (finite-dimensional) LMI formulation

Quartic level set: from 3D to 2D

$$\mathcal{F} = \{x, t \, : \, x^4 \leq t\}$$

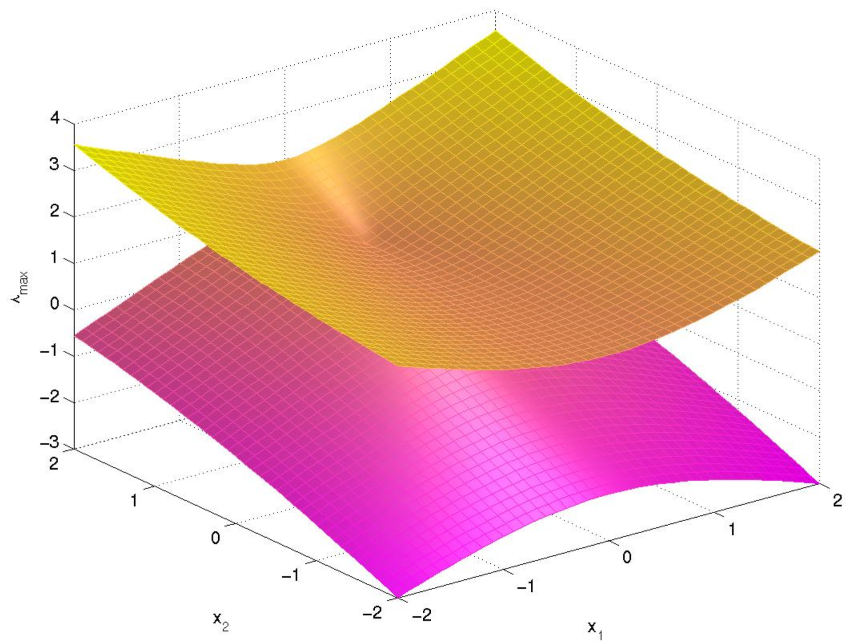




Largest eigenvalue

Function largest eigenvalue of a symmetric matrix
$\{X = X^T \in \mathbb{R}^{n\times n}, t \in \mathbb{R} \,:\, \lambda_{\max}(X) \leq t\}$ is LMI representable as

$$X \preceq tI_n$$



Eigenvalues of matrix $\begin{bmatrix} 1 & x_1 \\ x_1 & x_2 \end{bmatrix}$

Sums of largest eigenvalues

Let

$$S_k(X) = \sum_{i=1}^{k} \lambda_i(X), \quad k = 1, \ldots, n$$

denote the sum of the k largest eigenvalues
of an n-by-n symmetric matrix X

The epigraph $\{X = X^T \in \mathbb{R}^{n \times n}, t \in \mathbb{R} \ : \ S_k(X) \leq t\}$ is lifted-LMI representable as

$$\begin{array}{c} t - ks - \text{trace}\, Z \succeq 0 \\ Z \succeq 0 \\ Z - X + sI_n \succeq 0 \end{array}$$

where Z and s are liftings

Determinant of a PSD matrix

The determinant

$$\det(X) = \prod_{i=1}^{n} \lambda_i(X)$$

is not a convex function of X, but the function

$$f_q(X) = -\det^q(X), \quad X = X^T \succeq 0$$

is convex when $q \in [0,\ 1/n]$ is rational

The epigraph $\{X = X^T \in \mathbb{R}^{n \times n}, t \in \mathbb{R} \,:\, f_q(X) \leq t\}$ is lifted-LMI representable

$$\left[\begin{array}{cc} X & \Delta \\ \Delta^T & \operatorname{diag} \Delta \end{array}\right] \succeq 0$$
$$t \leq (\delta_1 \cdots \delta_n)^q$$

since we know that the latter constraint (hypograph of a concave monomial) is lifted-LMI representable

Here Δ is a lower triangular matrix of liftings with diagonal entries δ_i

Application: extremal ellipsoids

A little excursion in the world of ellipsoids and polytopes..

Various representations of an ellipsoid in $\mathbb{R}^n$

$$\begin{aligned} E &= \{x \in \mathbb{R}^n \ : \ x^T P x + 2x^T q + r \leq 0\} \\ &= \{x \in \mathbb{R}^n \ : \ (x - x_c)^T P (x - x_c) \leq 1\} \\ &= \{x = Qy + x_c \in \mathbb{R}^n \ : \ y^T y \leq 1\} \\ &= \{x \in \mathbb{R}^n \ : \ \|Rx - x_c\| \leq 1\} \end{aligned}$$

where

$$Q = R^{-1} = P^{-1/2} \succ 0$$

Ellipsoid volume

Volume of ellipsoid $E = \{Qy + x_c \ : \ y^T y \leq 1\}$

$$\mathbf{vol}\, E = k_n \det Q$$

where k_n is volume of n-dimensional unit ball

$$k_n = \begin{cases} \frac{2^{(n+1)/2}\pi^{(n-1)/2}}{n(n-2)!!} & \text{for } n \text{ odd} \\ \frac{2\pi^{n/2}}{n(n/2-1)!} & \text{for } n \text{ even} \end{cases}$$

n	1	2	3	4	5	6	7	8
k_n	2.00	3.14	4.19	4.93	5.26	5.17	4.72	4.06

Unit ball has maximum volume for $n = 5$

Outer and inner ellipsoidal approximations

Let $S \subset \mathbb{R}^n$ be a solid = a closed bounded convex set with nonempty interior

- the largest volume ellipsoid E_{in} contained in S is unique and satisfies $E_{\text{in}} \subset S \subset nE_{\text{in}}$
- the smallest volume ellipsoid E_{out} containing S is unique and satisfies $E_{\text{out}}/n \subset S \subset E_{\text{out}}$

These are Löwner-John ellipsoids

Factor n reduces to $\sqrt{n}$ if S is symmetric

How can these ellipsoids be computed ?

Ellipsoid in polytope

Let the intersection of hyperplanes

$$S = \{x \in \mathbb{R}^n \,:\, a_i^T x \le b_i,\; i = 1, \ldots, m\}$$

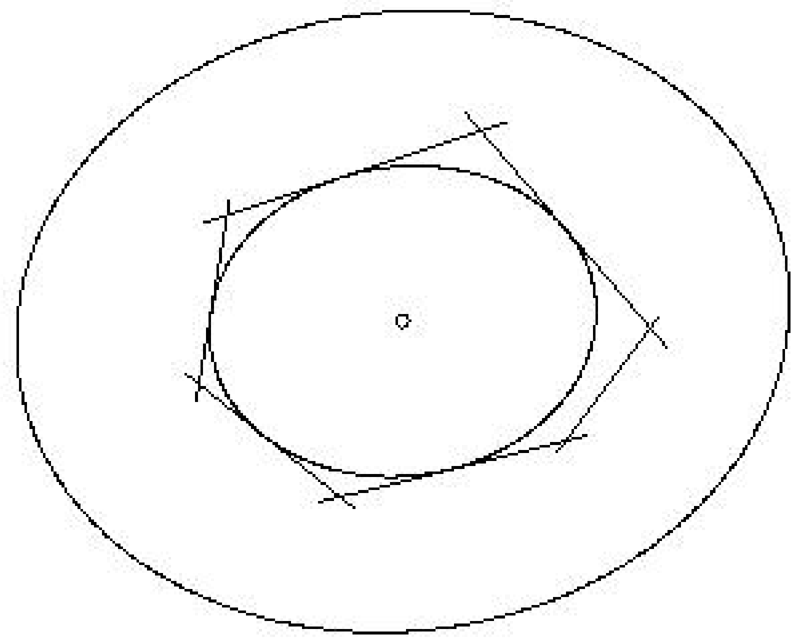
describe a polytope = bounded nonempty polyhedron

The largest volume ellipsoid contained in S is

$$E = \{Qy + x_c \,:\, y^T y \le 1\}$$

where Q, x_c are optimal solutions of the LMI problem

$$\begin{array}{ll} \max & \det^{1/n} Q \\ & Q \succeq 0 \\ & \|Qa_i\|_2 \le b_i - a_i^T x_c \end{array}$$



Polytope in ellipsoid

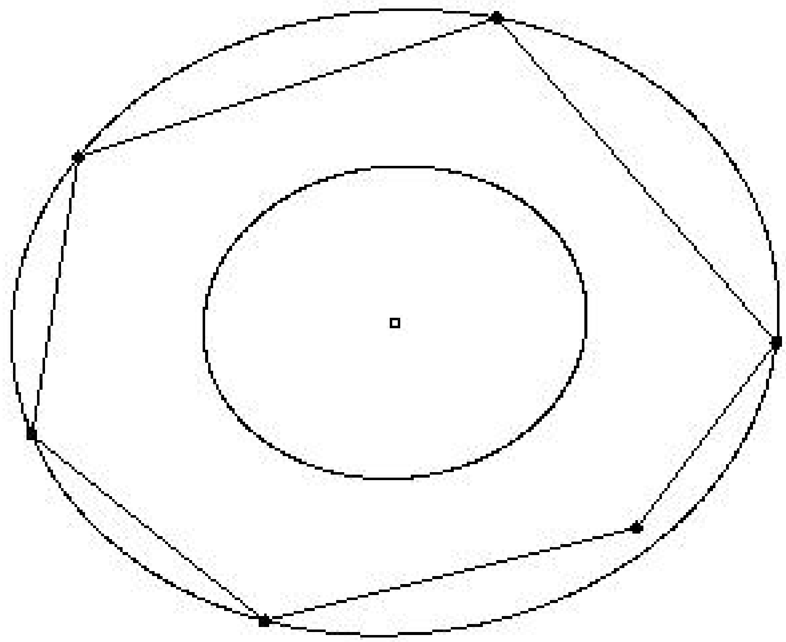
Let the convex hull of vertices $S = \text{conv}\{x_1, \ldots, x_m\}$ describe a polytope

The smallest volume ellipsoid containing S is

$$E = \{x \,:\, (x - x_c)^T P (x - x_c) \leq 1\}$$

where P, $x_c = -P^{-1}q$ are optimal solutions of the LMI problem

$$\begin{array}{ll} \max & t \\ & t \leq \det^{1/n} P \\ & \begin{bmatrix} P & q \\ q^T & r \end{bmatrix} \succeq 0 \\ & x_i^T P x_i + 2x_i^T q + r \leq 1 \end{array}$$



Sums of largest singular values

Let

$$\Sigma_k(X) = \sum_{i=1}^{k} \sigma_i(X), \quad k = 1, \ldots, n$$

denote the sum of the k largest singular values of an n-by-m matrix X

Then the epigraph $\{X \in \mathbb{R}^{n\times m}, t \in \mathbb{R} \,:\, \Sigma_k(X) \le t\}$ is lifted-LMI representable since

$$\sigma_i(X) = \lambda_i\left(\begin{bmatrix} 0 & X^T \\ X & 0 \end{bmatrix}\right)$$

and the sum of largest eigenvalues of a symmetric matrix is lifted-LMI representable

Positive polynomials

The set of univariate polynomials that are positive on the real axis is lifted-LMI representable in the coefficient space

Can be proved with cone duality (Nesterov) or with theory of moments (Lasserre) - more on that later

The even polynomial

$$p(s) = p_0 + p_1 s + \cdots + p_{2n} s^{2n}$$

satisfies $p(s) \geq 0$ for all $s \in \mathbb{R}$ if and only if

$$\begin{aligned} p_k &= \textstyle\sum_{i+j=k} X_{ij}, \qquad k = 0, 1, \ldots, 2n \\ &= \text{trace}\, H_k X \end{aligned}$$

for some lifting matrix $X = X^T \succeq 0$

Sum-of-squares decomposition

The expression of p_k with Hankel matrices H_k comes from

$$p(s) = [1 \quad s \quad \cdots \quad s^n] X [1 \quad s \quad \cdots \quad s^n]^\star$$

hence $X \succeq 0$ naturally implies $p(s) \geq 0$

Conversely, existence of X for any polynomial $p(s) \geq 0$ follows from the existence of a sum-of-squares (SOS) decomposition (with at most two elements) of

$$p(s) = \textstyle\sum_k q_k^2(s) \geq 0$$

Matrix X has entries $X_{ij} = \sum_k q_{k_i} q_{k_j}$

Seeking the lifting matrix amounts to seeking an SOS decomposition

Primal and dual formulations

Global minimization of polynomial $p(s) = \sum_{k=0}^{n} p_k s^k$
Global optimum p^*: maximum value of $\hat{p}$ such that $p(s) - \hat{p} \geq 0$

Primal LMI problem

$$\begin{array}{ll} \text{max} & \hat{p} = p_0 - \text{trace}\, H_0 X \\ \text{s.t.} & \text{trace}\, H_k X = p_k, \quad k = 1, \ldots, n \\ & X \succeq 0 \end{array}$$

Dual LMI problem

$$\begin{array}{ll} \text{min} & p_0 + \sum_{k=1}^{n} p_k y_k \\ \text{s.t.} & H_0 + \sum_{k=1}^{n} H_k y_k \succeq 0 \end{array}$$

with Hankel structure (moment matrix)

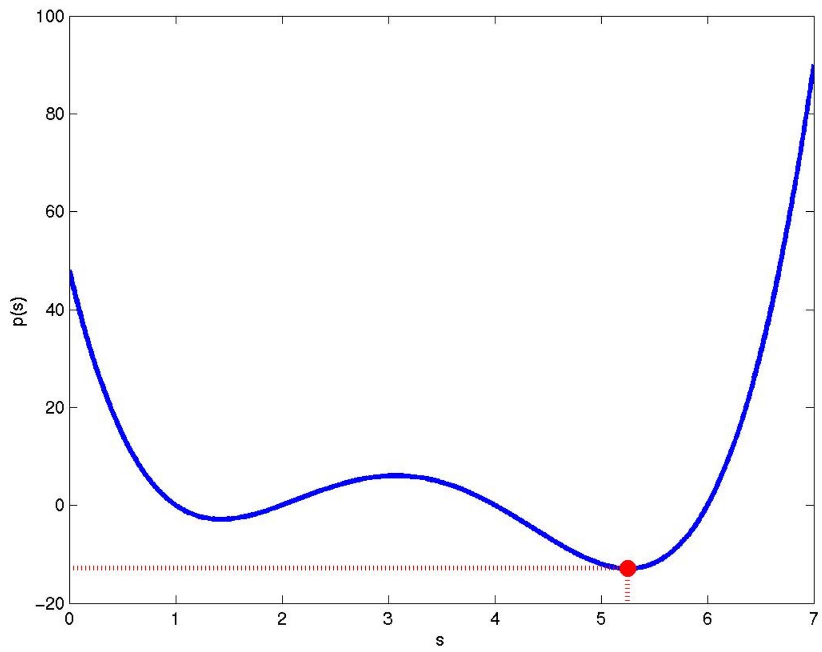
Positive polynomials and LMIs

Example: Global minimization of the polynomial

$$p(s) = 48 - 92s + 56s^2 - 13s^3 + s^4$$

Solving the dual LMI problem yields $p^* = p(5.25) = -12.89$

$$\begin{array}{ll} \min & 48 - 92y_1 + 56y_2 - 13y_3 + y_4 \\ \text{s.t.} & \begin{bmatrix} 1 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{bmatrix} \succeq 0 \end{array}$$



Complex LMIs

The complex valued LMI

$$F(x) = A(x) + jB(x) \succeq 0$$

is equivalent to the real valued LMI

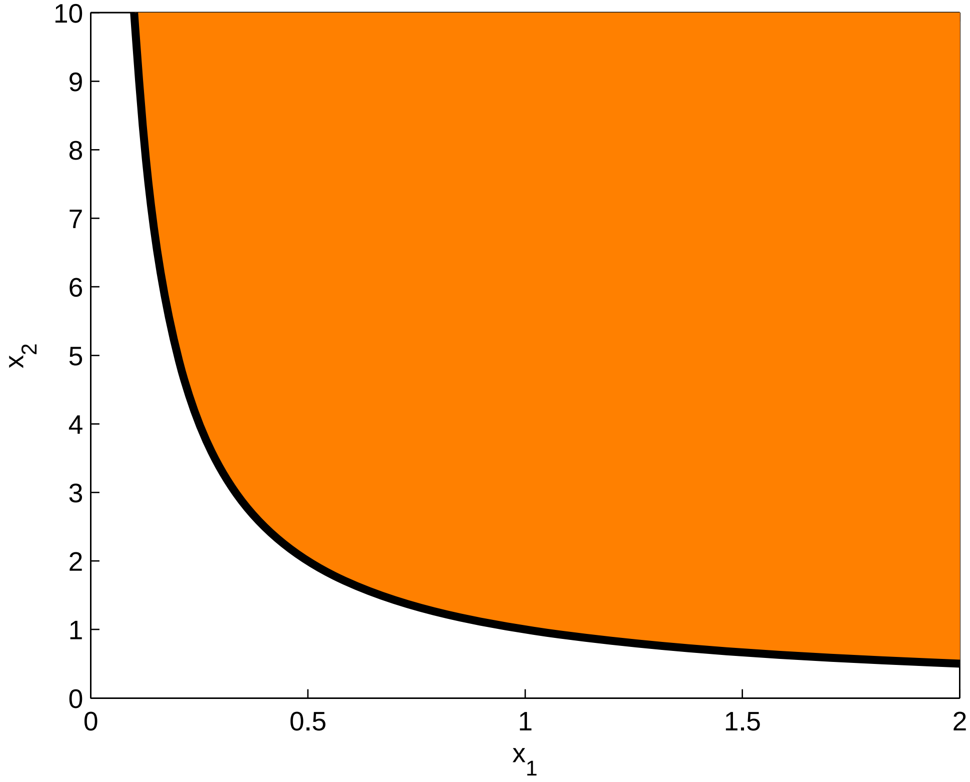
$$\begin{bmatrix} A(x) & B(x) \\ -B(x) & A(x) \end{bmatrix} \succeq 0$$

If there is a complex solution to the LMI
then there is a real solution to the same LMI

Note that matrix $A(x) = A^T(x)$ is symmetric
whereas $B(x) = -B^T(x)$ is skew-symmetric

INTERLUDE

LMI set or not ?



$$x_1 x_2 \geq 1 \text{ and } x_1 \geq 0$$

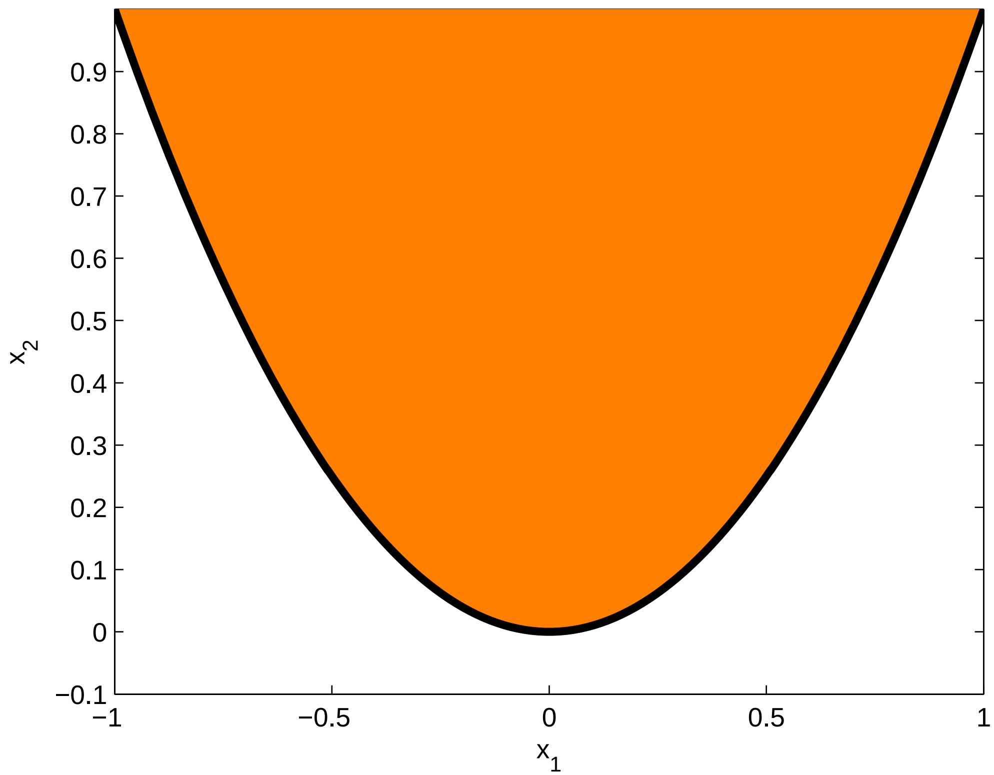
LMI

$$x_1 x_2 \geq 1 \text{ and } x_1 \geq 0$$

$$\Longleftrightarrow$$

$$\begin{bmatrix} x_1 & 1 \\ 1 & x_2 \end{bmatrix} \succeq 0$$

LMI set or not ?



$$x_2 \geq x_1^2$$

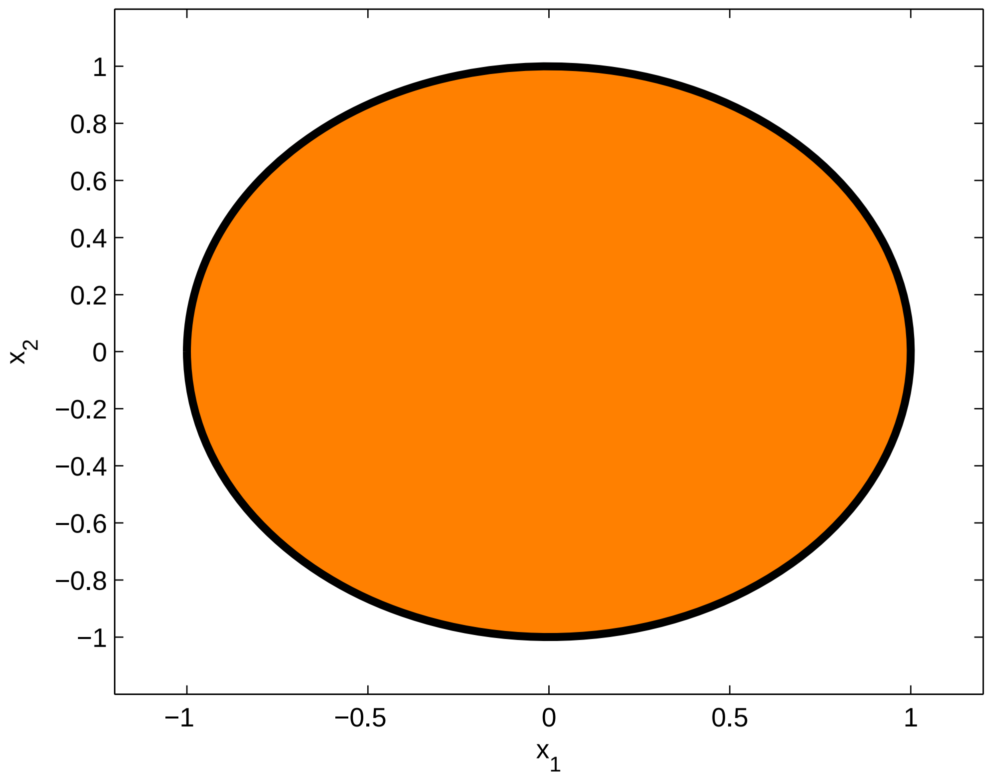
LMI

$$x_2 \geq x_1^2$$

$$\iff$$

$$\begin{bmatrix} 1 & x_1 \\ x_1 & x_2 \end{bmatrix} \succeq 0$$

LMI set or not ?



$$x_1^2 + x_2^2 \leq 1$$

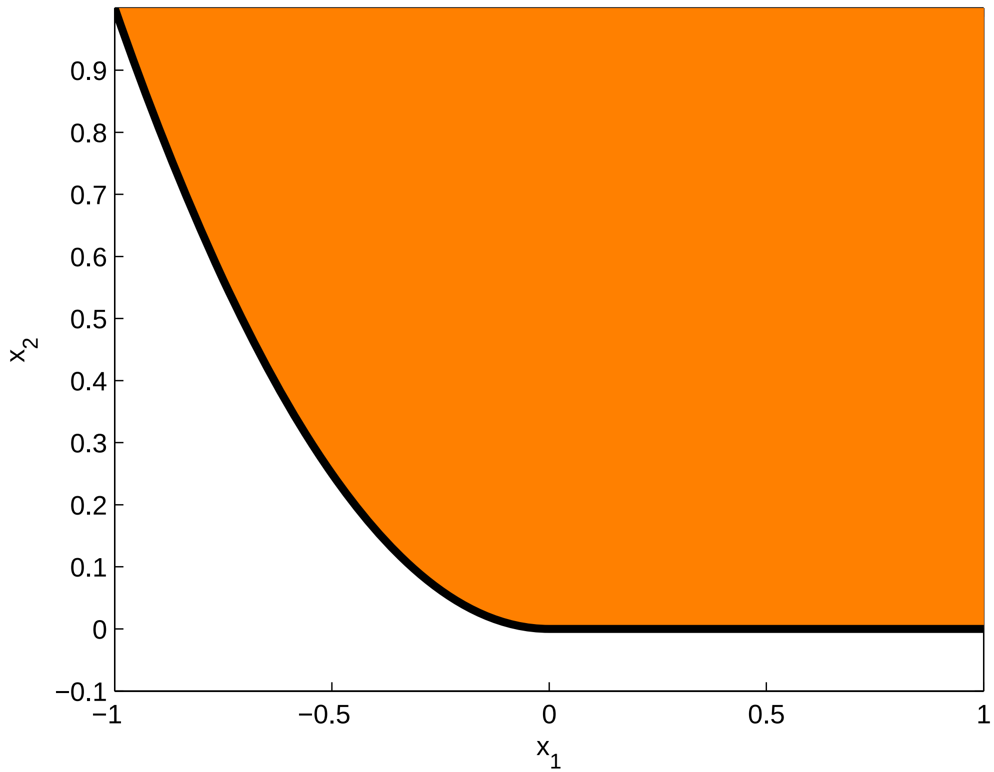
LMI

$$x_1^2 + x_2^2 \leq 1$$

$$\Longleftrightarrow$$

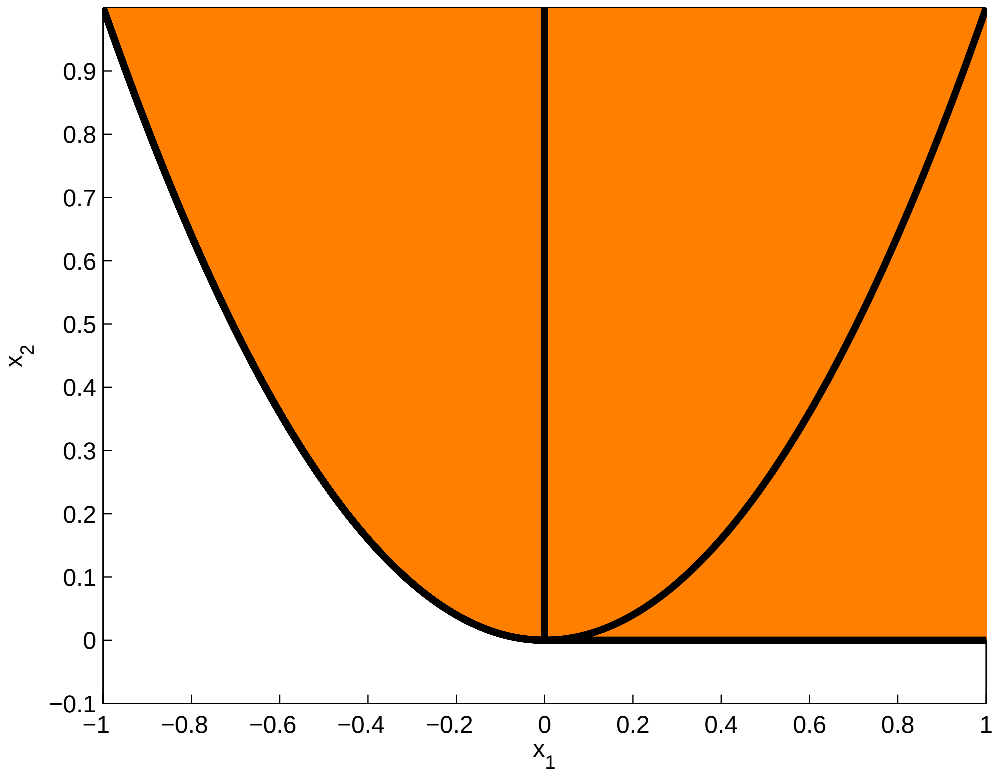
$$\begin{bmatrix} 1 + x_1 & x_2 \\ x_2 & 1 - x_1 \end{bmatrix} \succeq 0$$

LMI set or not ?



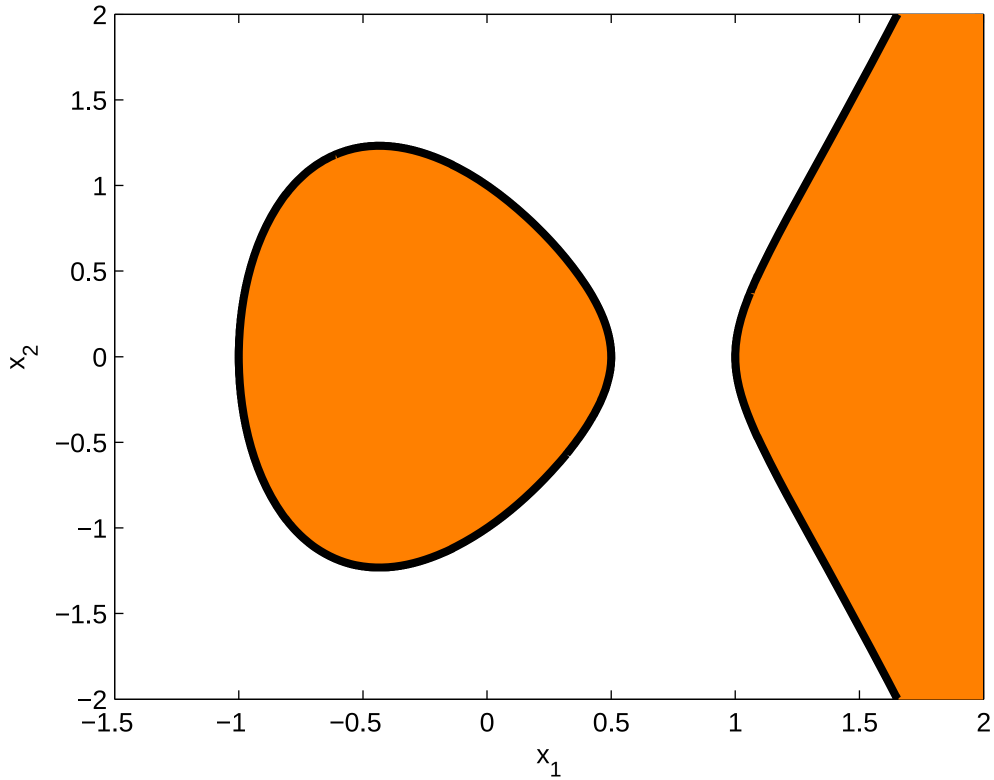
$$\{x \in \mathbb{R}^2 \ : \ t^4 + 2x_1t^2 + x_2 \geq 0, \ \forall t \in \mathbb{R}\}$$

NOT LMI: not basic semialgebraic



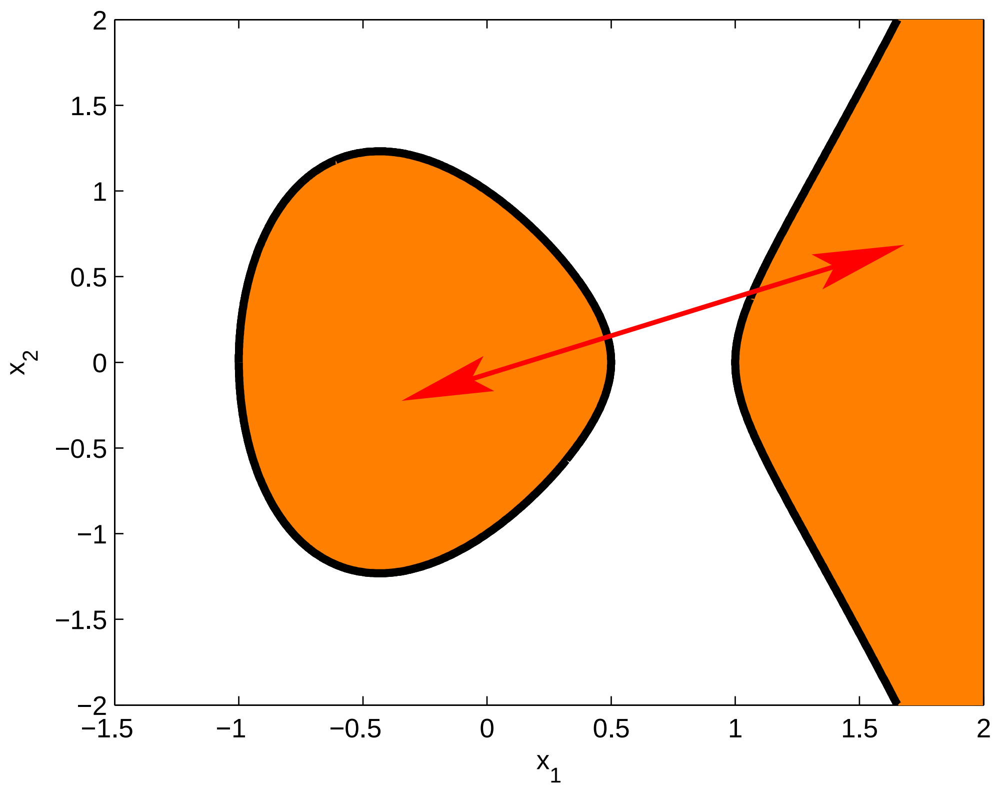
$$x_2 \geq x_1^2 \text{ or } x_1, x_2 \geq 0$$

LMI set or not ?



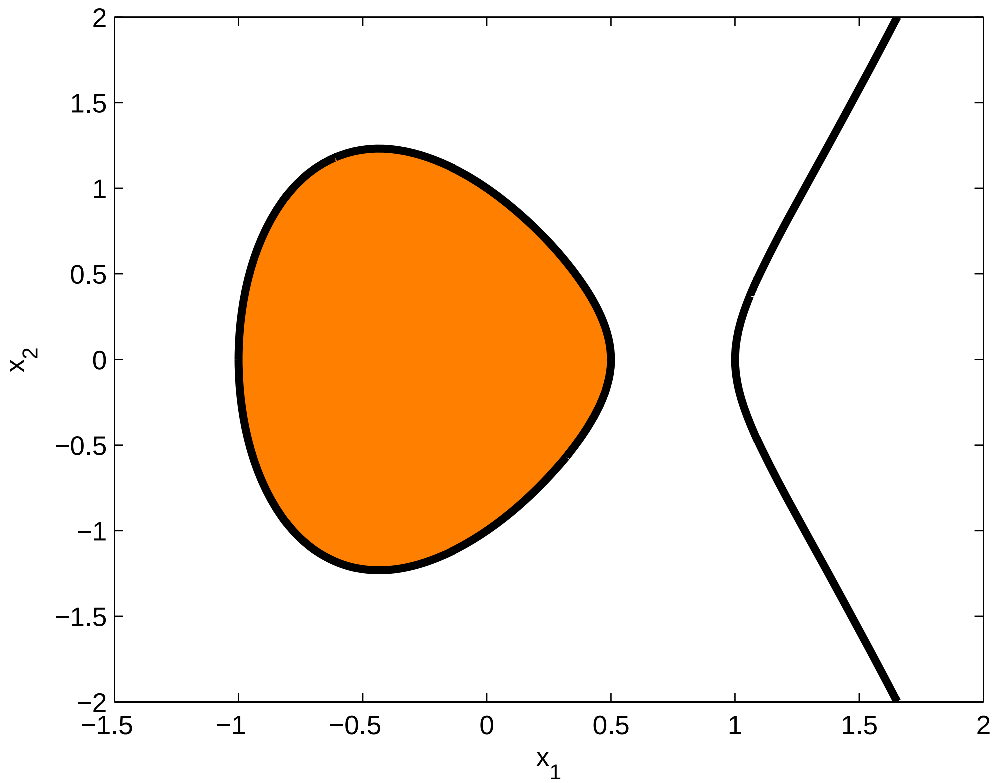
$$1 - 2x_1 - x_1^2 - x_2^2 + 2x_1^3 \geq 0$$

NOT LMI: not connected



$$1 - 2x_1 - x_1^2 - x_2^2 + 2x_1^3 \geq 0$$

LMI set or not ?



$$1 - 2x_1 - x_1^2 - x_2^2 + 2x_1^3 \geq 0 \text{ and } x_1 \leq \tfrac{1}{2}$$

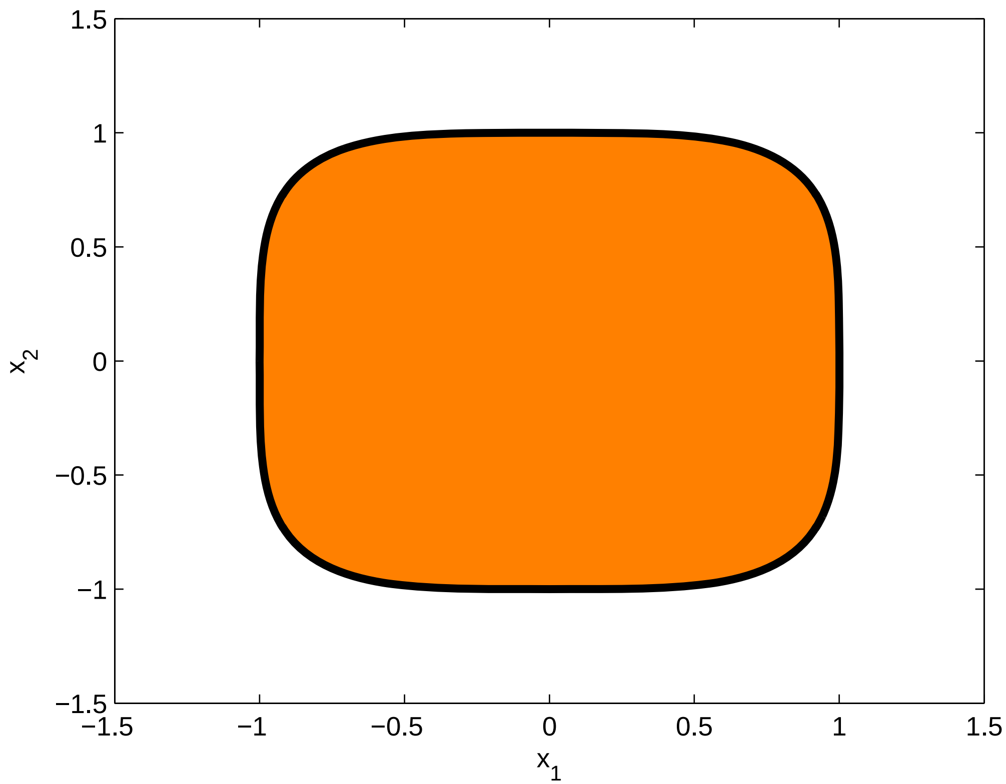
LMI

$$1 - 2x_1 - x_1^2 - x_2^2 + 2x_1^3 \geq 0 \text{ and } x_1 \leq \tfrac{1}{2}$$

$$\iff$$

$$\begin{bmatrix} 1 & x_1 & 0 \\ x_1 & 1 & x_2 \\ 0 & x_2 & 1 - 2x_1 \end{bmatrix} \succeq 0$$

LMI set or not ?



$$x_1^4 + x_2^4 \leq 1$$

NOT LMI

but projection of an LMI

$$\begin{bmatrix} 1+u_1 & u_2 & & & & \\ u_2 & 1-u_1 & & & & \\ & & 1 & x_1 & & \\ & & x_1 & u_1 & & \\ & & & & 1 & x_2 \\ & & & & x_2 & u_2 \end{bmatrix} \succeq 0$$

with two liftings u_1 and u_2