I.3. LMI DUALITY

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Primal and dual

For primal problem

$$
\begin{array}{rcll}
p^\star & = & \inf_x & g_0(x) \\
 & & \text{s.t.} & g_i(x) \leq 0
\end{array}
$$

define Lagrangian

$$L(x,z) = g_0(x) + \sum_i z_i g_i(x) = [g_0(x)\ g_1(x)\ g_2(x)\ \cdots][1\ z_1\ z_2\ \cdots]^T$$

and Lagrange dual function

$$f(z) = \inf_x L(x,z)$$

where z is a dual multiplier

Function f is always concave even if primal problem is nonconvex

Weak duality

Define dual problem

$$\begin{array}{rcll} d^\star & = & \text{sup}_z & f(z) \\ & & \text{s.t.} & z \geq 0 \end{array}$$

which is always convex since f is concave

Weak duality always holds: $p^\star \geq d^\star$ because

$$f(z) \leq g_0(x) + \sum_i z_i \underbrace{g_i(x)}_{\leq 0} \leq g_0(x)$$

for any primal feasible x and dual feasible z

The difference $p^\star - d^\star \geq 0$ is called duality gap

Strong duality

Sometimes, assumptions ensure that strong duality holds:

$$p^\star = d^\star$$

An example is Slater's constraint qualification assuming a strictly feasible convex primal (or dual) problem

Geometric interpretation of duality

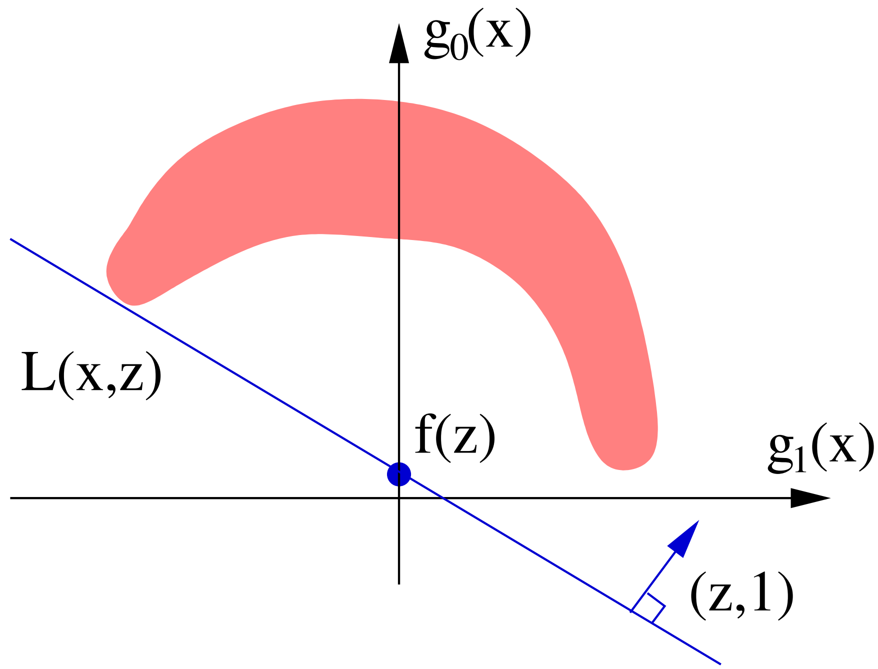
Consider the primal optimization problem

$$p^\star = \inf_x \; g_0(x) \quad \text{s.t.} \quad g_1(x) \le 0$$

with Lagrangian $L(x,z) = g_0(x) + z g_1(x)$
dual function $f(z) = \inf_x L(x,z)$
and dual problem

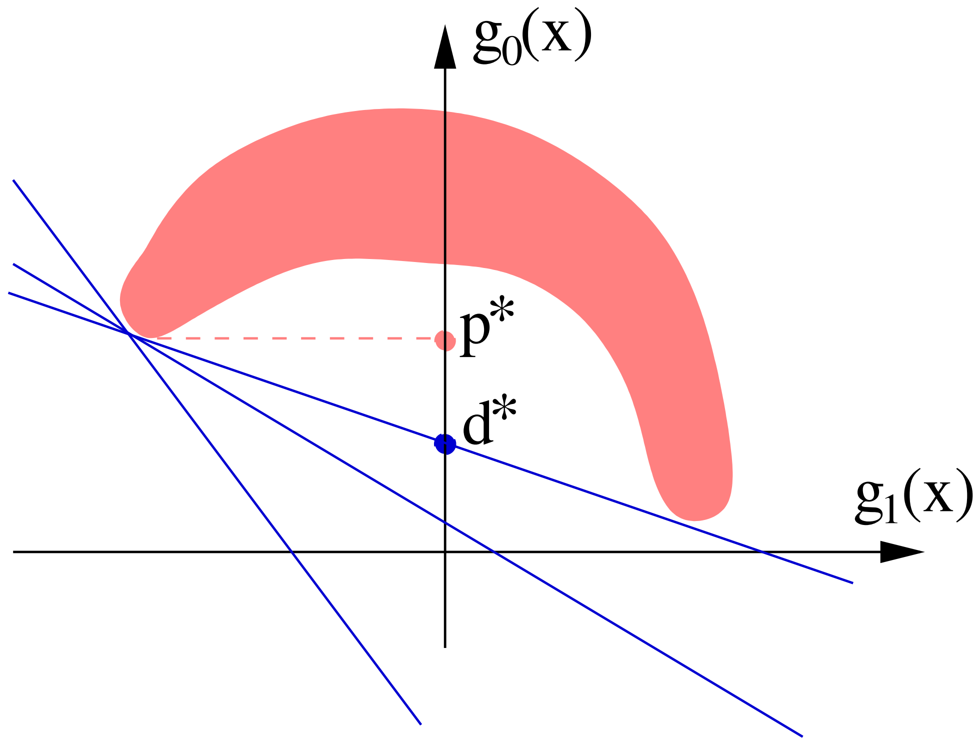
$$d^\star = \sup_z \; f(z) \quad \text{s.t.} \quad z \ge 0$$

Geometric duality



Lagrangian $L(x,z) = g_0(x) + z g_1(x)$ is a supporting line with (negative) slope $-z$, whose intersection with $g_1(x) = 0$ axis gives dual function $f(z) = \inf_x L(x,z)$

Geometric duality



Three supporting lines, including the optimum $z^\star$ yielding $d^\star < p^\star$
(duality gap = no strong duality here)

Complementary slackness

Suppose that strong duality holds, let $x^\star$ be primal optimal and $z^\star$ be dual optimal, then

$$\begin{array}{rcl} g_0(x^\star) & = & f(z^\star) \\ & = & \inf_x \left(g_0(x) + \sum_i z_i^\star g_i(x) \right) \\ & \le & g_0(x^\star) + \sum_i z_i^\star g_i(x^\star) \\ & \le & g_0(x^\star) \end{array}$$

from which it follows that $z_i^\star g_i(x^\star) = 0$

This is complementary slackness: $z_i^\star > 0 \implies g_i(x^\star) = 0$
or equivalently $g_i(x^\star) < 0 \implies z_i^\star = 0$

In words, the ith optimal multiplier is zero
unless the ith constraint is active at the optimum

KKT optimality conditions

Assuming that functions g_i are differentiable and that strong duality holds, then the gradient of Lagrangian $L(x, z^\star)$ over x vanishes at $x^\star$:

$$
\begin{gathered}
g_i(x^\star) \leq 0 \text{ (primal feasible)} \\
z_i^\star \geq 0 \text{ (dual feasible)} \\
z_i^\star g_i(x^\star) = 0 \text{ (complementary)} \\
\nabla g_0(x^\star) + \textstyle\sum_i z_i^\star \nabla g_i(x^\star) = 0
\end{gathered}
$$

Necessary Karush-Kuhn-Tucker conditions
satisfied by any primal and dual optimal pair

For convex problems, KKT conditions are also sufficient

Equality constraints

Multipliers corresponding to equality constraints are unconstrained:

$$\begin{array}{rcll} p^\star & = & \inf_x & g_0(x) \\ & & \text{s.t.} & h_j(x) = 0 \\ & & & g_i(x) \leq 0 \end{array}$$

Lagrangian $L(x, y, z) = g_0(x) + \sum_j y_j h_j(x) + \sum_i z_i g_i(x)$
dual function $f(y, z) = \inf_x L(x, y, z)$
dual problem

$$\begin{array}{rcll} d^\star & = & \sup_{y,z} & f(y, z) \\ & & \text{s.t.} & z \geq 0 \end{array}$$

no constraint on multiplier vector y

LP duality

Primal LP

$$\begin{array}{rll} p^\star & = & \inf_x \ c^T x \\ & & \text{s.t.} \ \ Ax = b \\ & & \qquad x \geq 0 \end{array}$$

dual function

$$\begin{aligned} f(y,z) &= \inf_x (c^T x + y^T(b - Ax) - z^T x) \\ &= \begin{cases} b^T y & \text{if } c - A^T y - z = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

Dual LP

$$\begin{array}{rll} d^\star & = & \sup_y \ b^T y \\ & & \text{s.t.} \ \ z = c - A^T y \geq 0 \end{array}$$

SDP duality

Primal SDP

$$
\begin{array}{rcll}
p^\star & = & \inf_X & \mathrm{trace}\, CX \\
 & & \text{s.t.} & \mathrm{trace}\, A_i X = b_i \\
 & & & X \succeq 0
\end{array}
$$

dual function

$$
\begin{array}{rcl}
f(y,Z) & = & \inf_X \left(\mathrm{trace}\, CX + \sum_i y_i (b_i - \mathrm{trace}\, A_i X) - \mathrm{trace}\, ZX\right) \\
 & = & \begin{cases} b^T y & \text{if } C - Z - \sum_i y_i A_i = 0 \\ -\infty & \text{otherwise} \end{cases}
\end{array}
$$

Dual SDP

$$
\begin{array}{rcll}
d^\star & = & \sup_y & b^T y \\
 & & \text{s.t.} & Z = C - \sum_i y_i A_i \succeq 0
\end{array}
$$

Example of SDP duality gap

Example
Consider the primal semidefinite program

$$\begin{array}{ll} \inf & x_1 \\ \text{s.t.} & \begin{bmatrix} 0 & x_1 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & 1+x_1 \end{bmatrix} \succeq 0 \end{array}$$

with dual

$$\begin{array}{ll} \sup & y_1 \\ \text{s.t.} & \begin{bmatrix} -y_2 & (1+y_1)/2 & -y_3 \\ (1+y_1)/2 & 0 & -y_4 \\ -y_3 & -y_4 & -y_1 \end{bmatrix} \succeq 0 \end{array}$$

In the primal necessarily $x_1 = 0$ (x_1 appears in a row with zero diagonal entry) so the primal optimum is $x_1 = 0$

Similarly, in the dual necessarily $(1+y_1)/2 = 0$ so the dual optimum is $y_1 = -1$

There is a nonzero duality gap here

Theorem of the alternatives

Consider primal feasibility problem

$$g_i(x) \geq 0$$

and dual feasibility problem

$$f(y) < 0, \;\; y \geq 0$$

with dual function $f(y) = \sup_x \sum_i y_i g_i(x)$

Dual feasible implies primal infeasible

Proof: if x^* is primal feasible then $f(y) = \sup_x \sum_i y_i g_i(x) \geq \sum_i y_i g_i(x^*)$ and hence $f(y) \geq 0$ for all $y \geq 0$

Separating hyperplanes in convex analysis

Can be generalized in the context of convex conic programming..

Farkas' lemma

When solving a primal/dual conic problem

inf	$c^T x$	sup	$b^T y$
s.t.	$Ax = b \quad x \in K$	s.t.	$c - A^T y \in K$

in the absence of a duality gap, then either

- x is optimal and y certifies optimality, i.e. $b^T y = c^T x$, or
- y is optimal and x certifies optimality, i.e. $c^T x = b^T y$, or
- there is no $x \in K$ with $Ax = b$ and this is certified by y, i.e. $b^T y > 0$ and $-A^T y \in K$, or
- there is no y such that $c - A^T y \in K$ and this is certified by x, i.e. $c^T x < 0$, $Ax = 0$, $x \in K$

Either we find a feasible point or
we certify that no such point exists

LMI duality

In the LMI formulation, the primal problem is actually in dual SDP form (confusing indeed..)

$$
\begin{array}{rcll}
p^\star & = & \inf_x & c^T x \\
 & & \text{s.t.} & F(x) = F_0 + \sum_i x_i F_i \succeq 0
\end{array}
$$

with dual LMI in primal SDP form

$$
\begin{array}{rcll}
d^\star & = & \sup_Z & \text{trace } -F_0 Z \\
 & & \text{s.t.} & \text{trace } F_i Z = c_i \\
 & & & Z \succeq 0
\end{array}
$$

Primal (resp. dual) not strictly feasible iff there exists a certificate of infeasibility provided by the dual (resp. primal)

Theorem of alternatives for LMIs

For the LMI mapping

$$F(x) = F_0 + \sum_i x_i F_i$$

Exactly one statement is true

- there exists x s.t. $F(x) \succ 0$
- there exists a nonzero $Z = Z^T \succeq 0$ s.t. trace $F_0 Z \leq 0$ and trace $F_i Z = 0$ for $i > 0$

Useful for detecting infeasibility of LMIs

Rich literature on theorems of alternatives for generalized inequalities, e.g. nonpolyhedral convex cones

S-procedure

S-procedure: frequently useful in robust and nonlinear control, is an outcome of the theorem of alternatives

There exists no nonzero complex vector x such that

$$x^* A_i x \geq 0,\ i = 1, \ldots, p$$

if there exist real numbers $y_i \geq 0$ such that

$$\sum_{i=1}^{p} y_i A_i \prec 0$$

If there exists x_0 such that $x_0^* A_i x_0 > 0$ for some i, the converse also holds

- when $p = 2$ for real quadratic forms
- when $p = 3$ for complex quadratic forms