I.2. WHAT IS AN LMI ?

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LMI - Linear Matrix Inequality

Canonical form

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0$$

Symmetric matrices F_i given, decision variables x_i to be found

Actually affine matrix constraint

Fundamental property: convex feasible set = closed LMI set

$$\{x \in \mathbb{R}^n \ : \ F(x) \succeq 0\}$$

Strict version $F(x) \succ 0$ defines open LMI set

Terminology coined out by Jan C. Willems in 1971

Lyapunov's LMI

Historically, the first LMIs appeared around 1890 when A. Lyapunov showed that the ODE

$$\frac{d}{dt}x(t) = \dot{x}(t) = Ax(t)$$

is exponentially asymptotically stable (all trajectories eventually converge to zero) iff there is a solution to the matrix inequalities

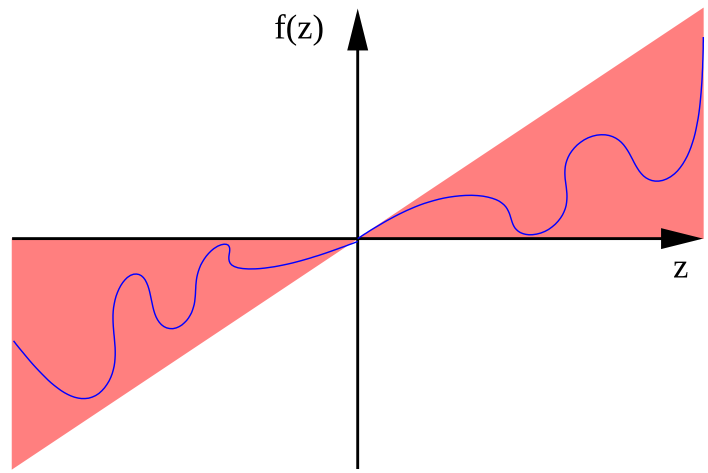
$$A^T P + PA \prec 0 \quad P = P^T \succ 0$$

which are linear in unknown matrix P

Some history

1940s: Lur'e, Postnikov et al applied Lyapunov's approach to control problems with nonlinearity in the actuator

$$\dot{x} = Ax + bf(x)$$



Sector-type nonlinearity

Polynomial (frequency dependent) inequalities

KYP lemma

1960s: Yakubovich, Popov, Kalman and also Anderson et al obtained the positive real lemma

Polynomial frequency inequalities reformulated as LMIs

Reduces solution of LMI to a simple graphical criterion (Popov, circle and Tsypkin criteria)

1970s: Willems focused on solving algebraic equations such as Lyapunov's or Riccati's equations (AREs), rather than LMIs

Then most of the work dedicated to numerical algebra, development of Matlab (1984), focus on AREs..

Mathematical programming

In parallel to these developments in control theory:

1979: ellipsoid algorithm of Khachiyan: polynomial bound on worst case iteration count for linear programming (LP)

1984: Karmarkar introduces interior-point (IP) methods for LP: improved complexity bound and efficiency

1988: Nesterov, Nemirovski extend IP methods for LMIs

1990: Alizadeh, Boyd, El Ghaoui, Jarre and Overton devise algorithms for nonsmooth convex eigenvalue functions

Mostly confidential research activities, until..

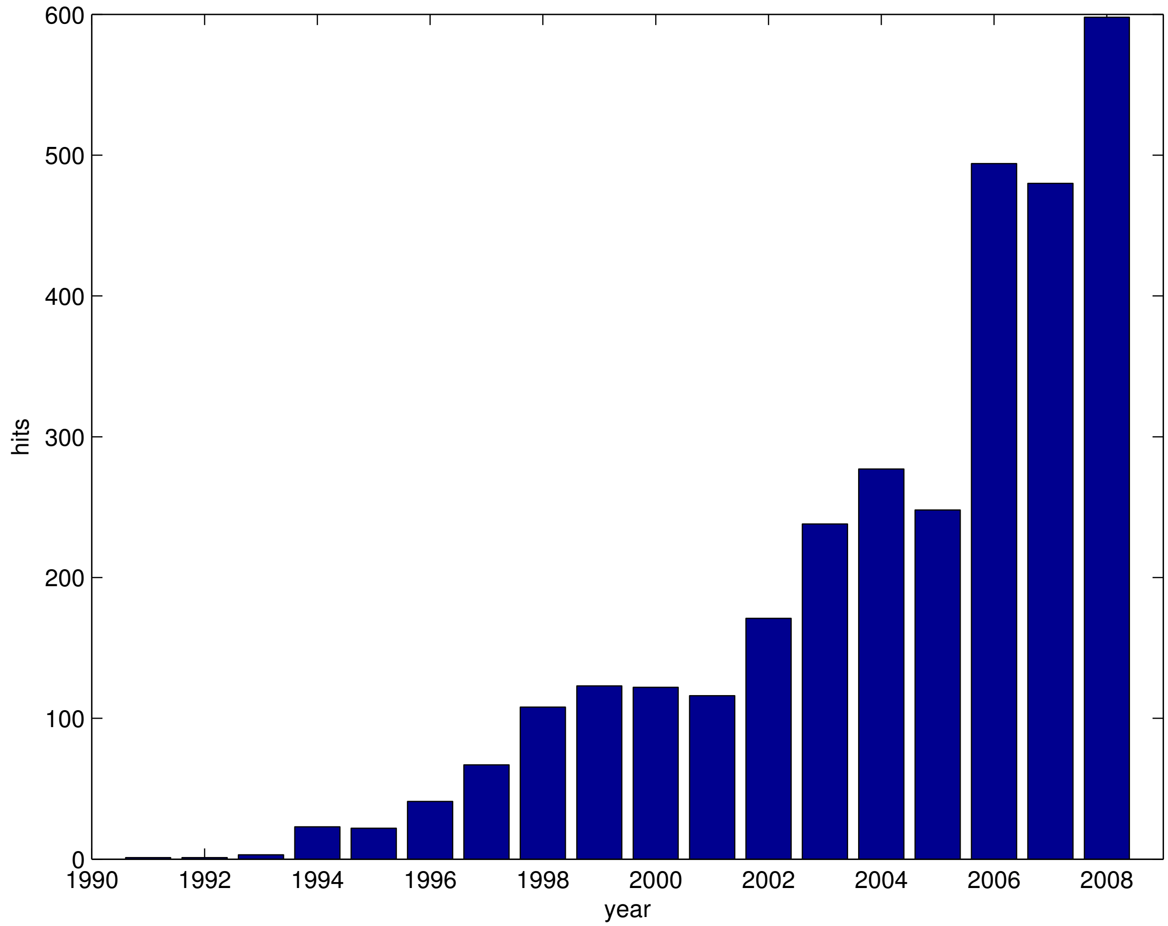
LMI boom

1994: Goemans and Williamson prove than LMI relaxations of MAXCUT (a problem of combinatorial optimization) provide solutions at least 88% the optimal value

1994: Research effort in control culminated in the LMI book by Boyd, El Ghaoui, Feron, Balakrishnan - Contributions also by Apkarian, Bernussou, Gahinet, Geromel, Peres and many others..

Strong impact in electrical and mechanical engineering

On 1 January 2009, the IEEE Xplore database provides 3133 hits with keyword LMI



IEEE Xplore hits with keyword LMI

LMI formalism

In a typical control LMI

$$A^T P + P A = F_0 + \sum_{i=1}^{n} x_i F_i \prec 0$$

individual matrix entries are decision variables

and hence control LMIs have a very specific structure

In mathematical programming terminology LMI = SDP

$$\begin{array}{ll} \inf & c^T x \\ \text{s.t.} & Ax = b \quad x \in K \end{array} \qquad \begin{array}{ll} \sup & b^T y \\ \text{s.t.} & c - A^T y \in K \end{array}$$

which is LP in convex cone K

SDP

LMI optimization is a generalization of linear programming (LP) to cone of positive semidefinite matrices

New branch of mathematical programming called semidefinite programming (SDP), with AMS MSC code 90C22 (since 2000)

Linear programming pioneered by
- Dantzig and its simplex algorithm (1947, ranked in the top 10 algorithms by SIAM Review in 2000)
- Kantorovich (co-winner of the 1975 Nobel prize in economics)

Unfortunately, SDP has not reached maturity of LP so far..

Embedded convex cones

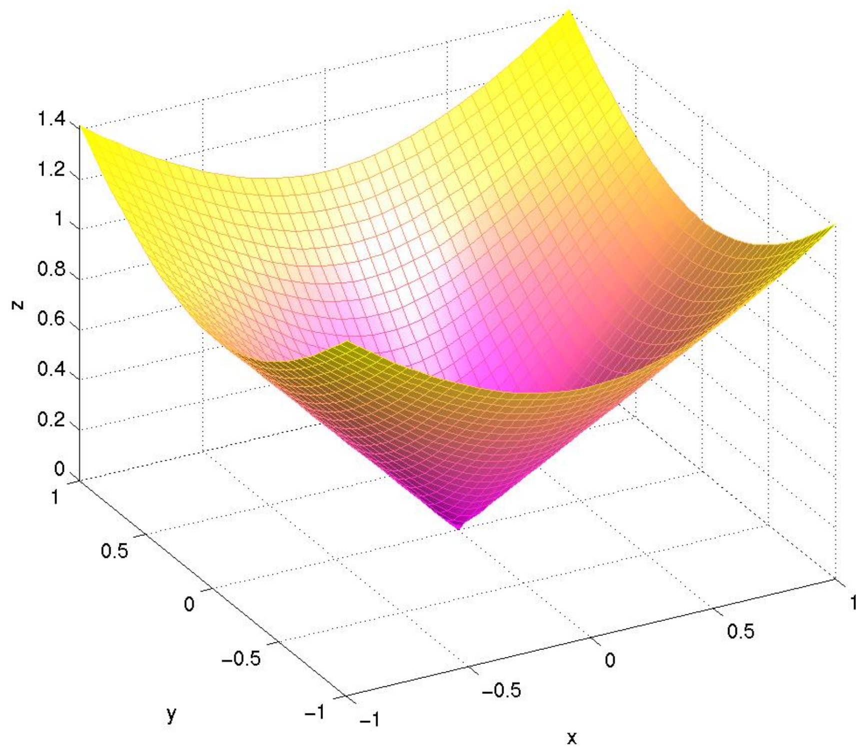
Conic programming = LP in convex cone K

- linear cone = positive orthant (LP)
- quadratic cone = Lorentz cone (QP)
- semidefinite cone (SDP)

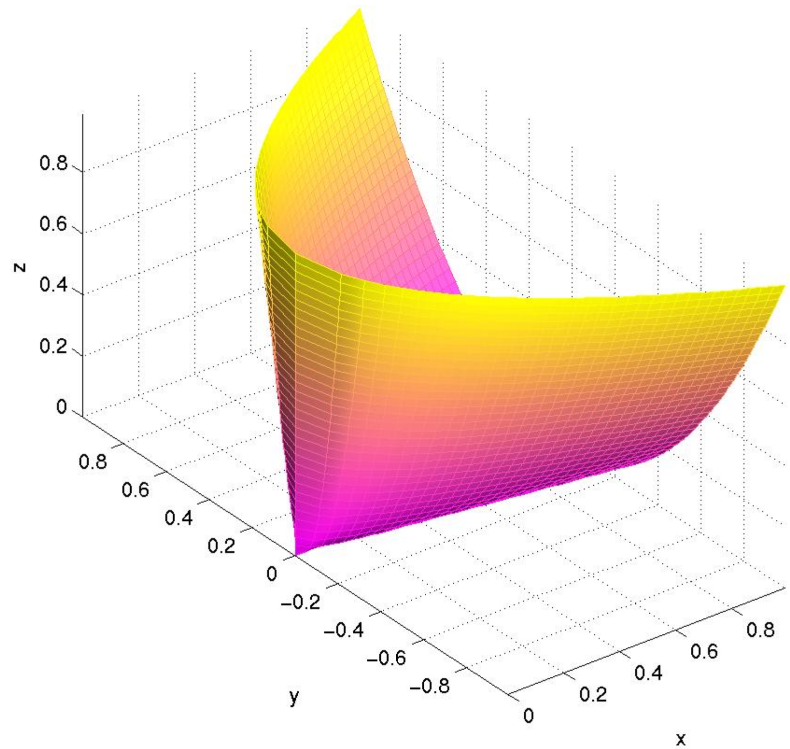
Hierarchy: LP cone $\subset$ QP cone $\subset$ SDP cone

Modelling paradigm: SDP = LMI is the most general
we will use direct products of these

Duality: LP, QP and SDP cones are self-dual, i.e. $K^* = K$



3D quadratic cone or Lorentz cone or ice-cream cone
$x^2 + y^2 \leq z^2, \ z \geq 0$



2D positive semidefinite cone

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \succeq 0 \iff x \geq 0, \; z \geq 0, \; xz \geq y^2$$