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# Orthonormal basis functions for modelling continuous-time systems

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**Abstract**

This paper studies continuous-time system model sets that are spanned by fixed pole orthonormal bases. The nature of these bases is such as to generalise the well-known Laguerre and two-parameter Kautz bases. The contribution of the paper is to establish that the obtained model sets are complete in all of the Hardy spaces $H_p(\Pi)$, $1 < p < \infty$, and the right half plane algebra $A(\Pi)$ provided that a mild condition on the choice of basis poles is satisfied. A characterisation of how modelling accuracy is affected by pole choice, as well as an application example of flexible structure modelling are also provided. © 1999 Elsevier Science B.V. All rights reserved.

**Zusammenfassung**

In diesem Artikel werden Modellmengen für Systeme in stetiger Zeit betrachtet, wobei diese Mengen von orthonormalen Basen mit fixierten Polen erzeugt werden. Diese Basen verallgemeinern die wohlbekannten Laguerre-Basen und die zweiparametrigen Kautz-Basen. In dieser Arbeit wird gezeigt, dass die erhaltenen Modellmengen in allen Hardy-Räumen $H_p(\Pi)$, $1 < p < \infty$, und in der Algebra $A(\Pi)$ der rechten Halbebene vollständig sind, vorausgesetzt, dass eine schwache Bedingung an die Pole der Basis erfüllt ist. Eine Charakterisierung des Einflusses der Polvorgabe auf die Modellgenauigkeit, sowie ein Anwendungsbeispiel der Modellierung einer flexiblen Struktur werden gegeben. © 1999 Elsevier Science B.V. All rights reserved.

**Résumé**

Cet article étudie des ensembles de modèles de systèmes à temps continu qui sont engendrés par des bases orthonormales à pôles fixes. La nature de ces bases est telle qu'elles généralisent les bases bien connues de Laguerre et de Keutz à deux paramètres. La contribution de cet article est d'établir que les ensembles de modèles obtenus sont complets dans tous les espaces de Hardy $H_p(\Pi)$, $1 < p < \infty$, ainsi que l'algèbre du demi-plan de droite $A(\Pi)$, pourvu qu'une condition douce sur le choix des pôles des bases soit satisfaite. Nous fournissons également une caractérisation de la façon dont la précision du modèle est affectée par le choix des pôles, ainsi qu'un exemple d'application de modélisation de structures flexibles. © 1999 Elsevier Science B.V. All rights reserved.

*Keywords:* Rational basis functions; Orthonormal; Completeness; Continuous-time systems

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**Nomenclature**

| | |
|---|---|
| $\boldsymbol{C}$ | field of complex numbers |
| $\boldsymbol{R}$ | field of real numbers |
| $\Pi$ | open right half-plane $\{s \in \boldsymbol{C}\colon \mathrm{Re}\{s\} > 0\}$ |
| $\bar{\Pi}$ | closed right half-plane $\{s \in \boldsymbol{C}\colon \mathrm{Re}\{s\} \geqslant 0\}$ |
| $\boldsymbol{D}$ | open unit disk $\{z \in \boldsymbol{C}\colon \lvert z\rvert < 1\}$ |
| $\boldsymbol{T}$ | unit circle $\{z \in \boldsymbol{C}\colon \lvert z\rvert = 1\}$ |
| $H_p(\Pi)$ | Hardy spaces of functions $f(s)$ analytic on $\Pi$ and such that $\lVert f\rVert_p^p = (1/2\pi)\sup_{x>0}\int_{-\infty}^{\infty}\lvert f(x+\mathrm{j}y)\rvert^p\,\mathrm{d}y < \infty$, $0 < p < \infty$ and $\lVert f\rVert_\infty = \sup_{s\in\Pi}\lvert f(s)\rvert < \infty$ |
| $A(\Pi)$ | right half-plane algebra $\{f\colon f \in H_\infty(\Pi)$ and continuous on $\bar{\Pi}\}$ |
| $A(\boldsymbol{D})$ | disk algebra $\{f\colon f$ analytic on $\boldsymbol{D}$ and continuous on $\bar{\boldsymbol{D}}\}$ |
| $\mathrm{sp}\,A$ | linear span of $A$ |
| $\bar{a}$ | complex conjugate of $a$ |

## 1. Introduction

The use of orthonormal bases for the purposes of approximation and analysis is fundamental to many areas of applied mathematics. In particular, in the areas of control theory, signal processing and system identification, there has long been interest in the use of the trigonometric (FIR), 'Laguerre', and 'two-parameter Kautz' bases [17,20,21]. More recently, in a discrete-time setting, this interest has been revived in a string of works [9–11,30,32,41,42,44] and this has led workers to consider orthonormal constructions which generalise the Laguerre and two-parameter Kautz cases [7,8,12,18,34]. One of these efforts presented in [3,31] considers the orthonormal basis functions defined on $\boldsymbol{D} \cup \boldsymbol{T}$ by a choice of numbers $\xi_n \in \boldsymbol{D}$, $n = 1,2,\ldots,$ as

$$\mathscr{B}_n(z) \triangleq \frac{\sqrt{1-|\xi_n|^2}}{1-\overline{\xi_n}z}\,\phi_{n-1}(z), \qquad \phi_n(z) \triangleq \prod_{k=1}^{n}\frac{z-\xi_k}{1-\overline{\xi_k}z}, \quad \phi_0(z) \triangleq 1. \tag{1}$$

In the special case of $\xi_n = \xi \in \boldsymbol{R}$ this becomes the discrete-time Laguerre basis, and in the special case of $\xi_n = \xi \in \boldsymbol{C}$ this provides the discrete-time two-parameter Kautz basis; see [31] for more details on the generalising aspects of definition (1).

In the case of considering continuous-time model descriptions, several important works have also recently appeared that employ continuous-time Laguerre and two-parameter Kautz bases [24,25,43] and rational wavelet bases [13]. The purpose of this paper is to make some further contribution in this area by considering some issues related to a natural generalisation of continuous-time Laguerre and Kautz bases.

The generalisation to be considered is analogous to the extension presented in (1). Namely, a set of basis functions $\{B_n(s)\}$ are treated which are defined by a choice of numbers $a_n \in \Pi$, $\forall n$ as

$$B_n(s) \triangleq \frac{\sqrt{2\,\mathrm{Re}\{a_n\}}}{s+a_n}\,\varphi_{n-1}(s), \qquad \varphi_n(s) \triangleq \prod_{k=1}^{n}\frac{s-\overline{a_k}}{s+a_k}, \quad \varphi_0(s) \triangleq 1. \tag{2}$$

We set $B_0(s) \equiv 1$. The rational basis functions $\{B_n\}_{n\geqslant 1}$ are orthonormal in $H_2(\Pi)$ with respect to the inner-product (see Section 3)

$$\langle B_n, B_m\rangle \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} B_n(\mathrm{j}\omega)\overline{B_m(\mathrm{j}\omega)}\,\mathrm{d}\omega = \begin{cases} 1; & m = n, \\ 0; & m \neq n. \end{cases}$$

Analogous to the discrete-time case, the continuous-time Laguerre basis (studied, for example, in [23,33,43]) is obtained as a special case of (2) by the choice $a_n = a \in \boldsymbol{R}$ and the continuous time two-parameter Kautz basis (studied in [43]) by the choice $a_n = a \in \boldsymbol{C}$.

It should be acknowledged that both the continuous and discrete-time generalisations shown in (2) and (1) enjoy a long history in both the pure mathematics [14,26,40] and engineering literature [19,28,36].

## 2. Main result

As mentioned in the introduction, an important motivation for the consideration of orthonormal parameterisations is for approximation purposes. In this setting, a dominant question must arise as to the quality of the approximation. Pertaining to this, one of the most fundamental properties that might be required is that linear combinations of the basis elements be capable of arbitrarily good approximation.

Put more precisely in the context of systems theory, this involves considering an element $f(s)$ living in a normed linear function space $(X, \|\cdot\|_X)$, and for arbitrary $\varepsilon > 0$ and for sufficiently large $n$ being able to find an element $g(s) \in \mathrm{sp}\{B_k(s)\}_{k=1}^{n}$ such that $\|f-g\|_X \leqslant \varepsilon$. Here $X$ is a complex vector space and the linear span is with respect to the field of complex numbers.

If this is in fact possible for arbitrarily small $\varepsilon$, then $\mathrm{sp}\{B_k(s)\}_{k\geqslant 1}$ is said to be 'complete' in $X$. The choice of the function space depends on the application of the approximate model, but for quadratic optimal control purposes or mean square optimal prediction purposes, the choice $H_2(\Pi)$ would be appropriate, while for robust control or estimation purposes, the choices $A(\Pi)$ or $H_p(\Pi)$ (for large $p$) would be suitable.

With regard to the discrete-time basis (1), the approximation issues have been addressed in [3] where the following result was obtained.

**Theorem 1** (Akçay and Ninness [3, Theorem 6 and Corollary 7]). *Consider the set of functions* $\{\mathscr{B}_k(z)\}$ *defined by* (1). *Then the set* $X = \mathrm{sp}\{\mathscr{B}_k(z)\}_{k\geqslant 1}$ *is complete in* $A(\boldsymbol{D})$ *and* $H_p(\boldsymbol{T})$ *for all* $1 \leqslant p < \infty$ *if and only if*

$$\sum_{k=1}^{\infty} (1 - |\xi_k|) = \infty. \tag{3}$$

The main result of this paper is to establish an analogous result for the continuous-time basis (2) as follows.

**Theorem 2.** *The model set spanned by the basis functions* $\{B_n(s)\}_{n\geqslant 0}$ *is complete in all of the spaces* $H_p(\Pi)$, $1 < p < \infty$, *and* $A(\Pi)$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{\mathrm{Re}\{a_n\}}{1+|a_n|^2} = \infty. \tag{4}$$

Condition (4) is satisfied by the Laguerre and two-parameter Kautz bases and the rational wavelets in [13]. In fact, it is a very mild condition. For example, when a sequence of poles with fixed real part is chosen, the only way it could be violated is that the imaginary parts of the poles must diverge to infinity faster than the linear rate. If a sequence of magnitude bounded poles are chosen, then in order to violate (4) the real parts of the chosen basis poles must approach to the imaginary axis very rapidly.

In contrast to the Laguerre and two-parameter Kautz bases, where all the poles are fixed at the same value, the general basis (2) enjoys increased flexibility of pole location. For example, slow and fast modes may coexist in the model structure. As a result, a fewer number of basis functions (and hence, for system identification applications, a fewer number of data) may be used without sacrificing modelling accuracy. This feature is quantified in Section 4.

Note that the conclusion of Theorem 2 may be extended to $H_1(\Pi)$. Moreover, it is possible to construct orthonormal model sets that are norm dense in $H_p(\Pi)$, $1 \leqslant p < \infty$, and have a prescribed asymptotic order [4]. In [4], it is also shown that the Fourier series formed by the general basis functions converge in all spaces $H_p(\Pi)$, $1 < p < \infty$.

## 3. Proof of Theorem 2

Before proceeding to the proof of Theorem 2, we shall demonstrate that the basis functions in (2) are indeed orthonormal. When $n > m$, we have by Cauchy's Integral Theorem [37]

$$\begin{aligned}
\langle B_n, B_m \rangle
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sqrt{4\,\mathrm{Re}\{a_n\}\,\mathrm{Re}\{a_m\}}}{(\mathrm{j}\omega + a_n)(\mathrm{j}\omega + a_m)} \frac{\varphi_{n-1}(\mathrm{j}\omega)}{\varphi_m(\mathrm{j}\omega)}\,\mathrm{d}\omega \\
&= -\frac{1}{2\pi\mathrm{j}}\int_{\mathrm{j}\boldsymbol{R}} \frac{\sqrt{4\,\mathrm{Re}\{a_n\}\,\mathrm{Re}\{a_m\}}}{(s + a_n)(s + a_m)} \frac{\varphi_{n-1}(s)}{\varphi_m(s)}\,\mathrm{d}s \\
&= -\lim_{r\to\infty}\left[\frac{1}{2\pi\mathrm{j}}\int_{-\mathrm{j}r}^{\mathrm{j}r} \frac{\sqrt{4\,\mathrm{Re}\{a_n\}\,\mathrm{Re}\{a_m\}}}{(s + a_n)(s + a_m)} \frac{\varphi_{n-1}(s)}{\varphi_m(s)}\,\mathrm{d}s \right. \\
&\qquad\qquad \left. + \,\mathrm{O}(r^{-1})\right]
\end{aligned}$$

$$= \lim_{r\to\infty}\left[\underbrace{\frac{1}{2\pi j}\oint_{\Gamma_r}\frac{\sqrt{4\,\mathrm{Re}\{a_n\}\,\mathrm{Re}\{a_m\}}}{(s+a_n)(s+a_m)}\frac{\varphi_{n-1}(s)}{\varphi_m(s)}\,\mathrm{d}s}_{=0} + \mathrm{O}(r^{-1})\right] = 0, \quad (5)$$

where the closed path $\Gamma_r$ consists of a segment of the imaginary axis and an $r$ radius semicircle in $\Pi$ centred at the origin and is traversed counter clockwise. When $n = m$, we have

$$\langle B_n, B_n\rangle = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\{a_n\}}{\omega^2 + 2\omega\,\mathrm{Im}\{a_n\} + |a_n|^2}\,\mathrm{d}\omega = 1,$$

where the second equality follows from the formula 3.3.16 in [1]

$$\int\frac{1}{ax^2+bx+c}\,\mathrm{d}x = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}.$$

Now we return to the proof of Theorem 2. Consideration is first given to the underlying space being $A(\Pi)$.

**Lemma 3.** *The linear span of the set* $\{B_n(s)\}_{n\geqslant 0}$ *with* $B_n(s)$ *defined in* (2) *is complete in* $A(\Pi)$ *if* (4) *holds.*

**Proof.** This will be established by first addressing the completeness of $\mathrm{sp}\{\varphi_n(s)\}_{n\geqslant 0}$ in $A(\Pi)$. Notice that since the bilinear map

$$s = \frac{1-z}{1+z}$$

preserves the supremum norms between $A(\Pi)$ and $A(\boldsymbol{D})$, the question of whether $\mathrm{sp}\{\varphi_n(s)\}_{n\geqslant 0}$ is complete in $A(\Pi)$ is equivalent to the question of the completeness of

$$\left\{\varphi_n\!\left(\frac{1-z}{1+z}\right)\right\}_{n\geqslant 0}$$

in $A(\boldsymbol{D})$. Let

$$\xi_n \triangleq \frac{1-\overline{a_n}}{1+\overline{a_n}}, \qquad \alpha_n \triangleq (-1)^n\prod_{l=1}^{n}\frac{1+\overline{a_l}}{1+a_l}. \quad (6)$$

Then provided $\mathrm{Re}\{a_n\} > 0$ for all $n$, $\xi_n\in\boldsymbol{D}$ for all $n$. Also

$$\varphi_n\!\left(\frac{1-z}{1+z}\right) = (-1)^n\prod_{l=1}^{n}\frac{1+\overline{a_l}}{1+a_l}\prod_{k=1}^{n}\frac{z-\xi_k}{1-\overline{\xi_k}z} = \alpha_n\phi_n(z).$$

Therefore, it is sufficient to establish the completeness of $\mathrm{sp}\{\phi_n(z)\}_{n\geqslant 0}$ in $A(\boldsymbol{D})$. To achieve this, consider the functions $\{\mathscr{B}_k(z)\}$ defined in (1). Then

$$\mathscr{B}_n(z) = \frac{\overline{\xi_n}\phi_n(z) + \phi_{n-1}(z)}{\sqrt{1-|\xi_n|^2}}; \quad n\geqslant 1$$

and hence $\mathrm{sp}\{\mathscr{B}_k(z)\}_{k=1}^{n}\subset \mathrm{sp}\{\phi_k(z)\}_{k=0}^{n}$. However by Theorem 1, $\mathrm{sp}\{\mathscr{B}_k(z)\}_{k\geqslant 1}$ is complete in $A(\boldsymbol{D})$ if and only if (3) is satisfied.

Now, since

$$1+|a_n|^2 \geqslant |1-a_n^2| = |1-a_n||1+a_n|$$

then by the definition in (6)

$$|\xi_n| = \left|\frac{1-\overline{a_n}}{1+\overline{a_n}}\right| \geqslant \frac{|1-a_n|^2}{1+|a_n|^2}$$

so that

$$\sum_{n=1}^{\infty}(1-|\xi_n|) \leqslant \sum_{n=1}^{\infty}\left(1-\frac{|1-a_n|^2}{1+|a_n|^2}\right) = 2\sum_{n=1}^{\infty}\frac{\mathrm{Re}\{a_n\}}{1+|a_n|^2}.$$

Conversely since

$$\left|\frac{1-\overline{a_n}}{1+\overline{a_n}}\right| = \left|\frac{1-a_n}{1+a_n}\right|\left|\frac{1+a_n}{1+a_n}\right| = \frac{|1-a_n^2|}{|1+a_n|^2} \leqslant \frac{1+|a_n^2|}{|1+a_n|^2}$$

then

$$\sum_{n=1}^{\infty}(1-|\xi_n|) \geqslant \sum_{n=1}^{\infty}\left(1-\frac{1+|a_n^2|}{|1+a_n|^2}\right) = 2\sum_{n=1}^{\infty}\frac{\mathrm{Re}\{a_n\}}{|1+a_n|^2}.$$

As well,

$$|1+a_n|^2 \leqslant 2(1+|a_n|^2)$$

so that

$$\sum_{n=1}^{\infty}\frac{\mathrm{Re}\{a_n\}}{1+|a_n|^2}<\sum_{n=1}^{\infty}(1-|\xi_n|)\leqslant 2\sum_{n=1}^{\infty}\frac{\mathrm{Re}\{a_n\}}{1+|a_n|^2}.$$

Therefore, (3) holds if and only if (4) holds implying that under the definition (6), then (4) is necessary and sufficient for $\mathrm{sp}\{\mathscr{B}_k(z)\}_{k\geqslant 1}$ to be complete in $A(\boldsymbol{D})$ and hence for $\mathrm{sp}\{\varphi_n(s)\}_{n\geqslant 0}$ to be complete in $A(\Pi)$. Summing the identity

$$\sqrt{2\mathrm{Re}\{a_k\}}B_k(s)=\varphi_{k-1}(s)-\varphi_k(s);\quad k\geqslant 1$$

over $k=1,\dots,n$ then provides

$$\sum_{k=1}^{n}\sqrt{2\mathrm{Re}\{a_k\}}B_k(s)=1-\varphi_n(s)=B_0(s)-\varphi_n(s).$$

Hence $\mathrm{sp}\{\varphi_n(s)\}_{n\geqslant 0}\subset\mathrm{sp}\{B_n(s)\}_{n\geqslant 0}$. This completes the proof. □

Next, the question of the $H_p(\Pi)$ sufficiency of (4) is considered.

**Lemma 4.** *Suppose that* (4) *is satisfied. Then* $\mathrm{sp}\{B_n\}_{n\geqslant 1}$ *is complete in* $H_p(\Pi)$ *for all* $1<p<\infty$.

**Proof.** Let $m$ denote the multiplicity of $a_1$. Suppose first that $m$ is finite. Then reorder the basis poles so that $a_1=a_2=\cdots=a_m$. We redefine the basis functions in (2) by

$$\tilde{B}_n(s)\triangleq\begin{cases}B_n(s), & n<m,\\ \left(\dfrac{s+a_1}{s-\overline{a_1}}\right)B_{n+1}(s), & n\geqslant m.\end{cases}\tag{7}$$

This modification of basis functions removes $a_m$ from the pole parameter set $\{a_k\}_{k\geqslant 1}$.

Let $Q_n$ denote the set containing all partial fraction expansion terms of $\tilde{B}_n$. For example, when $m>1$

$$Q_m\triangleq\left\{\frac{1}{s+a_1},\dots,\frac{1}{(s+a_1)^{m-1}},\frac{1}{s+a_{m+1}}\right\},$$

and so on. Let

$$Q\triangleq\bigcup_{n=1}^{\infty}Q_n.$$

Since

$$\sum_{k\neq m}\frac{\mathrm{Re}\{a_k\}}{1+|a_k|^2}=\infty,$$

$\mathrm{sp}\{\tilde{B}_n\}_{n\geqslant 0}$ is a complete set in $A(\Pi)$ by Lemma 3, and so is $\mathrm{sp}\{B_0\cup Q\}$ since

$$\mathrm{sp}\{\tilde{B}_n\}_{n\geqslant 0}\subset\mathrm{sp}\{B_0\cup Q\}.$$

Let $f\in H_p(\Pi)$ and $\varepsilon>0$. Approximate $f$ by a function $g\in A(\Pi)$ that has the properties

$$\lim_{|s|\to\infty}|s||g(s)|=0,\quad s\in\Pi,$$

and

$$\|f-g\|_p<\varepsilon.$$

This is possible since such functions form a dense subset of $H_p(\Pi)$ (see for example, Garnett [15, Corollary 3.3 in Chapter II]). Let $h(s)=(s+a_1)g(s)$. Then $h\in A(\Pi)$. Since $\mathrm{sp}\{B_0\cup Q\}$ is a complete set in $A(\Pi)$, there exists a $u\in\mathrm{sp}\{B_0\cup Q\}$ such that

$$\|h-u\|_\infty<\varepsilon$$

or

$$\left|g(s)-\frac{u(s)}{s+a_1}\right|<\frac{\varepsilon}{|s+a_1|},\quad s\in\Pi.$$

Hence

$$\left\|f-\frac{u}{s+a_1}\right\|_p<\varepsilon+\left\|\frac{1}{s+a_1}\right\|_p\varepsilon.$$

We have shown that the set

$$P\triangleq\left\{\frac{v(s)}{s+a_1}:\ v(s)\in\mathrm{sp}\{B_0\cup Q\}\right\}\tag{8}$$

is a dense subset of $H_p(\Pi)$ for all $1<p<\infty$. It remains to show that $P\subset\mathrm{sp}\{B_n\}_{n\geqslant 1}$. This will imply that $\mathrm{sp}\{B_n\}_{n\geqslant 1}$ is a complete set in $H_p(\Pi)$ for all $1<p<\infty$. To this end, let $v\in\mathrm{sp}\{B_0\cup Q\}$. Since $v\in\mathrm{sp}\{B_0\cup Q\}$, it can be written uniquely as a linear combination of the elements in $Q$ and $B_0$ as follows:

$$v(s) = c_0 + \sum_{k=1}^{\infty} \frac{c_k}{(s+a_k)^{N(k)}},$$

where $N(k)$ denotes the multiplicity of $a_k$ in the set $\{a_1,a_2,\dots,a_k\}$ and only a finite number of the coefficients $c_k$ are nonzero. Notice that $c_m = 0$. Then

$$\frac{v(s)}{s+a_1} = \frac{c_0}{s+a_1} + \cdots + \frac{c_{m-1}}{(s+a_1)^m} + \sum_{k=m+1}^{\infty} \frac{c_k}{(s+a_1)(s+a_k)^{N(k)}}.$$

Since $a_k \neq a_1$ for $k > m$, the terms $c_k/(s+a_1)(s+a_k)^{N(k)}$ admit further expansions

$$\frac{c_k}{(s+a_1)(s+a_k)^{N(k)}} = \frac{d_1}{s+a_1} + \frac{d_2}{s+a_k} + \cdots + \frac{d_{N(k)}}{(s+a_k)^{N(k)}}.$$

Hence

$$\frac{v(s)}{s+a_1} \in \operatorname{sp}\left\{\frac{1}{s+a_1},\dots,\frac{1}{(s+a_1)^m},\frac{1}{s+a_{m+1}},\dots\right\} = \operatorname{sp} F,$$

where

$$F \triangleq Q \cup \left\{\frac{1}{(s+a_1)^m}\right\}. \tag{9}$$

Thus $P \subset \operatorname{sp} F$. To complete the proof for $m < \infty$, we need to show that $\operatorname{sp} F \subset \operatorname{sp}\{B_n\}_{n\geqslant 1}$. Let $n$ be an arbitrary positive integer. Write the partial fraction expansions of the basis elements $B_1,B_2,\dots,B_n$ in the following linear equation form:

$$\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} = \begin{bmatrix} \alpha_{11} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix} \times \begin{bmatrix} \frac{1}{s+a_1} \\ \frac{1}{(s+a_2)^{N(2)}} \\ \vdots \\ \frac{1}{(s+a_n)^{N(n)}} \end{bmatrix}.$$

The degree of $B_k$ is $k$, which implies that $\alpha_{kk} \neq 0$ for all $k \leqslant n$ and thus the lower triangular matrix above is invertible. Hence, for $i = 1,\dots,n$

$$\frac{1}{(s+a_i)^{N(i)}} \in \operatorname{sp}\{B_k, k = 1,\dots,n\} \subset \operatorname{sp}\{B_k\}_{k\geqslant 1}.$$

Consequently $\operatorname{sp} F \subset \operatorname{sp}\{B_n\}_{n\geqslant 1}$.

Suppose now that $m = \infty$. Then for each $n$, define $Q_n$ as the set containing all partial fraction expansion terms of $B_n$. In this case, the proof above still applies with great simplifications since $P$ defined in (8) equals to $\operatorname{sp} Q$, and $F$ defined in (9) equals to $Q$ for all $m$.

**Proof of Theorem 2.** It only remains to establish the necessity of (4) for completeness in $H_p(\Pi)$ spaces. Suppose then that (4) fails to hold. Then in this case the finite Blaschke products $\lambda_n(s)$ defined by

$$\lambda_n(s) \triangleq \beta_n \varphi_n(s), \quad \beta_n \triangleq \prod_{k=1}^{n} \frac{|1-\overline{a_k}^2|}{1-\overline{a_k}^2}, \quad \beta_0(s) \triangleq 1$$

converge (as $n \to \infty$) uniformly on $\Pi$ to a nonzero function $\lambda(s) \in H_\infty(\Pi)$ which has zeros precisely at the points $\overline{a_n}$; see, for example, Garnett [15, Chapter II]. Therefore, the linear functional $F$ defined on $H_p(\Pi)$ for all $1 \leqslant p < \infty$ and $A(\Pi)$ by

$$F(h) \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} h(\mathrm{j}\omega) \frac{\overline{\lambda(\mathrm{j}\omega)}}{(-\mathrm{j}\omega+1)^2}\,\mathrm{d}\omega$$

is nontrivial and bounded. However, by Cauchy's Integral Theorem, it vanishes for any $B_n$ of the form (2) since

$$\overline{F(B_n)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-\sqrt{2\operatorname{Re}\{a_n\}}}{(\mathrm{j}\omega+1)^2(\mathrm{j}\omega-\overline{a_n})} \times \prod_{k=1}^{n-1} \frac{\mathrm{j}\omega+a_k}{\mathrm{j}\omega-\overline{a_k}} \prod_{i=1}^{\infty} \frac{|1-\overline{a_i}^2|}{1-\overline{a_i}^2} \frac{\mathrm{j}\omega-\overline{a_i}}{\mathrm{j}\omega+a_i}\,\mathrm{d}\omega$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Psi(j\omega)}{(j\omega+1)^2(j\omega+a_n)}\,d\omega$$

$$= -\frac{1}{2\pi j}\int_{j\mathbf{R}}\frac{\Psi(s)}{(s+1)^2(s+a_n)}\,ds$$

$$= -\lim_{r\to\infty}\left[\frac{1}{2\pi j}\int_{-jr}^{jr}\frac{\Psi(s)}{(s+1)^2(s+a_n)}\,ds + O(r^{-2})\right]$$

$$= \lim_{r\to\infty}\left[\underbrace{\frac{1}{2\pi j}\oint_{\Gamma_r}\frac{\Psi(s)}{(s+1)^2(s+a_n)}\,ds + O(r^{-2})}_{=0}\right]$$

$$= 0,$$

where

$$\Psi(s) \triangleq \sqrt{2\,\mathrm{Re}\{a_n\}}\prod_{k=1}^{n}\frac{|1-\overline{a_k}^2|}{1-\overline{a_k}^2}\prod_{i=n+1}^{\infty}\frac{|1-\overline{a_i}^2|}{1-\overline{a_i}^2}\,\frac{s-\overline{a_i}}{s+a_i}$$

is analytic on $\Pi$ and the closed path $\Gamma_r$ is as in (5). Similarly, $F(B_0)=0$. (Note that the remainder term above vanishes as $O(r^{-1})$ in this case). Hence by an application of the Hahn–Banach Theorem (see, for example [2, Section 30]), $\mathrm{sp}\{B_n\}_{n\geq 0}$ defined by (2) is not dense in any of the spaces $A(\Pi)$ and $H_p(\Pi)$, $1 \leqslant p < \infty$. This concludes the proof. □

It should be pointed out that the sufficiency part of Lemma 3 together with Lemma 4 gives the half of the sufficiency condition in Achieser [2, Section A.4] for the completeness of the model sets spanned by $B_0$ and the Cauchy kernels

$$B_a(s) \triangleq \frac{1}{s+a}, \quad a \in \{a_1, a_2, \ldots\}$$

in the Lebesgue spaces $L_p$, $1 < p < \infty$, and in the space of complex functions continuous on the imaginary axis including $\infty$. The linear span of the Cauchy kernels does not contain systems which have repeated poles whereas with the bases (2), a greater flexibility is utilised on the choice of basis poles.

### 3.1. Ensuring real-valued impulse response

Up until this point, the basis (2) has been considered with complete generality of pole location save for the condition (4). However, in any application involving the modelling of a physical process, it is necessary to ensure that the underlying modelled impulse response is real valued. If complex valued choices for $\{a_k\}$ are made in order to accommodate resonant characteristics, then this realness of impulse response is lost unless some restriction is placed on how linear combination of the basis functions is taken.

The purpose of this section is to illustrate how to use the basis formulation (2) in such a way that imposing realness of the weightings in the linear combination ensures realness of the underlying impulse response. This is achieved by requiring that if a *set* of numbers $\{a_1, a_2, \ldots, a_n\}$ used to define bases via (2) contains a complex valued element (say $a_k$), then it always also includes its conjugate $\overline{a_k}$.

To be more explicit on this point, suppose that the numbers $a_1, \ldots, a_{n-1}$ are real so that the basis functions $B_1, \ldots, B_{n-1}$ have real-valued impulse responses. We now wish to include a complex pole at $-a_n$. Then two new basis functions $B_n'$, $B_n''$ with real impulse responses should be formed as a linear combination of $B_n$ and $B_{n+1}$ generated by (2). These new functions then replace $B_n$ and $B_{n+1}$ in any modelling applications that require a real-valued impulse response.

The linear combination we are suggesting can be expressed as

$$\begin{bmatrix} B_n' \\ B_n'' \end{bmatrix} = \begin{bmatrix} c_0 & c_1 \\ c_0' & c_1' \end{bmatrix}\begin{bmatrix} B_n \\ B_{n+1} \end{bmatrix}, \quad c_0,\ c_0',\ c_1,\ c_1' \in \mathbf{C}. \tag{10}$$

Therefore, considering only $B_n'$ for the moment, if we choose complex poles in conjugate pairs as $-a_{n+1} = -\overline{a_n}$ then

$$B_n'(s) = \frac{\sqrt{2\,\mathrm{Re}\{a_n\}}(\beta s + \mu)}{s^2 + (a_n + \overline{a_n})s + |a_n|^2}\varphi_{n-1}(s), \tag{11}$$

where $\varphi_{n-1}(s)$ has real-valued impulse response and the real coefficients $\beta$, $\mu$ are related to the choice of $c_0$, $c_1$ by

$$c_0 = \frac{\overline{a_n}\beta + \mu}{2\overline{a_n}}, \qquad c_1 = \frac{\overline{a_n}\beta - \mu}{2\overline{a_n}}.$$

Therefore, to ensure a unit norm for $B_n'$ we must choose $\beta$ and $\mu$ according to the constraint that $|c_0|^2+|c_1|^2=1$ which becomes

$$x^{\mathrm{T}}Mx=2|a_n|^2, \tag{12}$$

where

$$x \triangleq (\beta,\mu)^{\mathrm{T}}, \qquad M \triangleq \begin{bmatrix} |a_n|^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now, suppose we make two pairs of choices: $x=(\beta,\mu)^{\mathrm{T}}$ giving a basis function $B_n'$ and $y=(\beta',\mu')^{\mathrm{T}}$ giving another basis function $B_n''$. These two choices correspond to two pairs of complex numbers $\{c_0,c_1\}$ and $\{c_0',c_1'\}$. The requirement $c_0\overline{c_0'}+c_1\overline{c_1'}=0$ ensuring orthogonality of $B_n'$ and $B_n''$ can be expressed as needing

$$x^{\mathrm{T}}My=0 \tag{13}$$

to hold, and in fact many solutions $x$ and $y$ to (12) and (13) will exist. To formulate them, suppose we begin by choosing any $x$ satisfying (12). All solutions to (12) are given by $\beta=\sqrt{2}\cos\theta$ and $\mu=|a_n|\sqrt{2}\sin\theta$, $0\leqslant\theta<2\pi$. Then for a fixed $\theta$, a unique $y$ that satisfies (12) and (13) is found by rotating $x$ ninety degrees in the normalised eigenspace of $M$:

$$y=M^{-1/2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}M^{1/2}x$$

or, to be more explicit

$$y=\sqrt{2}(-\sin\theta,\ |a_n|\cos\theta)^{\mathrm{T}}. \tag{14}$$

To summarise this discussion, if we want to include complex modes in a model structure, then we obtain two basis vectors $B_n'$ and $B_n''$ from two linear combinations of $B_n$ and $B_{n+1}$ that come from the unifying construction (2). Let $0\leqslant\theta<2\pi$. Then the basis functions $B_n'$ and $B_n''$ are found as

$$B_n'(s)=\frac{\sqrt{4\,\mathrm{Re}\{a_n\}}\,(s\cos\theta+|a_n|\sin\theta)}{s^2+(a_n+\overline{a_n})s+|a_n|^2}\varphi_{n-1}(s),$$

$$B_n''(s)=\frac{\sqrt{4\,\mathrm{Re}\{a_n\}}\,(-s\sin\theta+|a_n|\cos\theta)}{s^2+(a_n+\overline{a_n})s+|a_n|^2}\varphi_{n-1}(s).$$

These real-valued impulse response basis vectors $B_n'$ and $B_n''$ are then used for modelling instead of $B_n$ and $B_{n+1}$. If we require further basis functions with complex modes then we repeat the process in (10) by forming $B_{n+1}'$ and $B_{n+1}''$ from linear combinations of $B_{n+2}$ and $B_{n+3}$ and so on, and in this way arbitrary complex pole configurations may be accommodated. For example, when $a_n=\overline{a_{n+1}}=\cdots=a_{n+2m}=\overline{a_{n+2m+1}}$, with chosen $\theta=0$ the above basis construction process yields for $k=0,\dots,m$

$$B_{n+k}'(s)=\frac{\sqrt{2b}\,s}{s^2+bs+c}\left(\frac{s^2-bs+c}{s^2+bs+c}\right)^k\varphi_{n-1}(s),$$

$$B_{n+k}''(s)=\frac{\sqrt{2bc}}{s^2+bs+c}\left(\frac{s^2-bs+c}{s^2+bs+c}\right)^k\varphi_{n-1}(s)$$

where $b=2\,\mathrm{Re}\{a_n\}$ and $c=|a_n|^2$. With $n=1$ plugged in, this is the defining formula for the two-parameter Kautz functions in [43].

Having now illustrated how the constraint of realness of impulse response may be easily accommodated via constraining realness of linear combination weights, it remains to establish that this latter restriction does not destroy completeness properties. For this purpose, note that the basis functions of the form described above and the basis functions generated by (2) are related by a unitary block diagonal matrix of the form

$$U\triangleq\mathrm{diag}\left(1,\dots,1,\begin{bmatrix} c_0 & c_1 \\ c_0' & c_1' \end{bmatrix},\dots\right).$$

Let $U_n\in\boldsymbol{C}^{n\times n}$ denote truncations of $U$. For a given $G\in H_p(\Pi)$ and $\varepsilon>0$, via Theorem 2 there exists a $G_n\in H_p(\Pi)$ given by

$$G_n(s)=\sum_{k=1}^{n}\alpha_k B_k(s), \quad \alpha_k\in\boldsymbol{C}$$

such that $\|G-G_n\|_p\leqslant\varepsilon/2$. Then $G_n$ can be written as

$$\begin{aligned} G_n(s) &= \alpha^{\mathrm{T}}U_n^{-1}\Psi_n(s) \\ &= \sum_{k=1}^{n}\theta_k B_k'(s), \quad \theta_k\triangleq a_k+\mathrm{j}b_k; \quad a_k,b_k\in\boldsymbol{R}, \end{aligned}$$

where $\theta^{\mathrm{T}}\triangleq\alpha^{\mathrm{T}}U_n^{-1}$ and here $\Psi_n^{\mathrm{T}}\triangleq(B_1',\dots,B_n')$ refers to a set of basis functions that have real-valued impulse responses.

Now, assume that $G(s)$ has a real-valued impulse response so that $\overline{G(\mathrm{j}\omega)} = G(-\mathrm{j}\omega)$. Then, using the fact that $\|f\| = \|\bar{f}\| = \|f(-\mathrm{j}\omega)\|$ for any $f \in H_p(\Pi)$ provides

$$\|G - G_n\|_p \leqslant \varepsilon$$

$$\Rightarrow \left\| \bar{G} - \sum_{k=1}^{n} \overline{\theta_k}\, \overline{B'_k} \right\|_p \leqslant \varepsilon$$

$$\Rightarrow \left\| \overline{G(-\mathrm{j}\omega)} - \sum_{k=1}^{n} \overline{\theta_k}\, \overline{B'_k(-\mathrm{j}\omega)} \right\|_p \leqslant \varepsilon$$

$$\Rightarrow \left\| G - \sum_{k=1}^{n} \overline{\theta_k}\, B'_k \right\|_p \leqslant \varepsilon$$

and hence via the triangle inequality

$$\left\| G - \sum_{k=1}^{n} a_k B'_k \right\|_p = \left\| G - \sum_{k=1}^{n} \frac{\theta_k + \overline{\theta_k}}{2} B'_k \right\|_p$$

$$\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

so that indeed, arbitrarily accurate modelling of real impulse $G(s)$ is possible by taking real linear combinations of real impulse versions of the basis $\{B_k\}$.

## 4. Approximation of finite-dimensional systems

While the completeness result of Theorem 2 provides a theoretical pedigree for considering bases (2) for system approximation purposes, it leaves open the question of the quality of approximation for a finite number of bases. Addressing this will be the concern of this section, where a useful tool is to use the so-called 'reproducing kernel' $K_n(s,\mu)$ associated with the linear space $\mathrm{sp}\{B_k(s)\}_{k=1}^{n}$.

**Lemma 5.** *Consider the basis functions $\{B_k\}_{k=1}^{n}$ defined by (2). Then*

$$K_n(s,\mu) \triangleq \sum_{k=1}^{n} \overline{B_k(\mu)} B_k(s) = \frac{1 - \overline{\varphi_n(\mu)}\varphi_n(s)}{s + \bar{\mu}}.$$

**Proof.** The proof will be by induction. First, when $n = 1$

$$\overline{B_1(\mu)} B_1(s) = \frac{a_1 + \overline{a_1}}{(\bar{\mu} + \overline{a_1})(s + a_1)}$$

while

$$\frac{1 - \overline{\varphi_1(\mu)}\varphi_1(s)}{s + \bar{\mu}} = \left[ 1 - \frac{(\bar{\mu} - a_1)(s - \overline{a_1})}{(\bar{\mu} + \overline{a_1})(s + a_1)} \right] \frac{1}{s + \bar{\mu}}$$

$$= \frac{a_1 + \overline{a_1}}{(\bar{\mu} + \overline{a_1})(s + a_1)}$$

so that the result holds for $n = 1$. Now suppose it holds for $n > 1$. Then

$$K_n(s,\mu) = K_{n-1}(s,\mu) + \overline{B_n(\mu)} B_n(s)$$

$$= \frac{1 - \overline{\varphi_{n-1}(\mu)}\varphi_{n-1}(s)}{s + \bar{\mu}} + \frac{a_n + \overline{a_n}}{(\bar{\mu} + \overline{a_n})(s + a_n)} \overline{\varphi_{n-1}(\mu)}\varphi_{n-1}(s)$$

$$= \frac{1}{s + \bar{\mu}} - \left[ \frac{(\bar{\mu} + \overline{a_n})(s + a_n) - (a_n + \overline{a_n})(s + \bar{\mu})}{(s + \bar{\mu})(\bar{\mu} + \overline{a_n})(s + a_n)} \right] \times \overline{\varphi_{n-1}(\mu)}\varphi_{n-1}(s)$$

$$= \frac{1 - \overline{\varphi_n(\mu)}\varphi_n(s)}{s + \bar{\mu}}.$$

Therefore, by induction the result holds for all $n$. □

The utility of this result becomes apparent in the derivation of the following expression for the finite-order approximation error.

**Lemma 6.** *Suppose $f(s)$ is analytic on $\Pi$ and has a partial fraction expansion*

$$f(s) = \sum_{k=1}^{m} \frac{c_k}{s + \gamma_k}.$$

*Define $f_n(s)$ as an approximation to $f(s)$ obtained by projection onto* $\mathrm{sp}\{B_k(s)\}_{k=1}^{n}$:

$$f_n(s) \triangleq \sum_{k=1}^{n} \langle f, B_k \rangle B_k(s).$$

*Then*

$$|f(\mathrm{j}\omega) - f_n(\mathrm{j}\omega)| \leqslant \sum_{k=1}^{m} \left|\frac{c_k}{\mathrm{j}\omega + \gamma_k}\right| \prod_{l=1}^{n} \left|\frac{\gamma_k - a_l}{\gamma_k + \overline{a_l}}\right|. \qquad (15)$$

**Proof.** By the definition of $f_n(s)$ and for any $\mu \in \Pi$

$$f_n(\mu) = \sum_{k=1}^{n} \left(\frac{1}{2\pi \mathrm{j}}\int_{\mathrm{j}\boldsymbol{R}} f(s)\overline{B_k(s)}\,\mathrm{d}s\right) B_k(\mu)$$

$$= \frac{1}{2\pi \mathrm{j}} \oint_{\Gamma} f(s)\overline{K_n(s,\mu)}\,\mathrm{d}s,$$

where $\Gamma$ consists of the imaginary axis and an infinite radius semi-circle in the open right half-plane; it is traversed clockwise. Using this definition and Cauchy's integral formula gives, for $\mu \in \Pi$ arbitrary

$$f(\mu) = \frac{1}{2\pi \mathrm{j}} \oint_{\Gamma} \frac{f(s)}{\mu - s}\,\mathrm{d}s.$$

Therefore, by Lemma 5 and using the fact that $\bar{s} = -s$ for $s \in \mathrm{j}\boldsymbol{R}$

$$|f(\mu) - f_n(\mu)|$$

$$= \left|\frac{1}{2\pi \mathrm{j}} \oint_{\Gamma} \frac{f(s)}{\mu - s} \varphi_n(\mu)\overline{\varphi_n(s)}\,\mathrm{d}s\right|$$

$$= \left|\frac{\varphi_n(\mu)}{2\pi \mathrm{j}} \sum_{k=1}^{m} c_k \oint_{\Gamma} \frac{1}{(s+\gamma_k)(\mu - s)} \prod_{l=1}^{n} \frac{s + a_l}{s - \overline{a_l}}\,\mathrm{d}s\right|$$

$$= |\varphi_n(\mu)| \left|\sum_{k=1}^{m} c_k \frac{1}{(\mu + \gamma_k)} \prod_{l=1}^{n} \frac{\gamma_k - a_l}{\gamma_k + \overline{a_l}}\right|,$$

where in moving to the last line Cauchy's residue theorem was used to evaluate the integral after performing the change of variable $s \mapsto -s$. Taking the limit as $\mathrm{Re}\{\mu\} \to 0$ then gives the result.

The result exposes the dependence of the approximation error on the choice of poles $\{-a_n\}$ in the base $B_n(s)$. Namely, the closer the poles $\{-a_n\}$ are chosen to the poles $\{-\gamma_k\}$ of the function $f(s)$ being approxiated then the more accurate the approximation of $f(s)$ will be, and in such a way as to decrease exponentially with increasing $n$.

Certainly the error bound (15) gives strong motivation for the consideration of the general basis (2), since (in contrast to the Laguerre and Kautz cases where all the poles are fixed at the same value) the increased flexibility of pole location $\{-a_n\}$ will increase the possibility of making $|\gamma_k - a_l|$ small (for some $l$) for every $k$, and hence making the total product $\prod_{l=1}^{n} |\gamma_k - a_l||\gamma_k + \overline{a_l}|^{-1}$ as small as possible.

## 5. Application example

We conclude the paper by presenting an application example that illustrates the utility of the basis (2) for modelling purposes. The example involves measurements of the frequency response of a 58 cm long, 5 mm wide cantilevered piezo-electric laminate beam (for further details, see [29]). These measurements are shown as dots in Fig. 1. For the purposes of control (stiffness compensation using the piezo-electric actuators) a transfer function model that explains this frequency response is required. There are many ways in which this may be achieved [6,27,35,38,39], but for the purpose of illustrating the efficacy of the basis (2), the simple least-squares method introduced by Levy [22] to fit a model

$$G_n(s) = \frac{N(s)}{D(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + d_{n-1}s^{n-1} + \cdots + d_1 s + d_0}$$

to the frequency response measurements $\{G_k\}_{k=1}^{N}$ at the frequencies $\{\omega_k\}_{k=1}^{N}$ by means of minimising the cost

$$V_N = \sum_{k=1}^{N} |D(\mathrm{j}\,\omega_k)G_k - N(\mathrm{j}\,\omega_k)|^2$$

will be studied. Our purpose is not to present best possible results on this data set, but to illustrate by an example that the model parameterised by the orthonormal basis (2) should be preferred to polynomial models when one is concerned with numerical conditioning.

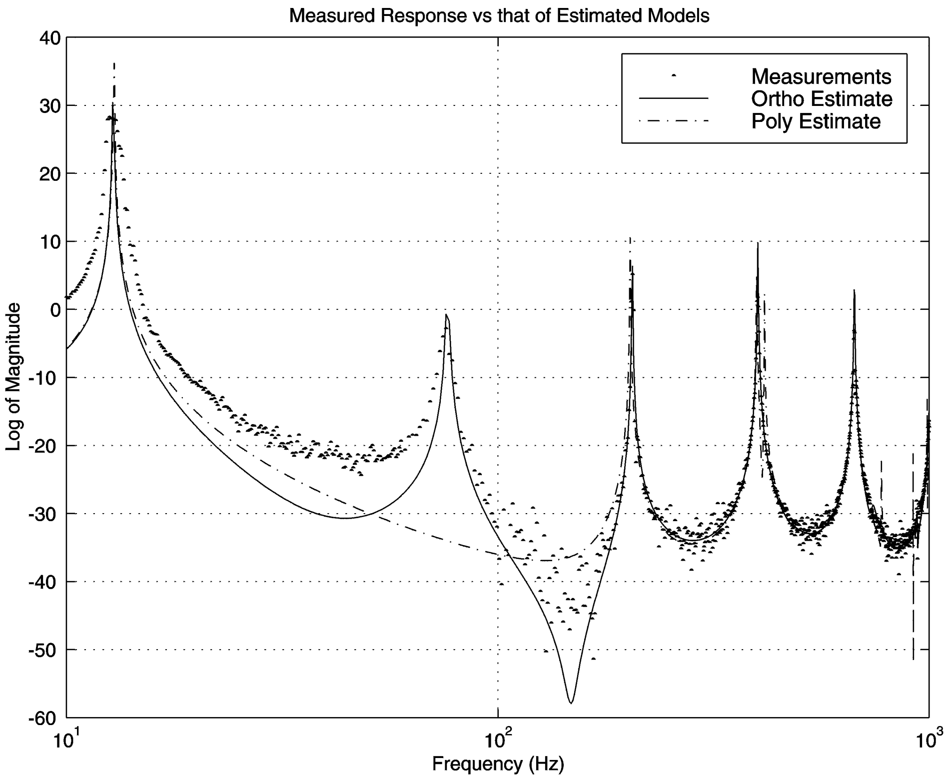


Fig. 1. Estimation using the polynomial and the orthonormal bases. The dots are the measurements, the solid line is the estimate using the basis (2) to parameterise the model, the dash–dot line is the estimate using the Tchebychev polynomials to parameterise the model. The Tchebychev and the polynomial estimates are identical.

As is well known [35], finding this estimate involves solving the so-called 'normal equations'

$$\begin{bmatrix}(\mathrm{j}\omega_1)^n G_1\\ \vdots \\ (\mathrm{j}\omega_N)^n G_N\end{bmatrix} = \underbrace{\begin{bmatrix} -(\mathrm{j}\omega_1)^{n-1}G_1, & \dots, & -G_1, (\mathrm{j}\omega_1)^n, & \dots, 1\\ & \vdots & \vdots & \\ -(\mathrm{j}\omega_N)^{n-1}G_N, & \dots, & -G_N, (\mathrm{j}\omega_N)^n, & \dots, 1\end{bmatrix}}_{\Phi} \times \begin{bmatrix} d_{n-1}\\ \vdots \\ d_0 \\ b_n \\ \vdots \\ b_0\end{bmatrix},$$

for which the numerical stability of the solution is highly dependent [16], on the conditioning of the matrix $\Phi^{\mathrm{T}}\Phi$. However this can be altered via reparameterisations of the model $G(s)$. For example, in [5] the parameterisation

$$N(s) = b_0 + \sum_{k=1}^{n} b_k p_k(s),$$
$$D(s) = s^n + \sum_{k=0}^{n-1} d_k p_k(s), \tag{16}$$

where each $p_k(s)$ is an order $k$ 'modified Tchebychev' polynomial ($p_0 = 1$) is suggested as a means of improving numerical conditioning.

In Fig. 1, the dash–dot line shows the results of using the above Tchebychev parameterisation to fit an $n = 18$th-order model to the observed frequency response. Note that the second resonance peak is completely missed, and that the resonance frequencies from the 3rd resonant mode onwards are significantly shifted.

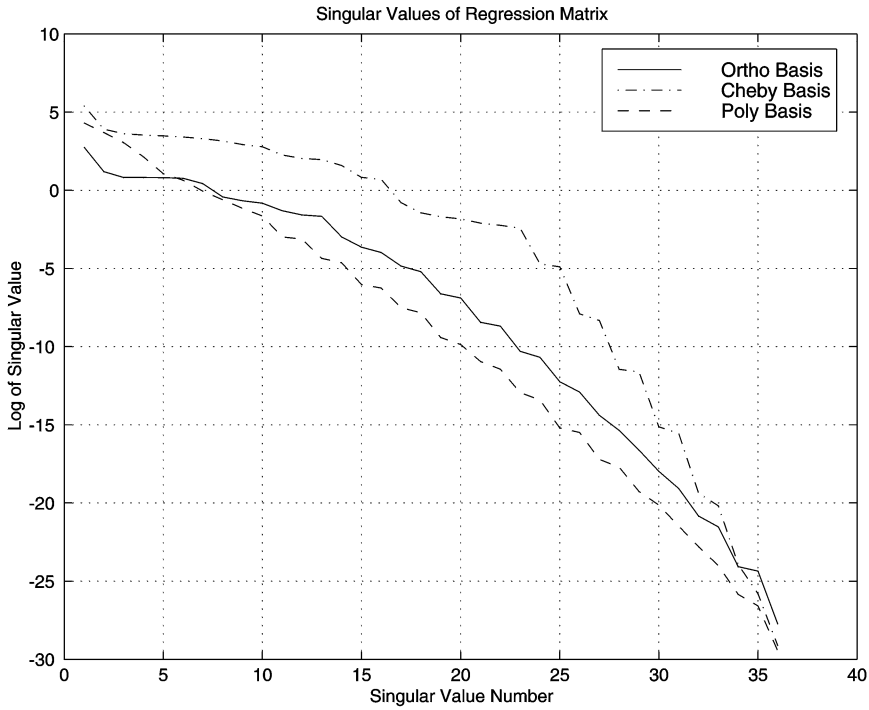


Fig. 2. The singular values of $\Phi$ using the polynomial (natural and the Tchebychev) and the rational orthonormal basis (2).

However, if the model is parameterised using the orthonormal basis (2) as

$$N(s) = b_0 + \sum_{k=1}^{n} b_k B_k(s),$$
$$D(s) = s^n + \sum_{k=1}^{n} d_k B_k(s), \tag{17}$$

with the pole choice $-a_k = -a = -2\omega_N$, then the ensuing 18th-order least-squares estimate is the solid line shown in Fig. 1, which now captures the second resonance peak, and correctly matches the resonance frequencies from the 3rd resonant mode onwards.

Since the model structures (16) and (17) both span the same manifold of rational models, the only explanation for the difference in results is that of differences in numerical conditioning. Fig. 2 shows the singular values of $\Phi$ for the three model parameterisation choices. (There are 36 singular values of $\Phi$ since the chosen model order is 18). Considering the log-scale employed, the parameterisation using the basis (17) enjoys a two order of magnitude better conditioning (the ratio of the largest to the smallest singular value) than either a Tchebychev polynomial, or conventional polynomial parameterisation.

As a consequence, for such applications of modelling resonant structures over large bandwidths, we suggest that the basis (2) should be employed in the interests of the resultant frequency response having no artifacts due to poor numerical conditioning.

## 6. Conclusion

This paper has provided a preliminary study of the approximation properties of a particular class of rational orthonormal bases that are suitable for continuous-time system modelling. The main result was to establish that the bases were capable of arbitrarily good approximation with respect to a wide variety of norms employed in the system-theoretic analysis of stable systems. The utility of the generalising nature of the particular bases considered here was also exposed by establishing that significantly improved finite-order approximation accuracy was possible by exploiting the flexibility in allowed pole position. This is in contrast to the more well known Laguerre and two-parameter Kautz bases, which are obtained as special cases of the bases considered here by choosing all the poles fixed at one location.

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## References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Dover, New York, 1965.

[2] N. Achieser, Theory of Approximation, Dover, New York, 1992.

[3] H. Akçay, B. Ninness, Rational basis functions for robust identification from frequency and time domain measurements, Automatica 34 (1998) 1101–1117.

[4] H. Akçay, B. Ninness, Orthonormal basis functions for continuous-time systems and $L_p$ convergence, Math. Control, Signals Systems (1999), to appear.

[5] D.S. Bayard, Multivariable frequency domain identification via 2-norm minimization, Proceedings of the American Control Conference, Chicago, IL, 1992, pp. 1253–1257.

[6] D.S. Bayard, High-order multivariable transfer function curve fitting: Algorithms, sparse matrix methods and experimental results, Automatica 30 (1994) 1439–1444.

[7] P. Bodin, T. Oliveira e Silva, B. Wahlberg, On the construction of orthonormal basis functions for system identification, Proceedings of 13th IFAC World Congress, San Francisco, 1996, pp. 291–296.

[8] J. Bokor, L. Gianone, Z. Szabo, Construction of generalised orthonormal bases in $\mathscr{H}_2$, Tech. Report, Computer and Automation Institute, Hungarian Academy of Sciences, 1995.

[9] W. Cluett, L. Wang, Frequency smoothing using Laguerre model, in: Proc. IEE-D 139 (1992) 88–96.

[10] G. Davidson, D. Falconer, Reduced complexity echo cancellation using orthonormal functions, IEEE Trans. Circuits Systems 38 (1991) 20–28.

[11] A.C. den Brinker, Laguerre-domain adaptive filters, IEEE Trans. Signal Process. 42 (1994) 953–956.

[12] N.F. Dudley Ward, J.R. Partington, Robust identification in the disc algebra using rational wavelets and orthonormal basis functions, Internat. J. Control 64 (1996) 409–423.

[13] N.F. Dudley Ward, J.R. Partington, A construction of rational wavelets and frames in Hardy–Sobolev spaces with applications to system modelling, SIAM J. Control Optim. 36 (1998) 654–679.

[14] B. Epstein, Orthogonal Families of Analytic Functions, Macmillan, New York, 1965.

[15] J.B. Garnett, Bounded Analytic Functions, Academic Press, New York, 1981.

[16] J. Golub, C. Van Loan, Matrix Computations, Johns Hopkins University Press, Baltimore, MD, 1989.

[17] J. Head, Approximation to transient by means of Laguerre series, Proc. Cambridge Philos. Soc. 52 (1956) 640–651.

[18] P. Heuberger, P.M.J. Van den Hof, O. Bosgra, A generalized orthonormal basis for linear dynamical systems, IEEE Trans. Automat. Control AC-40 (1995) 451–465.

[19] W.H. Kautz, Network synthesis for specified transient response, Tech. Report 209, Massachusetts Institute of Technology, Research Laboratory of Electronics, 1952.

[20] W.H. Kautz, Transient synthesis in the time domain, IRE Trans. Circuit Theory 1 (1954) 29–39.

[21] Y. Lee, Synthesis of electric networks by means of the Fourier transforms of Laguerre's functions, J. Math. Phys. XI (1933) 83–113.

[22] E.C. Levy, Complex curve fitting, IRE Trans. Automat. Control AC-4 (1959) 37–44.

[23] P. Mäkila, Approximation of stable systems by Laguerre filters, Automatica 26 (1990) 333–345.

[24] P. Mäkilä, Laguerre series approximation of infinite dimensional systems, Automatica 26 (1990) 985–995.

[25] P. Mäkilä, Laguerre methods and $H^\infty$ identification of continuous-time systems, Internat. J. Control 53 (1991) 689–707.

[26] F. Malmquist, Sur la détermination d'une classe de fonctions analytiques par leurs valeurs dans un ensemble donné de points, in: Proceedings of Comptes Rendus du Sixième Congrès des mathématiciens scandinaves, Copenhagen, 1925, pp. 253–259.

[27] T. McKelvey, H. Akçay, L. Ljung, Subspace-based multivariable system identification from frequency response data, IEEE Trans. Automat. Control AC-41 (1996) 960–979.

[28] J. Mendel, A unified approach to the synthesis of orthonormal exponential functions useful in systems analysis, IEEE Trans. Systems Sci. Cybernet. 2 (1966) 54–62.

[29] S.O.R. Moheimani, H.R. Pota, I.R. Petersen, Spatial control for active vibration control of piezoelectric laminates, Proceedings of 37th IEEE Conference on Decision and Control, Tampa, Florida, 1998, pp. 4308–4313.

[30] X. Nie, D. Raghuramireddy, R. Unbehauen, Orthogonal expansion of stable rational transfer functions, Electron. Lett. 27 (1991) 1492–1494.
[31] B. Ninness, F. Gustafsson, A unifying construction of orthonormal bases for system identification, IEEE Trans. Automat. Control 42 (1997) 515–521.
[32] Ü. Nurges, Laguerre models in problems of approximation and identification, in: Adaptive Systems, Plenum, 1987, pp. 346–352. Translated from Avtomatica i Telemekhanika 3 (March 1987) 88–96.
[33] J.R. Partington, Approximation of delay systems by Fourier–Laguerre series, Automatica 27 (1991) 569–572.
[34] H. Perez, S. Tsujii, A system identification algorithm using orthogonal functions, IEEE Trans. Signal Process. 38 (1991) 752–755.
[35] R. Pintelon, P. Guillaume, Y. Rolain, J. Schoukens, H. Van Hamme, Parametric identification of transfer functions in the frequency domain – a survey, IEEE Trans. Automat. Control 39 (1994) 2245–2260.
[36] D. Ross, Orthonormal exponentials, IEEE Trans. Commun. Electron. 71 (1964) 173–176.
[37] W. Rudin, Real and Complex Analysis, third ed., McGraw-Hill, New York, 1987.
[38] C.K. Sanathanan, J. Koerner, Transfer function synthesis as a ratio of two complex polynomials, IEEE Trans. Automat. Control AC-8 (1963) 56–58.
[39] M.D. Sidman, F.E. DeAngelis, G.C. Verghese, Parametric system identification on logarithmic frequency response data, IEEE Trans. Automat. Control AC-36 (1991) 1065–1070.
[40] S. Takenaka, On the orthogonal functions and a new formula of interpolation, Jpn. J. Math. II (1925) 129–145.
[41] B. Wahlberg, System identification using Laguerre models, IEEE Trans. Automat. Control AC-36 (1991) 551–562.
[42] B. Wahlberg, System identification using Kautz models, IEEE Trans. Automat. Control AC-39 (1994) 1276–1282.
[43] B. Wahlberg, P. Mäkilä, On approximation of stable linear dynamical systems using Laguerre and Kautz functions, Automatica 32 (1996) 693–708.
[44] C. Zervos, P. Belanger, G. Dumont, Controller tuning using orthonormal series identification, Automatica 24 (1988) 165–175.