Rational orthonormal basis functions:

an introduction

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Outline

1. Construction of rational orthonormal functions
   - State-space construction
2. Frequency-domain identification in $\mathcal{H}_\infty$
   - Problem formulation
   - $\mathcal{L}_\infty$ approximation by Fourier series
   - Nehari approximation

Notation

*: complex conjugate.

$T = \{z \in \mathbf{C} : |z| = 1\}$

$D^c = \{z \in \mathbf{C} : |z| > 1\} \cup \{\infty\}$.

$\mathcal{H}_2(D^c)$: the Hardy space of complex functions square integrable on T and analytic on D^c.

Inner product on $\mathcal{H}_2(D^c)$:

$$\langle X, Y \rangle = \frac{1}{2\pi i} \oint_T X(z) Y(z)^* (1/z^*) \frac{dz}{z}.$$

- $\|X\|^2 = \langle X, X \rangle$.

$F_1, F_2 \in \mathcal{H}_2(D^c)$ are orthonormal if $\|F_1\| = \|F_2\| = 1$ and $\langle F_1, F_2 \rangle = 0$.

Cross-Gramian (outer product):

$$[[X, Y]] = \frac{1}{2\pi i} \oint_T X(z) Y^*(1/z^*) \frac{dz}{z}.$$

- When X and Y have real-valued impulse responses, their cross-Gramian is a real valued matrix satisfying $[[X, Y]]^T = [[Y, X]]$.

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Objective

Construct state-space models

$$x(t+1) = Ax(t) + Bu(t),$$

where $u(t)$ is the input signal, and $x(t)$ is the state vector for which the transfer function from the input to states

$$V(z) = [F_1(z) \ \ \cdots \ \ F_n(z)]^T = (zI - A)^{-1}B$$

are orthonormal, *i.e.*, $[[V, V]] = I$.

- The poles of $\{F_k(z)\}$ are specified.

State covariance matrix:

Let $\bar{x}(t+1) = \bar{A}\bar{x}(t) + \bar{B}u(t)$ be a stable system with $u(t)$ a zero-mean white-noise process. Denote $\bar{P} = \mathrm{E}\{\bar{x}(t)\bar{x}^T(t)\}$.

- $\bar{P} = \bar{A}\bar{P}\bar{A}^T + \bar{B}\bar{B}^T$.
- $\bar{P}$ equals the controllability Gramian of $(\bar{A}, \bar{B})$.
- $\bar{P} = [[\bar{V}, \bar{V}]]$.

  Proof.

$$\begin{aligned}\bar{P} &= \frac{1}{2\pi i}\oint_{T}(zI-\bar{A})^{-1}\bar{B}\bar{B}^T(z^{-1}I-\bar{A}^T)^{-1}\frac{dz}{z} \\ &= \sum_{k=0}^{\infty}\bar{A}^k\bar{B}\bar{B}^T(\bar{A}^T)^k.\end{aligned}$$

Input balanced state-space realization:

Let $x(t) = T\bar{x}(t)$ where T is non-singular transformation matrix.

$$P = \mathrm{E}\{x(t)x(t)^T\} = T\bar{P}T^T.$$

The orthonormalization problem is to find T such that $P = I$.

Solution: pick any T satisfying $\bar{P}^{-1} = T^T T$.

Summary

- Consider the set spanned by the n linearly independent rational transfer functions $\{\bar{F}_1(z), \cdots, \bar{F}_n(z)\}$.

- Determine a controllable state-space model $\bar{x}(t+1) = \bar{A}\bar{x}(t) + \bar{B}u(t)$ with input-to-state transfer functions $\bar{V}(z) = [\bar{F}_1(z) \quad \cdots \quad \bar{F}_n(z)]^T = (zI - \bar{A})^{-1}\bar{B}$.
  - The eigenvalues of $\bar{A}$ equal the poles of $\{\bar{F}_1, \cdots, \bar{F}_n\}$.
- Determine the controllability Gramian by solving the Lyapunov equation $\bar{P} = \bar{A}\bar{P}\bar{A}^T + \bar{B}\bar{B}^T$.
- Find a square root T of $\bar{P}^{-1}$, *i.e.*, $T\bar{P}T^T = I$.
- Make a transformation of the states $x(t) = T\bar{x}(t)$:

$$x(t+1) = Ax(t) + Bu(t), A = T\bar{A}T^{-1}, B = T\bar{B}.$$

  - $V(z) = (zI - A)^{-}1B = T\bar{V}(z)$: $u \mapsto x$
  - $\{V_k(z)\}$ orthonormal set with $\mathrm{Sp}\{\bar{F}_k\}_{k=1}^n = \mathrm{Sp}\{V_k\}_{k=1}^n$.

Outline

$\mathcal{A}(D)$: the disk algebra of complex functions analytic on the open unit disk D with continuous extensions to T.

$$\|f\|_\infty = \sup\{|f(z)| : z \in D\}.$$

$\mathcal{C}(T)$: the set of complex functions continuous on T.

Given: corrupted frequency response measurements

$$E_N(z_k) = f(z_k) + \eta_k, \qquad k = 1, \cdots, N$$

where $f \in \mathcal{A}(D)$ and $\|\eta\|_\infty \le \varepsilon$.

Find: an algorithm that maps the data to a model $\widehat{f}_N \in \mathcal{A}(D)$ such that

$$\lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \sup_{\|\eta\|_\infty \le \varepsilon} \sup_{f\in\mathcal{A}(D)} \|f - \widehat{f}_N\|_\infty = 0.$$

- An algorithm with this property is called *convergent* and *robustly convergent* if it is not tuned to ε and priors on f.
- There is no linear robustly convergent algorithm!
- It is necessary to implement two-step methods.
  - Step 1: $\mathcal{L}_\infty$ approximation to data.
  - Step 2: Nehari approximation to this approximant.
- Approximation error in Step 1 can be controlled by choosing suitable summability kernels.

Outline

The Fourier coefficients of $f \in \mathcal{C}(T)$:

$$c_k(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{j\theta}) e^{-ik\theta} d\theta$$

- The discrete Fourier transform approximations:

$$c_N(k) = \frac{1}{N} \sum_{l=0}^{N-1} f(e^{i2\pi l/N}) e^{-2\pi ikl/N}, \qquad k = 0, \pm 1, \cdots .$$

The partial sums of the Fourier series expansion:

$$(S_n f)(\theta) = \sum_{k=-n}^{n} c_k(f) e^{ik\theta}, \qquad n = 0, 1, \cdots .$$

- The discrete partial sums approximations ($n < N$):

$$(S_{n,N} f)(\theta) = \sum_{|k| \le n} c_N(f) e^{ik\theta}, \qquad n = 0, 1, \cdots .$$

The Fourier coefficients obtained from the data:

$$\widehat{c}_N(k) = \frac{1}{N}\sum_{l=0}^{N-1} E_N(z_l)e^{2\pi ikl/N} = c_N(k) + \eta_N(k)$$

where

$$\eta_N(k) = \frac{1}{N}\sum_{l=0}^{N-1} \eta_l(z_l)e^{2\pi ikl/N}$$

A model?

$$f_{n,N}(\theta) = \sum_{k=-n}^{n} \widehat{c}_N(f)e^{ik\theta} = (S_{n,N}f)(\theta) + (S_{n,N}\eta)(\theta).$$

- $S_{n,N}f(\theta)$ diverges for some $f \in \mathcal{A}(D)$ and θ.

Discrete φ-summation

Assumption φ is a continuous, even, compactly supported function satisfying $\varphi(0) = 1$. The Fourier transform of φ is absolutely integrable:

$$\widehat{\varphi}(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(t)e^{-i\theta t}dt, \qquad \theta \in \mathbf{R}$$

and $\widehat{\varphi} \in \mathcal{L}_1(\mathbf{R})$.

Let

$$f_{n,N}^{\varphi}(\theta) = \sum_{k=-\infty}^{\infty} \varphi(k/n)\widehat{c}_N(k)e^{ik\theta}.$$

- The range of k is finite since φ is compactly supported. Moreover, $\widehat{c}_N(k)$ is periodic with period N.

Proposition Let φ be as in the assumption. Choose n such that $N/n \to \infty$ as N tends to infinity. Then for some $C_\varphi < \infty$,

$$1. \quad \lim_{n\to\infty} \|f^{\varphi}_{n,N} - f\|_\infty = 0, \qquad \text{for all } f \in \mathcal{A}(D),$$

$$2. \quad \sup_{n} \sup_{\|f\|_\infty \le 1} \|f^{\varphi}_{n,N}\| \le C_\varphi, \qquad \text{for all } f \in \mathcal{C}(T).$$

- The estimates $f^{\varphi}_{n,N}$ have the desired robust convergence property except the fact that they are not in $\mathcal{A}(D)$.

Examples:

- $\varphi_1(x) = 1 - |x|, \ |x| \leq 1$ yields the Cesàro means of f:

$$\sigma_n f = \frac{1}{n+1} \sum_{k=0}^{n} S_{k,N} f.$$

- The de la Vallée-Poussin means of f:

$$V_n f = 2\sigma_{2n+1} f - \sigma_n f$$

are obtained with

$$\varphi_2(x) = \begin{cases} 1, & |x| \leq 1/2; \\ 2(1 - |x|), & 1/2 \leq |x| \leq 1. \end{cases}$$

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The identified model is obtained by solving the Nehari problem:

$$\widehat{f}_N = \arg \min_{h \in \mathcal{H}_\infty(D)} \|f^{\varphi}_{n,N} - h\|_\infty .$$

- The overall identification error is bounded as

$$\|\widehat{f}_N - f\|_\infty \le 2\|f - f^{\varphi}_{n,N}\|_\infty + 2C_\varphi \varepsilon .$$

- The estimates $\widehat{f}_N \in \mathcal{A}(D)$ are robustly convergent.
- The convergence results and the error bounds hold for the general orthonormal bases as well (Szabó:2001).
- The first result in the proposition extends to complete rational orthormal bases (Ninness etal.:1998).