Frequency domain subspace-based identification of discrete-time power spectra

from uniformly spaced measurements

Hüseyin Akçay

Department of Electrical and Electronics Engineering
Anadolu University, Eskişehir, Turkey

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Outline

A subspace algorithm proposed by Van Overche etal. *(1997)) to identify state space models from given uniformly spaced power spectrum measurements.*

- When spectrum samples are on a uniform grid of frequencies, the inverse discrete-Fourier transform (IDFT) of the given power spectrum can be expressed in terms of the state-space parameters of the spectral factor.
- A realization theory based on McKelvey *etal.*:1996 is then devised to obtain the system matrices from this IDFT.
- Two methods ensuring positivity of the identified spectrum are proposed.

Consider the $l \times l$ dimensional square discrete time system:

$$\begin{aligned} x_{k+l} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k \end{aligned} \quad (1)$$

with $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times l}$, $C \in \mathbf{R}^{l\times n}$ and $D \in \mathbf{R}^{l\times l}$ nonsingular. The vector sequences u_k, $y_k \in \mathbf{R}^{l}$ are the input and output sequences, respectively.

- (1)–(2) stable and strictly minimum phase: all eigenvalues of A and $A - BD^{-1}C$ strictly inside the unit circle.
- $\{A, B\}$ and $\{A, C\}$ controllable and observable.
- A non-singular.

The system (1)–(2) is thus a minimal stochastic system.

The transfer function of the system:

$$G(z) = D + C(zI_n - A)^{-1}B.$$

The power spectrum associated with (l)–(2):

$$S(z) = G(z)G^T(z^{-1}). \qquad (2)$$

The system (l)–(2) is the innovation form, unity variance, minimum phase spectral factor associated with $S(z)$.

- (2) satisfies the positive realness condition

$$S(z) > 0, \qquad |z| = 1.$$

  imposing positivity constraints on the spectrum samples as well as on the identified power spectrum.

- Splitting of the power spectrum (Caines:1988)

$$S(z) = H(z) + H^T(z^{-1})$$

where

$$\begin{aligned} H(z) &= \Lambda_0/2 + C(zI_n - A)^{-1}G, \\ \Lambda_0 &= CPC^T + DD^T, \\ G &= APC^T + BD^T, \\ P &= APA^T + BB^T \quad \text{(discrete-time Lyapunov equation).} \end{aligned}$$

- The IDFT v_k of a given complex signal $V_k = V(e^{j2\pi k/2N})$, k= $O, \cdots, 2N-1$ is defined by

$$v_k = \frac{1}{2N}\sum_{r=0}^{2N-1} V_r e^{j2\pi rk/2N}).$$

The problem treated can now be stated as follows:

Given
$N+1$ matrices $S_k \in \mathbf{C}^{l \times l}$ of the power spectrum $S(z)$ evaluated at $N+1$ equidistant points over the unit circle:

$$S_k = S(e^{j2\pi k/2N}), \qquad k = 0, \cdots N,$$

Find

- The system matrices A, G, C, Λ_0 describing the power spectrum.
- The system matrices A, B, C, D describing the spectral factor (l)–(2).

Expand the $N+1$ given points S_k to $2N$ points as follows:

$$S_{N+k} = S_{N-k}^*, \qquad k = 1, \cdots, N-1.$$

From now on, S_k denotes this signal of length $2N$.

Theorem 1 (Inverse discrete Fourier transform)**.** With $M = (I_n - A^{2N})^{-1}$, the IDFT $s_k \in \mathbf{R}^{l \times l}$ of the given power spectrum S_k is given by

$$\begin{aligned} s_0 &= \Lambda_0 + CA^{2N}MG + G^T(A^T)^{2N}M^TC^T, \\ s_k &= CA^{k-1}MG + G^T(A^T)^{2N-k-1}M^TC^T, \;\; 1 \leq k < 2N. \end{aligned}$$

- The proof of this theorem is based on McKelvey *etal.*:1996.

The extended observability matrix $\Gamma_q \in \mathbf{R}^{lq \times n}$ and reversed extended observability matrix $\tilde{\Gamma}_r \in \mathbf{R}^{lr \times n}$ are defined as

$$\Gamma_q = \begin{pmatrix} C \\ \vdots \\ CA^{q-1} \end{pmatrix}, \qquad \tilde{\Gamma}_r = \Pi \Gamma_r$$

where $\Pi \in \mathbf{R}^{lN \times lN}$ is the permutation matrix given by

$$\Pi = \begin{pmatrix} 0 & \cdots & I_l \\ \vdots & \ddots & \vdots \\ I_l & \cdots & 0 \end{pmatrix}$$

The extended controllability matrix $\Delta_r \in \mathbf{R}^{n\times lr}$ and reversed extended controllability matrix $\tilde{\Delta}_q \in \mathbf{R}^{n\times lq}$ are defined as

$$\Delta_r = (G \ \cdots \ A^{r-1}), \qquad \tilde{\Delta}_q = \Delta_q \Pi.$$

- Since (l)–(2) is minimal, the matrices $\Gamma_q, \tilde{\Gamma}_r$ and $\Delta_r, \tilde{\Delta}_q$ are respectively of full column and row rank n.

Let

$$\mathcal{S} = \begin{pmatrix} s_1 & \cdots & s_r \\ \vdots & \ddots & \vdots \\ s_q & \cdots & s_{q+r-1} \end{pmatrix}$$

with $q, r \geq 2n$, $r + q < 2N$.

Theorem 2 The block-Hankel matrix $\mathcal{S}$ can be decomposed as

$$\mathcal{S} = \begin{pmatrix} \Gamma_q & \tilde{\Delta}_q^T \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & M^T \end{pmatrix} \begin{pmatrix} \Delta_r \\ \tilde{\Gamma}_r^T \end{pmatrix}$$

which leads to the following results:

- $\mathrm{rank}(\mathcal{S}) = 2n$.
- The column space of $\mathcal{S}$ can be expressed in terms of the system matrices as
  column space $\mathcal{S}$ = column space $(\Gamma_q \; \tilde{\Delta}_q^T)$.
- The row space of $\mathcal{S}$ can be expressed in terms of the system matrices as
  row space $\mathcal{S}$ = row space $\begin{pmatrix} \Delta_r \\ \tilde{\Gamma}_r^T \end{pmatrix}$

- A is determined by a procedure similar to in Akçay and Türkay:2004 (SVD followed by eigen decomposition).
- G, C, and A are determined in the same step of Van Overschee *etal.*:1997 whereas G, C, and Δ_0 are estimated from S_k by an LS procedure in Akçay and Türkay:2004.
- With $M = (I_n - A^{2N})^{-1}$,

$$\Delta_o = s_0 - CA^{2N-1}MG - G^T(A^T)^{2N-1}M^TC^T.$$

- B and D are determined as in Akçay and Türkay:2004.
- Consistency follows from McKelvey *etal.*:1996.

When S_k are not generated by a finite dimensional linear system or noise corrupted, there is no guarantee that the identified power spectrum, which is determined by A, G, C, and Λ_0 will be positive real.

- When the identified sequence is not positive real, the Riccati equation has no positive definite solution and the spectral factor cannot be computed.
- Two possible solutions to this problem are presented.
  - Both of these solutions start from given matrices A and C. The solutions then state how G and Λ_0 are determined through the solution of an optimization problem which guarantees a positive real identified power spectrum.

Outline

Given A and C, a positive real identified spectrum can be guaranteed by solving the following optimization problem:

Given the known transfer matrix

$$L(z) = \begin{pmatrix} C(zI_n - A)^{-1} & I_l \end{pmatrix},$$

solve

$$\min_{Q,S,R} \sum_{k=0}^{2N-1} \| S_k - L(e^{j2\pi k/2N}) \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} L(e^{-j2\pi k/2N}) \|_F^2$$

constrained to

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \geq 0.$$

- The system matrices G and Λ_0 can then be found by solving the set of equations:

$$P = APA^T + Q, \quad (3)$$
$$G = APC^T + S, \quad (4)$$
$$\Lambda_0 = CPC^T + R. \quad (5)$$

- The constraint guarantees that the resulting identified quadruple A, G, C, Λ_0 leads to a positive real spectrum.
- This optimization problem can be converted to an LMI.
- The drawback: the LMI software is not suited for large N.

Outline

Given A and C, a positive real identified spectrum can be guaranteed by solving the following NLS optimization:

$$\min_{B,D} \sum_{k=0}^{2N-1} \| S_k - L(e^{j2\pi k/2N}) \begin{pmatrix} B \\ D \end{pmatrix} \begin{pmatrix} B^T & D^T \end{pmatrix} L(e^{-j2\pi k/2N}) \|_F^2 .$$

- To insure a minimum phase model, (3)–(5) can be solved for G and Λ_0, after which a new B and D (guaranteeing a minimum phase model) can be computed through the solution of the Riccati equation.

Initial guesses of B and D

"Perturb Λ_0 to $\tilde{\Lambda}_0$ so that the the power spectrum associated with the resulting quadruple $\{A, G, C, \tilde{\Lambda}_0\}$ is positive real"

$$\tilde{\Lambda}_0 = \Lambda_0 + \tau I_l, \qquad \tau > 0.$$

- Taking τ large will trivially ensure positive realness of the power spectrum. However, we would like to keep τ as small as possible, which can be posed as an LMI:

$$\min \quad \tau \quad \text{subject to } \tau > 0, \;\; P > 0,$$

$$\begin{pmatrix} P - APA^T & G - APC^T \\ G^T - CPA^T & \Lambda_0 - CPC^T + \tau I_l \end{pmatrix} > 0.$$

- A general perturbation model $\tilde{\Lambda}_0 = \Lambda_0 + \tau$ where the Frobenius norm of τ is to be minimized can be introduced.
- The Riccati equation associated with $\{A, G, C, \tilde{\Lambda}_0\}$ can now be solved. This leads to matrices B^0 and D^0 which can serve as initial guesses for the NLS optimization problem.
- Both methods can be used with data sampled on a non-uniform grid of frequencies.
- The identification algorithm and the methods ensuring positivity of the spectrum are further studied in Hinnen *etal.*:2005.