Synthesis of Complete Orthonormal Fractional Basis Functions With Prescribed Poles

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Abstract—**In this paper, fractional orthonormal basis functions that generalize the well-known fixed pole rational basis functions are synthesized. For a range of noninteger differentiation orders and under mild restrictions on the choice of the basis poles, the synthesized basis functions are shown to be complete in the space of functions which are analytic on the open right-half plane and square-integrable on the imaginary axis. This presents an extension of the completeness results for the fractional Laguerre and Kautz bases to fractional orthonormal bases with prescribed pole locations.**

Index Terms—**Completeness, fractional calculus, orthonormal basis.**

I. Introduction

THE fractional calculus is a generalization of the traditional calculus that leads to similar concepts and tools but with a much wider applicability. The mathematical concept and formalism of fractional calculus originate from the works of Liouville [1] and Riemann [2]. For almost 300 years, it has remained an interesting, but abstract, mathematical concept. In recent years, fractional calculus has been taken up by scientists and engineers and applied in an increasing number of fields, namely, in the areas of thermal engineering, acoustics, electromagnetism, control, robotics, viscoelasticity, diffusion, turbulence, signal processing, and many others.

There are many linear systems with transfer functions that can be represented as fractional differential systems, that is, as functions $G(s)$ that involve fractional powers of the Laplace variable s. For instance, in the field of diffusion, recent work [3] generalized diffusion equations based on noninteger derivatives. In thermal diffusion, it was shown in [4] that in a semi-infinite homogeneous medium, the exact solution of the heat equation links thermal flux to a half-order derivative of the surface temperature on which the flux is applied. Expressing such a relation by the use of rational models would require a much higher number of parameters. Diffusion phenomena were investigated in semi-infinite planar, spherical, and cylindrical media in [5]–[7], where it was shown that the involved transfer functions use exponents of s that are multiples of 0.5. In electrochemical diffusion of charges in the electrode and the electrolyte, the most common physical model used in the literature is the Randles model [8], which uses Warburg impedance that involves an integrator of order 0.5. A fractal model for anomalous losses in ferromagnetic materials was used in [9]. In rheology, stress in a viscoelastic material is proportional to a noninteger derivative of deformation [10].

In the area of control, the idea of using fractional systems for modeling ideal loop transfer functions dates back to Bode [11]. He showed that the loop gain must have a frequency behavior described as a fractional order transfer function to reduce the effects of disturbances and uncertainties on the closed-loop system performance. Recently, in [12], the advantages of a fractional order controller known as *commande robust d'ordre non entier* (CRONE) with respect to classical devices were shown. Fractional proportional-integral-derivative controller applications were reported in [13]. System identification with fractional models was initiated in [14]. Recently, in [15], fractional models were used to identify thermal diffusive systems. An overview of system identification methods based on fractional models is presented in [16].

In signal processing, noninteger derivative was used in the synthesis of fractal noise [17]. The works of Mandelbrot on fractals led to a significant impact in several scientific areas. Presently, new themes are the object of active research such as fractional delay filtering [18], fractional splines, and wavelets [19]–[21]. In a similar line of thought, the concept of fractional Fourier transform [22] can be mentioned. This tool has mostly been applied in the field of optics. But, some applications to filtering, encoding, watermarking, and phase retrieval have appeared in the literature on signal analysis.

Of the greatest interest to the signal processing and control engineering communities is the fact that the fractional systems have both short- and long-term memories. Some basic properties of fractional systems such as stability [23], [24], observability, and controllability [25], the H_2-norm [26], and the H_∞-norm [27] have been investigated over the last ten years.

A fundamental idea in various areas of applied mathematics, control theory, signal processing, and system analysis is that of decomposing (perhaps infinite dimensional) descriptions of linear-time-invariant dynamics in terms of an orthonormal basis. This approach is of greatest utility when accurate system descriptions are achieved with only a small number of basis functions. In recognition of this, there has been much work [28], [29] over the past several decades and, with renewed interest, more recently [30]–[34] on the construction, analysis, and application of rational orthonormal bases suitable for providing linear system characterizations.

An important motivation for the consideration of orthonormal parameterizations is for approximation purposes. In this setting,

Manuscript received January 28, 2007; revised May 31, 2008. First published July 9, 2008; current version published September 17, 2008. The associate editor coordinating the review of this paper and approving it for publication was Prof. Patrice Abry.

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Digital Object Identifier 10.1109/TSP.2008.928163

a dominant question must arise as the quality of the approximation. Pertaining to this, one of the most fundamental properties that might be required is *completeness*. Formally, a model set A is complete in a space X if the closure of the linear span of A under the norm on X equals X.

In Laguerre model structures, prior knowledge of the relative stability of a transfer function is encoded in terms of a single basis pole. In the case of systems for which prior knowledge of a resonant mode exists, it is more appropriate to employ two-parameter Kautz bases. The well-known Laguerre and Kautz bases [31] are special cases of the general orthonormal bases [30] where the basis poles are again restricted to a finite set. In [33] and [34], model sets spanned by fixed pole orthonormal bases that generalize the Laguerre, two-parameter Kautz, and general orthonormal bases were investigated. These model sets were shown to be complete in $H_2(\Pi)$, the space of functions that are analytic on the open right-half plane denoted by Π and square integrable on the imaginary axis, provided that the chosen basis poles satisfy a mild condition. This generalization enjoys increased flexibility of pole location. As a result, a fewer number of basis functions may be used without sacrificing model accuracy.

Intuitively, one is led to the conclusion that the Laguerre functions can be extended to fractional differentiation orders by simply allowing their differentiation orders to be positive real numbers [35]. However, the classical Laguerre functions are divergent whenever their differentiation orders are noninteger [36]. The first complete fractional orthonormal basis, the so-called fractional Laguerre basis, was synthesized in [37]. This extension from the rational Laguerre basis to a fractional one provides a new class of fixed denominator models for system approximation and identification. A fractional orthogonal Kautz basis, which happens to be complete from the completeness of the fractional Laguerre functions in [37], was synthesized in [38].

The purpose of this paper is to generalize the results in [37] and [38] to fractional bases with infinitely many prescribed poles subject to mild restrictions on the choice of poles. This generalization is not straightforward. The key idea in [32]–[34] in showing completeness of the basis functions was to reparameterize the chosen model structures into a new one with equivalent fixed poles but for which the basis functions are orthonormal in $H_2(\Pi)$. Then, it was possible to derive analytic expressions for approximation errors of the rational basis functions in terms of the Blaschke products [39] formed by the basis poles. These analytic expressions yielded necessary and sufficient conditions for the completeness of the basis functions not only in $H_2(\Pi)$ but also in many spaces. It was also possible to express each basis function as a product of a Blaschke product with a first-order system.

This approach cannot be utilized in the synthesis of fixed pole fractional bases with infinitely many poles since fractional analogs of the Blaschke products cannot simply be defined by inserting s^γ in place of s due to the branch cut along the negative real line. The deficiency in defining fractional Blaschke products makes completeness study significantly harder for the fractional rationals because the orthonormality cannot be employed either as an implementional tool or as an analysis tool. The use of the conformal mapping technique in [37] is limited only to the synthesis of fractional Laguerre bases.

This paper is organized as follows. In Section II, mathematical background on the fractional derivatives and the fractional transfer functions is briefly reviewed. In Section III, fractional basis functions with prescribed poles are synthesized and shown to be complete in $H_2(\Pi)$. The orthonormalization of the synthesized basis functions is carried out in Section IV. In Section V, the impulse responses of the synthesized basis functions are studied. In Section VI, a numerical example is used to illustrate the basis synthesis scheme, and the impulse responses of the synthesized basis functions are computed. Section VII outlines future research directions and concludes this paper.

Notation

The field of the real and the complex numbers is denoted, respectively, by $\mathbf{R}$ and $\mathbf{C}$. The set of the positive numbers and its complement in $\mathbf{R}$ are denoted, respectively, by $\mathbf{R}_+$ and $\mathbf{R}_-$. The real and the imaginary parts of z are denoted, respectively, by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$. The complex conjugate of z is denoted by $\bar{z}$. The upper and the lower open half-planes are denoted, respectively, by Π_2 and Π_4, $\Pi_1 = \Pi$, and Π_3 denotes the open left half-plane.

Let $\mathcal{C}_\gamma$ denote the open sector defined by

$$\mathcal{C}_\gamma = \left\{ s \in \mathbf{C} : |\arg(s)| < \pi \left[1 - \left(\frac{\gamma}{2} \right) \right] \right\} \quad (0 < \gamma < 2).$$

Thus, $\mathcal{C}_0 = \mathbf{C} - \mathbf{R}_-$ and $\mathcal{C}_1 = \Pi$. As γ increases, $\mathcal{C}_\gamma$ decreases. Let $\mathbf{D}_\varepsilon(z)$ denote the open disk with center $z \in \mathbf{C}$ and radius $\varepsilon > 0$.

The Hardy spaces of functions $F(s)$ analytic on Π and such that $\|F\|_p < \infty$ $(0 < p \le \infty)$ are denoted by $H_p(\Pi)$, where

$$\|F\|_p = \begin{cases} \left[\frac{1}{2\pi} \sup_{x>0} \int_{-\infty}^{\infty} |F(x+jy)|^p dy \right]^{(1/p)}, & p \neq \infty \\ \sup_{s \in \Pi} |F(s)|, & p = \infty \end{cases}.$$

II. Fractional Linear Systems

In this section, we will review definitions and results of fractional calculus pertinent to our analysis. The readers are referred to [40] and the references therein for details.

A. Fractional Differential Equations

The inverse Laplace transform of $F(s)$ denoted by $f(t)$ is defined by

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds \quad (t > 0) \tag{1}$$

where $\sigma \in \mathbf{R}$ is inside the region of convergence. It is related to $F(s)$ by the Laplace transformation

$$F(s) = \int_0^\infty f(t) e^{-st} dt. \tag{2}$$

Note the following relation:

$$\int_0^\infty D_{\mathrm{d}}^\gamma f(t) e^{-st} dt = s^\gamma F(s) \text{ for } \mathrm{Re}(s) > 0 \tag{3}$$

where $D_{\mathrm{d}}^\gamma f(t)$ denotes the (direct) Grünwald-Letnikov fractional derivative of order γ of $f(t)$ [41].

The multivalued function s^γ becomes an analytic function in the complement of its branch cut line as soon as a branch cut line, i.e., $\mathbf{R}_-$, is specified. This choice is made for a causal system.

Let us define a linear system through a fractional differential equation in the form

$$\sum_{n=0}^{N} a_n D_{\mathrm{d}}^{\gamma_n} y(t) = \sum_{m=0}^{M} b_m D_{\mathrm{d}}^{\gamma_m} u(t) \tag{4}$$

where the differentiation orders γ_n are all positive and $a_N, b_M \neq 0$. Applying (2) to (4) and using (3), we obtain the transfer function of the system

$$F(s) = \frac{\sum_{m=0}^{M} b_m s^{\gamma_m}}{\sum_{n=0}^{N} a_n s^{\gamma_n}}. \tag{5}$$

The transfer function $F(s)$ is said to be *commensurable* of order $\gamma \in \mathbf{R}_+$ if γ_m, γ_n in (5) are integer multiples of γ and γ is the largest number with this property. Thus, a commensurable transfer function F is a rational function in s^γ; and, assuming $N > M$, by partial fraction expansion it can be decomposed as

$$F(s) = \sum_{k=1}^{L} \sum_{\ell=1}^{n_k} \frac{\alpha_{k,\ell}}{(s^\gamma + \lambda_k)^\ell} \tag{6}$$

for some complex numbers λ_k and positive integers n_k, where $\sum_{k=1}^{L} n_k = N$.

B. Stability of s^γ-Rational Functions

A system with transfer function $G(s)$ is said to be stable if $G \in H_\infty(\Pi)$. This means that the system defined by (4) with $\gamma_k = \gamma$ for all k maps bounded energy inputs $u(t)$ to bounded energy outputs $y(t)$. In fact, if this happens, then (4) maps magnitude bounded inputs to magnitude bounded outputs as well, that is, the fractional linear system (4) is bounded-input/bounded-output (BIBO) stable.

The stability of the fractional system defined by (4) with $\gamma_k = \gamma k$ for all k can be checked by checking γ and the arguments of λ_k denoted by $\arg(\lambda_k)$ in the partial fraction expansion of $F(s)$. Matignon [23] showed that (4) with $\gamma_k = \gamma k$ for all k is BIBO stable if and only if

$$0 < \gamma < 2 \text{ and } |\arg(\lambda_k)| < \pi\left(1 - \frac{\gamma}{2}\right). \tag{7}$$

Henceforth, we will restrict γ to the interval (0,2). In the course of deriving the completeness results, we will revisit this issue.

C. Fractional Orthonormal Bases

The partial fraction expansion (6) of a fractional linear system (4) with $\gamma_k = \gamma k$ for all k suggests approximating arbitrary functions in $H_2(\Pi)$ by linear combinations of the functions $(s^\gamma + \lambda_k)^{-\ell}$, $1 \leq \ell \leq n_k$; $k \geq 1$. There are several degrees of freedom and constraints in doing so. First of all, the stability constraint (7) has to be taken into account, which can be dealt with easily by suitably selecting the sequence λ_k for a fixed γ satisfying $\gamma \in (0, 2)$. Another degree of freedom comes from the choice of the parameters n_k.

For the sequence λ_k, we consider arbitrary choices and multiplicities subject to the argument restrictions in Section II-B. Thus, for all k, we assume $(s^\gamma + \lambda_k)^{-1} \in H_\infty(\Pi)$. It remains to satisfy $(s^\gamma + \lambda_k)^{-\ell} \in H_2(\Pi)$ so that their orthonormalized versions span a dense subset of $H_2(\Pi)$. Then, it suffices to let $\ell\gamma > (1/2)$ to assure $(s^\gamma + \lambda_k)^{-\ell} \in H_2(\Pi)$. Further details will be supplied later. Thus the problem studied is of synthesizing complete fractional orthonormal bases in $H_2(\Pi)$. The completeness problem boils down to deriving sufficient conditions in terms of the parameters λ_k and their multiplicities.

After establishing the completeness of the set $\left\{(s^\gamma + \lambda_k)^{-\ell} : 1 \leq \ell \leq n_k; k \geq 1\right\}$ in $H_2(\Pi)$, the next task is to orthonormalize this set. This is a nontrivial process due to the branch cut along the negative real axis. Since $s = 0$ is a branch point, the following inner products:

$$\left\langle (s^\gamma + \lambda_{k_1})^{-\ell_1}, (s^\gamma + \lambda_{k_2})^{-\ell_2} \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{[(j\omega)^\gamma + \lambda_{k_1}]^{\ell_1} \left[\overline{(j\omega)^\gamma} + \overline{\lambda_{k_2}}\right]^{\ell_2}} \tag{8}$$

are to be interpreted in a principal-value integral sense, where the value at $s = 0$ is defined by continuation. The inner products (8) can be calculated explicitly by the aid of the residue method. Note that

$$(j\omega)^\gamma = |\omega|^\gamma \begin{cases} e^{j\pi\gamma/2}, & \omega > 0 \\ e^{-j\pi\gamma/2}, & \omega < 0. \end{cases} \tag{9}$$

The final task is to obtain the impulse responses of the fractional orthonormal basis functions, which are complicated expressions due to the branch cut.

D. Motivation for the Completeness Study

Let us consider the simple fractional rational transfer function $(s^\gamma + \lambda_k)^{-\ell}$, $\lambda_k \in \mathbf{R}$. By taking its inverse Laplace transform, we obtain the impulse response of this system denoted by $\widehat{\phi}_{k\ell}(t)$, $t > 0$ as follows:

$$\widehat{\phi}_{k\ell}(t) = \sum_{p=0}^{1 \text{ or } 2} \sum_{q=0}^{\ell-1} c_{pq} t^q e^{\alpha_{pq} t} + \frac{1}{\pi} \sum_{i=1}^{\ell} \frac{\ell!}{i!(\ell - i)!} \int_0^\infty \frac{\lambda_k^{\ell-i} \sin(\pi\gamma i) x^{\gamma i} e^{-xt} dx}{[x^{2\gamma} + 2\lambda_k x^\gamma \cos(\pi\gamma) + \lambda_k^2]^\ell} \tag{10}$$

for some complex numbers c_{pq}, α_{pq}. This formula will be derived in a generalized setting. The above expression, as pointed out in [37], has two terms: the first term is the sum of the exponential modes originating from the poles of $(s^\gamma + \lambda_k)^{-\ell}$ and the second term is the combination of an infinite number of exponentials originating from the branch cut. The presence of the first term and the range of p as well as the numbers c_{pq}, α_{pq} depend on the values of γ and λ_k. This term has the character of a linear time-invariant dynamics and quickly dies out since $\mathrm{Re}(\alpha_{pq}) < 0$ for all p and q. The second term is more profound. In fact, from the definition of the gamma function

$$\Gamma(\beta) = \int_0^\infty e^{-z} z^{\beta-1} dz$$

we have, as $t \to \infty$

$$\widehat{\phi}_{k\ell}(t) \approx \sum_{i=1}^{\ell} \frac{\ell! \sin(\pi\gamma i)\Gamma(\gamma i+1)}{i!(\ell-i)!\pi\lambda_k^{\ell+i}} t^{-(\gamma i+1)}.$$

Thus, the fractional rationals appear to be more suitable than the rationals in modeling slowly decaying impulse responses.

III. Synthesis of Complete Fractional Bases

In this section, we synthesize complete fractional bases in $H_2(\Pi)$. As basis functions, we propose the following so-called generator functions:

$$\phi_{k\ell}(s) = \frac{(s^\gamma+\lambda)^{1-m}}{(s^\gamma+\lambda_k)^\ell}, \quad 1 \le \ell \le n_k;\ \ k \ge 1 \tag{11}$$

where $\lambda > 0$, $m \ge 2$ is a convergence factor to be fixed later; $\lambda_k \in \Pi$ are given complex numbers; and $\gamma \in (0,1)$ is a fixed number. The case $\gamma \in (1,2)$ will be dealt with later in this section, and the orthonormality issue is postponed to Section IV. We begin with a study of the generator poles. In this paper, it suffices to consider the complex function $\hat{\phi}$ defined for a given $z \in \mathbf{C}$ by

$$\hat{\phi}(s) = \frac{1}{s^\gamma + z}. \tag{12}$$

Lemma III.1: Let $\hat{\phi}$ be as in (12) with $\gamma \in (0,1)$. Write $s \in \mathcal{C}_0$ and $z \in \mathbf{C}$ as

$$s = re^{j\theta}, \quad |\theta| < \pi \text{ and } z = Re^{j\varphi}, \quad |\varphi| \le \pi. \tag{13}$$

Then, $\hat{\phi}$ has a simple pole in $\mathcal{C}_0$ at $z_\phi = R^{1/\gamma}e^{j(\varphi-\pi)/\gamma}$ for $\varphi > 0$ or at $z_\phi = R^{1/\gamma}e^{j(\varphi+\pi)/\gamma}$ for $\varphi < 0$ if and only if $\pi(1-\gamma) < |\varphi|$. If $\varphi = \pi(1-\gamma)$, $\hat{\phi}$ is analytic on $\mathcal{C}_0$, bounded on Π_2, and unbounded in $\Pi_4 \cap \mathbf{D}_\varepsilon(-R^{1/\gamma})$; and if $\varphi = -\pi(1-\gamma)$, $\hat{\phi}$ is analytic on $\mathcal{C}_0$, bounded on Π_4, and unbounded on $\Pi_2 \cap \mathbf{D}_\varepsilon(-R^{1/\gamma})$ for all $\varepsilon > 0$. If $|\varphi| < \pi(1-\gamma)$, $\hat{\phi}$ is bounded analytic on $\mathcal{C}_0$.

Proof: See Appendix A. ■

Suppose $\hat{\phi}$ is bounded analytic on Π. This means that $\hat{\phi}$ is the transfer function of a bounded-input/bounded-output stable continuous-time system. Then, $\hat{\phi}$ can have isolated singularities only on the open left half-plane. From Lemma III.1, there are only two possibilities: $\hat{\phi}$ is either analytic on $\mathcal{C}_0$ or has a simple pole with an argument $\theta = (\varphi-\pi)/\gamma$ for $\varphi > 0$ or $\theta = (\varphi+\pi)/\gamma$ for $\varphi < 0$. Since in the latter case $|\theta| > \pi/2$, we must then have $|\varphi| < \pi[1-(\gamma/2)]$. It follows that $z \in \mathcal{C}_\gamma$. Conversely, assume $z \in \mathcal{C}_\gamma$. Then, from Lemma III.1, $\hat{\phi}$ either is analytic on $\mathcal{C}_0$ or has a pole with argument satisfying $|\theta| > \pi/2$. Thus, $\hat{\phi}$ is bounded analytic on Π. Hence, $\hat{\phi}$ is the transfer function of a BIBO stable continuous-time system if and only if $z \in \mathcal{C}_\gamma$, which is a restatement of the stability result of Matignon [23].

Let us further constrain z to Π. When $z \in \Pi$, a pole in $\mathcal{C}_0$ may exist only if $\gamma > 1/2$; and this happens if φ satisfies $\pi(1-\gamma) < |\varphi| < \pi/2$. In this case, from Lemma III.1, the argument of a pole, if any, satisfies $2\gamma|\theta| > \pi$. Hence, $\hat{\phi}$ is analytic on $\mathcal{C}_{2-(1/\gamma)}$. If $0 < \gamma \le 1/2$, then, for all $z \in \Pi$, $\hat{\phi}$ is bounded analytic on $\mathcal{C}_0$.

The Blaschke products have played a major role in completeness and approximation properties of rational basis functions [33], [34]. Unfortunately, fractional analogs of the Blaschke products do not exist. As a result, basis orthonormality cannot be utilized in deriving completeness criteria for the generator functions (11). However, rather general sufficient conditions for the completeness of (11) in $H_2(\Pi)$ can be put forward using the Müntz–Szász theory.

The following lemma will be instrumental in establishing the first completeness result of this paper.

Lemma III.2: For a given $\gamma \in (0,1)$, let m be a fixed integer satisfying

$$2\gamma m > 1 + \gamma \tag{14}$$

and $\lambda > 0$. Define ψ on Π by

$$\psi(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\omega}{|(j\omega)^\gamma+\lambda|^{2m-2}\,|(j\omega)^\gamma+s|^2}. \tag{15}$$

Then

$$\Psi = \frac{1}{2\pi}\sup_{x>0}\int_{-\infty}^{\infty}\psi(x+jy)dy < \infty. \tag{16}$$

Proof: See Appendix B. ■

Let $\lceil x \rceil$ denote the smallest integer greater than x. Pick $m = \lceil 1/2\gamma \rceil$. Then, $\psi(s)$ exists for all $s \in \Pi$ and (14) is satisfied by the choice $m = \lceil 1+\gamma/2\gamma \rceil$. This increase is needed to establish $\Psi < \infty$. Observe that $\lceil 1+\gamma/2\gamma \rceil \le \lceil \gamma/2\gamma \rceil + 1$. In Section IV, we will enforce (14) in the calculation of the inner products of (11) by the residue method.

When $\gamma = 1$, it was shown in [33] that the generator functions (11) with $m = 1$ are complete in $H_2(\Pi)$ if and only if the chosen basis poles satisfy the criterion

$$\sum_{k=1}^{\infty} n_k \frac{\mathrm{Re}(\lambda_k)}{1+|\lambda_k|^2} = \infty. \tag{17}$$

Our first result in this section establishes that under the same criterion, the generator functions (11) are complete in $H_2(\Pi)$ if $0 < \gamma < 1$ and m satisfies (14).

The completeness proof uses the fact that in the special case $\lambda_k = \lambda$ for all k, (11) is complete in $H_2(\Pi)$ for all $\gamma \in (0,2)$. The proof of this simple fact is detailed in [37]. There are three ingredients in its proof. First, $(s^\gamma+\lambda)^{-1}$ transforms Π conformally onto the interior of a compact subset of Π. Secondly, $H_2(\Pi)$ may be replaced with a dense subset of it consists of functions decaying faster than $O(1/\sqrt{s})$ as $s \to \infty$. These two steps and the change of the variables as $z = s^\gamma + \lambda$ reduce the completeness problem on Π to a completeness problem on a compact subset of Π in the supremum norm. In the third step, Mergelyan theorem [49] is invoked.

Theorem III.3: Let $\gamma \in (0,1]$ be a fixed number and m be chosen as in (14). Consider the generator functions (11) defined by a choice of numbers $\lambda > 0$ and $\lambda_k \in \Pi$. Then, (11) is complete in $H_2(\Pi)$ if (17) holds.

Proof: See Appendix C. ■

The completeness condition (17) applies to all $0 < \gamma \le 1$. Recall that for a fixed $\gamma \in (0,1)$, restricting λ_k to Π for all k results in the generator functions (11) being analytic on $\mathcal{C}_{\max\{0,2-(1/\gamma)\}}$. This restriction might introduce some conservatism on the choice of λ_k. Nevertheless, (17) does not preclude the possibility of λ_k converging to zero slowly. For example, put $\lambda_k = 1/k$ for all k. Then, (17) is satisfied.

For $\gamma \in (1,2)$, let us reconsider the complex function (12). The results are summarized in the following.

Lemma III.4: Let $\hat{\phi}$ be as in (12) with $\gamma \in (1,2)$. Write $s \in \mathcal{C}_0$ and $z \in \mathbf{C}$ as

$$s = re^{j\theta}, \quad |\theta| < \pi \text{ and } z = Re^{j\varphi}, \quad |\varphi| \le \pi. \tag{18}$$

If $\pi(\gamma-1) < \varphi \le \pi$, $\hat{\phi}$ has a simple pole at $z_{\phi 1} = R^{1/\gamma}e^{j(\varphi-\pi)/\gamma}$ in $\mathcal{C}_0$ and no others. If $-\pi \le \varphi < -\pi(\gamma-1)$, the only pole of $\hat{\phi}$ in $\mathcal{C}_0$ is at $z_{\phi 2} = R^{1/\gamma}e^{j(\varphi+\pi)/\gamma}$ and simple. If $|\varphi| < \pi(\gamma-1)$, $\hat{\phi}$ has two simple poles at $z_{\phi 1}$ and $z_{\phi 2}$ in $\mathcal{C}_0$. If $\varphi = \pi(\gamma-1)$, $\hat{\phi}$ has a simple pole at $z_{\phi 1}$ in $\mathcal{C}_0$ and no others. Moreover, $\hat{\phi}$ is bounded in $\Pi_4 \cap \mathbf{D}_\varepsilon(-R^{1/\gamma})$ and unbounded in $\Pi_2 \cap \mathbf{D}_\varepsilon(-R^{1/\gamma})$ for all $\varepsilon > 0$. If $\varphi = -\pi(\gamma-1)$, $\hat{\phi}$ has a simple pole at $z_{\phi 2}$ in $\mathcal{C}_0$ and no others. Moreover, $\hat{\phi}$ is bounded in $\Pi_2 \cap \mathbf{D}_\varepsilon(-R^{1/\gamma})$ and unbounded in $\Pi_4 \cap \mathbf{D}_\varepsilon(-R^{1/\gamma})$ for all $\varepsilon > 0$.

If the BIBO stability is demanded on $\hat{\phi}$, then $z_{\phi 1}$ and/or $z_{\phi 2}$ in Lemma III.4 must satisfy the conditions $z_{\phi 1} \in \Pi_3$ and $z_{\phi 2} \in \Pi_3$. These conditions are guaranteed if and only if $z \in \mathcal{C}_\gamma$.

We propose the generator functions for the case $\gamma \in (1,2)$ as follows:

$$\eta_{k\ell}(s) = \frac{1}{(s^\gamma + \lambda_k)^\ell}, \quad 1 \le \ell \le n_k; \;\; k \ge 1 \tag{19}$$

where $\lambda_k \in \mathcal{C}_{\bar{\gamma}}$ for all k and some $\gamma < \bar{\gamma} < 2$. The following is a key lemma to extend the conclusion of Theorem III.3 for the case $1 < \gamma < 2$.

Lemma III.5: Let

$$\vartheta(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{1-e^{-s}}{(j\omega)^\gamma + s}\right|^2 d\omega, \quad s \in \mathcal{C}_{\bar{\gamma}} \tag{20}$$

where $\gamma \in (1,2)$ and $\gamma < \bar{\gamma} < 2$. Then

$$\Phi = \sup_{s\in\mathcal{C}_{\bar{\gamma}}} \vartheta(s) < \infty. \tag{21}$$

Proof: See Appendix D. ■

Theorem III.6: Let $\gamma \in (1,2)$ be a fixed number. Consider the generator functions (19) defined by a choice of the complex numbers $\lambda_k \in \mathcal{C}_{\bar{\gamma}}$ for all k and some $\gamma < \bar{\gamma} < 2$. Then, (19) is complete in $H_2(\Pi)$ if

$$\sum_{k=1}^{\infty} n_k \frac{r_k^a \cos(a\theta_k)}{1 + r_k^{2a}} = \infty \tag{22}$$

where $\lambda_k = r_k e^{j\theta_k}$ and $a = (2-\bar{\gamma})^{-1}$.

Proof: See Appendix E. ■

The completeness of the generator functions (19) was established by restricting λ_k to $\mathcal{C}_{\bar{\gamma}}$ for all k. This set is a proper subset of $\mathcal{C}_\gamma$. It is a difficult question to answer whether it is possible to relax this restriction. Nevertheless, as γ approaches one, $\bar{\gamma}$ can be forced to approach one. Then, (22) coincides with (17), demonstrating that the former is consistent with the boundary case $\gamma = 1$.

Lemmas III.1 and III.4 will be used in Section V to calculate the impulse responses of the generator functions by the residue method. In the application of the residue method, one needs to know the residues of the poles inside the chosen contour. In addition to this, in our problem the line integrals along the upper and the lower edges of $\mathbf{R}_-$ need to be calculated. If λ_k happens to satisfy $|\arg(\lambda_k)| = \pi|1-\gamma|$, Lemmas III.1 and III.4 tell us that these line integrals cannot closely be approximated by contour integrals just above and below the branch cut line.

If $|\arg(\lambda_k)| = \pi(1-\gamma)$ and $\gamma \in (0,1)$, from the restriction $\lambda_k \in \Pi$, γ satisfies $\gamma > (1/2)$. However, in this case, (14) is satisfied with $m = 2$, which implies that $\phi_{k\ell} \in H_1(\Pi)$ for all k, ℓ. Clearly, $\eta_{k\ell} \in H_1(\Pi)$ for all k, ℓ. In this case, the impulse responses of the generator functions can directly be calculated from (1) with $\sigma = 0$.

IV. Orthonormal Basis Generation

In this section, we study the problem of constructing an orthonormal set from (11) or (19) with the same linear span. Derivation of explicit formulas is cumbersome due to many possibilities for the parameters γ and λ_k. Instead, we limit the study to presentation of a scheme which has three stages. The solution for a particular problem can be worked out using this scheme.

A. Calculating Inner Products of the Generator Functions

Let us demonstrate that the inner product (8) can effectively be calculated in closed form by the residue method. To this end, first by the change of variables $x = \omega^\gamma$, we get

$$\langle \phi_{k_1\ell_1}, \phi_{k_2\ell_2} \rangle = I(\lambda_{k_1}, \lambda_{k_2}, \ell_1, \ell_2) + I(\overline{\lambda_{k_2}}, \overline{\lambda_{k_1}}, \ell_2, \ell_1) \tag{23}$$

where $\widetilde{\lambda}_{k_1} = \lambda_{k_1}e^{-j(\pi\gamma/2)}$, $\widetilde{\lambda}_{k_2} = \overline{\lambda_{k_2}}\, e^{j(\pi\gamma/2)}$, $\widetilde{\lambda}_{k_3} = \lambda e^{-j(\pi\gamma/2)}$, $\widetilde{\lambda}_{k_4} = \lambda e^{j(\pi\gamma/2)}$, and

$$I(\lambda_{k_1}, \lambda_{k_2}, \ell_1, \ell_2) = \frac{e^{j(\pi\gamma/2)(\ell_2-\ell_1)}}{2\pi\gamma} \cdot \int_0^\infty f(x; \lambda_{k_1}, \lambda_{k_2}, \ell_1, \ell_2)dx. \tag{24}$$

In (24), $f(x;\cdot)$ is the restriction of the complex function

$$f(s;\cdot) = \frac{(s+\widetilde{\lambda}_{k_3})^{1-m}(s+\widetilde{\lambda}_{k_4})^{1-m}}{(s+\widetilde{\lambda}_{k_1})^{\ell_1}(s+\widetilde{\lambda}_{k_2})^{\ell_2}} s^{(1-\gamma/\gamma)} \tag{25}$$

defined on the domain $\{s \in \mathbf{C} : |s| > 0, 0 < \arg s < 2\pi\}$ to the upper edge of $\mathbf{R}_+$. With $m = 1$ plugged in (24), (23) holds for $\langle \eta_{k_1\ell_1}, \eta_{k_2\ell_2} \rangle$ as well.

To calculate (24) by the residue method, we consider the path shown in Fig. 1, where C_ρ and C_R denote, respectively, the circles $|z| = \rho$ and $|z| = R$ $(\rho < 1 < R)$. Since $\lambda \in \mathbf{R}_+$ and $\lambda_{k_1}, \lambda_{k_2} \in \Pi$ or $\lambda_{k_1}, \lambda_{k_2} \in \mathcal{C}_{\bar{\gamma}}$ for $\gamma \in (1,2)$, note that $\widetilde{\lambda}_{k_q} \notin \mathbf{R}_-$, $q = 1, \ldots, 4$. If $\ell_1 + \ell_2 + 2m - 2 - (1/\gamma) > 1$, which is (14) when $\ell_1 = \ell_2 = 1$, the contour integrals of f on

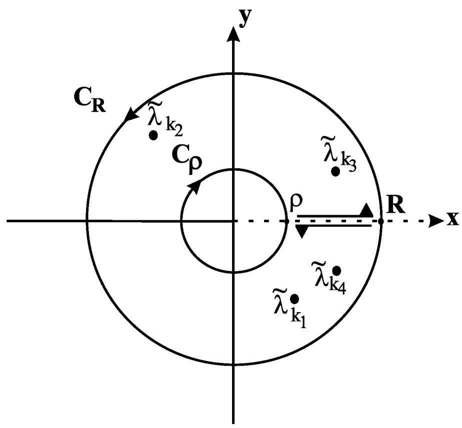

Fig. 1. Path for $f(s)$ with a cut on $\mathbf{R}_-$.

C_ρ and C_R vanish as $\rho \to 0$ and $R \to \infty$, and the residue theorem yields

$$\sum_{q=1}^{4} \mathrm{Res}_{s=-\widetilde{\lambda}_{k_q}} f(s;\cdot) = \frac{1-e^{j2\pi((1-\gamma)/\gamma)}}{2\pi j} \int_0^\infty f(x;\cdot)dx. \tag{26}$$

The residue at $-\widetilde{\lambda}_{k_1}$ is given by

$$\mathrm{Res}_{s=-\widetilde{\lambda}_{k_1}} f(s;\cdot) = \left[(s+\widetilde{\lambda}_{k_1})^{-\ell_1} f(s;\cdot)\right]^{(\ell_1-1)}_{s=-\widetilde{\lambda}_{k_1}}. \tag{27}$$

The chain rule of differentiation is applied several times to evaluate this residue. The rest of the residue evaluations are similar. These calculations demonstrate that (8) can be evaluated in closed form. The other case $\gamma \in (1,2)$ is simpler since a convergence factor is not needed.

B. Basis Functions With Real-Valued Impulse Responses

Up until this point, the generator functions have been considered with complete generality of pole location save for the completeness condition (17) or (22). However, in any application involving the modelling of a physical process, it is necessary that the underlying modelled impulse response be real valued. The purpose of this section is to illustrate how to modify the generator functions (11) and (19) to ensure the realness of the underlying impulse response. This is achieved by requiring that, for each n, the set $\{\lambda_1, \ldots, \lambda_n\}$ always contains complex conjugates of its elements. Since we have yet to impose orthonormality, the required modification to ensure the realness of the impulse response is simple.

To be more explicit on this, consider, for example, the generator functions (11) for some fixed $\lambda_k \in \Pi - \mathbf{R}$, $\lambda \in \mathbf{R}$, and positive integers m, k, l. The discussion in the sequel covers also the case $1 < \gamma < 2$, as can be seen by letting $m = 1$ in (11). Recalling $\lambda \in \mathbf{R}$, let

$$\check{f}_1(s) = \phi_{kl}(s) \text{ and } \check{f}_2(s) = \frac{(s^\gamma+\lambda)^{1-m}}{(s^\gamma+\overline{\lambda_k})^\ell}. \tag{28}$$

Write λ_k in the polar coordinates as $\lambda_k = R_k e^{j\varphi_k}$ with $|\varphi_k| \leq \pi$. Then, from the binomial expansion formula, we have

$$\check{f}_1 + \check{f}_2 = \frac{2\sum_{q=0}^{\ell}\binom{\ell}{q} s^{q\gamma} R_k^{\ell-q}\cos((\ell-q)\varphi_k)}{(s^\gamma+\lambda)^{m-1}\left(s^{2\gamma}+2R_k s^\gamma\cos\varphi_k + R_k^2\right)^\ell}$$

and

$$j\left[\check{f}_1 - \check{f}_2\right] = \frac{2\sum_{q=0}^{\ell}\binom{\ell}{q} s^{q\gamma} R_k^{\ell-q}\sin((\ell-q)\varphi_k)}{(s^\gamma+\lambda)^{m-1}(s^{2\gamma}+2R_k s^\gamma\cos\varphi_k + R_k^2)^\ell}.$$

Now, let $\check{g}_1 = \check{f}_1 + \check{f}_2$ and $\check{g}_2 = j(\check{f}_1 - \check{f}_2)$. Then, $\check{g}_1$ and $\check{g}_2$ are real-rational functions of s^γ. Hence, they have real-valued impulse responses as desired. Moreover, linear independence of $\check{f}_1$ and $\check{f}_2$ implies linear independence of $\check{g}_1$ and $\check{g}_2$.

Let us enumerate the sequence $\phi_{k\ell}$ or $\eta_{k\ell}$ and denote the corresponding sequence of basis functions with real-valued impulse responses defined above by χ_q, $q \geq 1$. This means that for a given index q, there exists a unique pair (k,l) such that χ_q either equals $\phi_{k\ell}$ or equals one of $\check{g}_1$ and $\check{g}_2$, where $\check{f}_1$ and $\check{f}_2$ are defined by $\check{f}_1 = (1/2)(\check{g}_1 - j\check{g}_2)$ and $\check{f}_2 = (1/2)(\check{g}_1 + j\check{g}_2)$ satisfy (28) for some λ and λ_k in the pole parameter set. Thus, we complete the second stage of the orthonormalization procedure.

C. Gram–Schmidt Orthonormalization

Let χ_q be the sequence of basis functions with real-valued impulse responses constructed in Section IV-B. We apply the iterative Gram–Schmidt procedure to orthonormalize this sequence. It starts with B_1 defined as $B_1 = \chi_1/\|\chi_1\|_2$. Assuming that $B_1, \ldots, B_{n-1}$ form an orthonormal set obtained from $\chi_1, \ldots, \chi_{n-1}$, the problem is to find a B_n constructed from $B_1, \ldots, B_{n-1}, \chi_n$ and orthogonal to $B_1, \ldots, B_{n-1}$. Furthermore, B_n is constrained to have a unity norm.

These requirements can be expressed as a set of $n-1$-linear equations:

$$\langle B_n, B_q\rangle = 0, \quad q = 1, \ldots, n-1 \tag{29}$$

where $\alpha_q \in \mathbf{R}$, $q = 1, \ldots, n$, are unknowns to be determined in the linear combination

$$B_n = \sum_{q=1}^{n-1} \alpha_q B_q + \alpha_n \chi_n \tag{30}$$

plus the normality condition $\|B_n\|_2 = 1$. It is easy to see that (29) yields the solution

$$B_n = \alpha_n \left(\chi_n - \sum_{q=1}^{n-1} \langle \chi_n, B_q\rangle B_q\right). \tag{31}$$

Then, α_n is determined from the normality condition and back-substituted in (31).

Thus, the problem of orthonormalizing the sequence χ_n is reduced to the problem of computing the inner products $\langle \chi_n, B_q\rangle$,

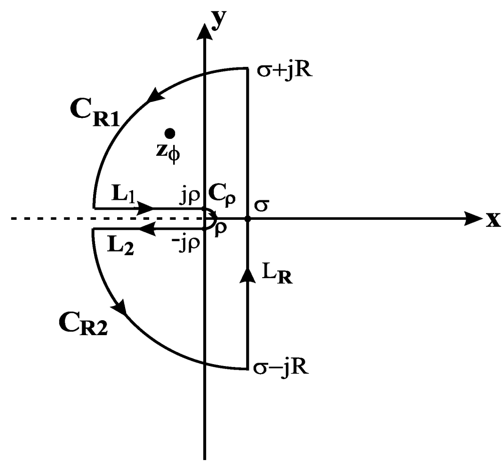

Fig. 2. Bromwich path for $\phi_{k\ell}(s)$ with a cut on $\mathbf{R}_{-}$.

$q = 1, 2, \ldots, n-1$, in (31). But the recursive formula (30) expresses each B_n as a linear combination of $\chi_1, \ldots, \chi_n$. Hence, the problem of orthonormalizing χ_n is equivalent to evaluating the inner products $\langle \chi_q, \chi_r \rangle$ for all $q, r \geq 1$. Recall that a given χ_q equals to one of $\phi_{k\ell}$ (or $\eta_{k\ell}$), $\breve{g}_1$, and $\breve{g}_2$. Then, $\langle \chi_q, \chi_r \rangle$ is the sum of at most four inner products of the generators functions (11) or (19); and the formulas derived in Section IV to calculate such inner products can be used. This completes the third stage of the scheme to orthonormalize the generator functions.

V. Impulse Responses of the Basis Functions

The aim of this section is to discuss how to obtain analytically the impulse responses of the orthonormal basis functions B_n constructed in Section IV.

Recall that the impulse response of B_n denoted by $b_n(t)$ is defined by

$$b_n(t) = \frac{1}{2\pi j} \lim_{R \to \infty} \int_{L_R} e^{st} B_n(s) ds \quad (t > 0) \qquad (32)$$

where $\sigma > 0$ and L_R is the vertical line segment from $s = \sigma - jR$ to $s = \sigma + jR$ shown in Fig. 2. We will not attempt to obtain b_n explicitly for two reasons: first, B_n is a linear combination of n complex functions with real-valued impulse responses, which themselves are linear combinations of either one or two generator functions of type (11) or (19), secondly and most compelling, even simple fractional systems such as $(s^\gamma + \lambda_k)^{-\ell}$ have impulse responses that can only be expressed in the integral form (10). Time-domain simulation of fractional systems is currently an active research area [42]–[48]. We limit the discussion to calculating only the complex-valued generator impulse response $\tilde{\phi}_{k\ell}$ of a fractional system with transfer function $\phi_{k\ell}$.

The calculation of $\tilde{\phi}_{k\ell}$ by the residue method will be studied for the case $\gamma \in (0, 1)$. In the application of the residue method, the so-called Bromwich path for $\phi_{k\ell}(s)$ with a cut on $\mathbf{R}_{-}$ is shown in Fig. 2. Since $\lambda > 0$ and $\lambda_k \in \Pi$, from Lemma III.1, $\phi_{k\ell}(s)$ either has no pole in $\mathcal{C}_0$ or has a pole at $z_\phi \in \Pi_3 \cap \mathcal{C}_0$ with multiplicity ℓ. It is also assumed that $|\arg(\lambda_k)| \neq \pi(1 - \gamma)$. From Lemma III.1, this implies that $\phi_{k\ell}$ is bounded on the horizontal slit $\{s \in \mathbf{C} : |\mathrm{Im}(s)| < \varepsilon\} \cap \Pi_3$ for some $\varepsilon > 0$.

It is easy to show that as $R \to \infty$ and $\rho \to 0$, the contour integrals on C_{R1}, C_{R2}, and C_ρ vanish. Assuming that $\phi_{k\ell}(s)$ has poles in $\mathcal{C}_0$, then an application of the residue theorem and the definition (32) yields

$$\begin{aligned}\tilde{\phi}_{k\ell}(t) = & \sum_{z_\phi \in \mathcal{C}_0} \mathrm{Res}_{s=z_\phi} \left[e^{st} \phi_{k\ell}(s) \right] \\ & - \frac{e^{-j\pi\gamma(\ell+m-1)}}{2\pi j} \int_0^\infty \frac{(x^\gamma + \lambda e^{-j\pi\gamma})^{1-m}}{(x^\gamma + \lambda_k e^{-j\pi\gamma})^\ell} \\ & \cdot e^{-xt} dx \\ & + \frac{e^{j\pi\gamma(\ell+m-1)}}{2\pi j} \int_0^\infty \frac{(x^\gamma + \lambda e^{j\pi\gamma})^{1-m}}{(x^\gamma + \lambda_k e^{j\pi\gamma})^\ell} \\ & \cdot e^{-xt} dx. \end{aligned} \qquad (33)$$

The boundedness of $\phi_{k\ell}$ on $\{s \in \mathbf{C} : |\mathrm{Im}(s)| < \varepsilon\} \cap \Pi_3$ was used in obtaining the second and the third terms in (33) by letting L_1 and L_2 approach to the negative real axis. If ϕ_{kl} has no poles in $\mathcal{C}_0$, the residue term in (33) is dropped.

Putting $m = 1$ in (33) and assuming $\lambda_k > 0$, after some algebra the sum of the second and the third terms in (33) is seen to be the second term in (10). Therefore, the second and third terms in (33) are in the simplest possible form. Note from Lemma III.1 that if $|\arg(\lambda_k)| < \pi(1 - \gamma)$, then $\phi_{k\ell}(s)$ has no singularities inside the Bromwich path; and $\tilde{\phi}_{k\ell}(t)$ equals the sum of the second and third terms in (33).

The residue at $s = z_\phi$ is calculated as follows:

$$\begin{aligned}\mathrm{Res}_{s=z_\phi} & \left[e^{st} \phi_{k\ell}(s) \right] \\ & = \left[e^{st} (s - z_\phi)^\ell \phi_{k\ell} \right]_{s=z_\phi}^{(\ell-1)} \\ & = \sum_{r=0}^{\ell-1} \binom{\ell-1}{r} \left[(s - z_\phi)^\ell \phi_{k\ell}(s) \right]_{s=z_\phi}^{(\ell-1-r)} t^r e^{tz_\phi} \end{aligned} \qquad (34)$$

which is a linear combination of the exponential functions $t^r e^{tz_\phi}$, $r \geq 0$, thus verifying the presence of the first term in (10). The evaluation of the above derivatives is quite involved.

Recall that $|\arg(\lambda_k)| = \pi(1 - \gamma)$ implies $\gamma > (1/2)$, $m = 2$, and $\phi_{k\ell} \in H_1(\Pi)$. Then, $\tilde{\phi}_{k\ell}(t)$ can be calculated from (32) with $\sigma = 0$. The case $\gamma \in (1, 2)$ is simpler since there is no need for a convergence factor; and the above formulas are still valid under the assumption $|\arg(\lambda_k)| \neq \pi(\gamma - 1)$. The number and location of the poles are specified in Lemma III.4. If $|\arg(\lambda_k)| < \pi(\gamma - 1)$, it is necessary to evaluate the residue at $z_{\phi 1}$ and $z_{\phi 2}$. Since there is no convergence problem when $\gamma > 1$, alternatively (32) with $\sigma = 0$ can be used to calculate the impulse response. However, (32) is the only option when $|\arg(\lambda_k)| = \pi(\gamma - 1)$.

An interesting approach to analytically calculate the inverse Laplace transforms of the basis functions is suggested in [43]. In [43], first $\eta_{k1}(s)$ is expanded in descending powers of s^γ by long division. Then, this series is term-by-term inverse transformed to calculate the impulse response of η_{k1} denoted by $\widetilde{\eta}_{k1}$. The series representation is not unique. For example, in [42], $\widetilde{\eta}_{k1}(t)$ is obtained as a series of Mittag–Leffler functions.

VI. Numerical Example

In this section, we illustrate the basis synthesis scheme outlined in Section IV by a numerical example. Let

$$\phi_1(s) = \frac{1}{s^{1.5} + e^{j(\pi/6)}}, \quad \phi_2(s) = \frac{1}{s^{1.5} + e^{-j(\pi/6)}}$$

be two given generator functions. We are to construct two orthonormal basis functions from ϕ_1 and ϕ_2 and compute their impulse responses as well. The basis functions with real-valued impulse responses are easily found as

$$\begin{aligned}\breve{g}_1(s) &= \phi_1(s) + \phi_2(s) = \frac{2s^{1.5} + \sqrt{3}}{s^3 + \sqrt{3}s^{1.5} + 1}\\ \breve{g}_2(s) &= j[\phi_1(s) - \phi_2(s)] = \frac{1}{s^3 + \sqrt{3}s^{1.5} + 1}.\end{aligned}$$

The difficult part is the orthonormalization of $\breve{g}_1$ and $\breve{g}_2$. To this end, first we compute the inner products $\langle \phi_k, \phi_l \rangle$, $k, l = 1, 2$. Proceeding as in the derivation of (23)–(25), we get

$$\langle \phi_1, \phi_2 \rangle = \frac{1}{1.5\pi} \int_0^\infty \frac{x^{-(1/3)} dx}{(x + e^{-j(7\pi/12)})(x + e^{j(11\pi/12)})}.$$

From (26) and (27), we have the equation at the bottom of the page. Likewise

$$\begin{aligned}\|\phi_1\|_1^2 = \|\phi_2\|_2^2 &= \frac{1}{3\sin\left(\frac{\pi}{3}\right)} \left[\frac{\sin\left(\frac{11\pi}{36}\right)}{\sin\left(\frac{11\pi}{12}\right)} + \frac{\sin\left(\frac{7\pi}{36}\right)}{\sin\left(\frac{7\pi}{12}\right)} \right]\\ &= 1.4468.\end{aligned}$$

The Gram–Schmidt procedure applied to $\breve{g}_1$ and $\breve{g}_1$ yields two basis functions that are orthogonal to each other

$$\tilde{\chi}_1 = \breve{g}_1, \quad \tilde{\chi}_2 = \breve{g}_1 - \frac{\|\breve{g}_1\|_2^2}{\langle \breve{g}_1, \breve{g}_2 \rangle} \breve{g}_2.$$

The inner products of $\breve{g}_1$ and $\breve{g}_2$ are computed as

$$\begin{aligned}\langle \breve{g}_1, \breve{g}_2 \rangle &= -2\mathrm{Im}\langle \phi_1, \phi_2 \rangle = 0.9896\\ \|\breve{g}_1\|_2^2 &= 2\|\phi_1\|_2^2 + 2\mathrm{Re}\langle \phi_1, \phi_2 \rangle = 4.0730\\ \|\breve{g}_2\|_2^2 &= 2\|\phi_1\|_2^2 - 2\mathrm{Re}\langle \phi_1, \phi_2 \rangle = 1.7142.\end{aligned}$$

Thus

$$\begin{aligned}\|\tilde{\chi}_1\|_2 &= \|\breve{g}_1\|_2 = 2.0182,\\ \|\tilde{\chi}_2\|_2^2 &= -\|\breve{g}_1\|_2^2 + \frac{\|\breve{g}_1\|_2^4}{\langle \breve{g}_1, \breve{g}_2 \rangle^2} \|\breve{g}_2\|_2^2 = 24.9653.\end{aligned}$$

It follows that the following basis functions:

$$\chi_1(s) = 0.4955\breve{g}_1, \quad \chi_2(s) = 0.2001\breve{g}_1 - 0.8237\breve{g}_2 \tag{35}$$

are orthonormal and their linear span equals the linear span of ϕ_1 and ϕ_2. More explicitly, χ_1 and χ_2 can be written as

$$\begin{aligned}\chi_1(s) &= \frac{0.9910s^{1.5} + 0.8582}{s^3 + \sqrt{3}s^{1.5} + 1}\\ \chi_2(s) &= \frac{0.4002s^{1.5} - 0.4771}{s^3 + \sqrt{3}s^{1.5} + 1}.\end{aligned}$$

If the impulse responses of $\breve{g}_1$ and $\breve{g}_2$ are known, then the impulse responses of χ_1 and χ_2 denoted by $\hat{\chi}_1$ and $\hat{\chi}_2$, respectively, can be computed from (35) by superposition. The former impulse responses are also computed by superposition from the impulse responses of ϕ_1 and ϕ_2 denoted, respectively, by h_1 and h_2. From Lemma III.4, notice that ϕ_1 has two poles in Π_3 at $z_{11} = e^{-j(5\pi/9)}$, $z_{12} = e^{j(7\pi/9)}$; and ϕ_2 has two poles at the conjugate points $z_{21} = e^{j(5\pi/9)}$, $z_{22} = e^{-j(7\pi/9)}$. For ϕ_1, the residue term in (33) is computed from (34) as

$$\begin{aligned}\sum_{k=1}^{2} &\frac{e^{z_{1k}t}}{(s^{1.5} + e^{j(\pi/6)})'_{s=z_{1k}}}\\ &= \frac{1}{1.5} e^{-t\cos(4\pi/9) - j[t\sin(4\pi/9) - (5\pi/18)]}\\ &\quad + \frac{1}{1.5} e^{-t\cos(2\pi/9) + j[t\sin(2\pi/9) - 7\pi/18]}.\end{aligned}$$

A similar computation is made for ϕ_2. Thus

$$\begin{aligned}h_1(t) = &-\frac{1}{\pi}\int_0^\infty \frac{x^{1.5}}{x^3 + e^{j(\pi/3)}} e^{-xt} dx\\ &+ \frac{1}{1.5}\{e^{-t\cos(4\pi/9) - j[t\sin(4\pi/9) - (5\pi/18)]}\\ &\quad + e^{-t\cos(2\pi/9) + j[t\sin(2\pi/9) - (7\pi/18)]}\}\\ h_2(t) = &-\frac{1}{\pi}\int_0^\infty \frac{x^{1.5}}{x^3 + e^{-j(\pi/3)}} e^{-xt} dx\\ &+ \frac{1}{1.5}\{e^{-t\cos(2\pi/9) - j[t\sin(2\pi/9) - (7\pi/18)]}\\ &\quad + e^{-t\cos(4\pi/9) + j[t\sin(4\pi/9) - (5\pi/18)]}\}.\end{aligned}$$

Hence

$$\begin{aligned}\hat{\chi}_1(t) = &\, 0.6607e^{-0.1736t}\cos(0.9848t - 0.8727)\\ &+ 0.6607e^{-0.7660t}\cos(0.6428t - 1.2217)\\ &- \int_0^\infty \frac{0.3154x^{4.5} + 0.1577x^{1.5}}{x^6 + x^3 + 1} e^{-xt} dx\end{aligned}$$

$$\begin{aligned}\langle \phi_1, \phi_2 \rangle &= \frac{\frac{2\pi j}{1.5\pi}}{1 - e^{-j(2\pi/3)}} \left[\frac{(-e^{-j(7\pi/12)})^{-(1/3)}}{e^{j(11\pi/12)} - e^{-j(7\pi/12)}} + \frac{(-e^{j(11\pi/12)})^{-(1/3)}}{e^{-j(7\pi/12)} - e^{j(11\pi/12)}} \right]\\ &= \frac{4}{\sqrt{27}} e^{-j(2\pi/9)} = 0.5897 - j0.4948\end{aligned}$$

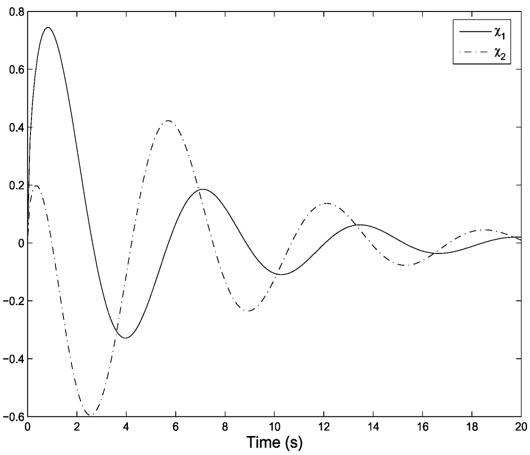

Fig. 3. Impulse responses of the synthesized basis functions.

$$\begin{aligned}\hat{\chi}_2(t) =& -1.1302e^{-0.1736t}\sin(0.9848t - 1.1110)\\ &+ 1.1302e^{-0.7660t}\sin(0.6428t - 0.9834)\\ &- \int_0^\infty \frac{0.1274x^{4.5} - 0.3904x^{1.5}}{x^6 + x^3 + 1} e^{-xt}dx.\end{aligned}$$

The periodic modes in both responses are linear combinations of two damped sinusoids, which quickly die off. Fig. 3 shows the impulse responses of χ_1 and χ_2. As expected from the initial value theorem, both responses start at zero. Note that

$$\phi_k(0+) = \lim_{s\to\infty} s\phi_k(s) = 0, \quad k = 1, 2.$$

VII. Conclusion

In this paper, fractional orthonormal basis functions with prescribed poles were synthesized. These basis functions were shown to be complete in $H_2(\Pi)$ under mild restrictions on the choice of the basis poles. This result enables one to approximate systems in $H_2(\Pi)$—in particular, the systems with both short and long memories, by convergent Fourier series of the fractional orthonormal basis functions of this paper.

The work initiated in this paper can be continued in several directions. First, completeness properties of the synthesized bases in different spaces—for example, the spaces in which the rational orthonormal bases have been shown to be complete—should be investigated. The convergence and the approximation properties of the Fourier series formed by the fractional orthonormal basis functions over some known subsets of these spaces need to be explored. It is worth studying the completeness problem for fractional incommensurable rationals with prescribed poles. Fast and reliable numerical methods are needed to evaluate the so-called memory integrals in (33). Then, it will be possible to quickly calculate time responses of the synthesized basis functions to arbitrary inputs.

Appendix A
Proof of Lemma III.1

Suppose that $\hat{\phi}$ has a pole at $z_\phi = re^{j\theta}$ in $\mathcal{C}_0$. Then

$$z_\phi^\gamma + z = e^{j\varphi}\left[r^\gamma e^{j(\gamma\theta-\varphi)} + R\right] = 0.$$

Hence, $r = R^{1/\gamma}$ and $|\gamma\theta - \varphi| = \pi$. Since $|\gamma\theta| < \gamma\pi$ for all θ, $\varphi \neq 0$. Moreover, $\varphi > 0$ implies $\theta \leq \pi$ and $\varphi < 0$ implies $\theta \geq 0$. It follows that

$$\gamma\theta - \varphi = \begin{cases} -\pi, & \text{if } \varphi > 0\\ \pi, & \text{if } \varphi < 0. \end{cases} \tag{36}$$

Thus, if there exists a pole at z_ϕ, its argument must satisfy

$$\theta = \begin{cases} \frac{(\varphi-\pi)}{\gamma}, & \varphi > 0\\ \frac{(\varphi+\pi)}{\gamma}, & \varphi < 0 \end{cases} \tag{37}$$

which implies from $|\theta| < \pi$ that $|\varphi| > \pi(1-\gamma)$.

Conversely, assume that $\pi(1-\gamma) < \varphi$. Set $\theta = (\varphi - \pi)/\gamma$. Then, θ satisfies the inequalities $-\pi < \theta \leq 0$. Therefore, it is consistent with the phase restriction in (13). Moreover, $\gamma\theta - \varphi = -\pi$. It follows that $R^{1/\gamma}e^{j\theta}$ is a pole. The proof of the other case $\varphi < -\pi(1-\gamma)$ is similar. The uniqueness follows from (37). This pole is simple since $\{s^\gamma + z\}'_{s=z_\phi} = \gamma z_\phi^{\gamma-1} \neq 0$. This completes the proof of the first claim.

For the second claim, assume first $\varphi = \pi(1-\gamma)$ and write $s^\gamma + z$ as

$$s^\gamma + z = e^{j\pi(1-\gamma)}[R - r^\gamma e^{j\gamma(\theta+\pi)}]. \tag{38}$$

From (38), $s^\gamma + z \to j2R\sin(\pi\gamma) \neq 0$ as $s \to -R^{1/\gamma}$ provided that s lies in Π_2. As $s \to -R^{1/\gamma}$ in Π_4, which implies $\theta \to -\pi$, $s^\gamma + z \to 0$. The proof of the case $\varphi = -\pi(1-\gamma)$ is similar.

To complete the proof, assume $|\varphi| < \pi(1-\gamma)$. Then, from

$$|\gamma\theta - \varphi| < \pi\gamma + \pi(1-\gamma) = \pi$$

we conclude that $\hat{\phi}$ is bounded in $\mathcal{C}_0$. The first part shows that it can not have a pole in $\mathcal{C}_0$ either. It follows that $\hat{\phi}$ is bounded analytic on $\mathcal{C}_0$.

Appendix B
Proof of Lemma III.2

For each fixed $s \in \Pi$, assuming $2\gamma m > 1$, $\psi(s)$ is well defined since

$$|(j\omega)^\gamma + s|^2 \geq w^{2\gamma}\cos^2\left(\frac{\pi\gamma}{2}\right) + x^2. \tag{39}$$

To derive (39), first note from (9) that

$$|(j\omega)^\gamma + x + jy|^2 = \begin{cases} \left[\omega^\gamma\cos\left(\frac{\pi\gamma}{2}\right) + x\right]^2 + \left[\omega^\gamma\sin\left(\frac{\pi\gamma}{2}\right) + y\right]^2, & \omega > 0\\ \left[|\omega|^\gamma\cos\left(\frac{\pi\gamma}{2}\right) + x\right]^2 + \left[-|\omega|^\gamma\sin\left(\frac{\pi\gamma}{2}\right) + y\right]^2, & \omega < 0 \end{cases}.$$

Then, an application of the inequality $x\cos(\pi\gamma/2) \geq 0$ valid for all $0 \leq \gamma \leq 1$ and $s \in \Pi$ to the above equation yields (39). Suppose $2\gamma m > 1$. For all $s \in \Pi$

$$\frac{1}{2\pi}\int_{-\infty}^\infty \psi(x+jy)dy = \frac{1}{4\pi^2}\int_{-\infty}^\infty \frac{d\omega}{|(j\omega)^\gamma + \lambda|^{2m-2}} \cdot \int_{-\infty}^\infty \frac{dy}{|(j\omega)^\gamma + x + jy|^2}. \tag{40}$$

Let $\Omega = |\omega|^\gamma \cos(\pi\gamma/2) + x$. By the change of variables $\Omega t = -|\omega|^\gamma \sin(\pi\gamma/2) + y$ $(\omega < 0)$ or $\Omega t = \omega^\gamma \sin(\pi\gamma/2) + y$ $(\omega > 0)$, we get for all $x > 0$

$$\int_{-\infty}^{\infty} \frac{dy}{|(j\omega)^\gamma + x + jy|^2} = \frac{1}{\Omega}\int_{-\infty}^{\infty} \frac{dt}{1+t^2} = \frac{\pi}{|\omega|^\gamma \cos\left(\frac{\pi\gamma}{2}\right) + x}. \quad (41)$$

The restriction $\gamma \in (0,1)$ ensures $\Omega > 0$ for all $s \in \Pi$. If for a given $\gamma \in (0,1)$, an m satisfying (14) is chosen, then from (40), (41), and (39), we have for all $x > 0$

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(x+jy)dy \\
&\leq \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\left[\omega^{2\gamma}\cos^2\left(\frac{\pi\gamma}{2}\right) + \lambda^2\right]^{1-m} d\omega}{|\omega|^\gamma \cos\left(\frac{\pi\gamma}{2}\right)} \\
&= \frac{1}{4\pi}\int_{-1}^{1} \frac{\left[\omega^{2\gamma}\cos^2\left(\frac{\pi\gamma}{2}\right) + \lambda^2\right]^{1-m} d\omega}{|\omega|^\gamma \cos\left(\frac{\pi\gamma}{2}\right)} \\
&\quad + \frac{1}{4\pi}\int_{|\omega|>1} \frac{\left[\omega^{2\gamma}\cos^2\left(\frac{\pi\gamma}{2}\right) + \lambda^2\right]^{1-m} d\omega}{|\omega|^\gamma \cos\left(\frac{\pi\gamma}{2}\right)} \\
&\leq \frac{1}{2\pi}\int_0^1 \frac{\lambda^{2-2m} d\omega}{\omega^\gamma \cos\left(\frac{\pi\gamma}{2}\right)} + \frac{1}{2\pi}\int_1^{\infty} \frac{\omega^{(1-2m)\gamma} d\omega}{\cos^{2m-1}\left(\frac{\pi\gamma}{2}\right)} \\
&= \frac{\lambda^{2-2m}}{2\pi(1-\gamma)\cos\left(\frac{\pi\gamma}{2}\right)} + \frac{\left[\cos^{2m-1}\left(\frac{\pi\gamma}{2}\right)\right]^{-1}}{2\pi[(2m-1)\gamma - 1]}.
\end{aligned}$$

Taking supremum of the left-hand side with respect to $x > 0$, (16) is obtained.

Appendix C
Proof of Theorem III.3

Let $\mathcal{B} = \{\phi_{k\ell} : 1 \leq \ell \leq n_k; k \geq 1\}$. The proof is by contradiction. So, we assume there exists a nontrivial $h \in H_2(\Pi)$ orthogonal to $\mathcal{B}$

$$\int_{-\infty}^{\infty} \frac{[(j\omega)^\gamma + \lambda_k]^{-\ell}\overline{h(j\omega)}}{[(j\omega)^\gamma + \lambda]^{m-1}} d\omega = 0, \quad \ell = 1, \ldots, n_k \quad (42)$$

$k \geq 1$. Then, we define a complex function f on Π by

$$f(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\overline{h(j\omega)} d\omega}{[(j\omega)^\gamma + \lambda]^{m-1}[(j\omega)^\gamma + s]}. \quad (43)$$

By an application of the Cauchy–Schwarz inequality

$$|f(s)| \leq \|h\|_2\sqrt{\psi(s)}, \quad s \in \Pi.$$

Hence, for all $s \in \Pi$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x+jy)|^2 dy \leq \frac{\|h\|_2^2}{2\pi}\int_{-\infty}^{\infty} \psi(x+jy)dy. \quad (44)$$

Thus, from Lemma III.2 and (44)

$$\|f\|_2 \leq \|h\|_2\sqrt{\Psi}. \quad (45)$$

Moreover, an application of the bounded convergence theorem to (43) shows that f is continuous on Π. Next, we apply Morera's theorem (see, for example, [49, Theorem 10.17]) to show that f is analytic on Π. In the application of Morera's theorem, the change of the integration orders is justified by Fubini's theorem and the fact that $[(j\omega)^\gamma + s]^{-1}$ (as a function of s) is analytic on Π is used. Thus, from (45), we have $f \in H_2(\Pi)$.

Now, repeatedly differentiate f

$$f^{(n)}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(-1)^n n!\overline{h(j\omega)} d\omega}{[(j\omega)^\gamma + s]^{n+1}[(j\omega)^\gamma + \lambda]^{m-1}} \quad (46)$$

for $n \geq 0$ and evaluate the derivatives at $s = \lambda$ to get

$$f^{(n)}(\lambda) = \frac{(-1)^n n!}{2\pi}\int_{-\infty}^{\infty} \frac{\overline{h(j\omega)} d\omega}{[(j\omega)^\gamma + \lambda]^{m+n}}. \quad (47)$$

Note that the orthogonality relations can be written as

$$f^{(\ell)}(\lambda_k) = 0, \quad 0 \leq \ell < n_k; \ \ k \geq 1.$$

Now, we are ready to finish the proof of our assertion. To this end, let $g = f \circ \Lambda$, where Λ is the bilinear map

$$s = \Lambda(z) = \frac{1-z}{1+z}. \quad (48)$$

Let $\alpha_k = \Lambda(\lambda_k)$ for $k \geq 1$. Then, $g \in H_2(\mathbf{D}_1(0))$ (see, for example, [39, Theorem 11.1]). By the chain rule of differentiation, observe that $g^{(\ell)}(\alpha_k)$ is a linear combination of the first ℓ derivatives of $f(s)$ at $s = \lambda_k$ and hence equals zero. Thus

$$g^{(\ell)}(\alpha_k) = 0, \quad 0 \leq \ell < n_k; \ \ k \geq 1.$$

Since

$$\sum_{k=1}^{\infty} \frac{n_k \mathrm{Re}(\lambda_k)}{1+|\lambda_k|^2} < \sum_{k=1}^{\infty} n_k(1-|\alpha_k|) \leq 2\sum_{k=1}^{\infty} \frac{n_k \mathrm{Re}(\lambda_k)}{1+|\lambda_k|^2} \quad (49)$$

the zeros of g satisfy $\sum_{k=1}^{\infty} n_k(1-|\alpha_k|) = \infty$. (The proof of the inequalities in (49) is contained in the proof of Lemma 3 in [33]). This implies $g(z) = 0$ for all $z \in \mathbf{D}_1(0)$ (see, for example, the corollary to [49, Th. 15.23]). Hence, f vanishes on Π. In particular

$$f^{(n)}(\lambda) = 0, \quad n = 0, 1, 2, \ldots. \quad (50)$$

The equations in (50) imply that the linear span of the functions $(s^\gamma + \lambda)^{-n}, \ n \geq m$ is not dense in $H_2(\Pi)$, which is a contradiction [37]. It follows that $span(\mathcal{B})$ is dense in $H_2(\Pi)$ for all $\gamma \in (0,1)$ and m satisfying (14).

Appendix D
Proof of Lemma III.5

Let $s = re^{j\theta} \in \mathcal{C}_{\bar{\gamma}}$. Then

$$|(j\omega)^\gamma + s|^2 = \begin{cases} \omega^{2\gamma} + r^2 + 2\omega^\gamma r\cos\left(\frac{\pi\gamma}{2} - \theta\right), & \omega > 0 \\ \omega^{2\gamma} + r^2 + 2|\omega|^\gamma r\cos\left(\frac{\pi\gamma}{2} + \theta\right), & \omega < 0. \end{cases}$$

Hence

$$|(j\omega)^\gamma + s|^2 \geq \omega^{2\gamma} + r^2 - 2|\omega|^\gamma r \cos\left[\frac{\pi(\bar{\gamma}-\gamma)}{2}\right]. \quad (51)$$

Let $\mu = \cos[\pi(\bar{\gamma}-\gamma)/2]$. Then, $\mu \in (0,1)$. Rewrite (51)

$$|(j\omega)^\gamma + s|^2 \geq (|\omega|^\gamma - r\mu)^2 + (1-\mu^2)r^2. \quad (52)$$

Let us first consider the case $\mathcal{C}_{\bar{\gamma}} \cap \mathbf{D}_1(0)$ in (20). In $\mathbf{D}_1(0)$, $1-e^{-s}$ is represented by

$$1 - e^{-s} = s + \zeta(s)s^2 \quad (53)$$

where $\zeta(s)$ satisfies $|\zeta(s)| \leq e$. Then, from (52) and (53), we have (54) shown at the bottom of the page, where the first inequality at the top is due to the symmetry of the set $\mathcal{C}_{\bar{\gamma}} \cap \mathbf{D}_1(0)$ with respect to $\mathbf{R}$ and the first and the second equalities from the top have followed from the change of variables $t = \omega^\gamma$ and $t - r\mu = r\sqrt{1-\mu^2}\,u$, respectively.

For the case $\mathcal{C}_{\bar{\gamma}} \cap [\mathbf{D}_1(0)]^{\mathrm{c}}$ in (16), where $[\mathbf{D}_1(0)]^{\mathrm{c}}$ denotes the complement of $\mathbf{D}_1(0)$ in $\mathbf{C}$, first note the inequality $|1-e^{-s}| < 2$ valid for all $s \in \Pi$. Then

$$\begin{aligned}
\pi \sup_{s\in\mathcal{C}_{\bar{\gamma}}\cap[\mathbf{D}_1(0)]^{\mathrm{c}}} \vartheta(s)
&\leq \sup_{s\in\mathcal{C}_{\bar{\gamma}}\cap[\mathbf{D}_1(0)]^{\mathrm{c}}} \int_0^\infty \left|\frac{1-e^{-s}}{(j\omega)^\gamma + s}\right|^2 d\omega \\
&< 2\sup_{r\geq 1}\int_0^\infty \frac{d\omega}{(\omega^\gamma - r\mu)^2 + (1-\mu^2)r^2} \\
&= \frac{2}{\gamma}\sup_{r\geq 1}\frac{r^{(1/\gamma)-2}}{\sqrt{1-\mu^2}} \\
&\quad \cdot \int_{-\left(\mu/\sqrt{1-\mu^2}\right)}^\infty \frac{\left(\mu + u\sqrt{1-\mu^2}\right)^{(1-\gamma/\gamma)} du}{1+u^2} \\
&= \frac{2}{\gamma\sqrt{1-\mu^2}} \\
&\quad \cdot \int_{-\left(\mu/\sqrt{1-\mu^2}\right)}^\infty \frac{du}{(\mu + u\sqrt{1-\mu^2})^{(\gamma-1/\gamma)}(1+u^2)} \\
&< \frac{2}{\gamma\sqrt{1-\mu^2}}\left[\frac{\gamma\mu^{(1/\gamma)}}{\sqrt{1-\mu^2}} + \frac{\pi}{2}\mu^{-(\gamma-1/\gamma)}\right].
\end{aligned} \quad (55)$$

Thus, combining (54) and (55), we get

$$\Phi < \frac{(1+e)^2}{\pi\gamma\sqrt{1-\mu^2}}\left[\frac{\gamma\mu^{(1/\gamma)}}{\sqrt{1-\mu^2}} + \frac{\pi}{2}\mu^{-(\gamma-1/\gamma)}\right] < \infty$$

what we were set to prove.

Appendix E
Proof of Theorem III.6

Let us first show the completeness of the functions

$$\varsigma_{k\ell}(s) = \frac{1-e^{-\lambda_k}}{(s^\gamma + \lambda_k)^\ell}, \quad 1 \leq \ell \leq n_k;\ \ k \geq 1. \quad (56)$$

Let $\widetilde{\mathcal{B}} = \{\varsigma_{k\ell} : 1 \leq \ell \leq n_k; k \geq 1\}$. The completeness of the functions (56) is shown by contradiction. So, we assume that there exists a nontrivial $\tau \in H_2(\Pi)$ orthogonal to $\widetilde{\mathcal{B}}$

$$\int_{-\infty}^\infty \frac{1-e^{-\lambda_k}}{[(j\omega)^\gamma + \lambda_k]^\ell}\overline{\tau(j\omega)}d\omega = 0, \quad 1 \leq \ell \leq n_k \quad (57)$$

$$\begin{aligned}
\pi \sup_{s\in\mathcal{C}_{\bar{\gamma}}\cap\mathbf{D}_1(0)} \vartheta(s) &\leq \sup_{s\in\mathcal{C}_{\bar{\gamma}}\cap\mathbf{D}_1(0)} \int_0^\infty \left|\frac{1-e^{-s}}{(j\omega)^\gamma + s}\right|^2 d\omega \\
&\leq (1+e)^2 \sup_{r<1}\int_0^\infty \frac{r^2 d\omega}{(\omega^\gamma - r\mu)^2 + (1-\mu^2)r^2} \\
&= \frac{(1+e)^2}{\gamma}\sup_{r<1}\int_0^\infty \frac{r^2 dt}{t^{(\gamma-1/\gamma)}\left[(t-r\mu)^2 + (1-\mu^2)r^2\right]} \\
&= \sup_{r<1}\frac{(1+e)^2 r^{(1/\gamma)}}{\gamma\sqrt{1-\mu^2}}\int_{-\left(\mu/\sqrt{1-\mu^2}\right)}^\infty \frac{\left(\mu + u\sqrt{1-\mu^2}\right)^{(1-\gamma/\gamma)}}{1+u^2}du \\
&< \frac{(1+e)^2}{\gamma\sqrt{1-\mu^2}}\left[\int_{-\left(\mu/\sqrt{1-\mu^2}\right)}^0 \frac{du}{\left(\mu + u\sqrt{1-\mu^2}\right)^{(\gamma-1/\gamma)}} + \frac{1}{\mu^{(\gamma-1/\gamma)}}\int_0^\infty \frac{du}{1+u^2}\right] \\
&= \frac{(1+e)^2}{\gamma\sqrt{1-\mu^2}}\left[\int_0^\mu \frac{v^{-(\gamma-1/\gamma)}dv}{\sqrt{1-\mu^2}} + \frac{\pi}{2}\mu^{-(\gamma-1/\gamma)}\right] \\
&= \frac{(1+e)^2}{\gamma\sqrt{1-\mu^2}}\left[\frac{\gamma\mu^{(1/\gamma)}}{\sqrt{1-\mu^2}} + \frac{\pi}{2}\mu^{-(\gamma-1/\gamma)}\right]
\end{aligned} \quad (54)$$

$k \geq 1$. Then, we define a complex function χ on $\mathcal{C}_{\bar{\gamma}}$ by

$$\chi(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-s}}{(j\omega)^{\gamma} + s} \overline{\tau(j\omega)} d\omega, \quad s \in \mathcal{C}_{\bar{\gamma}}. \tag{58}$$

The existence of the above integral follows from Lemma III.5 by an application of the Cauchy–Schwarz inequality to (58). Thus

$$|\chi(s)| \leq \|\tau\|_2 \sqrt{\vartheta(s)}, \quad s \in \mathcal{C}_{\bar{\gamma}}. \tag{59}$$

Hence, from Lemma III.5 and (59)

$$\sup_{s \in \mathcal{C}_{\bar{\gamma}}} |\chi(s)| \leq \|\tau\|_2 \sqrt{\Phi}. \tag{60}$$

The same argument in the proof of Theorem III.3 can be used to show that χ is analytic on $\mathcal{C}_{\bar{\gamma}}$. Thus, from (60), we have $\chi \in H_\infty(\mathcal{C}_{\bar{\gamma}})$.

Now, repeatedly differentiate χ for $n = 1, 2, \ldots$

$$\chi^{(n)}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \left\{ [(j\omega)^{\gamma} + s]^{-1} \right\}^{(m)} \cdot (1 - e^{-s})^{(n-m)} \overline{\tau(j\omega)} d\omega \tag{61}$$

and evaluate the derivatives at $s = \lambda$ to obtain

$$\chi^{(n)}(\lambda) = \begin{cases} (1 - e^{-\lambda}) c_1, & n = 0 \\ e^{-\lambda} \sum\limits_{m=0}^{n} \frac{(-1)^{n-1} n!}{(n-m)!} c_{m+1}, & n \geq 1 \end{cases} \tag{62}$$

where

$$c_m(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\overline{\tau(j\omega)} d\omega}{[(j\omega)^{\gamma} + \lambda]^m}, \quad m = 1, 2, \ldots. \tag{63}$$

Note from (63) that the orthogonality relations in (57) can be written as

$$(1 - e^{-\lambda_k}) c_\ell(\lambda_k) = 0, \quad 1 \leq \ell \leq n_k; \ k \geq 1. \tag{64}$$

Since $|e^{-\lambda_k}| < 1$ for all k, (64) implies

$$c_\ell(\lambda_k) = 0, \quad 1 \leq \ell \leq n_k; \ k \geq 1. \tag{65}$$

Then, (65) and (62) imply

$$\chi^{(\ell)}(\lambda_k) = 0, \quad 0 \leq \ell < n_k; \ k \geq 1.$$

Let

$$\beta_k = \lambda_k^a, \quad k = 1, 2, \ldots$$

and define a complex function ν as a composition of χ with $w^{2-\bar{\gamma}}$

$$\nu(w) = \chi\left(w^{2-\bar{\gamma}}\right), \quad w \in \Pi.$$

Then, $\beta_k \in \Pi$ for all k and the map $s = w^{2-\bar{\gamma}}$ sends Π conformally onto $\mathcal{C}_{\bar{\gamma}}$. Moreover, $\nu \in H_\infty(\Pi)$. Since $\lambda_k \neq 0$, $w^{2-\bar{\gamma}}$ has derivatives of all orders at λ_k. By the chain rule of differentiation, observe that $\nu^{(\ell)}(\beta_k)$ is a linear combination of the first ℓ derivatives of $\chi(s)$ at $s = \lambda_k$ and hence equals zero. Thus

$$\nu^{(\ell)}(\beta_k) = 0, \quad 0 \leq \ell < n_k; \ k \geq 1.$$

Since $\mathrm{Re}(\beta_k) = r_k^a \cos(a\theta_k)$ and $|\beta_k| = r_k^a$, the zeros of ν satisfy

$$\sum_{k=1}^{\infty} n_k \frac{\mathrm{Re}(\beta_k)}{1 + |\beta_k|^2} = \infty. \tag{66}$$

Now, let $\kappa = \nu \circ \Lambda$ and $\gamma_k = \Lambda(\beta_k)$ for $k \geq 1$, where Λ is the bilinear map (48). Then, $\kappa \in H_\infty(\mathbf{D}_1(0))$. The argument in the proof of Theorem III.6 shows that

$$\kappa^{(\ell)}(\gamma_k) = 0, \quad 0 \leq \ell < n_k; \ k \geq 1.$$

Furthermore, from (66) and (49), the zeros of κ satisfy $\sum_{k=1}^{\infty} n_k(1 - |\gamma_k|) = \infty$. Then, the corollary to [49, Theorem 15.23] tells us that κ vanishes on $\mathbf{D}_1(0)$. Consequently, ν vanishes on Π. This implies that χ vanishes on $\mathcal{C}_{\bar{\gamma}}$. In particular, for a fixed $\lambda \in \mathcal{C}_{\bar{\gamma}}$

$$\chi^{(n)}(\lambda) = 0, \quad n = 0, 1, 2, \ldots. \tag{67}$$

The set of (62) can be solved recursively for c_n. Then, from (67), we have $c_n(\lambda) = 0$ for all n. This implies that the linear span of the functions $(s^{\gamma} + \lambda)^{-n}, n \geq 1$ is not dense in $H_2(\Pi)$, which is a contradiction [37]. It follows that span$(\widetilde{\mathcal{B}})$ is dense in $H_2(\Pi)$ for all $\gamma \in (1, 2)$. Since $1 - e^{-\lambda_k} \neq 0$ for all k, span$(\widetilde{\mathcal{B}})$ equals to the linear span of the functions (19).

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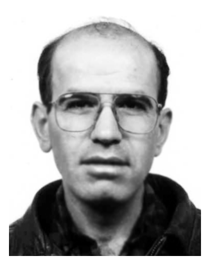

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