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Discrete-time system modelling in L_p with orthonormal basis functions

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Abstract

In this paper, model sets for linear time-invariant systems spanned by fixed pole orthonormal bases are investigated. The obtained model sets are shown to be complete in $L_p(\boldsymbol{T})$ $(1 < p < \infty)$, the Lebesque spaces of functions on the unit circle $\boldsymbol{T}$, and in $C(\boldsymbol{T})$, the space of periodic continuous functions on $\boldsymbol{T}$. The L_p norm error bounds for estimating systems in $L_p(\boldsymbol{T})$ by the partial sums of the Fourier series formed by the orthonormal functions are computed for the case $1 < p < \infty$. Some inequalities on the mean growth of the Fourier series are also derived. These results have application in estimation and model reduction. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The decomposing description of linear time-invariant infinite-dimensional dynamics in terms of an orthonormal basis is an important part of modern Systems Theory and has a long history in modelling and identification of dynamical systems dating back to the classical work of Lee [11] and Wiener [27]. This approach is of greatest utility when accurate system descriptions are achieved with a small number of basis functions. The development of suitable basis functions that reflect the dominant characteristics of the system has attracted considerable interest [17,19,21–26,13,14,9,2–5].

In particular, in the areas of control theory, signal processing and system identification, there has long been interest in the use of the finite-impulse response, the *Laguerre*, and the *two-parameter Kautz* functions to model stable linear dynamical systems [11,10,8]. The Laguerre and the Kautz models are special cases of the general orthonormal basis functions in [9], where the poles of the system transfer function are restricted to a finite set. The general orthonormal basis functions are generalized by the rational orthonormal basis functions with fixed poles considered in detail in [14,2–4].

In [2] the rational orthonormal basis functions were shown to be complete in the disk algebra provided that the chosen basis poles satisy a mild condition and more recently in [4], it was established that the Fourier series formed by the rational orthonormal basis functions converges in the Hardy spaces.

In this paper, a similar completeness result is obtained for the spaces $L_p(\boldsymbol{T})$ $(1 \leqslant p < \infty)$ and $C(\boldsymbol{T})$. As the orthonormal system, we consider a set of complex-valued rational functions $\{B_n\}$ defined by a choice of numbers z_n and x_n in the open unit $\boldsymbol{D}$ as $B_0 = \sqrt{1 - |z_0|^2}/(1 - \overline{z_0}z)$ and for $n = 1, 2, \ldots$

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$$B_n = \frac{\sqrt{1-|z_n|^2}}{1-\overline{z_n}z}\phi_n, \qquad \phi_n = \prod_{j=0}^{n-1} \frac{z-z_j}{1-\overline{z_j}z}, \tag{1}$$

$$B_{-n} = \frac{\sqrt{1-|x_n|^2}}{z-x_n}\phi'_{n-1}, \qquad \phi'_n = \prod_{j=1}^{n} \frac{1-\overline{x_j}z}{z-x_j}, \tag{2}$$

where $\phi'_0 = 1$. The orthonormality is with respect to the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta.$$

We will establish the following completeness result.

Theorem 1. *The linear span of the functions* $\{B_n\}$ *defined by* (1)–(2) *are everywhere dense in* $L_p(\boldsymbol{T})$ $(1 < p < \infty)$ *as well as in* $C(\boldsymbol{T})$ *if and only if*

$$\sum_{n=0}^{\infty}(1-|z_n|) = \infty, \qquad \sum_{n=1}^{\infty}(1-|x_n|) = \infty. \tag{3}$$

This result has interesting applications on the robust recovery of functions in $L_p(\boldsymbol{T})$ from noise-corrupted evaluations on the unit circle. An abstract framework that solves this type of problems is outlined in [15]. In the modelling of physical systems, it is necessary to ensure that the modelled impulse response is real valued. This issue is addressed in Section 5.

The next result concerns the Fourier series of integrable functions on $\boldsymbol{T}$ with respect to the orthonormal system (1)–(2) whose partial sums are defined by

$$\mathscr{S}_n f(e^{i\theta}) = \sum_{k=-n}^{n} \langle f, B_k\rangle B_k(e^{i\theta}). \tag{4}$$

The L_p norm errors of the estimate (4) are computed quite accurately for the case $1 < p < \infty$. In establishing this, an essential role is played by the *Blaschke* products in (1)–(2). Relations between projection operators, conjugate functions, and the Fourier series are also displayed. Having computed the error bounds for the partial sums of the Fourier series (4), we provide bounds on the mean growth of the Fourier coefficients $\{\langle f, B_k\rangle\}$ and derive the so-called Hausdorff–Young inequalities.

Finally, a simulation example is given to illustrate the use of the basis functions defined by (1)–(2) for modelling.

2. Completeness of the orthonormal system

We will represent $\mathscr{S}_n f$ in terms of two Cauchy integrals of f when $f(e^{i\theta})$ is the restriction to $\boldsymbol{T}$ of a complex function which is analytic on a region that contains $\boldsymbol{T}$. This representation facilitates a simple proof of Theorem 1. The analysis of the estimate (4) will be based on these formulae. To this end, first we have the following lemma.

Lemma 2 (Christoffel–Darboux formulae)**.**

$$\sum_{k=0}^{n} \overline{B_k(\zeta)}B_k(z) = \frac{1-\overline{\phi_{n+1}(\zeta)}\phi_{n+1}(z)}{1-\bar{\zeta}z}, \quad z\bar{\zeta} \neq 1, \tag{5}$$

$$\sum_{k=-n}^{-1} \overline{B_k(\zeta)}B_k(z) = \frac{1-\overline{\phi'_n(\zeta)}\phi'_n(z)}{\bar{\zeta}z-1}, \quad z\bar{\zeta} \neq 1. \tag{6}$$

Proof. The formulae (5)–(6) are well known in the literature on approximation. A concise proof of (5) is by induction; for $n = 0$,

$$\frac{1-\overline{\phi_1(\zeta)}\phi_1(z)}{1-\bar{\zeta}z} = \frac{1-|z_0|^2}{(1-z_0\bar{\zeta})(1-\overline{z_0}z)} = \overline{B_0(\zeta)}B_0(z)$$

while for $n > 0$

$$\frac{1-\overline{\phi_{n+1}(\zeta)}\phi_{n+1}(z)}{1-\bar{\zeta}z} = \sum_{k=0}^{n-1} \overline{B_k(\zeta)}B_k(z) + (1-|z_n|^2)\frac{\overline{\phi_n(\zeta)}\phi_n(z)}{(1-z_n\bar{\zeta})(1-\overline{z_n}z)}.$$

The proof of (6) follows from (5) by the transformations and back transformations $z \mapsto 1/z$, $\zeta \mapsto 1/\zeta$, $x_j \mapsto \overline{z_{j-1}}$, $j = 1, 2, \ldots, n$. □

Hence from (5)–(6), we get for the two components of the sum in (4)

$$\sum_{k=0}^{n} \langle f, B_k\rangle B_k(z) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{\phi_{n+1}(z)}{2\pi i}\int_{\gamma_0} \frac{f(\zeta)\,d\zeta}{(\zeta-z)\phi_{n+1}(\zeta)} \tag{7}$$

$$\sum_{k=-n}^{-1} \langle f, B_k\rangle B_k(z) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(\zeta)}{z-\zeta}\,d\zeta - \frac{\phi'_n(z)}{2\pi i}\int_{\gamma_0} \frac{f(\zeta)\,d\zeta}{(z-\zeta)\phi'_n(\zeta)} \tag{8}$$

where $\gamma_0(s) = e^{is}$ $(0 \leqslant s \leqslant 2\pi)$.

Let $A(r_1,r_2)$ be the annulus $\{z: r_1<|z|<r_2\}$, where $r_1<1$ and $r_2>1$ are two given positive numbers. Suppose that $f(z)$ is analytic in a region that contains $A(r_1,r_2)$. Then the following Cauchy formula is valid on $A(r_1,r_2)$:

$$f(z)=\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{\zeta-z}d\zeta+\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta-z}d\zeta \tag{9}$$

where

$$\gamma_1(s)=r_1e^{-is},\qquad \gamma_2(s)=r_2e^{is}\quad(0\leqslant s\leqslant 2\pi).$$

The integrands in (7) are meromorphic functions on $A(r_1,r_2)$ whose singularities are inside γ_0 and are encircled once by the contours γ_0 and γ_2. Hence by the residue theorem [18, Theorem 10.42]

$$\sum_{k=0}^{n}\langle f,B_k\rangle B_k(z)=\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta-z}\,d\zeta-\frac{\phi_{n+1}(z)}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{(\zeta-z)\phi_{n+1}(\zeta)}\,d\zeta$$

and letting $z\to e^{i\theta}$, we obtain

$$\sum_{k=0}^{n}\langle f,B_k\rangle B_k(e^{i\theta})=\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta-e^{i\theta}}\,d\zeta-\frac{\phi_{n+1}(e^{i\theta})}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{(\zeta-e^{i\theta})\phi_{n+1}(\zeta)}\,d\zeta. \tag{10}$$

Since the integrands in (8) are analytic on $A(r_1,r_2)$, their integrals on the cycle $\gamma_0\cup\gamma_1$ must vanish by the Cauchy theorem. Hence

$$\sum_{k=-n}^{-1}\langle f,B_k\rangle B_k(z)=\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{\zeta-z}\,d\zeta-\frac{\phi_n'(z)}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{(\zeta-z)\phi_n'(\zeta)}\,d\zeta$$

and letting $z\to e^{i\theta}$, we get

$$\sum_{k=-n}^{-1}\langle f,B_k\rangle B_k(e^{i\theta})=\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{\zeta-e^{i\theta}}\,d\zeta-\frac{\phi_n'(e^{i\theta})}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{(\zeta-e^{i\theta})\phi_n'(\zeta)}\,d\zeta. \tag{11}$$

Thus from (9) and (11)

$$f(e^{i\theta})-\mathscr{S}_nf(e^{i\theta})=\frac{\phi_{n+1}(e^{i\theta})}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{(\zeta-e^{i\theta})\phi_{n+1}(\zeta)}\,d\zeta+\frac{\phi_n'(e^{i\theta})}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{(\zeta-e^{i\theta})\phi_n'(\zeta)}\,d\zeta. \tag{12}$$

The third step is to bound $f-\mathscr{S}_nf$. First we have the following lemma.

Lemma 3.

$$\sup_{z\in\gamma_2}\frac{1}{|\phi_{n+1}(z)|}\leqslant\exp\left(-\frac{r_2-1}{2r_2}\sum_{j=0}^{n}(1-|z_j|)\right), \tag{13}$$

$$\sup_{z\in\gamma_1}\frac{1}{|\phi_n'(z)|}\leqslant\exp\left(-\frac{1-r_1}{2}\sum_{j=1}^{n}(1-|x_j|)\right). \tag{14}$$

Proof. Let $\bar{w}=z^{-1}$. Then

$$\frac{1}{|\phi_{n+1}(z)|}=|\phi_{n+1}(w)|\leqslant\prod_{j=0}^{n}\left|\frac{w-z_j}{1-\overline{z_j}w}\right|. \tag{15}$$

Let $w=re^{i\theta}$ and $z_j=R_je^{i\theta_j}$ denote the polar decompositions of w and z_j. Then a simple algebraic manipulation yields

$$\left|\frac{w-z_j}{1-\overline{z_j}w}\right|^2\leqslant 1-(1-r)(1-R_j)\leqslant\exp(-(1-r)(1-R_j)), \tag{16}$$

where the last inequality follows from the fact that $e^{-x}\geqslant 1-x$ for all x. Consideration of (15) and (16) with $r=1/r_2$ completes the proof of (13). The proof of (14) is similar. □

Hence from Lemma 3 and the integral formulation of the approximation error (12)

$$\|f-\mathscr{S}_nf\|_\infty\leqslant\sup_{z\in A(r_1,r_2)}|f(z)|\frac{r_2}{r_2-1}\exp\left(-\frac{r_2-1}{2r_2}\sum_{j=0}^{n}(1-|z_j|)\right)+\sup_{z\in A(r_1,r_2)}|f(z)|\frac{r_1}{1-r_1}\exp\left(-\frac{1-r_1}{2}\sum_{j=1}^{n}(1-|x_j|)\right).$$

Now we complete the proof of the sufficiency. Let $f \in L_p(\boldsymbol{T})$. Recall that the trigonometric system $\{e^{\pm ik\theta}\}$ is closed in $C(\boldsymbol{T})$ (Weierstrass' second theorem) and hence in $L_p(\boldsymbol{T})$ since $C(\boldsymbol{T})$ is a dense subset of $L_p(\boldsymbol{T})$. Thus, we may assume without restriction that $f(e^{i\theta})$ is a trigonometric polynomial. Since f extends to an analytic function on the punctured plane $\boldsymbol{A}(0,\infty)$, it follows from the above inequality with $r_1 = 1/2$ and $r_2 = 2$ that

$$\lim_{n\to\infty} \sup_{0\leqslant\theta\leqslant 2\pi} |f(e^{i\theta}) - \mathscr{S}_n f(e^{i\theta})| = 0$$

provided that the conditions in (3) are satisfied. This proves the sufficiency.

For the necessity, assume that

$$\sum_{n=0}^{\infty} (1 - |z_n|) < \infty.$$

Then the unimodulated finite Blaschke products $\phi_n(z)$ in (1) converge uniformly on $\boldsymbol{D}$ to a Blaschke product

$$\phi(z) = \prod_{n=0}^{\infty} \frac{z_n - z}{1 - \overline{z_n} z} \frac{|z_n|}{z_n}$$

(with the convention $|z_n|/z_n = 1$ when $z_n = 0$) which has zeros precisely at the points z_n. In this case, the linear functional Φ defined on $L_p(\boldsymbol{T})$ $(1 \leqslant p < \infty)$ and $C(\boldsymbol{T})$ by $\Phi(f) = \langle f, \phi\rangle$ is clearly nontrivial and also bounded. However by Cauchy's theorem it also vanishes at every B_n as

$$\overline{\Phi(B_n)} = (-1)^{n+1} \prod_{k=0}^{n} \frac{|z_k|}{z_k} \frac{1}{2\pi i} \int_{\gamma_0} \frac{\sqrt{1-|z_n|^2}}{1 - \overline{z_n}\,\zeta} \times \prod_{k=n+1}^{\infty} \frac{z_k - \zeta}{1 - \overline{z_k}\,\zeta} \frac{|z_k|}{z_k}\, d\zeta = 0.$$

With the same reasoning we have $\Phi(B_n) = 0$ for all $n < 0$. Hence the linear span of the sets $\{B_n(e^{i\theta})\}$ is not dense in the spaces $C(\boldsymbol{T})$ and $L_p(\boldsymbol{T})$ $(p \geqslant 1)$. The other case

$$\sum_{n=1}^{\infty} (1 - |x_n|) < \infty$$

is similar and it suffices to consider the Blaschke product

$$\phi'(z) = \prod_{n=1}^{\infty} \frac{1 - \overline{x_n} z}{x_n - z} \frac{x_n}{|x_n|}$$

which is analytic on $\boldsymbol{A}(1,\infty)$ and has common zeros with the functions $B_n(z)$, $n < 0$ and the linear functional Φ' defined on $L_p(\boldsymbol{T})$ $(1 \leqslant p < \infty)$ and $C(\boldsymbol{T})$ by $\Phi'(f) = \langle f, \phi'/z\rangle$.

Remark 4. In Achieser [1], Theorem 1 is proven for the rational functions in the form

$$\left\{ \frac{1}{e^{i\theta} - z_n} \right\}_{n=1}^{\infty} \quad (0 \leqslant \theta \leqslant 2\pi),$$

where $\{z_n\}$ is a given sequence of distinct complex numbers satisfying $|z_n| \neq 1$. These functions do not include the exponentials $\{e^{\pm in\theta}\}$ whereas the orthonormal functions defined by (1)–(2) include them in the special case $z_n = x_n = 0$ for all n.

The proof in Achieser builds on the solution of a certain extremal problem. When suited for the basis functions in (1)–(2), this extremal problem directly yields Theorem 1. We omit the details. Our proof on the other hand is based on the integral formulation of the approximation error.

The completeness conditions (3) are very mild. For example, removing a finite number of pole parameters z_n and x_n does not destroy the completeness as the same conditions still apply. This stability property is not seen in the bases spanned by the complex exponentials $\{e^{i\lambda_n\theta}\}$ where $\{\lambda_n\}$ is a sequence of real or complex numbers. For example if $\{\lambda_n\}$ satisfies

$$|\lambda_n - n| \leqslant \frac{1}{2p}, \quad n = 0, \pm 1, \pm 2, \ldots$$

then $\{e^{i\lambda_n\theta}\}$ is complete in $L_p(\boldsymbol{T})$ $(1 < p < \infty)$ (Kadec's $\frac{1}{4}$-theorem). However, the constant $1/2p$ cannot be replaced by any larger number.

3. Mean convergence of the Fourier series

In this section we show that the Fourier series formed by the orthonormal functions in (1)–(2) converges in the spaces $L_p(\boldsymbol{T})$ $(1 < p < \infty)$.

Let $S_n f$ denote the partial sums of the Fourier series of an integrable function f with respect to the exponential functions $\{e^{\pm ik\theta}\}$. It is well-known fact that every $f \in L_p(\boldsymbol{T})$ $(p \geqslant 1)$ has a Fourier series converging in $L_p(\boldsymbol{T})$ if and only if the operators S_n are uniformly bounded.

Now assume that $\sup_n ||S_n|| < \infty$ and consider the operators P_n which maps $\sum_{-\infty}^{\infty} c_k e^{ik\theta} \in X$ to $\sum_0^n c_k e^{ik\theta}$. The identity

$$P_{2n} f(e^{i\theta}) = e^{in\theta} S_n(e^{-in\theta} f) \tag{17}$$

shows that $\sup_n ||P_n|| < \infty$. Hence for each $f \in L_p(\boldsymbol{T})$, the sequence $P_n f$ converges in the norm and let $\mathscr{P}_+ f$ denote the limit, which is the projection of f as $\sum_{-\infty}^{\infty} c_k e^{ik\theta} \mapsto \sum_0^{\infty} c_k e^{ik\theta}$. In particular,

$||\mathscr{P}_+|| < \infty$. This implies that the complementary projection $\mathscr{P}_- : f(\mathrm{e}^{\mathrm{i}\theta}) \mapsto \sum_{k=-\infty}^{-1} c_k \mathrm{e}^{\mathrm{i}k\theta}$ is also bounded.

Let F denote the Cauchy integral of f defined as

$$F(z) = \frac{1}{2\pi\mathrm{i}} \int_{\gamma_0} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta, \quad \gamma_0 = \mathrm{e}^{\mathrm{i}\theta}\ (0 \leqslant \theta \leqslant 2\pi). \tag{18}$$

On the domains seperated by $\boldsymbol{T}$, $F(z)$ is analytic. Observe that the Cauchy integral of $\mathscr{P}_- f$ vanishes on $\boldsymbol{D}$. (This follows from the boundedness of $||\mathscr{P}_-||$ and the denseness of the trigonometric polynomials in $L_p(\boldsymbol{T})$ $(1 \leqslant p < \infty)$.) Thus, F equals to the Cauchy integral of $\mathscr{P}_+ f$ on $\boldsymbol{D}$. This implies that $F(z)$ converges to $\mathscr{P}_+ f(\mathrm{e}^{\mathrm{i}\theta})$ for almost every $\mathrm{e}^{\mathrm{i}\theta} \in \boldsymbol{T}$ as $z \to \mathrm{e}^{\mathrm{i}\theta}$ nontangentially in $\boldsymbol{D}$. Hence in (7) letting $z \to \mathrm{e}^{\mathrm{i}\theta}$ nontangentially in $\boldsymbol{D}$, we get almost everywhere on $\boldsymbol{T}$

$$\sum_{k=0}^{n} \langle f, B_k \rangle B_k(\mathrm{e}^{\mathrm{i}\theta}) = \mathscr{P}_+ f(\mathrm{e}^{\mathrm{i}\theta}) - \phi_{n+1}(\mathrm{e}^{\mathrm{i}\theta}) \mathscr{P}_+ \left[\frac{f}{\phi_{n+1}}\right](\mathrm{e}^{\mathrm{i}\theta}). \tag{19}$$

Next consider the Cauchy integral (18) on $\boldsymbol{A}(1,\infty)$, the complement of the closed unit disk. The conjugation and the change of variables $\zeta = \mathrm{e}^{\mathrm{i}t}$ yield

$$\overline{F(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\overline{f(\mathrm{e}^{\mathrm{i}t})}}{1 - \bar{z}\,\mathrm{e}^{\mathrm{i}t}}\,\mathrm{d}t. \tag{20}$$

In (20) substituting $\bar{w} = 1/z$ and changing the variables as $\mathrm{e}^{\mathrm{i}t} = \zeta$ we obtain

$$-\frac{1}{\omega} \overline{F\left(\frac{1}{\bar{w}}\right)} = \frac{1}{2\pi\mathrm{i}} \int_{\gamma_0} \frac{\overline{f(\zeta)}}{\zeta(\zeta - w)}\,\mathrm{d}\zeta, \quad w \in \boldsymbol{D}.$$

This is recognized as the previously considered situation where $f(\mathrm{e}^{\mathrm{i}\theta})$ is replaced by $\mathrm{e}^{-\mathrm{i}\theta}\,\overline{f(\mathrm{e}^{\mathrm{i}\theta})}$. Consequently for almost every $\mathrm{e}^{\mathrm{i}\theta} \in \boldsymbol{T}$, as $w \to \mathrm{e}^{\mathrm{i}\theta}$ nontangentially in $\boldsymbol{D}$

$$-\frac{1}{\omega} \overline{F\left(\frac{1}{\bar{w}}\right)} \to \mathscr{P}_+ \left[\frac{\overline{f(\mathrm{e}^{\mathrm{i}\theta})}}{\mathrm{e}^{\mathrm{i}\theta}}\right]$$

which implies $F(z) \to -\mathscr{P}_- f(\mathrm{e}^{\mathrm{i}\theta})$ as $z \to \mathrm{e}^{\mathrm{i}\theta}$ nontangentially in $\boldsymbol{A}(1,\infty)$. Thus in (8) letting $z \to \mathrm{e}^{\mathrm{i}\theta}$ nontangentially in $\boldsymbol{A}(1,\infty)$ we get almost everywhere on $\boldsymbol{T}$

$$\sum_{k=-n}^{-1} \langle f, B_k \rangle B_k(\mathrm{e}^{\mathrm{i}\theta}) = \mathscr{P}_- f(\mathrm{e}^{\mathrm{i}\theta}) - \phi_n'(\mathrm{e}^{\mathrm{i}\theta}) \mathscr{P}_- \left[\frac{f}{\phi_n'}\right](\mathrm{e}^{\mathrm{i}\theta}). \tag{21}$$

Hence from (19) and (21)

$$f - \mathscr{S}_n f = \phi_{n+1} \mathscr{P}_+(f/\phi_{n+1}) + \phi_n' \mathscr{P}_-(f/\phi_n') \quad \text{a.e. } \boldsymbol{T} \tag{22}$$

Let $\tilde{f}(\mathrm{e}^{\mathrm{i}\theta})$ denote the conjugate of $f(\mathrm{e}^{\mathrm{i}\theta})$. Recall that f and $\tilde{f}$ are recovered almost everywhere on $\boldsymbol{T}$ by taking nontangential limits of $u(z)$ and $\tilde{u}(z)$ defined by

$$(u + \mathrm{i}\tilde{u})(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z} f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t \tag{23}$$

as $z \to \mathrm{e}^{\mathrm{i}\theta}$. Let $\mathscr{F}$ denote the map $f \mapsto f + \mathrm{i}\tilde{f}$. Noting that $c_0 = \langle f, 1 \rangle$, the operators $\mathscr{P}_+$ and $\mathscr{P}_-$ can be written as

$$\mathscr{P}_+ f = \tfrac{1}{2}(\mathscr{F} f + \langle f, 1 \rangle), \tag{24}$$

$$\mathscr{P}_- f = \overline{\mathscr{P}_+(\bar{f})} - c_0 = \tfrac{1}{2}(\overline{\mathscr{F}(\bar{f})} - \langle f, 1 \rangle).$$

Thus from (22) and the equalities $\langle f/\phi_k, 1 \rangle = \langle f, \phi_k \rangle$ for all k

$$f - \mathscr{S}_n f = \frac{\phi_{n+1}}{2} \mathscr{F}\left(\frac{f}{\phi_{n+1}}\right) + \frac{\phi_n'}{2} \overline{\mathscr{F}\left(\frac{\bar{f}}{\bar{\phi}_n'}\right)} + \frac{\phi_{n+1}}{2} \langle f, \phi_{n+1} \rangle - \frac{\phi_n'}{2} \langle f, \phi_n' \rangle.$$

Hence

$$||f - \mathscr{S}_n f||_p \leqslant (1 + ||\mathscr{F}||)\,||f||_p, \quad f \in L_p(\boldsymbol{T}). \tag{25}$$

We started with the assumption $\sup_n ||S_n|| < \infty$ and concluded via to the boundedness of $\mathscr{P}_+$ that $||\mathscr{F}|| < \infty$. The converse is also true by the equalities (24) and (17).

Let X_n denote the linear space spanned by the sets $\{B_k(\mathrm{e}^{\mathrm{i}t})\}_{k=-n}^{n}$ and define

$$e_n(f; L_p(\boldsymbol{T})) = \min_{g \in X_n} ||g - f||_p, \quad f \in L_p(\boldsymbol{T}). \tag{26}$$

Thus $e_n(f; L_p(\boldsymbol{T}))$ is the best approximation error of $f \in L_p(\boldsymbol{T})$ by functions in X_n. Since (1)–(2) is closed in $L_p(\boldsymbol{T})$ $(1 \leqslant p < \infty)$ and $C(\boldsymbol{T})$, the quantity $e_n(f; L_p(\boldsymbol{T}))$ defined by (25) monotonically tends to zero as $n \to \infty$.

Let f be a given function in $L_p(\boldsymbol{T})$ and let g be the minimizing solution in (26). Let $\psi = f - g$ denote the approximation error. Observe that $\mathscr{S}_n g = g$ since $g \in X_n$. Due to the linearity of $\mathscr{S}_n$ notice also that $\mathscr{S}_n \psi = \mathscr{S}_n f - \mathscr{S}_n g$. Thus from (25)

$$\begin{aligned} ||f - \mathscr{S}_n f||_p &= ||\psi - \mathscr{S}_n \psi||_p \\ &\leqslant (1 + ||\mathscr{F}||)\, e_n(f; L_p(\boldsymbol{T})). \end{aligned} \tag{27}$$

The error bound in (27) expressed in terms of $||\mathscr{F}||$ is rather tight and without further assumptions on f and the orthonormal system (1)–(2) it does not seem possible to improve upon. In the special case $f \in H_p(\boldsymbol{T})$, the Hardy space of functions g which are analytic on $\boldsymbol{D}$ and such that $g(e^{i\theta}) \in L_p(\boldsymbol{T})$, we have instead of (27)

$$||f - \mathscr{S}_n f||_p \leqslant \tfrac{1}{2}(1 + ||\mathscr{F}||)\, e_n(f; H_p(\boldsymbol{T})) \tag{28}$$

where $\mathscr{S}_n f = \sum_{k=0}^{n} \langle f, B_k \rangle B_k$.

We need the following lemma to compute an upper bound for $||\mathscr{F}||$.

Lemma 5. *Let $f = f_{\mathrm{R}} + i f_{\mathrm{I}}$ where f_{R} and f_{I} are real-valued functions. Then*

$$||f_{\mathrm{R}}||_p + ||f_{\mathrm{I}}||_p \leqslant B_p\, ||f||_p, \tag{29}$$

where

$$B_p = \begin{cases} 2^{1/2}, & 1 \leqslant p \leqslant 2, \\ 2^{(p-1)/p}, & p \geqslant 2. \end{cases} \tag{30}$$

Proof. Note the following inequalities whose proofs can be found for example in [6, Section 4.2]:

$$2^{p-1}(a^p + b^p) \leqslant (a+b)^p \leqslant a^p + b^p, \quad 0 < p \leqslant 1,$$

$$a^p + b^p \leqslant (a+b)^p \leqslant 2^{p-1}(a^p + b^p), \quad p \geqslant 1,$$

where a and b are two arbitrary nonnegative numbers. Put $a = ||f_{\mathrm{R}}||_p$ and $b = ||f_{\mathrm{I}}||_p$ in the above inequalities. Then for $1 \leqslant p \leqslant 2$

$$\begin{aligned} (||f_{\mathrm{R}}||_p + ||f_{\mathrm{I}}||_p)^p &\leqslant 2^{p-1} \int ([f_{\mathrm{R}}^2]^{p/2} + [f_{\mathrm{I}}^2]^{p/2}) \\ &\qquad (p \geqslant 1) \\ &\leqslant 2^{p-1} 2^{1-p/2} \int (f_{\mathrm{R}}^2 + f_{\mathrm{I}}^2)^{p/2} \quad (p/2 \leqslant 1) \\ &= 2^{p/2}\, ||f||_p^p \end{aligned}$$

while for $p \geqslant 2$

$$\begin{aligned} (||f_{\mathrm{R}}||_p + ||f_{\mathrm{I}}||_p)^p &\leqslant 2^{p-1} \int ([f_{\mathrm{R}}^2]^{p/2} + [f_{\mathrm{I}}^2]^{p/2}) \\ &\qquad (p \geqslant 1) \\ &\leqslant 2^{p-1} \int (f_{\mathrm{R}}^2 + f_{\mathrm{I}}^2)^{p/2} \quad (p/2 \geqslant 1) \\ &= 2^{p-1}\, ||f||_p^p. \quad \square \end{aligned}$$

When p equals to 1 or 2, the top equality in (29) is attained for complex-valued functions in the form $f = (1+i) f_{\mathrm{R}}$. Observe that when $p = \infty$, the bottom equality is attained by complex-valued functions with real and imaginary parts disjointly supported on $\boldsymbol{T}$.

If $1 < p < \infty$ and f is real-valued function, it is known [7] that

$$||\mathscr{F} f||_p \leqslant C_p\, ||f||_p, \tag{31}$$

where C_p is the best possible constant given by

$$C_p = \begin{cases} [\cos(\pi/2p)]^{-1}, & 1 < p \leqslant 2, \\ [\sin(\pi/2p)]^{-1}, & 2 < p < \infty. \end{cases} \tag{32}$$

Write f as $f = f_{\mathrm{R}} + i f_{\mathrm{I}}$ where f_{R} and f_{I} are real valued. Then from (31) and (29) due to the linearity of $\mathscr{F}$

$$\begin{aligned} ||\mathscr{F} f||_p &\leqslant ||\mathscr{F} f_{\mathrm{R}}||_p + ||\mathscr{F} f_{\mathrm{I}}||_p \\ &\leqslant C_p (||f_{\mathrm{R}}||_p + ||f_{\mathrm{I}}||_p) \\ &\leqslant C_p B_p\, ||f||_p. \end{aligned} \tag{33}$$

Using (27) and (33), the following result can now be established.

Theorem 6. *Consider the partial sums of the Fourier series defined by* (4). *Let $e_n(f; L_p(\boldsymbol{T}))$, B_p, and C_p be as in* (26), (30), *and* (32). *Then for all $1 < p < \infty$ and $f \in L_p(\boldsymbol{T})$*

$$||f - \mathscr{S}_n f||_p \leqslant (1 + B_p C_p)\, e_n(f; L_p(\boldsymbol{T})) \tag{34}$$

and if the conditions in (3) *are satisfied*

$$\lim_{n \to \infty} ||f - \mathscr{S}_n f||_p = 0.$$

From (28), (33), (30), and (32), observe that $||f - \mathscr{S}_n f||_2 \leqslant (3/2)\, e_n(f; H_2(\boldsymbol{T}))$ while the best value is $e_n(f; H_2(\boldsymbol{T}))$.

The inequality (34) shows that the approximation error of the Fourier series is in the order of the best achievable error for every choice of orthonormal system of functions when the approximated function lies in $L_p(\boldsymbol{T})$ $(1 < p < \infty)$. The choice of orthonormal functions on the other hand depends on the class of functions being approximated. This subject is not investigated here.

In Theorem 6, the spaces $L_1(\boldsymbol{T})$ and $C(\boldsymbol{T})$ can not be included since the projection operator $\mathscr{P}_+$ is not bounded on these spaces.

4. Mean growth of the Fourier coefficients

In this section we will derive two inequalities which are analogous to the Hausdorff–Young inequalities for the trigonometric basis $\{e^{\pm int}\}$.

Theorem 7. *Let* $1 \leqslant p \leqslant 2$ *and let* q *be the conjugate exponent, that is,* $q = p/(p-1)$. *Suppose that the basis defined by* (1)–(2) *is uniformy bounded, i.e.*

$$\sup_n \{|z_n|, |x_n|\} = r < 1. \tag{35}$$

If $f \in L_p(\boldsymbol{T})$ *then*

$$\left(\sum_{n=-\infty}^{\infty} |\langle f, B_n\rangle|^q\right)^{1/q} \leqslant \left(\frac{1+r}{1-r}\right)^{(q-2)/2q} ||f||_p. \tag{36}$$

If $\{a_n\} \in \ell_p$ *then there exists a function* $f \in L_q(\boldsymbol{T})$ *such that* $a_n = \langle f, B_n\rangle$. *Moreover,*

$$||f||_q \leqslant \left(\frac{1+r}{1-r}\right)^{(q-2)/2q} \left(\sum_{n=-\infty}^{\infty} |a_n|^p\right)^{1/p}. \tag{37}$$

Proof. The mapping $\mathscr{S} : f \mapsto \{\langle f, B_n\rangle\}$ is a transformation of functions on the measure space $(\boldsymbol{T}, dt)$ into functions on (Z, dn), Z being the group of integers and dn the so-called counting measure. The norm of the mapping as $L_1(\boldsymbol{T}) \mapsto \ell_\infty$ is

$$||\mathscr{S}||_{L_1,\ell_\infty} = \sup_{||f||_1 \leqslant 1} ||\langle f, B_n\rangle||_\infty = \sqrt{\frac{1+r}{1-r}}.$$

The mapping $\mathscr{S}$ is an isometry of $L_2(\boldsymbol{T})$ onto ℓ_2. Hence $||\mathscr{S}||_{L_2,\ell_2} = 1$. Then by the Riesz–Thorin interpolation theorem [16, Theorem IX.17] the mapping $\mathscr{S}$ from $L_p(\boldsymbol{T})$ into ℓ_q is bounded as

$$||\mathscr{S}||_{L_p,\ell_q} \leqslant ||\mathscr{S}||^{(q-2)/q}_{L_1,\ell_\infty} ||\mathscr{S}||^{2/q}_{L_2,\ell_2} = \left(\frac{1+r}{1-r}\right)^{(q-2)/2q}.$$

This proves (36). The proof of (37) is again by interpolation. For this consider the mapping $\mathscr{T} : \{a_n\} \mapsto f(t) = \sum a_n B_n(e^{it})$. If $\{a_n\} \in \ell_1$ then $f(t) = \sum a_n B_n(e^{it}) \in C(\boldsymbol{T})$ and $\langle f, B_n\rangle = a_n$. Moreover

$$||\mathscr{T}||_{\ell_1,L_\infty} = \sup_{||a||_1 \leqslant 1} ||f||_\infty = \sqrt{\frac{1+r}{1-r}}.$$

The equality $||T||_{\ell_2,L_2} = 1$ is obvious. Thus (37) follows from

$$||\mathscr{T}||_{\ell_p,L_q} \leqslant ||\mathscr{T}||^{(q-2)/q}_{\ell_1,L_\infty} ||\mathscr{T}||^{2/q}_{\ell_2,L_2} = \left(\frac{1+r}{1-r}\right)^{(q-2)/2q}.$$

□

Theorem 7 cannot be extended to the case $p > 2$. For example with the trigonometric basis $z_n = x_n = 0$ for all n, there exist continuous functions f such that

$$\sum_{k=-\infty}^{\infty} |\langle f, e^{ik\theta}\rangle|^{2-\varepsilon} = \infty \quad \text{for all } \varepsilon > 0.$$

The uniformly bounded basis assumption can be relaxed if $f(e^{i\theta})$ extends to a function that is analytic on a region which contains $\boldsymbol{T}$.

In the next result, we restrict the attention to $H_p(\boldsymbol{D})$.

Corollary 8. *Let* $1 \leqslant p \leqslant 2$. *Suppose that* $\sup_n |z_n| = r < 1$. *Then*

$$\left(\sum_{n=0}^{\infty} |\langle f, B_n\rangle|^q\right)^{1/q} \leqslant \left(\frac{1+r}{1-r}\right)^{(q-2)/2q} ||f||_p, \quad f \in H_p(\boldsymbol{D}). \tag{38}$$

If $c = \{c_0, c_1, \ldots\} \in \ell_p$, *then there exists a function* $f \in H_q(\boldsymbol{D})$ *such that* $c_n = \langle f, B_n\rangle$. *Moreover,*

$$||f||_q \leqslant \left(\frac{1+r}{1-r}\right)^{(q-2)/2q} ||c||_p. \tag{39}$$

Proof. Let $f \in H_p(\boldsymbol{D})$. Then $f(e^{i\theta}) \in L_p(\boldsymbol{T})$. Notice that $\langle f, B_n\rangle = 0$ for all $n < 0$ since $\{B_n\}_{n\geqslant 0}$ is a basis for $H_p(\boldsymbol{D})$. Thus (38) follows from (36) in Theorem 7. Conversely, if $c \in \ell_p$ $(1 \leqslant p \leqslant 2)$, then $c \in \ell_2$ and $\sum_{k=0}^n c_k B_k$ converges to some $f \in H_2(\boldsymbol{T})$. The numbers c_n are the Fourier coefficients of $f(e^{i\theta})$. The inequality (37) in Theorem 7 tells us that $f(e^{i\theta}) \in L_q(\boldsymbol{T})$, which implies $f \in H_q(\boldsymbol{D})$. □

5. Modelling of physical systems

Up to now, we have not imposed any restriction on pole location save for the conditions in (3). However, in any appplication involving the modelling of a physical system, it is necessary to ensure that the underlying modelled impulse response is real valued. A requirement is that the sets $\{z_0, z_1, \ldots, z_n\}$ and $\{x_1, x_2, \ldots, x_n\}$ used to define basis via (1)–(2) always contain complex conjugates. Then the constraint of realness of impulse response is easily accommodated by taking suitable linear combinations of the basis functions (1)–(2). The idea in the following basis construction is taken from [14].

Suppose that $z_0, \ldots, z_{n-1}$ are real so that the basis functions $B_0, \ldots, B_{n-1}$ have real-valued impulse responses. Now we wish to include a complex pole at

$1/\overline{z_n}$. Then two new basis functions $\tilde{B}_n, \tilde{B}_{n+1}$ with real impulse responses should be formed as a linear combination of B_n and B_{n+1} generated by (1) with $z_{n+1} = \overline{z_n}$. These new functions then replace B_n and B_{n+1}. The suggested linear combination can be expressed as

$$\begin{bmatrix} \tilde{B}_n \\ \tilde{B}_{n+1} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \begin{bmatrix} B_n \\ B_{n+1} \end{bmatrix}. \tag{40}$$

Considering only $\tilde{B}_n$ for the moment given by

$$\tilde{B}_n(z) = \frac{\sqrt{1-|z_n|^2}(\beta z + \mu)}{1-(z_n+\overline{z_n})z + |z_n|^2 z^2}\phi_n(z),$$

where $\phi_n(z)$ has real-valued impulse response and the real coefficients β, μ are related to the choice of c_1, c_2 by

$$c_1 = \frac{\mu + \beta z_n}{1-z_n^2}, \qquad c_2 = \frac{\mu z_n + \beta}{1-z_n^2},$$

to ensure a unit norm for $\tilde{B}_n$, β and μ must be chosen according to the constraint that $|c_1|^2+|c_2|^2=1$ which becomes

$$x^{\mathrm{T}}Mx = |1-z_n^2|^2, \tag{41}$$

where

$$x = (\beta,\mu)^{\mathrm{T}}, \qquad M = \begin{bmatrix} 1+|z_n|^2 & z_n+\overline{z_n} \\ z_n+\overline{z_n} & 1+|z_n|^2 \end{bmatrix}.$$

Now, suppose we make two pairs of choices: $x = (\beta,\mu)^{\mathrm{T}}$ giving a basis function $\tilde{B}_n$ and $y = (\beta',\mu')^{\mathrm{T}}$ giving another basis function $\tilde{B}_{n+1}$. These two choices correspond to two pairs of complex numbers $\{c_1, c_2\}$ and $\{c_3, c_4\}$. The requirement $c_1\overline{c_3}+c_2\overline{c_4}=0$ ensuring orthogonality of $\tilde{B}_n$ and $\tilde{B}_{n+1}$ can be expressed as

$$x^{\mathrm{T}}My = 0. \tag{42}$$

All solutions to (41) are given by

$$x = \frac{1}{\sqrt{2}}\begin{bmatrix} |1-z_n|\cos\theta + |1+z_n|\sin\theta \\ |1-z_n|\cos\theta - |1+z_n|\sin\theta \end{bmatrix}, \quad 0 \leqslant \theta < 2\pi.$$

Then for a fixed θ, a unique y that satisfies (41) and (42) is found by substituting $\theta + \pi/2$ above:

$$y = -\frac{1}{\sqrt{2}}\begin{bmatrix} |1-z_n|\sin\theta - |1+z_n|\cos\theta \\ |1-z_n|\sin\theta + |1+z_n|\cos\theta \end{bmatrix}.$$

Let $\theta = 0$. Then the basis functions $\tilde{B}_n$ and $\tilde{B}_{n+1}$ are found as

$$\tilde{B}_n(z) = \frac{2^{-1/2}(1-|z_n|^2)^{1/2}|1-z_n|(z+1)}{1-(z_n+\overline{z_n})z+|z_n|^2z^2}\phi_n(z),$$

$$\tilde{B}_{n+1}(z) = \frac{2^{-1/2}(1-|z_n|^2)^{1/2}|1+z_n|(z-1)}{1-(z_n+\overline{z_n})z+|z_n|^2z^2}\phi_n(z).$$

These real-valued impulse response basis vectors $\tilde{B}_n$ and $\tilde{B}_{n+1}$ are then used for modelling instead of B_n and B_{n+1}. If we require further basis functions with complex modes then we repeat the process in (40) by forming $\tilde{B}_{n+2}$ and $\tilde{B}_{n+3}$ from linear combinations of B_{n+2} and B_{n+3} and so on, and in this way arbitrary complex pole configurations may be accommodated.

For example, when $z_n = \overline{z_{n+1}} = \cdots = z_{n+2m} = \overline{z_{n+2m+1}}$, the above basis construction process yields for $j = 0, \ldots, m$,

$$\tilde{B}_{n+2j}(z) = \frac{a(z+1)}{1-bz+cz^2}\left(\frac{z^2-bz+c}{1-bz+cz^2}\right)^j \phi_n(z), \tag{43}$$

$$\tilde{B}_{n+2j+1}(z) = \frac{a_1(z-1)}{1-bz+cz^2}\left(\frac{z^2-bz+c}{1-bz+cz^2}\right)^j \phi_n(z) \tag{44}$$

where $b = z_n + \overline{z_n}$, $c = |z_n|^2$, and

$$a = \sqrt{\frac{(1-c)(1-b+c)}{2}},$$

$$a_1 = \sqrt{\frac{(1-c)(1+b+c)}{2}}. \tag{45}$$

With $n = 0$, this is the defining formula for the two-parameter Kautz functions.

The old basis functions B_n and B_{n+1} can be written in terms of the new basis functions as

$$\begin{bmatrix} B_n \\ B_{n+1} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \dfrac{1-z_n}{|1-z_n|} & -\dfrac{1+z_n}{|1+z_n|} \\ \dfrac{1-z_n}{|1-z_n|} & \dfrac{1+z_n}{|1+z_n|} \end{bmatrix}\begin{bmatrix} \tilde{B}_n \\ \tilde{B}_{n+1} \end{bmatrix}.$$

From this, we derive the following identity:

$$\langle f, B_n\rangle B_n + \langle f, B_{n+1}\rangle B_{n+1} = \langle f, \tilde{B}_n\rangle \tilde{B}_n + \langle f, \tilde{B}_{n+1}\rangle \tilde{B}_{n+1}. \tag{46}$$

Having shown how to construct new basis functions with real-valued impulse responses from the basis functions B_n, $n = 0, 1, \ldots,$ we will next study the same problem for the basis functions in (2). For the new basis functions $\tilde{B}_{-n}$ and $\tilde{B}_{-n-1}$, we seek a linear transformation of the old basis functions B_{-n} and B_{-n-1} expressed as

$$\begin{bmatrix} \tilde{B}_{-n}(z) \\ \tilde{B}_{-n-1}(z) \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}\begin{bmatrix} B_{-n}(z) \\ B_{-n-1}(z) \end{bmatrix}. \tag{47}$$

The substitutions $\overline{x_j} \mapsto z_{j-1}, \forall j$ and $z \mapsto z^{-1}$ transform this problem to the previously considered case.

Thus when $x_n = \overline{x_{n+1}} = \cdots = x_{n+2m} = \overline{x_{n+2m+1}}$, we have for $j = 0, \ldots, m$,

$$\tilde{B}_{-n-2j}(z) = \frac{a'(1+z)}{z^2 - b'z + c'} \left(\frac{1 - b'z + c'z^2}{z^2 - b'z + c'} \right)^j \phi_n'(z),$$

$$\tilde{B}_{-n-2j-1}(z) = \frac{a_1'(1-z)}{z^2 - b'z + c'} \left(\frac{1 - b'z + c'z^2}{z^2 - b'z + c'} \right)^j \phi_n'(z),$$

where $b' = x_n + \overline{x_n}$, $c' = |x_n|^2$ and a', a_1' are computed from the formulae in (45) with b' and c'. Furthermore

$$\begin{bmatrix} B_{-n} \\ B_{-n-1} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \dfrac{1 - \overline{x_n}}{|1 - x_n|} & -\dfrac{1 + \overline{x_n}}{|1 + x_n|} \\ \dfrac{1 - \overline{x_n}}{|1 - x_n|} & \dfrac{1 + \overline{x_n}}{|1 + x_n|} \end{bmatrix} \begin{bmatrix} \tilde{B}_{-n} \\ \tilde{B}_{-n-1} \end{bmatrix}$$

which leads to

$$\begin{aligned} &\langle f, B_{-n} \rangle B_{-n} + \langle f, B_{-n-1} \rangle B_{-n-1} \\ &\quad = \langle f, \tilde{B}_{-n} \rangle \tilde{B}_{-n} + \langle f, \tilde{B}_{-n-1} \rangle \tilde{B}_{-n-1}. \end{aligned} \tag{48}$$

The unitary equivalance of the bases $\{B_k\}$ and $\{\tilde{B}_k\}$ shows that the latter is complete in $L_p(\boldsymbol{T})$ $(1 < p < \infty)$ and $C(\boldsymbol{T})$ if the conditions in (3) hold. Moreover from (46) and (48),

$$\mathscr{S}_n f = \sum_{k=-n}^{n} \langle f, \tilde{B}_k \rangle \tilde{B}_k = \tilde{\mathscr{S}}_n f$$

whenever the sequence $\{z_0, z_1, x_1, \ldots, z_n, x_n\}$ contains complex conjugates as well. In this case, if f has a real-valued impulse response, then both $\mathscr{S}_n f$ and $\tilde{\mathscr{S}}_n f$ will have real-valued impulse responses. This identity shows also that approximation properties of $\mathscr{S}_n f$ and $\tilde{\mathscr{S}}_n f$ are identical.

6. Example

In this section, we use a simulation example to illustrate the use of the basis functions defined by (1) for modelling. We consider the identification of a fifth order system with poles (in the usual stability notion) 0.95 ± 0.20i, 0.85 ± 0.10i, 0.55 and zeros 0.96 ± 0.28i, 0.96 ± 0.17i. The transfer function of the system is normalized so that its H_∞ norm satisfies $||G||_\infty = 1$. This system was studied in [20] to illustrate the use of the generalized orthonormal basis functions for the time-domain identification.

We assume $N = 500$ frequency response measurements

$$E_k = G(e^{i\omega_k}) + \eta_k, \quad k = 1, \ldots, N \tag{49}$$

are available where ω_k are equally spaced on the interval $[0, 3]$ and the disturbances η_k are bounded random variables as

$$\eta_k = 0.1 e^{i\alpha_k},$$

where α_k are independent and uniformly distributed random variables in the interval $[0, 2\pi]$. Note that by this choice of frequencies, frequency response are not on a uniform grid of frequencies.

The basis functions in (1) were chosen with $z_0 = 0$ and

$$z_k = \begin{cases} 0.2, & k \text{ odd}, \\ 0.9, & k \text{ even}. \end{cases}$$

This simple choice represents both slow and fast dynamics in the model structure via to the Laguerre functions. We will estimate G from the data (49) by two algorithms.

In the first algorithm, a high-order model is computed from the data (49) by the simple least-squares method as

$$\tilde{G}_N(z) = \sum_{k=0}^{100} [\Phi^\dagger E]_k B_k(z), \tag{50}$$

where $\Phi^\dagger$ is the Moore–Penrose pseudoinverse of Φ defined by

$$\Phi^\dagger = (\Phi^* \Phi)^{-1} \Phi^*$$

and

$$\Phi(\omega) = \begin{bmatrix} 1 & \cdots & B_{100}(e^{i\omega_1}) \\ \vdots & \ddots & \vdots \\ 1 & \cdots & B_{100}(e^{i\omega_N}) \end{bmatrix}. \tag{51}$$

The estimated linear-in parameters model was reduced to a fifth-order final model by using the subspace-based identification algorithm in [12] for model reduction purpose. The input to the algorithm in [12] were 2048 equally spaced frequency response data on $[0, 2\pi]$. Note that this amounts to evaluating Φ on a uniform grid of 2048 frequencies for which fast algorithms are known to exist. The size of the Hankel matrix in the subspace algorithm was chosen 128 by 128. The returned models by this algorithm are almost balanced and they converge to balanced truncations of the approximated system as the number of the supplied data tends to infinity. The step prior to forming a Hankel matrix was 2048-point inverse fast Fourier transform.

In Fig. 1, the magnitudes of E, $\tilde{G}_N(e^{i\omega})$, the final model transfer function denoted by $\hat{G}_N(e^{i\omega})$, and the measured errors $\tilde{G}_N(e^{i\omega}) - E$, $\hat{G}_N(e^{i\omega}) - E$ are plotted.

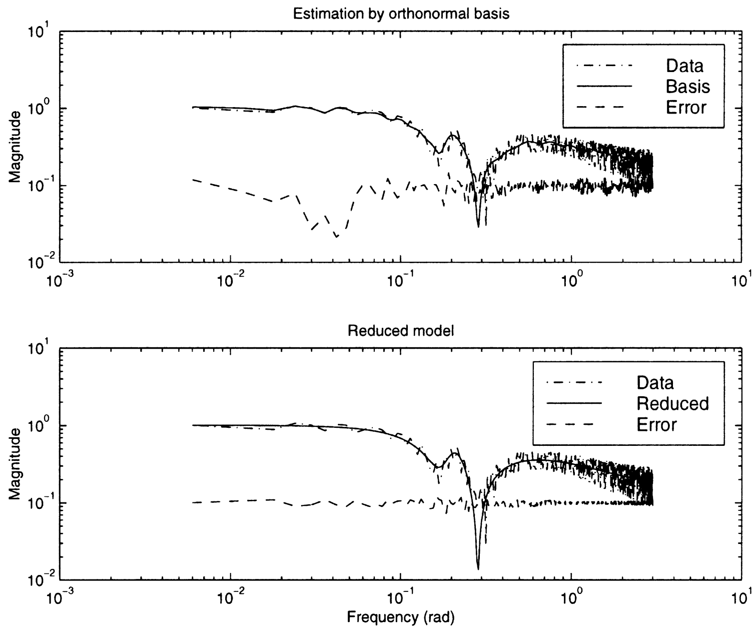

Fig. 1. The magnitude plots of E, $\tilde{G}_N(e^{i\omega})$, $\tilde{G}_N(e^{i\omega}) - E$ (on the top) and E, $\hat{G}_N(e^{i\omega})$, $\hat{G}_N(e^{i\omega}) - E$ (on the bottom) using the linear estimate in (50).

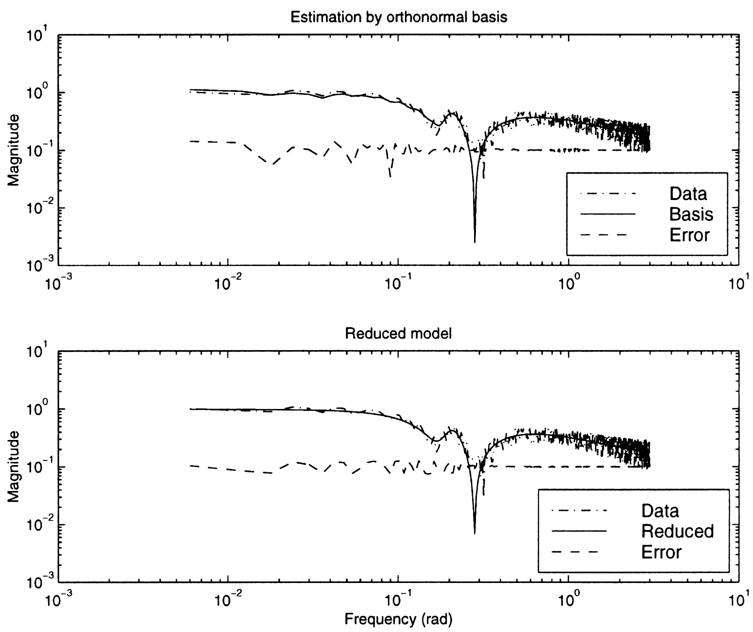

Fig. 2. The magnitude plots of E, $\tilde{G}_N(e^{i\omega})$, $\tilde{G}_N(e^{i\omega}) - E$ (on the top) and E, $\hat{G}_N(e^{i\omega})$, $\hat{G}_N(e^{i\omega}) - E$ (on the bottom) using the min–max estimate in (52).

The poles of $\hat{G}_N$ are $0.95 \pm 0.19i$, $0.85 \pm 0.11i$, 0.54 and the four significant zeros are $0.97 \pm 0.17i$, $0.96 \pm 0.28i$. They all agree well with the system poles and zeros.

Next we will compare this algorithm with the minimax algorithm in [2]. In the minimax algorithm, the coefficient vector $\hat{\Lambda} \in \boldsymbol{R}^{101}$ in the linearly parameterized model

$$\hat{G}_N(z) = \sum_{k=0}^{101} \hat{\Lambda}_k B_k(z) \tag{52}$$

is obtained by solving the following min–max problem:

$$\hat{\Lambda} = \arg \min_{\Lambda \in \boldsymbol{R}^{101}} \left\| \begin{bmatrix} \Phi_\mathrm{R} \\ \Phi_\mathrm{I} \end{bmatrix} \Lambda - \begin{bmatrix} E_\mathrm{R} \\ E_\mathrm{I} \end{bmatrix} \right\|_\infty, \tag{53}$$

where E_R and E_I are respectively the real and imaginary parts of E in (49) and Φ_R and Φ_I are the real and imaginary parts of Φ.

The min–max solution in (53) is obtained from the following linear programming problem:

$$\min_{\mu} [O \ 1] \begin{bmatrix} \Lambda \\ \mu \end{bmatrix}$$

subject to

$$\begin{bmatrix} \Phi_\mathrm{R} - J \\ \Phi_\mathrm{I} - J \\ -\Phi_\mathrm{R} - J \\ -\Phi_\mathrm{I} - J \end{bmatrix} \begin{bmatrix} \Lambda \\ \mu \end{bmatrix} \leqslant \begin{bmatrix} E_\mathrm{R} \\ E_\mathrm{I} \\ -E_\mathrm{R} \\ -E_\mathrm{I} \end{bmatrix}$$

where $O \in \boldsymbol{R}^{1\times 101}$ and $J \in \boldsymbol{R}^{N\times 1}$ are respectively row and column vectors of zeros and ones. This program is implemented by the ***lp*** command in the MATLAB's Optimization Toolbox.

In Fig. 2, the simulation results are plotted for the minimax algorithm. We followed the same model reduction procedure as in the previous algorithm. The poles of the final model are $0.95 \pm 0.20i$, $0.87 \pm 0.10i$, 0.54 and the four significant zeros are $0.96 \pm 0.17i$, $0.96 \pm 0.28i$. They are in very good agreement with the system poles and zeros. This increase in accuracy was offset by the fact that computing (52) took about two orders of magnitude more time than needed to compute (50).

7. Conclusions

In this paper completeness and approximation properties of a general class of fixed pole rational orthonormal basis functions in the $L_p(\boldsymbol{T})$ $(1 < p < \infty)$ and $C(\boldsymbol{T})$ spaces were studied and a fairly complete analysis of the convergence properties of the Fourier series formed by the orthonormal basis functions was carried out for the case $(1 < p < \infty)$.

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